

Pensieve header: An isolated attempt to compute the multiplication tensor of QU.

The implementation of QU is modified from pensieve://Projects/SL2Portfolio/SL2PortfolioProgram.nb.

Initialization / Utilities

```
In[*]:=
$k = 1; $U = QU; $E := {$k};
$trim := {e^{k-} /; k > $k -> 0};
SetAttributes[{SS, SST}, HoldAll];
q_{h-} := e^{\gamma \epsilon \hbar};
(* Upper to lower and lower to Upper: *)
U21 = {B_{i-}^{p-} -> e^{-p \hbar \gamma} b_i, B_{i-}^{p-} -> e^{-p \hbar \gamma} b, T_{i-}^{p-} -> e^{p \hbar} t_i, T_{i-}^{p-} -> e^{p \hbar} t, \mathcal{A}_{i-}^{p-} -> e^{p \gamma \alpha_i}, \mathcal{A}_{i-}^{p-} -> e^{p \gamma \alpha}};
12U = {e_{-}^{c-} b_{i-} + d_{-} -> B_{i-}^{-c / (\hbar \gamma)} e^d, e_{-}^{c-} b + d_{-} -> B^{-c / (\hbar \gamma)} e^d,
e_{-}^{c-} t_{i-} + d_{-} -> T_{i-}^{c / \hbar} e^d, e_{-}^{c-} t + d_{-} -> T^{c / \hbar} e^d,
e_{-}^{c-} \alpha_{i-} + d_{-} -> \mathcal{A}_{i-}^{c / \gamma} e^d, e_{-}^{c-} \alpha + d_{-} -> \mathcal{A}^{c / \gamma} e^d,
e^{\mathcal{E}-} -> e^{\text{Expand}@\mathcal{E}}};
SS[\mathcal{E}_-, op_] := Collect[Normal@Series[\mathcal{E}, {\epsilon, 0, $k}], \epsilon, op];
SS[\mathcal{E}_-] := SS[\mathcal{E}_-, Together];
SST[\mathcal{E}_-, op___] := SS[\mathcal{E}_- / . U21, op];
Simp[\mathcal{E}_-, op_] := Collect[\mathcal{E}_-, _CU | _QU, op];
Simp[\mathcal{E}_-] := Simp[\mathcal{E}_-, SS[#, Expand] &];
SimpT[\mathcal{E}_-] := Collect[\mathcal{E}_-, _CU | _QU, SST[#, Expand] &];
K\delta /: K\delta_{i-, j-} := If[i == j, 1, 0];
c_Integer_{k_Integer} := c + O[\epsilon]^{k+1};
```

```
In[*]:=
CF[\mathcal{E}_-] := ExpandDenominator@
ExpandNumerator@Together[Expand[\mathcal{E}_-] /. e^{x-} e^{y-} -> e^{x+y} /. e^{x-} -> e^{CF[x]}];
```

```
In[*]:=
Unprotect[SeriesData];
SeriesData /: CF[sd_SeriesData] := MapAt[CF, sd, 3];
SeriesData /: Expand[sd_SeriesData] := MapAt[Expand, sd, 3];
SeriesData /: Simplify[sd_SeriesData] := MapAt[Simplify, sd, 3];
SeriesData /: Together[sd_SeriesData] := MapAt[Together, sd, 3];
SeriesData /: Collect[sd_SeriesData, specs___] := MapAt[Collect[#, specs] &, sd, 3];
Protect[SeriesData];
```

Self-Pair (SP):

```
In[*]:=
SP_{\{}}[P_-] := P; SP_{\{\mathcal{E}_- \rightarrow x_-, ps_\_\_\}}[P_-] := Expand[P // SP_{\{ps_\}}] /. f_{-} \cdot \zeta^{d-} -> \partial_{\{x, d\}} f
```

DeclareAlgebra

```
In[*]:= Unprotect[NonCommutativeMultiply]; Attributes[NonCommutativeMultiply] = {};
(NCM = NonCommutativeMultiply)[x_] := x;
NCM[x_, y_, z_] := (x ⓧ y) ⓧ z;
0 ⓧ _ = _ ⓧ 0 = 0;
(x_Plus) ⓧ y_ := (# ⓧ y) & /@ x; x_ ⓧ (y_Plus) := (x ⓧ #) & /@ y;
B[x_, x_] = 0; B[x_, y_] := x ⓧ y - y ⓧ x;
B[x_, y_, e_] := B[x, y, e] = B[x, y];
```

```
In[*]:= DeclareAlgebra[U_Symbol, opts__Rule] := Module[
{gp, sr, g, cp, M, pow, k = 0,
gs = Generators /. {opts},
cs = Centrals /. {opts} /. Centrals -> {}
},
U[Generators] = gs;
(#_ = U@#) & /@ gs;
gp = Alternatives @@ gs; gp = gp | gp_ (* gens *)
sr = Flatten@Table[{g -> ++k, g_i -> {i, k}}, {g, gs}]; (* sorting -> *)
cp = Alternatives @@ cs; (* cents *)
SetAttributes[M, HoldRest]; M[0, _] = 0; M[a_, x_] := a x;
CE[ε_] := Collect[ε, _U, Expand] /. e^x_ -> e^Expand[x] /. $trim;
U_i_[ε_] := ε /. {t : cp -> t_i, u_U -> (#_ &) /@ u};
U_i_[NCM[]] = pow[ε_, 0] = U@{} = 1_U = U[];
B[U@(x_)_i_, U@(y_)_i_] := U_i_@B[U@x, U@y];
B[U@(x_)_i_, U@(y_)_j_] /; i != j := 0;
B[U@y_, U@x_] := CE[-B[U@x, U@y]];
x_ ⓧ (c_. 1_U) := CE[c x]; (c_. 1_U) ⓧ x_ := CE[c x];
(a_. U[xx_, x_]) ⓧ (b_. U[y_, yy_]) := If[OrderedQ[{x, y} /. sr],
CE@M[a b /. $trim, U[xx, x, y, yy]],
U@xx ⓧ CE@M[a b /. $trim, U@y ⓧ U@x + B[U@x, U@y, $E]] ⓧ U@yy];
U@{c_. * (l : gp)^n_, r___} /; FreeQ[c, gp] := CE[c U@Table[l, {n}] ⓧ U@{r}];
U@{c_. * l : gp, r___} := CE[c U[l] ⓧ U@{r}];
U@{c_, r___} /; FreeQ[c, gp] := CE[c U@{r}];
U@{L_Plus, r___} := CE[U@{#, r} & /@ L];
U@{L_, r___} := U@{Expand[L], r};
U[ε_NonCommutativeMultiply] := U /@ ε;
0_U[specs___, poly_] := Module[{sp, null, vs, us},
sp = Replace[{specs}, L_List -> L_null, {1}];
vs = Join @@ (First /@ sp);
us = Join @@ (sp /. L_s_ -> (l /. x_i_ -> x_s));
CE[Total[
CoefficientRules[poly, vs] /. (p_ -> c_) -> c U@{us^p}
]] /. x_null -> x];
```

```

 $\mathcal{O}_U[\text{specs\_}, \mathbb{E}[L\_ , Q\_ , P\_]] := \mathcal{O}_U[\text{specs}, \text{SS@Normal}[P e^{L+Q}]];$ 
 $\text{pow}[\mathcal{E}_ , n\_ ] := \text{pow}[\mathcal{E}_ , n - 1] \otimes \mathcal{E}_;$ 
 $\text{S}_U[\mathcal{E}_ , \text{ss\_Rule}] := \text{CE@Total}[\text{CoefficientRules}[\mathcal{E}_ , \text{First} / @ \{\text{ss}\}] /. \text{ } \\ \text{ } (p\_ \rightarrow c\_ ) \Rightarrow c \text{ NCM} @ \text{MapThread}[\text{pow}, \{\text{Last} / @ \{\text{ss}\}, p\}]]];$ 
 $\sigma_{rs\_} [c\_ . * u\_ U] := (c / . (t : cp)_{j\_} \Rightarrow t_{j / . \{rs\}}) U[\text{List} @ @ (u / . v_{-j\_} \Rightarrow v_{j / . \{rs\}})];$ 
 $m_{j \rightarrow k\_} [c\_ . * u\_ U] := \text{CE}[(c / . (t : cp)_{j\_} \rightarrow t_k) \text{DeleteCases}[u, \_j | k] \otimes \\ U @ @ \text{Cases}[u, w_{-j} \Rightarrow w_k] \otimes U @ @ \text{Cases}[u, \_k]];]$ 
 $U / : c\_ . * u\_ U * v\_ U := \text{CE}[c u \otimes v];$ 
 $S_{i\_} [c\_ . * u\_ U] := \text{CE}[(c / . S_i[U, \text{Centrals}]) \text{DeleteCases}[u, \_i] \otimes \\ U_i[\text{NCM} @ @ \text{Reverse} @ \text{Cases}[u, x_{-i} \Rightarrow S @ U @ x]]];$ 
 $\Delta_{i \rightarrow j\_ , k\_} [c\_ . * u\_ U] := \text{CE}[(c / . \Delta_{i \rightarrow j, k}[U, \text{Centrals}]) \text{DeleteCases}[u, \_i] \otimes \\ (\text{NCM} @ @ \text{Cases}[u, x_{-i} \Rightarrow \sigma_{1 \rightarrow j, 2 \rightarrow k} @ \Delta @ U @ x] / . \text{NCM}[] \rightarrow U[])];]$ 

```

$\text{In}[*]:= \text{S}_{\text{QU}}[(a^2 + a) b, a \rightarrow \text{QU}[y, x], b \rightarrow 7 \text{QU}[]]$

$\text{Out}[*]=$

$(14 - 7 T) \text{QU}[y, x] + 14 T \in \text{QU}[y, a, x] + (7 + 7 \in) \text{QU}[y, y, x, x]$

Meta-Operations

$\text{In}[*]:=$

```

 $\sigma_{rs\_} [\mathcal{E}_{\text{Plus}}] := \sigma_{rs} / @ \mathcal{E};$ 
 $m_{j \rightarrow j\_} = \text{Identity}; m_{j \rightarrow k\_} [0] = 0;$ 
 $m_{j \rightarrow k\_} [\mathcal{E}_{\text{Plus}}] := \text{Simp}[m_{j \rightarrow k} / @ \mathcal{E};]$ 
 $m_{is\_ , i\_ , j \rightarrow k\_} [\mathcal{E}_] := m_{j \rightarrow k} @ m_{is, i \rightarrow j} @ \mathcal{E};$ 
 $S_{i\_} [\mathcal{E}_{\text{Plus}}] := \text{Simp}[S_i / @ \mathcal{E};]$ 
 $\Delta_{is\_} [\mathcal{E}_{\text{Plus}}] := \text{Simp}[\Delta_{is} / @ \mathcal{E};]$ 

```

Implementing $\text{CU} = \mathcal{U}(\text{sl}_2^{\vee \epsilon})$

Verify σ and Δ ! Also Generalize Δ to $\Delta_{i, j_1, j_2, \dots}$.

$\text{In}[*]:=$

```

 $\text{DeclareAlgebra}[\text{CU}, \text{Generators} \rightarrow \{y, a, x\}, \text{Centrals} \rightarrow \{t\}];$ 
 $B[a_{\text{CU}}, y_{\text{CU}}] = -\gamma y_{\text{CU}}; B[x_{\text{CU}}, a_{\text{CU}}] = -\gamma x_{\text{CU}};$ 
 $B[x_{\text{CU}}, y_{\text{CU}}] = 2 \in a_{\text{CU}} - t 1_{\text{CU}};$ 
 $(S @ y_{\text{CU}} = -y_{\text{CU}}; S @ a_{\text{CU}} = -a_{\text{CU}}; S @ x_{\text{CU}} = -x_{\text{CU}});$ 
 $S_{i\_} [\text{CU}, \text{Centrals}] = \{t_i \rightarrow -t_i\};$ 
 $\Delta @ y_{\text{CU}} = \text{CU} @ y_1 + \text{CU} @ y_2; \Delta @ a_{\text{CU}} = \text{CU} @ a_1 + \text{CU} @ a_2; \Delta @ x_{\text{CU}} = \text{CU} @ x_1 + \text{CU} @ x_2;$ 
 $\Delta_{i \rightarrow j\_ , k\_} [\text{CU}, \text{Centrals}] = \{t_i \rightarrow t_j + t_k\};$ 

```

Implementing $QU = \mathcal{U}_q(\mathfrak{sl}_2^{\vee \epsilon})$

```
In[*]:=  $\hbar = \gamma = 1;$ 
DeclareAlgebra[QU, Generators  $\rightarrow \{y, a, x\}$ , Centrals  $\rightarrow \{t, T\}$ ];
B[aQU, yQU] = - $\gamma$  yQU; B[xQU, aQU] = - $\gamma$  QU@x;
B[xQU, yQU] := SS[q $\hbar$  - 1] QU@{y, x} + OQU[{a}, SS[(1 - T e-2 $\epsilon$  a  $\hbar$ ) /  $\hbar$ ]];
(S@yQU := OQU[{a, y}, SS[-T-1 e $\hbar \epsilon$  a y]]; S@aQU = -aQU; S@xQU := OQU[{a, x}, SS[-e $\hbar \epsilon$  a x]];
Si_[QU, Centrals] = {ti  $\rightarrow$  -ti, Ti  $\rightarrow$  Ti-1};
 $\Delta$ @yQU := OQU[{y1, a1}, {y2}, SS[y1 + T1 e- $\hbar \epsilon$  a1 y2]];
 $\Delta$ @aQU = QU@a1 + QU@a2;  $\Delta$ @xQU := OQU[{a1, x1}, {x2}, SS[x1 + e- $\hbar \epsilon$  a1 x2]];
 $\Delta$ i $\rightarrow$ j $\_,k\_$ [QU, Centrals] = {ti  $\rightarrow$  tj + tk, Ti  $\rightarrow$  Tj Tk};
```

```
In[*]:= B[xQU, yQU]
```

```
Out[*]=
```

(1 - T) QU[] + 2 T $\in$ QU[a] + ϵ QU[y, x]

tSW

Logoi from Pensieve://Talks/Toulouse-1705/DogmaDemo.nb and from Pensieve://Talks/Sydney-1708/ExtraDetails@@.nb.

Goal. In either U , compute $F = e^{-\eta y} e^{\xi x} e^{\eta y} e^{-\xi x}$. First compute $G = e^{\xi x} y e^{-\xi x}$, a finite sum. Now F satisfies the ODE $\partial_\eta F = \partial_\eta (e^{-\eta y} e^{\eta G}) = -yF + FG$ with initial conditions $F(\eta = 0) = 1$. So we set it up and solve:

```
In[*]:= SWxy[U_, kk_] :=
  SWxy[U, kk] = Block[{ $\$U = U$ ,  $\$k = kk$ ,  $\$p = kk$ }, Module[{G, F, fs, f, bs, e, b, es},
    G = Simp[Table[ $\xi^k / k!$ , {k, 0,  $\$k + 1$ }] . NestList[Simp[B[xU, #]] &, yU,  $\$k + 1$ ]];
    fs = Flatten@Table[f1,i,j,k[ $\eta$ ], {1, 0,  $\$k$ }, {i, 0, 1}, {j, 0, 1}, {k, 0, 1}];
    F = fs . (bs = fs /. fL $\_,i\_,j\_,k\_-$ [ $\eta$ ]  $\Rightarrow$   $\epsilon^L U @ \{y^i, a^j, x^k\}$ );
    es = Flatten[
      Table[Coefficient[e, b] == 0, {e, {F - 1U /.  $\eta \rightarrow 0$ , F  $\boxtimes$  G - yU  $\boxtimes$  F -  $\partial_\eta F$ }}, {b, bs}]];
    F = F /. DSolve[es, fs,  $\eta$ ][[1]];
     $\mathbb{E}$ [0,
       $\xi x + \eta y + (U /. \{CU \rightarrow -t \eta \xi, QU \rightarrow \eta \xi (1 - T) / \hbar\})$ ,
      F + 0 $\$k$  /. {e $\rightarrow$  1, U  $\rightarrow$  Times}
    ] /. (v :  $\eta \mid \xi \mid t \mid T \mid y \mid a \mid x$ )  $\rightarrow$  v1
  ]];
tSWxy $\_,i\_,j\rightarrow k\_-$  := SWxy[$U, $k] /. { $\xi_1 \rightarrow \xi_i, \eta_1 \rightarrow \eta_j, (v : t \mid T \mid y \mid a \mid x)_1 \rightarrow v_k$ };
tSWxa $\_,i\_,j\rightarrow k\_-$  :=  $\mathbb{E}[\alpha_j a_k, e^{-\gamma \alpha_j} \xi_i x_k, 1]$ ;
tSWay $\_,i\_,j\rightarrow k\_-$  :=  $\mathbb{E}[\alpha_i a_k, e^{-\gamma \alpha_i} \eta_j y_k, 1]$ ;
```

Exponentials as needed.

Task. Define $\text{Exp}_{U_i,k}[\xi, P]$ which computes $e^{\xi \mathcal{O}(P)}$ to ϵ^k in the algebra U_i , where ξ is a scalar, X is x_i or y_i , and P is an ϵ -dependent near-docile element, giving the answer in $\mathbb{E}$ -form. Should satisfy

$$U @ \text{Exp}_{U,k}[\xi, P] == \mathbb{S}_U[e^{\xi x}, x \rightarrow \mathcal{O}(P)].$$

Methodology. If $P_0 := P_{\epsilon=0}$ and $\mathcal{O}^{\xi \mathcal{O}(P)} = \mathcal{O}(e^{\xi P_0} F(\xi))$, then $F(\xi=0) = 1$ and we have:

$$\mathcal{O}(e^{\xi P_0} (P_0 F(\xi) + \partial_\xi F)) = \mathcal{O}(\partial_\xi e^{\xi P_0} F(\xi)) = \partial_\xi \mathcal{O}(e^{\xi P_0} F(\xi)) = \partial_\xi \mathcal{O}^{\xi \mathcal{O}(P)} = \mathcal{O}^{\xi \mathcal{O}(P)} \mathcal{O}(P) = \mathcal{O}(e^{\xi P_0} F(\xi)) \mathcal{O}(P).$$

This is an ODE for F . Setting inductively $F_k = F_{k-1} + \epsilon^k \varphi$ we find that $F_0 = 1$ and solve for φ .

```
In[*]:= (* Bug: The first line is valid only if  $\mathcal{O}(e^{P_0}) = e^{\mathcal{O}(P_0)}$ . *)
(* Bug:  $\xi$  must be a symbol. *)
Exp_{U,i}_,0[\xi_, P_] := Module[{LQ = Normal@P /.  $\epsilon \rightarrow 0$ },
  E[\xi LQ /. (x | y)_i  $\rightarrow 0$ ,  $\xi$  LQ /. (t | a)_i  $\rightarrow 0, 1$ ]];
Exp_{U,i}_,k[\xi_, P_] := Block[{$U = U, $k = k},
  Module[{P0,  $\varphi$ ,  $\varphi_S$ , F, j, rhs, at0, at\xi},
    P0 = Normal@P /.  $\epsilon \rightarrow 0$ ;
     $\varphi_S$  =
      Flatten@Table[\varphi_{j1,j2,j3}[\xi], {j2, 0, k}, {j1, 0, 2k+1-j2}, {j3, 0, 2k+1-j2-j1}];
    F = Normal@Last@Exp_{U,i}_,k-1[\xi, P] +  $\epsilon^k \varphi_S$ . (\varphi_S /. \varphi_{js\_}[\xi] := Times@@{y_i, a_i, x_i}^{js});
    rhs = Normal@
      Last@m_{i,j} \rightarrow i [E[\xi P0 /. (x | y)_i  $\rightarrow 0$ ,  $\xi$  P0 /. (t | a)_i  $\rightarrow 0$ , F + 0_k] m_{i,j} \rightarrow j @ E[0, 0, P + 0_k]];
    at0 = (# == 0) & /@ Flatten@CoefficientList[F - 1 /.  $\xi \rightarrow 0$ , {y_i, a_i, x_i}];
    at\xi = (# == 0) & /@ Flatten@CoefficientList[(\partial_\xi F) + P0 F - rhs, {y_i, a_i, x_i}];
    E[\xi P0 /. (x | y)_i  $\rightarrow 0$ ,  $\xi$  P0 /. (t | a)_i  $\rightarrow 0$ , F + 0_k] /.
    DSolve[And@@(at0 \cup at\xi),  $\varphi_S$ , \xi][[1]]]
```

In[*]:= \$k = 1

Out[*]=

1

In[*]:= \$Basis = {a_{QU}, x_{QU}, y_{QU}};

Module[{a, b}, Table[{a, b} $\rightarrow$ CE[B[a, b]], {a, \$Basis}, {b, \$Basis}]] // MatrixForm

Out[*]//MatrixForm=

$$\begin{pmatrix} \{QU[a], QU[a]\} \rightarrow 0 & \{QU[a], QU[x]\} \rightarrow \gamma QU[x] & \\ \{QU[x], QU[a]\} \rightarrow -\gamma QU[x] & \{QU[x], QU[x]\} \rightarrow 0 & \{QU[x], QU[x]\} \rightarrow 0 \\ \{QU[y], QU[a]\} \rightarrow \gamma QU[y] & \{QU[y], QU[x]\} \rightarrow \left(-\frac{1}{\hbar} + \frac{T}{\hbar}\right) QU[] - 2 T \in QU[a] - \gamma \in \hbar QU[y, x] & \end{pmatrix}$$

```

In[*]:= $Basis = {aQU, xQU, yQU};
Module[{a, b},
  DeleteCases[
    Flatten[Table[{a, b, c} → CE[B[a, b] + B[b, a]], {a, $Basis}, {b, $Basis}]], _ → 0]
]
Module[{a, b, c},
  DeleteCases[Flatten[Table[{a, b, c} → CE[B[a, B[b, c]] + B[b, B[c, a]] + B[c, B[a, b]]],
    {a, $Basis}, {b, $Basis}, {c, $Basis}]], _ → 0]
]
$ABasis = {aQU, xQU, yQU, QU[y, y], QU[y, a], QU[y, x], QU[a, a], QU[a, x], QU[x, x]};
Module[{a, b, c},
  DeleteCases[Flatten[Table[{a, b, c} → CE[a ⊗ CE[b ⊗ c] - CE[a ⊗ b] ⊗ c],
    {a, $ABasis}, {b, $ABasis}, {c, $ABasis}]], _ → 0]
]

Out[*]=
{}

Out[*]=
{}

Out[*]=
{}

In[*]:= Module[{a, b, c},
  Table[{a, b, c} → CE[a ⊗ CE[b ⊗ c]], {a, $ABasis}, {b, $ABasis}, {c, $ABasis}]
];

In[*]:= B[xQU, yQU] // CE
Out[*]=

$$\left(\frac{1}{\hbar} - \frac{T}{\hbar}\right) QU[] + 2 T \in QU[a] + \gamma \in \hbar QU[y, x]$$


In[*]:= B[xQU, B[xQU, yQU]] // CE
Out[*]=

$$(\gamma \in -3 T \gamma \in) QU[x]$$


In[*]:= B[xQU, B[xQU, B[xQU, yQU]]] // CE
Out[*]=
0

In[*]:= B[xQU, B[xQU, B[xQU, B[xQU, yQU]]]] // CE
Out[*]=
0

In[*]:= B[xQU, B[xQU, B[xQU, B[xQU, B[xQU, yQU]]]]] // Expand
Out[*]=
0

In[*]:= B[xQU, B[xQU, B[xQU, B[xQU, B[xQU, B[xQU, yQU]]]]]] // Expand
Out[*]=
0

```

```
In[*]:= adPower[x_, k_, Env_] [c_ * y_QU] := CE[c adPower[x, k, Env] [y]];
adPower[x_, k_, Env_] [y_Plus] := adPower[x, k, Env] /@ y;
adPower[_ , 0, Env_] [y_] := y;
adPower[x_, k_, Env_] [0] = 0;
adPower[x_, k_, Env_] [y_QU] :=
  adPower[x, k, Env] [y] = CE@B[adPower[x, k - 1, Env] [y], x];
adPower[x_, k_] [y_] := adPower[x, k, $E] [y];
```

```
In[*]:= adPower[x_QU, 1] [y_QU]
```

```
Out[*]=
```

$$\left(-\frac{1}{\hbar} + \frac{T}{\hbar}\right) \text{QU}[] - 2 T \in \text{QU}[a] - \gamma \in \hbar \text{QU}[y, x]$$

```
In[*]:= adPower[x_QU, 2] [y_QU]
```

```
Out[*]=
```

$$(\gamma \in -3 T \gamma \in) \text{QU}[x]$$

```
In[*]:= adPower[x_QU, 3] [y_QU]
```

```
Out[*]=
```

$$0$$

```
In[*]:= adPower[x_QU, 4] [y_QU]
```

```
Out[*]=
```

$$0$$

```
In[*]:= adPower[x_QU, 5] [y_QU]
```

```
Out[*]=
```

$$0$$

```
In[*]:= NilQ[x_QU] = NilQ[y_QU] = True; NilQ[a_QU] = False;
PotQ[c_ * x_QU] := ~ NilQ[x]; (* Potent Q *)
NilQ[c_ * x_QU] := NilQ[x];
PotQ[x_Plus] := And @@ (PotQ /@ (List @@ x));
NilQ[x_Plus] := And @@ (NilQ /@ (List @@ x));
```

```
In[*]:= Ad[0] = Identity;
Ad[c_ . a_QU] [z_] := z /. w_QU => e^{c \gamma (Count[w,y]-Count[w,x])} w;
Ad[c_ . QU[]] [z_] := z;
```

```
In[*]:= Ad[x_] [y_] /; NilQ[x] := Expand@Module[{k = 0, s = 0},
  While[adPower[x, k] [y] != 0,
    s += adPower[x, k] [y] / k!;
    ++k
  ];
  s
]
```

In[*]:= **Ad**[**x**_{QU}][**y**_{QU}]

Out[*]=

$$-QU[] + T QU[] - 2 T \in QU[a] + \frac{1}{2} \in QU[x] - \frac{3}{2} T \in QU[x] + QU[y] - \in QU[y, x]$$

In[*]:= **\$k**

Out[*]=

1

In[*]:= **Ad**[**c** **a**_{QU}]/@{**QU**[], **x**_{QU}, **y**_{QU}, **QU**[**x**, **y**]}

Out[*]=

{**QU**[], $e^{-c \gamma} QU[x]$, $e^{c \gamma} QU[y]$, **QU**[**x**, **y**]}

In[*]:= **Ad**[(**α1** + **α2** + **ε** **fa**[**λ**]) **a**_{QU}]/@{**QU**[], **x**_{QU}, **y**_{QU}, **QU**[**x**, **y**]}

Out[*]=

{**QU**[], $e^{-\gamma (\alpha 1 + \alpha 2 + \epsilon fa[\lambda])} QU[x]$, $e^{\gamma (\alpha 1 + \alpha 2 + \epsilon fa[\lambda])} QU[y]$, **QU**[**x**, **y**]}

In[*]:= **Normal@Series**[$e^{-\gamma (\alpha 1 + \alpha 2 + \epsilon fa[\lambda])}$, {**ε**, **θ**, **\$k**}]

Out[*]=

$$e^{-(\alpha 1 + \alpha 2) \gamma} - e^{-(\alpha 1 + \alpha 2) \gamma} \gamma \in fa[\lambda]$$

In[*]:= **CE**[

Ad[(**α1** + **α2** + **ε** **fa**[**λ**]) **a**_{QU}]/@{**QU**[], **x**_{QU}, **y**_{QU}, **QU**[**x**, **y**]}/. $e^{\epsilon} \rightarrow \text{Normal@Series}[e^{\epsilon}, \{\epsilon, \theta, \$k\}]$]

Out[*]=

{**QU**[], ($e^{-\alpha 1 \gamma - \alpha 2 \gamma} - e^{-\alpha 1 \gamma - \alpha 2 \gamma} \gamma \in fa[\lambda]$) **QU**[**x**], ($e^{\alpha 1 \gamma + \alpha 2 \gamma} + e^{\alpha 1 \gamma + \alpha 2 \gamma} \gamma \in fa[\lambda]$) **QU**[**y**], **QU**[**x**, **y**]}

In[*]:= **Ad**[**a**_{QU}][**Ad**[**x**_{QU}][**y**_{QU}]]

Out[*]=

$$-\frac{QU[]}{\hbar} + \frac{T QU[]}{\hbar} - 2 T \in QU[a] + \frac{1}{2} e^{-\gamma} \gamma \in QU[x] - \frac{3}{2} e^{-\gamma} T \gamma \in QU[x] + e^{\gamma} QU[y] - \gamma \in \hbar QU[y, x]$$

In[*]:= **λTangent**[] = **θ**;

λTangent[**xs**___, **x**_] := **CE**[**Ad**[**x**][**λTangent**[**xs**]] + **∂_λ****x**];

λTangent[**xs_List**] := **λTangent**@@**xs**

In[*]:= **λTangent**[**λ** **y**_{QU}, **λ** **a**_{QU}, **λ** **x**_{QU}]

Out[*]=

$$\left(-\frac{e^{\gamma \lambda} \lambda}{\hbar} + \frac{e^{\gamma \lambda} T \lambda}{\hbar} \right) QU[] + (1 - 2 e^{\gamma \lambda} T \in \lambda) QU[a] +$$

$$\left(1 + \gamma \lambda + \frac{1}{2} e^{\gamma \lambda} \gamma \in \lambda^2 - \frac{3}{2} e^{\gamma \lambda} T \gamma \in \lambda^2 \right) QU[x] + e^{\gamma \lambda} QU[y] - e^{\gamma \lambda} \gamma \in \lambda \hbar QU[y, x]$$

In[*]:= **Seepage**_U[{ }][**c**_] = **c**;

Seepage_U[**ξs_List**][**P**_] := **Module**[{**gs** = **U**[**Generators**], **lx**},

lx = **gs**[[**Length**@**ξs**]];

Collect[**P**, **lx**, (**Ad**[**Last**[**ξs**] **lx**_U][**0**_U[**gs**, **Seepage**_U[**Most**@**ξs**]@#]] /. **U** → **Times**) &]

In[*]:= **Seepage_{QU}**[{ η , α , ξ }] [y]

Out[*]=

$$e^\alpha y - e^\alpha \xi + e^\alpha T \xi - 2 a e^\alpha T \in \xi - e^\alpha x y \in \xi + \frac{1}{2} e^\alpha x \in \xi^2 - \frac{3}{2} e^\alpha T x \in \xi^2$$

Handwritten notes and diagrams:

$E(l_\lambda = \sum_{i=1}^n x_i + \sum_{j=1}^m y_j) \quad E(\sum_w f_w(\lambda) w = r_\lambda)$
 $l_0 = 0 \quad S(y) \otimes S(y) \xrightarrow{\quad} S(y) \quad \downarrow \quad \downarrow$
 $U_q(y) \otimes U_q(y) \xrightarrow{\quad} U_q(y) \quad \downarrow \quad \downarrow$
 $L_\lambda = E(l_\lambda) // \mathcal{O} // m \quad L_{\lambda, \lambda_\lambda}$
 $R_\lambda = E(l_\lambda) // \mathcal{O}$
 $L_\lambda^{-1} \partial_\lambda L_\lambda \stackrel{\text{computed by}}{\lambda T T} \stackrel{\text{red}}{=} R_\lambda^{-1} \partial_\lambda R_\lambda = E_2[l_\lambda^{-1}] E_2[\partial_\lambda l_\lambda] = \text{key comb}$
 $E[l_\lambda^{-1} \partial_\lambda] = (F, i, A)$
 $\partial_\lambda [l_\lambda^{-1} \partial_\lambda] = \partial_\lambda [l_\lambda^{-1} \partial_\lambda]$

$\Gamma_\lambda = \sum_{i=1}^n x_i + \sum_{j=1}^m y_j$
 $\Gamma_\lambda = \sum_{i=1}^n x_i + \sum_{j=1}^m y_j$
 $R_\lambda = \mathcal{O}(E(l_\lambda)) \cdot S(e^{l_\lambda})$
 $\partial_\lambda R_\lambda = \mathcal{O}(E(l_\lambda)) S(e^{l_\lambda}) \cdot S(e^{l_\lambda}) + \mathcal{O}(E(l_\lambda)) \partial_\lambda S(e^{l_\lambda})$
 $S_{\text{Seepage}}(\{f_i\} X) = S(X)$
 $R_\lambda^{-1} \partial_\lambda R_\lambda = S(l_\lambda)^{-1} (S(f_i x_i) S(e^{l_\lambda}) + \partial_\lambda S(e^{l_\lambda}))$
 $= S(e^{l_\lambda}) S(f_i x_i) S(e^{l_\lambda}) + S(e^{l_\lambda}) \partial_\lambda S(e^{l_\lambda})$

In[*]:= **L λ = λ Tangent**[λ η 1 y_{QU} , α 1 a_{QU} , λ ξ 1 x_{QU} , λ η 2 y_{QU} , α 2 a_{QU} , λ ξ 2 x_{QU}]

Out[*]=

$$\begin{aligned}
 & \left(-e^{\alpha_1} \eta_1 \lambda \xi_1 + e^{\alpha_1} T \eta_1 \lambda \xi_1 + \eta_2 \lambda \xi_1 - T \eta_2 \lambda \xi_1 + \frac{1}{2} e^{\alpha_1} \in \eta_1 \eta_2 \lambda^3 \xi_1^2 - 2 e^{\alpha_1} T \in \eta_1 \eta_2 \lambda^3 \xi_1^2 + \right. \\
 & \quad \frac{3}{2} e^{\alpha_1} T^2 \in \eta_1 \eta_2 \lambda^3 \xi_1^2 - e^{\alpha_1 + \alpha_2} \eta_1 \lambda \xi_2 + e^{\alpha_1 + \alpha_2} T \eta_1 \lambda \xi_2 - e^{\alpha_2} \eta_2 \lambda \xi_2 + e^{\alpha_2} T \eta_2 \lambda \xi_2 + \\
 & \quad e^{\alpha_1 + \alpha_2} \in \eta_1 \eta_2 \lambda^3 \xi_1 \xi_2 - 4 e^{\alpha_1 + \alpha_2} T \in \eta_1 \eta_2 \lambda^3 \xi_1 \xi_2 + 3 e^{\alpha_1 + \alpha_2} T^2 \in \eta_1 \eta_2 \lambda^3 \xi_1 \xi_2 - \\
 & \quad \left. \frac{1}{2} e^{\alpha_2} \in \eta_2^2 \lambda^3 \xi_1 \xi_2 + 2 e^{\alpha_2} T \in \eta_2^2 \lambda^3 \xi_1 \xi_2 - \frac{3}{2} e^{\alpha_2} T^2 \in \eta_2^2 \lambda^3 \xi_1 \xi_2 \right) QU[] + \\
 & \quad (-2 e^{\alpha_1} T \in \eta_1 \lambda \xi_1 + 2 T \in \eta_2 \lambda \xi_1 - 2 e^{\alpha_1 + \alpha_2} T \in \eta_1 \lambda \xi_2 - 2 e^{\alpha_2} T \in \eta_2 \lambda \xi_2) QU[a] + \\
 & \quad \left(e^{-\alpha_2} \xi_1 + \frac{1}{2} e^{\alpha_1 - \alpha_2} \in \eta_1 \lambda^2 \xi_1^2 - \frac{3}{2} e^{\alpha_1 - \alpha_2} T \in \eta_1 \lambda^2 \xi_1^2 + \xi_2 + e^{\alpha_1} \in \eta_1 \lambda^2 \xi_1 \xi_2 - \right. \\
 & \quad 3 e^{\alpha_1} T \in \eta_1 \lambda^2 \xi_1 \xi_2 - \in \eta_2 \lambda^2 \xi_1 \xi_2 + 3 T \in \eta_2 \lambda^2 \xi_1 \xi_2 + \frac{1}{2} e^{\alpha_1 + \alpha_2} \in \eta_1 \lambda^2 \xi_2^2 - \\
 & \quad \left. \frac{3}{2} e^{\alpha_1 + \alpha_2} T \in \eta_1 \lambda^2 \xi_2^2 + \frac{1}{2} e^{\alpha_2} \in \eta_2 \lambda^2 \xi_2^2 - \frac{3}{2} e^{\alpha_2} T \in \eta_2 \lambda^2 \xi_2^2 \right) QU[x] + \\
 & \quad \left(e^{\alpha_1 + \alpha_2} \eta_1 + e^{\alpha_2} \eta_2 - e^{\alpha_1 + \alpha_2} \in \eta_1 \eta_2 \lambda^2 \xi_1 + 3 e^{\alpha_1 + \alpha_2} T \in \eta_1 \eta_2 \lambda^2 \xi_1 + \right. \\
 & \quad \left. \frac{1}{2} e^{\alpha_2} \in \eta_2^2 \lambda^2 \xi_1 - \frac{3}{2} e^{\alpha_2} T \in \eta_2^2 \lambda^2 \xi_1 \right) QU[y] + \\
 & \quad (-e^{\alpha_1} \in \eta_1 \lambda \xi_1 + \in \eta_2 \lambda \xi_1 - e^{\alpha_1 + \alpha_2} \in \eta_1 \lambda \xi_2 - e^{\alpha_2} \in \eta_2 \lambda \xi_2) QU[y, x]
 \end{aligned}$$

```
In[*]:= r0λ = {fy0[λ] + ε fy1[λ], α1 + α2 + ε fa1[λ], fx0[λ] + ε fx1[λ]};
pλ = (ft0[λ] + ε ft1[λ]) t QU[] + (ε fyx1[λ]) QU[y, x];
rλ = pλ + r0λ.QU /@ (QU@Generators)
```

```
Out[*]=
t (ft0[λ] + ε ft1[λ]) QU[] + (α1 + α2 + ε fa1[λ]) QU[a] +
(fx0[λ] + ε fx1[λ]) QU[x] + (fy0[λ] + ε fy1[λ]) QU[y] + ε fyx1[λ] QU[y, x]
```

```
In[*]:= SS[epλ/.QU→Times]
```

```
Out[*]=
et ft0[λ] + et ft0[λ] ∈ (t ft1[λ] + x y fyx1[λ])
```

```
In[*]:= InvertQU[ε_] := Module[{A, B, β}, (* ε=A QU[] + ε B *)
  A = Coefficient[ε, 1QU] /. ε → 0;
  B = CE[ε-1 (ε - A 1QU)];
  SQU[SS[(A + ε β)-1], β → B]
]
```

```
In[*]:= ε = OQU[{y, a, x}, CF@SeepageQU[r0λ][SS[epλ/.QU→Times]] /. eε -> Normal@Series[eε, {ε, 0, $k}]]
```

```
Out[*]=
(et ft0[λ] + et ft0[λ] t ∈ ft1[λ]) QU[] +
(-eα1+α2+t ft0[λ] ∈ fx0[λ] fyx1[λ] + eα1+α2+t ft0[λ] T ∈ fx0[λ] fyx1[λ]) QU[x] +
eα1+α2+t ft0[λ] ∈ fyx1[λ] QU[y, x]
```

```
In[*]:= InvertQU[ε]
```

```
Out[*]=
(e-t ft0[λ] - e-t ft0[λ] t ∈ ft1[λ]) QU[] +
(eα1+α2-t ft0[λ] ∈ fx0[λ] fyx1[λ] - eα1+α2-t ft0[λ] T ∈ fx0[λ] fyx1[λ]) QU[x] -
eα1+α2-t ft0[λ] ∈ fyx1[λ] QU[y, x]
```

```
In[*]:= CE[InvertQU[ε] ⊗ ε]
```

```
Out[*]=
QU[]
```

```
In[*]:= = S(eh)-1 · S(∂λ rλ0) · eh + ∂λ eh
```

```
Out[*]=
= S(eh)-1 · S(∂λ rλ0) · eh + ∂λ eh
```

```
In[*]:= Pλ =
  OQU[{y, a, x}, CF@SeepageQU[r0λ][SS[epλ/.QU→Times]] /. eε -> Normal@Series[eε, {ε, 0, $k}]]];
InvertQU[Pλ]
```

```
Out[*]=
(e-t ft0[λ] - e-t ft0[λ] t ∈ ft1[λ]) QU[] +
(eα1+α2-t ft0[λ] ∈ fx0[λ] fyx1[λ] - eα1+α2-t ft0[λ] T ∈ fx0[λ] fyx1[λ]) QU[x] -
eα1+α2-t ft0[λ] ∈ fyx1[λ] QU[y, x]
```

In[*]:= $R\lambda = \mathbf{CE@SS}[\mathbf{Invert}_{\mathbf{QU}}[\mathbf{P}\lambda] \otimes \mathbf{O}_{\mathbf{QU}}[\{\mathbf{y}, \mathbf{a}, \mathbf{x}\},$

$\mathbf{Seepage}_{\mathbf{QU}}[\mathbf{r0}\lambda][(\partial_{\lambda}(\mathbf{r0}\lambda.\mathbf{QU@Generators}))(\mathbf{P}\lambda / . \mathbf{QU} \rightarrow \mathbf{Times}) + \partial_{\lambda}(\mathbf{P}\lambda / . \mathbf{QU} \rightarrow \mathbf{Times})]]]$

Out[*]=

$$\begin{aligned}
& \mathbf{QU}[\mathbf{y}, \mathbf{x}, \mathbf{x}] \left(-e^{\alpha_1 + \alpha_2} \in \mathbf{fyx1}[\lambda] \mathbf{fx0}'[\lambda] + e^{2\alpha_1 + 2\alpha_2} \in \mathbf{fyx1}[\lambda] \mathbf{fx0}'[\lambda] \right) + \\
& \mathbf{QU}[\mathbf{x}, \mathbf{x}] \left(-e^{2\alpha_1 + 2\alpha_2} \in \mathbf{fx0}[\lambda] \mathbf{fyx1}[\lambda] \mathbf{fx0}'[\lambda] + e^{2\alpha_1 + 2\alpha_2} \mathbf{T} \in \mathbf{fx0}[\lambda] \mathbf{fyx1}[\lambda] \mathbf{fx0}'[\lambda] \right) + \\
& \mathbf{QU}[\mathbf{a}] \left(\in \mathbf{fa1}'[\lambda] - 2 e^{\alpha_1 + \alpha_2} \mathbf{T} \in \mathbf{fx0}[\lambda] \mathbf{fy0}'[\lambda] \right) + \\
& \mathbf{QU}[\mathbf{y}, \mathbf{y}, \mathbf{x}] \left(-e^{2\alpha_1 + 2\alpha_2} \in \mathbf{fyx1}[\lambda] \mathbf{fy0}'[\lambda] + e^{3\alpha_1 + 3\alpha_2} \in \mathbf{fyx1}[\lambda] \mathbf{fy0}'[\lambda] \right) + \\
& \mathbf{QU}[\mathbf{y}] \left(e^{\alpha_1 + \alpha_2} \mathbf{fy0}'[\lambda] + e^{\alpha_1 + \alpha_2} \in \mathbf{fa1}[\lambda] \mathbf{fy0}'[\lambda] - \right. \\
& \quad \left. e^{2\alpha_1 + 2\alpha_2} \in \mathbf{fyx1}[\lambda] \mathbf{fy0}'[\lambda] + e^{2\alpha_1 + 2\alpha_2} \mathbf{T} \in \mathbf{fyx1}[\lambda] \mathbf{fy0}'[\lambda] + e^{\alpha_1 + \alpha_2} \in \mathbf{fy1}'[\lambda] \right) + \\
& \mathbf{QU}[] \left(\mathbf{t} \mathbf{ft0}'[\lambda] + \mathbf{t} \in \mathbf{ft1}'[\lambda] - e^{\alpha_1 + \alpha_2} \mathbf{fx0}[\lambda] \mathbf{fy0}'[\lambda] + e^{\alpha_1 + \alpha_2} \mathbf{T} \mathbf{fx0}[\lambda] \mathbf{fy0}'[\lambda] - \right. \\
& \quad e^{\alpha_1 + \alpha_2} \in \mathbf{fa1}[\lambda] \mathbf{fx0}[\lambda] \mathbf{fy0}'[\lambda] + e^{\alpha_1 + \alpha_2} \mathbf{T} \in \mathbf{fa1}[\lambda] \mathbf{fx0}[\lambda] \mathbf{fy0}'[\lambda] - \\
& \quad e^{\alpha_1 + \alpha_2} \in \mathbf{fx1}[\lambda] \mathbf{fy0}'[\lambda] + e^{\alpha_1 + \alpha_2} \mathbf{T} \in \mathbf{fx1}[\lambda] \mathbf{fy0}'[\lambda] + e^{2\alpha_1 + 2\alpha_2} \in \mathbf{fx0}[\lambda] \mathbf{fyx1}[\lambda] \mathbf{fy0}'[\lambda] - \\
& \quad 2 e^{2\alpha_1 + 2\alpha_2} \mathbf{T} \in \mathbf{fx0}[\lambda] \mathbf{fyx1}[\lambda] \mathbf{fy0}'[\lambda] + e^{2\alpha_1 + 2\alpha_2} \mathbf{T}^2 \in \mathbf{fx0}[\lambda] \mathbf{fyx1}[\lambda] \mathbf{fy0}'[\lambda] - \\
& \quad \left. e^{\alpha_1 + \alpha_2} \in \mathbf{fx0}[\lambda] \mathbf{fy1}'[\lambda] + e^{\alpha_1 + \alpha_2} \mathbf{T} \in \mathbf{fx0}[\lambda] \mathbf{fy1}'[\lambda] \right) + \\
& \mathbf{QU}[\mathbf{y}, \mathbf{x}] \left(-e^{\alpha_1 + \alpha_2} \mathbf{t} \in \mathbf{fyx1}[\lambda] \mathbf{ft0}'[\lambda] + e^{2\alpha_1 + 2\alpha_2} \mathbf{t} \in \mathbf{fyx1}[\lambda] \mathbf{ft0}'[\lambda] - e^{\alpha_1 + \alpha_2} \in \mathbf{fx0}[\lambda] \mathbf{fy0}'[\lambda] + \right. \\
& \quad e^{2\alpha_1 + 2\alpha_2} \in \mathbf{fx0}[\lambda] \mathbf{fyx1}[\lambda] \mathbf{fy0}'[\lambda] - 2 e^{3\alpha_1 + 3\alpha_2} \in \mathbf{fx0}[\lambda] \mathbf{fyx1}[\lambda] \mathbf{fy0}'[\lambda] - e^{2\alpha_1 + 2\alpha_2} \mathbf{T} \in \mathbf{fx0}[\lambda] \\
& \quad \left. \mathbf{fyx1}[\lambda] \mathbf{fy0}'[\lambda] + 2 e^{3\alpha_1 + 3\alpha_2} \mathbf{T} \in \mathbf{fx0}[\lambda] \mathbf{fyx1}[\lambda] \mathbf{fy0}'[\lambda] + e^{2\alpha_1 + 2\alpha_2} \in \mathbf{fyx1}'[\lambda] \right) + \mathbf{QU}[\mathbf{x}] \\
& \left(\in \mathbf{fx0}[\lambda] \mathbf{fa1}'[\lambda] - e^{2\alpha_1 + 2\alpha_2} \mathbf{t} \in \mathbf{fx0}[\lambda] \mathbf{fyx1}[\lambda] \mathbf{ft0}'[\lambda] + e^{2\alpha_1 + 2\alpha_2} \mathbf{t} \mathbf{T} \in \mathbf{fx0}[\lambda] \mathbf{fyx1}[\lambda] \mathbf{ft0}'[\lambda] + \right. \\
& \quad \mathbf{fx0}'[\lambda] - e^{\alpha_1 + \alpha_2} \in \mathbf{fyx1}[\lambda] \mathbf{fx0}'[\lambda] + e^{\alpha_1 + \alpha_2} \mathbf{T} \in \mathbf{fyx1}[\lambda] \mathbf{fx0}'[\lambda] + \\
& \quad \in \mathbf{fx1}'[\lambda] + \frac{1}{2} e^{\alpha_1 + \alpha_2} \in \mathbf{fx0}[\lambda]^2 \mathbf{fy0}'[\lambda] - \frac{3}{2} e^{\alpha_1 + \alpha_2} \mathbf{T} \in \mathbf{fx0}[\lambda]^2 \mathbf{fy0}'[\lambda] + \\
& \quad e^{3\alpha_1 + 3\alpha_2} \in \mathbf{fx0}[\lambda]^2 \mathbf{fyx1}[\lambda] \mathbf{fy0}'[\lambda] - 2 e^{3\alpha_1 + 3\alpha_2} \mathbf{T} \in \mathbf{fx0}[\lambda]^2 \mathbf{fyx1}[\lambda] \mathbf{fy0}'[\lambda] + \\
& \quad e^{3\alpha_1 + 3\alpha_2} \mathbf{T}^2 \in \mathbf{fx0}[\lambda]^2 \mathbf{fyx1}[\lambda] \mathbf{fy0}'[\lambda] - e^{\alpha_1 + \alpha_2} \in \mathbf{fx0}[\lambda] \mathbf{fyx1}'[\lambda] - \\
& \quad \left. e^{2\alpha_1 + 2\alpha_2} \in \mathbf{fx0}[\lambda] \mathbf{fyx1}'[\lambda] + e^{\alpha_1 + \alpha_2} \mathbf{T} \in \mathbf{fx0}[\lambda] \mathbf{fyx1}'[\lambda] + e^{2\alpha_1 + 2\alpha_2} \mathbf{T} \in \mathbf{fx0}[\lambda] \mathbf{fyx1}'[\lambda] \right)
\end{aligned}$$

In[*]:= **diff** = **CE**[**L** λ - **R** λ]

Out[*]=

$$\begin{aligned}
& \text{QU}[y, x, x] \left(e^{\alpha_1 + \alpha_2} \in \text{fyx1}[\lambda] \text{fx0}'[\lambda] - e^{2\alpha_1 + 2\alpha_2} \in \text{fyx1}[\lambda] \text{fx0}'[\lambda] \right) + \\
& \text{QU}[x, x] \left(e^{2\alpha_1 + 2\alpha_2} \in \text{fx0}[\lambda] \text{fyx1}[\lambda] \text{fx0}'[\lambda] - e^{2\alpha_1 + 2\alpha_2} \text{T} \in \text{fx0}[\lambda] \text{fyx1}[\lambda] \text{fx0}'[\lambda] \right) + \\
& \text{QU}[a] \left(-2 e^{\alpha_1} \text{T} \in \eta_1 \lambda \xi_1 + 2 \text{T} \in \eta_2 \lambda \xi_1 - 2 e^{\alpha_1 + \alpha_2} \text{T} \in \eta_1 \lambda \xi_2 - \right. \\
& \quad \left. 2 e^{\alpha_2} \text{T} \in \eta_2 \lambda \xi_2 - e \text{fa1}'[\lambda] + 2 e^{\alpha_1 + \alpha_2} \text{T} \in \text{fx0}[\lambda] \text{fy0}'[\lambda] \right) + \\
& \text{QU}[y, y, x] \left(e^{2\alpha_1 + 2\alpha_2} \in \text{fyx1}[\lambda] \text{fy0}'[\lambda] - e^{3\alpha_1 + 3\alpha_2} \in \text{fyx1}[\lambda] \text{fy0}'[\lambda] \right) + \\
& \text{QU}[y] \left(e^{\alpha_1 + \alpha_2} \eta_1 + e^{\alpha_2} \eta_2 - e^{\alpha_1 + \alpha_2} \in \eta_1 \eta_2 \lambda^2 \xi_1 + 3 e^{\alpha_1 + \alpha_2} \text{T} \in \eta_1 \eta_2 \lambda^2 \xi_1 + \right. \\
& \quad \frac{1}{2} e^{\alpha_2} \in \eta_2^2 \lambda^2 \xi_1 - \frac{3}{2} e^{\alpha_2} \text{T} \in \eta_2^2 \lambda^2 \xi_1 - e^{\alpha_1 + \alpha_2} \text{fy0}'[\lambda] - e^{\alpha_1 + \alpha_2} \in \text{fa1}[\lambda] \text{fy0}'[\lambda] + \\
& \quad \left. e^{2\alpha_1 + 2\alpha_2} \in \text{fyx1}[\lambda] \text{fy0}'[\lambda] - e^{2\alpha_1 + 2\alpha_2} \text{T} \in \text{fyx1}[\lambda] \text{fy0}'[\lambda] - e^{\alpha_1 + \alpha_2} \in \text{fy1}'[\lambda] \right) + \\
& \text{QU}[] \left(-e^{\alpha_1} \eta_1 \lambda \xi_1 + e^{\alpha_1} \text{T} \eta_1 \lambda \xi_1 + \eta_2 \lambda \xi_1 - \text{T} \eta_2 \lambda \xi_1 + \frac{1}{2} e^{\alpha_1} \in \eta_1 \eta_2 \lambda^3 \xi_1^2 - 2 e^{\alpha_1} \text{T} \in \eta_1 \eta_2 \lambda^3 \xi_1^2 + \right. \\
& \quad \frac{3}{2} e^{\alpha_1} \text{T}^2 \in \eta_1 \eta_2 \lambda^3 \xi_1^2 - e^{\alpha_1 + \alpha_2} \eta_1 \lambda \xi_2 + e^{\alpha_1 + \alpha_2} \text{T} \eta_1 \lambda \xi_2 - e^{\alpha_2} \eta_2 \lambda \xi_2 + e^{\alpha_2} \text{T} \eta_2 \lambda \xi_2 + \\
& \quad e^{\alpha_1 + \alpha_2} \in \eta_1 \eta_2 \lambda^3 \xi_1 \xi_2 - 4 e^{\alpha_1 + \alpha_2} \text{T} \in \eta_1 \eta_2 \lambda^3 \xi_1 \xi_2 + 3 e^{\alpha_1 + \alpha_2} \text{T}^2 \in \eta_1 \eta_2 \lambda^3 \xi_1 \xi_2 - \\
& \quad \frac{1}{2} e^{\alpha_2} \in \eta_2^2 \lambda^3 \xi_1 \xi_2 + 2 e^{\alpha_2} \text{T} \in \eta_2^2 \lambda^3 \xi_1 \xi_2 - \frac{3}{2} e^{\alpha_2} \text{T}^2 \in \eta_2^2 \lambda^3 \xi_1 \xi_2 - \text{tft0}'[\lambda] - \text{t} \in \text{ft1}'[\lambda] + \\
& \quad e^{\alpha_1 + \alpha_2} \text{fx0}[\lambda] \text{fy0}'[\lambda] - e^{\alpha_1 + \alpha_2} \text{T} \text{fx0}[\lambda] \text{fy0}'[\lambda] + e^{\alpha_1 + \alpha_2} \in \text{fa1}[\lambda] \text{fx0}[\lambda] \text{fy0}'[\lambda] - \\
& \quad e^{\alpha_1 + \alpha_2} \text{T} \in \text{fa1}[\lambda] \text{fx0}[\lambda] \text{fy0}'[\lambda] + e^{\alpha_1 + \alpha_2} \in \text{fx1}[\lambda] \text{fy0}'[\lambda] - e^{\alpha_1 + \alpha_2} \text{T} \in \text{fx1}[\lambda] \text{fy0}'[\lambda] - \\
& \quad e^{2\alpha_1 + 2\alpha_2} \in \text{fx0}[\lambda] \text{fyx1}[\lambda] \text{fy0}'[\lambda] + 2 e^{2\alpha_1 + 2\alpha_2} \text{T} \in \text{fx0}[\lambda] \text{fyx1}[\lambda] \text{fy0}'[\lambda] - \\
& \quad \left. e^{2\alpha_1 + 2\alpha_2} \text{T}^2 \in \text{fx0}[\lambda] \text{fyx1}[\lambda] \text{fy0}'[\lambda] + e^{\alpha_1 + \alpha_2} \in \text{fx0}[\lambda] \text{fy1}'[\lambda] - e^{\alpha_1 + \alpha_2} \text{T} \in \text{fx0}[\lambda] \text{fy1}'[\lambda] \right) + \\
& \text{QU}[y, x] \left(-e^{\alpha_1} \in \eta_1 \lambda \xi_1 + e \eta_2 \lambda \xi_1 - e^{\alpha_1 + \alpha_2} \in \eta_1 \lambda \xi_2 - e^{\alpha_2} \in \eta_2 \lambda \xi_2 + e^{\alpha_1 + \alpha_2} \text{t} \in \text{fyx1}[\lambda] \text{ft0}'[\lambda] - \right. \\
& \quad e^{2\alpha_1 + 2\alpha_2} \text{t} \in \text{fyx1}[\lambda] \text{ft0}'[\lambda] + e^{\alpha_1 + \alpha_2} \in \text{fx0}[\lambda] \text{fy0}'[\lambda] - e^{2\alpha_1 + 2\alpha_2} \in \text{fx0}[\lambda] \text{fyx1}[\lambda] \text{fy0}'[\lambda] + \\
& \quad 2 e^{3\alpha_1 + 3\alpha_2} \in \text{fx0}[\lambda] \text{fyx1}[\lambda] \text{fy0}'[\lambda] + e^{2\alpha_1 + 2\alpha_2} \text{T} \in \text{fx0}[\lambda] \text{fyx1}[\lambda] \text{fy0}'[\lambda] - \\
& \quad \left. 2 e^{3\alpha_1 + 3\alpha_2} \text{T} \in \text{fx0}[\lambda] \text{fyx1}[\lambda] \text{fy0}'[\lambda] - e^{2\alpha_1 + 2\alpha_2} \in \text{fyx1}'[\lambda] \right) + \\
& \text{QU}[x] \left(e^{-\alpha_2} \xi_1 + \frac{1}{2} e^{\alpha_1 - \alpha_2} \in \eta_1 \lambda^2 \xi_1^2 - \frac{3}{2} e^{\alpha_1 - \alpha_2} \text{T} \in \eta_1 \lambda^2 \xi_1^2 + \xi_2 + e^{\alpha_1} \in \eta_1 \lambda^2 \xi_1 \xi_2 - \right. \\
& \quad 3 e^{\alpha_1} \text{T} \in \eta_1 \lambda^2 \xi_1 \xi_2 - e \eta_2 \lambda^2 \xi_1 \xi_2 + 3 \text{T} \in \eta_2 \lambda^2 \xi_1 \xi_2 + \frac{1}{2} e^{\alpha_1 + \alpha_2} \in \eta_1 \lambda^2 \xi_2^2 - \\
& \quad \frac{3}{2} e^{\alpha_1 + \alpha_2} \text{T} \in \eta_1 \lambda^2 \xi_2^2 + \frac{1}{2} e^{\alpha_2} \in \eta_2 \lambda^2 \xi_2^2 - \frac{3}{2} e^{\alpha_2} \text{T} \in \eta_2 \lambda^2 \xi_2^2 - e \text{fx0}[\lambda] \text{fa1}'[\lambda] + \\
& \quad e^{2\alpha_1 + 2\alpha_2} \text{t} \in \text{fx0}[\lambda] \text{fyx1}[\lambda] \text{ft0}'[\lambda] - e^{2\alpha_1 + 2\alpha_2} \text{t} \text{T} \in \text{fx0}[\lambda] \text{fyx1}[\lambda] \text{ft0}'[\lambda] - \text{fx0}'[\lambda] + \\
& \quad e^{\alpha_1 + \alpha_2} \in \text{fyx1}[\lambda] \text{fx0}'[\lambda] - e^{\alpha_1 + \alpha_2} \text{T} \in \text{fyx1}[\lambda] \text{fx0}'[\lambda] - e \text{fx1}'[\lambda] - \frac{1}{2} e^{\alpha_1 + \alpha_2} \in \text{fx0}[\lambda]^2 \text{fy0}'[\lambda] + \\
& \quad \frac{3}{2} e^{\alpha_1 + \alpha_2} \text{T} \in \text{fx0}[\lambda]^2 \text{fy0}'[\lambda] - e^{3\alpha_1 + 3\alpha_2} \in \text{fx0}[\lambda]^2 \text{fyx1}[\lambda] \text{fy0}'[\lambda] + 2 e^{3\alpha_1 + 3\alpha_2} \text{T} \in \text{fx0}[\lambda]^2 \\
& \quad \text{fyx1}[\lambda] \text{fy0}'[\lambda] - e^{3\alpha_1 + 3\alpha_2} \text{T}^2 \in \text{fx0}[\lambda]^2 \text{fyx1}[\lambda] \text{fy0}'[\lambda] + e^{\alpha_1 + \alpha_2} \in \text{fx0}[\lambda] \text{fyx1}'[\lambda] + \\
& \quad \left. e^{2\alpha_1 + 2\alpha_2} \in \text{fx0}[\lambda] \text{fyx1}'[\lambda] - e^{\alpha_1 + \alpha_2} \text{T} \in \text{fx0}[\lambda] \text{fyx1}'[\lambda] - e^{2\alpha_1 + 2\alpha_2} \text{T} \in \text{fx0}[\lambda] \text{fyx1}'[\lambda] \right)
\end{aligned}$$

In[*]:= **diff /. $\epsilon \rightarrow 0$**

Out[*]=

$$\begin{aligned} & \text{QU}[x] \left(e^{-\alpha^2} \xi_1 + \xi_2 - \text{fx0}'[\lambda] \right) + \text{QU}[y] \left(e^{\alpha^1 + \alpha^2} \eta_1 + e^{\alpha^2} \eta_2 - e^{\alpha^1 + \alpha^2} \text{fy0}'[\lambda] \right) + \\ & \text{QU}[] \left(-e^{\alpha^1} \eta_1 \lambda \xi_1 + e^{\alpha^1} T \eta_1 \lambda \xi_1 + \eta_2 \lambda \xi_1 - T \eta_2 \lambda \xi_1 - e^{\alpha^1 + \alpha^2} \eta_1 \lambda \xi_2 + e^{\alpha^1 + \alpha^2} T \eta_1 \lambda \xi_2 - \right. \\ & \left. e^{\alpha^2} \eta_2 \lambda \xi_2 + e^{\alpha^2} T \eta_2 \lambda \xi_2 - t \text{ft0}'[\lambda] + e^{\alpha^1 + \alpha^2} \text{fx0}[\lambda] \text{fy0}'[\lambda] - e^{\alpha^1 + \alpha^2} T \text{fx0}[\lambda] \text{fy0}'[\lambda] \right) \end{aligned}$$

In[*]:= **{sol1} = DSolve[Coefficient[diff /. $\epsilon \rightarrow 0$, x_{QU}] == 0 $\wedge$ fx0[0] == 0, fx0, λ]**

Out[*]=

$$\left\{ \left\{ \text{fx0} \rightarrow \text{Function} \left[\{\lambda\}, e^{-\alpha^2} \lambda \left(\xi_1 + e^{\alpha^2} \xi_2 \right) \right] \right\} \right\}$$

In[*]:= **CE[diff /. sol1 /. $\epsilon \rightarrow 0$]**

Out[*]=

$$\begin{aligned} & \text{QU}[y] \left(e^{\alpha^1 + \alpha^2} \eta_1 + e^{\alpha^2} \eta_2 - e^{\alpha^1 + \alpha^2} \text{fy0}'[\lambda] \right) + \\ & \text{QU}[] \left(-e^{\alpha^1} \eta_1 \lambda \xi_1 + e^{\alpha^1} T \eta_1 \lambda \xi_1 + \eta_2 \lambda \xi_1 - T \eta_2 \lambda \xi_1 - e^{\alpha^1 + \alpha^2} \eta_1 \lambda \xi_2 + \right. \\ & \left. e^{\alpha^1 + \alpha^2} T \eta_1 \lambda \xi_2 - e^{\alpha^2} \eta_2 \lambda \xi_2 + e^{\alpha^2} T \eta_2 \lambda \xi_2 - t \text{ft0}'[\lambda] + e^{\alpha^1} \lambda \xi_1 \text{fy0}'[\lambda] - \right. \\ & \left. e^{\alpha^1} T \lambda \xi_1 \text{fy0}'[\lambda] + e^{\alpha^1 + \alpha^2} \lambda \xi_2 \text{fy0}'[\lambda] - e^{\alpha^1 + \alpha^2} T \lambda \xi_2 \text{fy0}'[\lambda] \right) \end{aligned}$$

In[*]:= **{sol2} = DSolve[Coefficient[CE[diff /. sol1 /. $\epsilon \rightarrow 0$], y_{QU}] == 0 $\wedge$ fy0[0] == 0, fy0, λ]**

Out[*]=

$$\left\{ \left\{ \text{fy0} \rightarrow \text{Function} \left[\{\lambda\}, e^{-\alpha^1} \left(e^{\alpha^1} \eta_1 + \eta_2 \right) \lambda \right] \right\} \right\}$$

In[*]:= **CE[diff /. sol1 /. sol2 /. $\epsilon \rightarrow 0$]**

Out[*]=

$$\text{QU}[] \left(2 \eta_2 \lambda \xi_1 - 2 T \eta_2 \lambda \xi_1 - t \text{ft0}'[\lambda] \right)$$

In[*]:= **{sol3} = DSolve[Coefficient[CE[diff /. sol1 /. sol2 /. $\epsilon \rightarrow 0$], QU[]] == 0 $\wedge$ ft0[0] == 0, ft0, λ]**

Out[*]=

$$\left\{ \left\{ \text{ft0} \rightarrow \text{Function} \left[\{\lambda\}, -\frac{(-1 + T) \eta_2 \lambda^2 \xi_1}{t} \right] \right\} \right\}$$

In[*]:= **CE[diff /. sol1 /. sol2 /. sol3] /. $\epsilon \rightarrow 0$**

Out[*]=

$$0$$

In[*]:= **CE[diff /. sol1 /. sol2 /. sol3]**

Out[*]=

$$\begin{aligned}
& \left(e^{2\alpha_1} \in \lambda \xi^1 f_{yx1}[\lambda] - e^{2\alpha_1} T \in \lambda \xi^1 f_{yx1}[\lambda] + 2 e^{2\alpha_1+\alpha_2} \in \lambda \xi^1 \xi^2 f_{yx1}[\lambda] - \right. \\
& \quad \left. 2 e^{2\alpha_1+\alpha_2} T \in \lambda \xi^1 \xi^2 f_{yx1}[\lambda] + e^{2\alpha_1+2\alpha_2} \in \lambda \xi^2 f_{yx1}[\lambda] - e^{2\alpha_1+2\alpha_2} T \in \lambda \xi^2 f_{yx1}[\lambda] \right) QU[x, x] + \\
& \left(e^{\alpha_1} \in \xi^1 f_{yx1}[\lambda] - e^{2\alpha_1+\alpha_2} \in \xi^1 f_{yx1}[\lambda] + e^{\alpha_1+\alpha_2} \in \xi^2 f_{yx1}[\lambda] - e^{2\alpha_1+2\alpha_2} \in \xi^2 f_{yx1}[\lambda] \right) QU[y, x, x] + \\
& \left(e^{2\alpha_1+2\alpha_2} \in \eta^1 f_{yx1}[\lambda] - e^{3\alpha_1+3\alpha_2} \in \eta^1 f_{yx1}[\lambda] + e^{\alpha_1+2\alpha_2} \in \eta^2 f_{yx1}[\lambda] - e^{2\alpha_1+3\alpha_2} \in \eta^2 f_{yx1}[\lambda] \right) \\
& \quad QU[y, y, x] + QU[a] (4 T \in \eta^2 \lambda \xi^1 - e fa1'[\lambda]) + \\
& QU[y] \left(-e^{\alpha_1+\alpha_2} \in \eta^1 \eta^2 \lambda^2 \xi^1 + 3 e^{\alpha_1+\alpha_2} T \in \eta^1 \eta^2 \lambda^2 \xi^1 + \frac{1}{2} e^{\alpha_2} \in \eta^2 \lambda^2 \xi^1 - \right. \\
& \quad \frac{3}{2} e^{\alpha_2} T \in \eta^2 \lambda^2 \xi^1 - e^{\alpha_1+\alpha_2} \in \eta^1 fa1[\lambda] - e^{\alpha_2} \in \eta^2 fa1[\lambda] + e^{2\alpha_1+2\alpha_2} \in \eta^1 f_{yx1}[\lambda] - \\
& \quad \left. e^{2\alpha_1+2\alpha_2} T \in \eta^1 f_{yx1}[\lambda] + e^{\alpha_1+2\alpha_2} \in \eta^2 f_{yx1}[\lambda] - e^{\alpha_1+2\alpha_2} T \in \eta^2 f_{yx1}[\lambda] - e^{\alpha_1+\alpha_2} \in fy1'[\lambda] \right) + \\
& QU[] \left(\frac{1}{2} e^{\alpha_1} \in \eta^1 \eta^2 \lambda^3 \xi^1 - 2 e^{\alpha_1} T \in \eta^1 \eta^2 \lambda^3 \xi^1 + \frac{3}{2} e^{\alpha_1} T^2 \in \eta^1 \eta^2 \lambda^3 \xi^1 + \right. \\
& \quad e^{\alpha_1+\alpha_2} \in \eta^1 \eta^2 \lambda^3 \xi^1 \xi^2 - 4 e^{\alpha_1+\alpha_2} T \in \eta^1 \eta^2 \lambda^3 \xi^1 \xi^2 + 3 e^{\alpha_1+\alpha_2} T^2 \in \eta^1 \eta^2 \lambda^3 \xi^1 \xi^2 - \\
& \quad \frac{1}{2} e^{\alpha_2} \in \eta^2 \lambda^3 \xi^1 \xi^2 + 2 e^{\alpha_2} T \in \eta^2 \lambda^3 \xi^1 \xi^2 - \frac{3}{2} e^{\alpha_2} T^2 \in \eta^2 \lambda^3 \xi^1 \xi^2 + e^{\alpha_1} \in \eta^1 \lambda \xi^1 fa1[\lambda] - \\
& \quad e^{\alpha_1} T \in \eta^1 \lambda \xi^1 fa1[\lambda] + e^{\alpha_2} \in \eta^2 \lambda \xi^1 fa1[\lambda] - T \in \eta^2 \lambda \xi^1 fa1[\lambda] + e^{\alpha_1+\alpha_2} \in \eta^1 \lambda \xi^2 fa1[\lambda] - \\
& \quad e^{\alpha_1+\alpha_2} T \in \eta^1 \lambda \xi^2 fa1[\lambda] + e^{\alpha_2} \in \eta^2 \lambda \xi^2 fa1[\lambda] - e^{\alpha_2} T \in \eta^2 \lambda \xi^2 fa1[\lambda] + e^{\alpha_1+\alpha_2} \in \eta^1 fx1[\lambda] - \\
& \quad e^{\alpha_1+\alpha_2} T \in \eta^1 fx1[\lambda] + e^{\alpha_2} \in \eta^2 fx1[\lambda] - e^{\alpha_2} T \in \eta^2 fx1[\lambda] - e^{2\alpha_1+\alpha_2} \in \eta^1 \lambda \xi^1 f_{yx1}[\lambda] + \\
& \quad 2 e^{2\alpha_1+\alpha_2} T \in \eta^1 \lambda \xi^1 f_{yx1}[\lambda] - e^{2\alpha_1+\alpha_2} T^2 \in \eta^1 \lambda \xi^1 f_{yx1}[\lambda] - e^{\alpha_1+\alpha_2} \in \eta^2 \lambda \xi^1 f_{yx1}[\lambda] + \\
& \quad 2 e^{\alpha_1+\alpha_2} T \in \eta^2 \lambda \xi^1 f_{yx1}[\lambda] - e^{\alpha_1+\alpha_2} T^2 \in \eta^2 \lambda \xi^1 f_{yx1}[\lambda] - e^{2\alpha_1+2\alpha_2} \in \eta^1 \lambda \xi^2 f_{yx1}[\lambda] + \\
& \quad 2 e^{2\alpha_1+2\alpha_2} T \in \eta^1 \lambda \xi^2 f_{yx1}[\lambda] - e^{2\alpha_1+2\alpha_2} T^2 \in \eta^1 \lambda \xi^2 f_{yx1}[\lambda] - e^{\alpha_1+2\alpha_2} \in \eta^2 \lambda \xi^2 f_{yx1}[\lambda] + \\
& \quad 2 e^{\alpha_1+2\alpha_2} T \in \eta^2 \lambda \xi^2 f_{yx1}[\lambda] - e^{\alpha_1+2\alpha_2} T^2 \in \eta^2 \lambda \xi^2 f_{yx1}[\lambda] - t \in ft1'[\lambda] + e^{\alpha_1} \in \lambda \xi^1 fy1'[\lambda] - \\
& \quad \left. e^{\alpha_1} T \in \lambda \xi^1 fy1'[\lambda] + e^{\alpha_1+\alpha_2} \in \lambda \xi^2 fy1'[\lambda] - e^{\alpha_1+\alpha_2} T \in \lambda \xi^2 fy1'[\lambda] \right) + QU[y, x] \\
& \left(2 e^{\alpha_1+\alpha_2} \in \eta^1 \lambda \xi^1 f_{yx1}[\lambda] + 2 e^{3\alpha_1+2\alpha_2} \in \eta^1 \lambda \xi^1 f_{yx1}[\lambda] + e^{2\alpha_1+\alpha_2} T \in \eta^1 \lambda \xi^1 f_{yx1}[\lambda] - \right. \\
& \quad 2 e^{3\alpha_1+2\alpha_2} T \in \eta^1 \lambda \xi^1 f_{yx1}[\lambda] + e^{\alpha_1+\alpha_2} \in \eta^2 \lambda \xi^1 f_{yx1}[\lambda] - e^{\alpha_1+\alpha_2} T \in \eta^2 \lambda \xi^1 f_{yx1}[\lambda] - \\
& \quad e^{2\alpha_1+2\alpha_2} \in \eta^1 \lambda \xi^2 f_{yx1}[\lambda] + 2 e^{3\alpha_1+3\alpha_2} \in \eta^1 \lambda \xi^2 f_{yx1}[\lambda] + e^{2\alpha_1+2\alpha_2} T \in \eta^1 \lambda \xi^2 f_{yx1}[\lambda] - \\
& \quad 2 e^{3\alpha_1+3\alpha_2} T \in \eta^1 \lambda \xi^2 f_{yx1}[\lambda] - e^{\alpha_1+2\alpha_2} \in \eta^2 \lambda \xi^2 f_{yx1}[\lambda] + 2 e^{2\alpha_1+3\alpha_2} \in \eta^2 \lambda \xi^2 f_{yx1}[\lambda] + \\
& \quad \left. e^{\alpha_1+2\alpha_2} T \in \eta^2 \lambda \xi^2 f_{yx1}[\lambda] - 2 e^{2\alpha_1+3\alpha_2} T \in \eta^2 \lambda \xi^2 f_{yx1}[\lambda] - e^{2\alpha_1+2\alpha_2} \in fyx1'[\lambda] \right) + \\
& QU[x] \left(-\frac{1}{2} e^{-\alpha_2} \in \eta^2 \lambda^2 \xi^1 + \frac{3}{2} e^{-\alpha_2} T \in \eta^2 \lambda^2 \xi^1 - 2 e^{\alpha_2} \in \eta^2 \lambda^2 \xi^1 \xi^2 + 6 T \in \eta^2 \lambda^2 \xi^1 \xi^2 + \right. \\
& \quad e^{\alpha_1} \in \xi^1 f_{yx1}[\lambda] - e^{\alpha_1} T \in \xi^1 f_{yx1}[\lambda] - e^{3\alpha_1+\alpha_2} \in \eta^1 \lambda^2 \xi^1 f_{yx1}[\lambda] + \\
& \quad 2 e^{3\alpha_1+\alpha_2} T \in \eta^1 \lambda^2 \xi^1 f_{yx1}[\lambda] - e^{3\alpha_1+\alpha_2} T^2 \in \eta^1 \lambda^2 \xi^1 f_{yx1}[\lambda] + e^{2\alpha_1+\alpha_2} \in \eta^2 \lambda^2 \xi^1 f_{yx1}[\lambda] - \\
& \quad 2 e^{2\alpha_1+\alpha_2} T \in \eta^2 \lambda^2 \xi^1 f_{yx1}[\lambda] + e^{2\alpha_1+\alpha_2} T^2 \in \eta^2 \lambda^2 \xi^1 f_{yx1}[\lambda] + e^{\alpha_1+\alpha_2} \in \xi^2 f_{yx1}[\lambda] - \\
& \quad e^{\alpha_1+\alpha_2} T \in \xi^2 f_{yx1}[\lambda] - 2 e^{3\alpha_1+2\alpha_2} \in \eta^1 \lambda^2 \xi^1 \xi^2 f_{yx1}[\lambda] + 4 e^{3\alpha_1+2\alpha_2} T \in \eta^1 \lambda^2 \xi^1 \xi^2 f_{yx1}[\lambda] - \\
& \quad 2 e^{3\alpha_1+2\alpha_2} T^2 \in \eta^1 \lambda^2 \xi^1 \xi^2 f_{yx1}[\lambda] - e^{3\alpha_1+3\alpha_2} \in \eta^1 \lambda^2 \xi^2 f_{yx1}[\lambda] + \\
& \quad 2 e^{3\alpha_1+3\alpha_2} T \in \eta^1 \lambda^2 \xi^2 f_{yx1}[\lambda] - e^{3\alpha_1+3\alpha_2} T^2 \in \eta^1 \lambda^2 \xi^2 f_{yx1}[\lambda] - e^{2\alpha_1+3\alpha_2} \in \eta^2 \lambda^2 \xi^2 f_{yx1}[\lambda] + \\
& \quad 2 e^{2\alpha_1+3\alpha_2} T \in \eta^2 \lambda^2 \xi^2 f_{yx1}[\lambda] - e^{2\alpha_1+3\alpha_2} T^2 \in \eta^2 \lambda^2 \xi^2 f_{yx1}[\lambda] - e^{-\alpha_2} \in \lambda \xi^1 fa1'[\lambda] - \\
& \quad e^{\alpha_1} \in \lambda \xi^2 fa1'[\lambda] - e^{\alpha_1} \in \lambda \xi^1 fy1'[\lambda] + e^{2\alpha_1+\alpha_2} \in \lambda \xi^1 fy1'[\lambda] - \\
& \quad e^{\alpha_1} T \in \lambda \xi^1 fy1'[\lambda] - e^{2\alpha_1+\alpha_2} T \in \lambda \xi^1 fy1'[\lambda] + e^{\alpha_1+\alpha_2} \in \lambda \xi^2 fy1'[\lambda] + \\
& \quad \left. e^{2\alpha_1+2\alpha_2} \in \lambda \xi^2 fy1'[\lambda] - e^{\alpha_1+\alpha_2} T \in \lambda \xi^2 fy1'[\lambda] - e^{2\alpha_1+2\alpha_2} T \in \lambda \xi^2 fy1'[\lambda] \right)
\end{aligned}$$

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In[ ]:= {sol4} =
  DSolve[Coefficient[CE[diff /. sol1 /. sol2 /. sol3], QU[y, x]] == 0 ^ fyx1[0] == 0, fyx1, λ]
Out[ ]=

$$\left\{ \left\{ \text{fyx1} \rightarrow \text{Function} \left[ \{ \lambda \}, \right. \right. \right.$$


$$\left. \left( 2 e^{-\alpha_1 - \alpha_2 - \frac{1}{2}} e^{-\alpha_1 - \alpha_2} (-1 + T) \lambda^2 \left( -e^{\alpha_1} \eta_1 \xi_1 + 2 e^{2\alpha_1 + \alpha_2} \eta_1 \xi_1 + \eta_2 \xi_1 - e^{\alpha_1 + \alpha_2} \eta_1 \xi_2 + 2 e^{2\alpha_1 + 2\alpha_2} \eta_1 \xi_2 - e^{\alpha_2} \eta_2 \xi_2 + 2 e^{\alpha_1 + 2\alpha_2} \eta_2 \xi_2 \right) \right. \right.$$


$$\left. \left( -1 + e^{\frac{1}{2}} e^{-\alpha_1 - \alpha_2} (-1 + T) \lambda^2 \left( -e^{\alpha_1} \eta_1 \xi_1 + 2 e^{2\alpha_1 + \alpha_2} \eta_1 \xi_1 + \eta_2 \xi_1 - e^{\alpha_1 + \alpha_2} \eta_1 \xi_2 + 2 e^{2\alpha_1 + 2\alpha_2} \eta_1 \xi_2 - e^{\alpha_2} \eta_2 \xi_2 + 2 e^{\alpha_1 + 2\alpha_2} \eta_2 \xi_2 \right) \right) \right.$$


$$\left. \left. \eta_2 \xi_1 \right) \right/ \left( (-1 + T) \left( -e^{\alpha_1} \eta_1 \xi_1 + 2 e^{2\alpha_1 + \alpha_2} \eta_1 \xi_1 + \eta_2 \xi_1 - \right. \right.$$


$$\left. \left. e^{\alpha_1 + \alpha_2} \eta_1 \xi_2 + 2 e^{2\alpha_1 + 2\alpha_2} \eta_1 \xi_2 - e^{\alpha_2} \eta_2 \xi_2 + 2 e^{\alpha_1 + 2\alpha_2} \eta_2 \xi_2 \right) \right) \right] \right\}$$

  

In[ ]:= CE[diff /. sol1 /. sol2 /. sol3 /. sol4]
Out[ ]=

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