Pensieve header: Analyzing Reidemeister moves by hand.

Integration

As in Talks/Beijing-2407

```
In[∘]:= CF[ω_. ℰ_𝔼, specs___] := CF[ω, specs] CF[#, specs] & /@ ℰ;
        CF[ℰ_List] := CF /@ ℰ;
        CF[q_, vp_, h_] := Module[{H}, Expand@Collect[q, vp, H] /. H → h];
        CF[q_, vp_] := CF[q, vp, Factor];
        CF[q_] := CF[q, ϵ | (p | x | π | ξ)_ | (p | x | π | ξ)₊ | (p | x | π | ξ)₋];
```

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In[∘]:= 𝔼 /: 𝔼[A_] 𝔼[B_] := 𝔼[A + B];
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In[∘]:= $π = Identity;
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```
In[∘]:= Unprotect[Integrate];
        ∫ω_. 𝔼[L_] ⅆ(vs_List) := Module[{n, L0, Q, Δ, G, Z0, Z, λ, DZ, DDZ, FZ, a, b},
           n = Length@vs; L0 = L /. ϵ → 0;
           Q = Table[(-∂_{vs⟦a⟧,vs⟦b⟧} L0) /. Thread[vs → 0] /. (p | x)__ → 0, {a, n}, {b, n}];
           If[(Δ = Det[Q]) == 0, Return@"Degenerate Q!"];
           Z = Z0 = CF@$π[L + vs.Q.vs / 2]; G = Inverse[Q];
           FixedPoint[((DZ = Table[∂_v Z, {v, vs}];
              DDZ = Table[∂_u DZ, {u, vs}];
              FZ = Sum[G⟦a, b⟧ (DDZ⟦a, b⟧ + DZ⟦a⟧ DZ⟦b⟧), {a, n}, {b, n}] / 2;
              Z = CF[Z0 + ∫_0^λ $π[FZ] ⅆλ]) &, Z];
           PowerExpand@Factor[ω Δ^(-1/2)] 𝔼[CF[Z /. λ → 1 /. Thread[vs → 0]]]];
        Protect[Integrate];
```

```
In[∘]:= ∫𝔼[- μ x^2 / 2 + ⅈ ξ x] ⅆ{x}
```

Out[∘]=

$$\frac{\mathbb{E}\left[-\frac{\xi^2}{2\,\mu}\right]}{\sqrt{\mu}}$$

```
In[∘]:= FofG = ∫𝔼[- μ (x - a)^2 / 2 + ⅈ ξ x] ⅆ{x}
```

Out[∘]=

$$\frac{\mathbb{E}\left[\frac{\mathrm{i}\,(2\,a\,\mu+\mathrm{i}\,\xi)\,\xi}{2\,\mu}\right]}{\sqrt{\mu}}$$

In[∘]:=

$$\int \text{FofG}\;\mathbb{E}[-i\;\xi\;x]\;d\{\xi\}$$

Out[∘]=

$$\mathbb{E}\left[-\frac{1}{2}\;(a-x)^2\;\mu\right]$$

In[∘]:=

$$L = -\frac{1}{2}\;\{x_1,\,x_2\}.\begin{pmatrix}a & b\\ b & c\end{pmatrix}.\{x_1,\,x_2\} + \{\xi_1,\,\xi_2\}.\{x_1,\,x_2\};$$

$$Z12 = \int \mathbb{E}[L]\;d\{x_1,\,x_2\}$$

Out[∘]=

$$\frac{\mathbb{E}\left[\frac{c\;\xi_1^2}{2\,(-b^2+a\,c)}+\frac{b\;\xi_1\;\xi_2}{b^2-a\,c}+\frac{a\;\xi_2^2}{2\,(-b^2+a\,c)}\right]}{\sqrt{-b^2+a\,c}}$$

In[∘]:=

$$\left\{Z1 = \int \mathbb{E}[L]\;d\{x_1\},\;Z12 == \int Z1\;d\{x_2\}\right\}$$

Out[∘]=

$$\left\{\frac{\mathbb{E}\left[-\frac{(-b^2+a\,c)\;x_2^2}{2\,a}-\frac{b\;x_2\;\xi_1}{a}+\frac{\xi_1^2}{2\,a}+x_2\;\xi_2\right]}{\sqrt{a}},\;\text{True}\right\}$$

In[∘]:=

$$\$\pi = \text{Normal}\left[\# + O[\epsilon]^{13}\right]\;\&;\;\int \mathbb{E}\left[-\phi^2/2+\epsilon\;\phi^3/6\right]\;d\{\phi\}$$

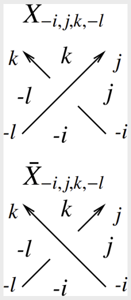
Out[∘]=

$$\mathbb{E}\left[\frac{5\,\epsilon^2}{24}+\frac{5\,\epsilon^4}{16}+\frac{1105\,\epsilon^6}{1152}+\frac{565\,\epsilon^8}{128}+\frac{82\,825\,\epsilon^{10}}{3072}+\frac{19\,675\,\epsilon^{12}}{96}\right]$$

The Quadratic

Matrices from SSAlexander4Tangles.nb

In[∘]:=



$$\left(\begin{array}{l} M = \begin{pmatrix}-1 & -1 & 2 & 0\\ 0 & 0 & 0 & 0\\ 0 & 1 & -1 & 0\\ 1 & 0 & -1 & 0\end{pmatrix}\\ \overline{M} = \begin{pmatrix}1 & -1 & 0 & 0\\ 0 & 0 & 0 & 0\\ -2 & 1 & 1 & 0\\ 1 & 0 & -1 & 0\end{pmatrix}\end{array}\right);$$

In[∘]:=

$$Q_{i\_,j\_,k\_,l\_} := \{p_i,\,p_j,\,p_k,\,p_l\}.\left(t\,M - t^{-1}\,M^{\mathsf{T}}\right).\{x_i,\,x_j,\,x_k,\,x_l\};$$

$$\overline{Q}_{i\_,j\_,k\_,l\_} := \{p_i,\,p_j,\,p_k,\,p_l\}.\left(t\,\overline{M} - t^{-1}\,\overline{M}^{\mathsf{T}}\right).\{x_i,\,x_j,\,x_k,\,x_l\};$$

In[∘]:=
```
CF[Q_{i,j,k,l}]
```

Out[∘]=

$$-\frac{(-1+t)\,(1+t)\,p_i\,x_i}{t}+\frac{p_j\,x_i}{t}-\frac{2\,p_k\,x_i}{t}+t\,p_l\,x_i-t\,p_i\,x_j+t\,p_k\,x_j+2\,t\,p_i\,x_k-\frac{p_j\,x_k}{t}-\frac{(-1+t)\,(1+t)\,p_k\,x_k}{t}-t\,p_l\,x_k-\frac{p_i\,x_l}{t}+\frac{p_k\,x_l}{t}$$

In[∘]:=
```
{CF[Q_{i,j,k,l} /. x_ → x], CF[Q_{i,j,k,l} /. p_ → p]}
```

Out[∘]=
```
{0, 0}
```

In[∘]:=
```
Collect[Q_{i,j,k,l}, p_, Simplify]
```

Out[∘]=

$$\frac{p_j\,(x_i-x_k)}{t}+t\,p_l\,(x_i-x_k)+\frac{p_k\,\left(-2\,x_i+t^2\,x_j+x_k-t^2\,x_k+x_l\right)}{t}+p_i\left(\left(\frac{1}{t}-t\right)x_i-t\,x_j+2\,t\,x_k-\frac{x_l}{t}\right)$$

In[∘]:=
```
Collect[Q_{i,j,k,l}, x_, Simplify]
```

Out[∘]=

$$\frac{\left(p_i-t^2\,p_i+p_j-2\,p_k+t^2\,p_l\right)x_i}{t}+t\,(-p_i+p_k)\,x_j+\left(2\,t\,p_i-\frac{p_j}{t}+\left(\frac{1}{t}-t\right)p_k-t\,p_l\right)x_k+\frac{(-p_i+p_k)\,x_l}{t}$$

In[∘]:=
```
CF[Q_{i,j,k,l} /. {p_i → (p_+ + p_-) / 2, p_k → (p_+ - p_-) / 2, x_i → (x_+ + x_-) / 2, x_k → (x_+ - x_-) / 2}]
```

Out[∘]=

$$-\frac{(-1+t)\,(1+t)\,p_-\,x_-}{t}-\frac{\left(1+t^2\right)x_-\,p_+}{2\,t}+\frac{\left(1+t^2\right)p_-\,x_+}{2\,t}+\frac{x_-\,p_j}{t}+t\,x_-\,p_l-t\,p_-\,x_j-\frac{p_-\,x_l}{t}$$

In[∘]:=
```
CF[Q̄_{i,j,k,l} /. {p_i → (p_+ + p_-) / 2, p_k → (p_+ - p_-) / 2, x_i → (x_+ + x_-) / 2, x_k → (x_+ - x_-) / 2}]
```

Out[∘]=

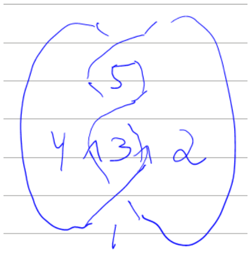
$$\frac{(-1+t)\,(1+t)\,p_-\,x_-}{t}-\frac{\left(1+t^2\right)x_-\,p_+}{2\,t}+\frac{\left(1+t^2\right)p_-\,x_+}{2\,t}+\frac{x_-\,p_j}{t}+t\,x_-\,p_l-t\,p_-\,x_j-\frac{p_-\,x_l}{t}$$

In[∘]:=
```
CF[Q_{i,j,k,l} - Q̄_{i,j,k,l} /. {p_i → (p_+ + p_-) / 2, p_k → (p_+ - p_-) / 2, x_i → (x_+ + x_-) / 2, x_k → (x_+ - x_-) / 2}]
```

Out[∘]=

$$-\frac{2\,(-1+t)\,(1+t)\,p_-\,x_-}{t}$$

The Trefoil



In[∘]:= $L = CF[Q_{1,2,3,4} + Q_{3,2,5,4} + Q_{5,2,1,4}]$

Out[∘]=

$$-\frac{2\,(-1+t)\,(1+t)\,p_1\,x_1}{t}-\frac{2\,p_3\,x_1}{t}+2\,t\,p_5\,x_1+2\,t\,p_1\,x_3-\frac{2\,(-1+t)\,(1+t)\,p_3\,x_3}{t}-\frac{2\,p_5\,x_3}{t}-\frac{2\,p_1\,x_5}{t}+2\,t\,p_3\,x_5-\frac{2\,(-1+t)\,(1+t)\,p_5\,x_5}{t}$$

In[∘]:= $\{CF[(\partial_{p_1} L) + (\partial_{p_3} L) + (\partial_{p_5} L)],\ CF[(\partial_{x_1} L) + (\partial_{x_3} L) + (\partial_{x_5} L)]\}$

Out[∘]=

$\{0,\ 0\}$

In[∘]:= $\int \mathbb{E}[L]\ \mathbb{d}\{p_1,\ p_3,\ x_1,\ x_3\}$

Out[∘]=

$$\frac{t^2\,\mathbb{E}[0]}{4\,\left(1-t^2+t^4\right)}$$

In[∘]:= $CF[L,\ (p\mid x)_5]$

Out[∘]=

$$\frac{2\,p_5\,\left(t^2\,x_1-x_3\right)}{t}-\frac{2\,\left(-p_1\,x_1+t^2\,p_1\,x_1+p_3\,x_1-t^2\,p_1\,x_3-p_3\,x_3+t^2\,p_3\,x_3\right)}{t}+\frac{2\,\left(-p_1+t^2\,p_3\right)\,x_5}{t}-\frac{2\,(-1+t)\,(1+t)\,p_5\,x_5}{t}$$

In[∘]:= $CF[L\ /.\ \{p_5 \to p_+ - p_1 - p_3,\ x_5 \to x_+ - x_1 - x_3\}]$

Out[∘]=

$$-\frac{2\,(-1+t)\,(1+t)\,p_+\,x_+}{t}+\frac{2\,\left(-2+t^2\right)\,x_+\,p_1}{t}+\frac{2\,\left(-1+2\,t^2\right)\,x_+\,p_3}{t}+\frac{2\,\left(-1+2\,t^2\right)\,p_+\,x_1}{t}-\frac{6\,(-1+t)\,(1+t)\,p_1\,x_1}{t}-6\,t\,p_3\,x_1+\frac{2\,\left(-2+t^2\right)\,p_+\,x_3}{t}+\frac{6\,p_1\,x_3}{t}-\frac{6\,(-1+t)\,(1+t)\,p_3\,x_3}{t}$$

In[∘]:= $CF[L\ /.\ \{p_5 \to p_+ - p_1 - p_3,\ x_5 \to x_+ - x_1 - x_3,\ p_3 \to p_0 - p_1,\ x_3 \to x_0 - x_1\}]$

Out[∘]=

$$-\frac{2\,(-1+t)\,(1+t)\,p_+\,x_+}{t}+2\,t\,x_+\,p_0-\frac{4\,x_+\,p_1}{t}+\frac{2\,(-1+t)\,(1+t)\,x_+\,p_3}{t}-\frac{2\,p_+\,x_0}{t}-\frac{2\,(-1+t)\,(1+t)\,p_0\,x_0}{t}+4\,t\,p_1\,x_0+\frac{2\,p_3\,x_0}{t}+4\,t\,p_+\,x_1-\frac{4\,p_0\,x_1}{t}-\frac{8\,(-1+t)\,(1+t)\,p_1\,x_1}{t}-4\,t\,p_3\,x_1+\frac{2\,(-1+t)\,(1+t)\,p_+\,x_3}{t}-2\,t\,p_0\,x_3+\frac{4\,p_1\,x_3}{t}-\frac{2\,(-1+t)\,(1+t)\,p_3\,x_3}{t}$$

In[∘]:= $\int \mathbb{E}[L]\ \mathbb{d}\{p_1,\ p_5,\ x_1,\ x_5\}$

Out[∘]=

$$\frac{t^2\,\mathbb{E}[0]}{4\,\left(1-t^2+t^4\right)}$$

In[∘]:=

$$\text{CF}[Q_{1,2,3,4}+Q_{3,2,5,4}+Q_{5,2,1,4}\text{ /. }\{p_1\to -p_3-p_5,\ x_1\to -x_3-x_5\}]$$

Out[∘]=

$$-\frac{6\,(-1+t)\,(1+t)\,p_3\,x_3}{t}-6\,t\,p_5\,x_3+\frac{6\,p_3\,x_5}{t}-\frac{6\,(-1+t)\,(1+t)\,p_5\,x_5}{t}$$

Re-sectioning

In[∘]:=

$$\{\text{CF}[Q_{i,j,k,l}\text{ /. }(p\mid x)_i\to 0],\ \text{CF}[Q_{i,j,k,l}\text{ /. }(p\mid x)_k\to 0]\}$$

Out[∘]=

$$\Big\{t\,p_k\,x_j-\frac{p_j\,x_k}{t}-\frac{(-1+t)\,(1+t)\,p_k\,x_k}{t}-t\,p_l\,x_k+\frac{p_k\,x_l}{t},\ -\frac{(-1+t)\,(1+t)\,p_i\,x_i}{t}+\frac{p_j\,x_i}{t}+t\,p_l\,x_i-t\,p_i\,x_j-\frac{p_i\,x_l}{t}\Big\}$$

In[∘]:=

$$\{\text{lhs},\text{rhs}\}=\Big\{\int\mathbb{E}[Q_{i,j,k,l}\text{ /. }(p\mid x)_i\to 0]\,\text{ⅆ}\{p_k,x_k\},\ \int\mathbb{E}[Q_{i,j,k,l}\text{ /. }(p\mid x)_k\to 0]\,\text{ⅆ}\{p_i,x_i\}\Big\}$$

Out[∘]=

$$\Big\{-\frac{\mathrm{i}\,t\,\mathbb{E}\Big[-\frac{t\,p_j\,x_j}{(-1+t)\,(1+t)}-\frac{t^3\,p_l\,x_j}{(-1+t)\,(1+t)}-\frac{p_j\,x_l}{(-1+t)\,t\,(1+t)}-\frac{t\,p_l\,x_l}{(-1+t)\,(1+t)}\Big]}{-1+t^2},\ -\frac{\mathrm{i}\,t\,\mathbb{E}\Big[-\frac{t\,p_j\,x_j}{(-1+t)\,(1+t)}-\frac{t^3\,p_l\,x_j}{(-1+t)\,(1+t)}-\frac{p_j\,x_l}{(-1+t)\,t\,(1+t)}-\frac{t\,p_l\,x_l}{(-1+t)\,(1+t)}\Big]}{-1+t^2}\Big\}$$

In[∘]:=

$$\text{lhs}==\text{rhs}$$

Out[∘]=

True

In[∘]:=

$$f=\xi_j\,x_j+\pi_j\,p_j+\xi_l\,x_l+\pi_l\,p_l;$$

$$\{\text{lhs},\text{rhs}\}=$$

$$\Big\{\int\mathbb{E}[(Q_{i,j,k,l}+f)\text{ /. }(p\mid x)_i\to 0]\,\text{ⅆ}\{p_k,x_k\},\ \int\mathbb{E}[(Q_{i,j,k,l}+f)\text{ /. }(p\mid x)_k\to 0]\,\text{ⅆ}\{p_i,x_i\}\Big\}$$

$$\text{lhs}==\text{rhs}$$

Out[∘]=

$$\Big\{-\frac{1}{-1+t^2}\,\mathrm{i}\,t\,\mathbb{E}\Big[p_j\,\pi_j+p_l\,\pi_l-\frac{t\,p_j\,x_j}{(-1+t)\,(1+t)}-\frac{t^3\,p_l\,x_j}{(-1+t)\,(1+t)}-\frac{p_j\,x_l}{(-1+t)\,t\,(1+t)}-\frac{t\,p_l\,x_l}{(-1+t)\,(1+t)}+x_j\,\xi_j+x_l\,\xi_l\Big],$$

$$-\frac{1}{-1+t^2}\,\mathrm{i}\,t\,\mathbb{E}\Big[p_j\,\pi_j+p_l\,\pi_l-\frac{t\,p_j\,x_j}{(-1+t)\,(1+t)}-\frac{t^3\,p_l\,x_j}{(-1+t)\,(1+t)}-\frac{p_j\,x_l}{(-1+t)\,t\,(1+t)}-\frac{t\,p_l\,x_l}{(-1+t)\,(1+t)}+x_j\,\xi_j+x_l\,\xi_l\Big]\Big\}$$

Out[∘]=

True

In[∘]:=

$$f = \xi_j\, x_j + \pi_j\, p_j + \xi_l\, x_l + \pi_l\, p_l + \pi\, (p_i - p_k) + \xi\, (x_i - x_k);$$

$$\{lhs, rhs\} = \left\{\int \mathbb{E}[(Q_{i,j,k,l} + f) \,/.\, (p \mid x)_i \to 0]\, \mathbb{d}\{p_k, x_k\},\ \int \mathbb{E}[(Q_{i,j,k,l} + f) \,/.\, (p \mid x)_k \to 0]\, \mathbb{d}\{p_i, x_i\}\right\}$$

$$lhs == rhs$$

Out[∘]=

$$\begin{aligned}\Big\{&-\frac{1}{-1+t^2}\,\mathrm{i}\,t\,\mathbb{E}\Big[\frac{\pi\,t\,\xi}{(-1+t)\,(1+t)} + \frac{\pi\,p_j}{(-1+t)\,(1+t)} + \frac{\pi\,t^2\,p_l}{(-1+t)\,(1+t)} + \\ &\quad p_j\,\pi_j + p_l\,\pi_l - \frac{t^2\,\xi\,x_j}{(-1+t)\,(1+t)} - \frac{t\,p_j\,x_j}{(-1+t)\,(1+t)} - \frac{t^3\,p_l\,x_j}{(-1+t)\,(1+t)} - \\ &\quad \frac{\xi\,x_l}{(-1+t)\,(1+t)} - \frac{p_j\,x_l}{(-1+t)\,t\,(1+t)} - \frac{t\,p_l\,x_l}{(-1+t)\,(1+t)} + x_j\,\xi_j + x_l\,\xi_l\Big], \\ &-\frac{1}{-1+t^2}\,\mathrm{i}\,t\,\mathbb{E}\Big[\frac{\pi\,t\,\xi}{(-1+t)\,(1+t)} + \frac{\pi\,p_j}{(-1+t)\,(1+t)} + \frac{\pi\,t^2\,p_l}{(-1+t)\,(1+t)} + p_j\,\pi_j + \\ &\quad p_l\,\pi_l - \frac{t^2\,\xi\,x_j}{(-1+t)\,(1+t)} - \frac{t\,p_j\,x_j}{(-1+t)\,(1+t)} - \frac{t^3\,p_l\,x_j}{(-1+t)\,(1+t)} - \\ &\quad \frac{\xi\,x_l}{(-1+t)\,(1+t)} - \frac{p_j\,x_l}{(-1+t)\,t\,(1+t)} - \frac{t\,p_l\,x_l}{(-1+t)\,(1+t)} + x_j\,\xi_j + x_l\,\xi_l\Big]\Big\}\end{aligned}$$

Out[∘]=

True

In[∘]:=

$$f = \xi_j\, x_j + \pi_j\, p_j + \xi_l\, x_l + \pi_l\, p_l + \pi\, (p_i - p_k) + \xi\, (x_i - x_k);$$

$$\{lhs, rhs\} = \left\{\int \mathbb{E}\left[\left(\overline{Q}_{i,j,k,l} + f\right) \,/.\, (p \mid x)_i \to 0\right] \mathbb{d}\{p_k, x_k\},\ \int \mathbb{E}\left[\left(\overline{Q}_{i,j,k,l} + f\right) \,/.\, (p \mid x)_k \to 0\right] \mathbb{d}\{p_i, x_i\}\right\}$$

$$lhs == rhs$$

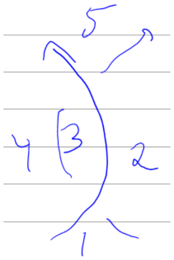
Out[∘]=

$$\begin{aligned}\Big\{&-\frac{1}{-1+t^2}\,\mathrm{i}\,t\,\mathbb{E}\Big[-\frac{\pi\,t\,\xi}{(-1+t)\,(1+t)} - \frac{\pi\,p_j}{(-1+t)\,(1+t)} - \frac{\pi\,t^2\,p_l}{(-1+t)\,(1+t)} + \\ &\quad p_j\,\pi_j + p_l\,\pi_l + \frac{t^2\,\xi\,x_j}{(-1+t)\,(1+t)} + \frac{t\,p_j\,x_j}{(-1+t)\,(1+t)} + \frac{t^3\,p_l\,x_j}{(-1+t)\,(1+t)} + \\ &\quad \frac{\xi\,x_l}{(-1+t)\,(1+t)} + \frac{p_j\,x_l}{(-1+t)\,t\,(1+t)} + \frac{t\,p_l\,x_l}{(-1+t)\,(1+t)} + x_j\,\xi_j + x_l\,\xi_l\Big], \\ &-\frac{1}{-1+t^2}\,\mathrm{i}\,t\,\mathbb{E}\Big[-\frac{\pi\,t\,\xi}{(-1+t)\,(1+t)} - \frac{\pi\,p_j}{(-1+t)\,(1+t)} - \frac{\pi\,t^2\,p_l}{(-1+t)\,(1+t)} + p_j\,\pi_j + \\ &\quad p_l\,\pi_l + \frac{t^2\,\xi\,x_j}{(-1+t)\,(1+t)} + \frac{t\,p_j\,x_j}{(-1+t)\,(1+t)} + \frac{t^3\,p_l\,x_j}{(-1+t)\,(1+t)} + \\ &\quad \frac{\xi\,x_l}{(-1+t)\,(1+t)} + \frac{p_j\,x_l}{(-1+t)\,t\,(1+t)} + \frac{t\,p_l\,x_l}{(-1+t)\,(1+t)} + x_j\,\xi_j + x_l\,\xi_l\Big]\Big\}\end{aligned}$$

Out[∘]=

True

R2b



In[∘]:=

$$\mathbf{Q}_{1,2,3,4}+\overline{\mathbf{Q}}_{3,2,5,4}\ //\ \mathbf{CF}$$

Out[∘]=

$$-\frac{(-1+t)\,(1+t)\,p_1\,x_1}{t}+\frac{p_2\,x_1}{t}-\frac{2\,p_3\,x_1}{t}+t\,p_4\,x_1-t\,p_1\,x_2+t\,p_5\,x_2+2\,t\,p_1\,x_3-2\,t\,p_5\,x_3-\frac{p_1\,x_4}{t}+\frac{p_5\,x_4}{t}-\frac{p_2\,x_5}{t}+\frac{2\,p_3\,x_5}{t}-t\,p_4\,x_5+\frac{(-1+t)\,(1+t)\,p_5\,x_5}{t}$$

In[∘]:=

$$\mathbf{CF}\left[\mathbf{Q}_{1,2,3,4}+\overline{\mathbf{Q}}_{3,2,5,4},\ (\mathbf{p}\mid\mathbf{x})_3\right]$$

Out[∘]=

$$2\,t\,(p_1-p_5)\,x_3-\frac{2\,p_3\,(x_1-x_5)}{t}-\frac{-p_1\,x_1+t^2\,p_1\,x_1-p_2\,x_1-t^2\,p_4\,x_1+t^2\,p_1\,x_2-t^2\,p_5\,x_2+p_1\,x_4-p_5\,x_4+p_2\,x_5+t^2\,p_4\,x_5+p_5\,x_5-t^2\,p_5\,x_5}{t}$$

In[∘]:=

$$\mathbf{Expand}\left[\mathbf{Q}_{1,2,3,4}+\overline{\mathbf{Q}}_{3,2,5,4}\ /.\ \{p_5\to p_1,\ x_5\to x_1\}\right]$$

Out[∘]=

$$0$$

In[∘]:=

$$\mathbf{CF}\left[\mathbf{Q}_{1,2,3,4}+\overline{\mathbf{Q}}_{3,2,5,4}\ /.\ (\mathbf{p}\mid\mathbf{x})_3\to 0\right]$$

Out[∘]=

$$-\frac{(-1+t)\,(1+t)\,p_1\,x_1}{t}+\frac{p_2\,x_1}{t}+t\,p_4\,x_1-t\,p_1\,x_2+t\,p_5\,x_2-\frac{p_1\,x_4}{t}+\frac{p_5\,x_4}{t}-\frac{p_2\,x_5}{t}-t\,p_4\,x_5+\frac{(-1+t)\,(1+t)\,p_5\,x_5}{t}$$

In[∘]:=

$$\mathbf{CF}\left[\mathbf{Q}_{1,2,3,4}+\overline{\mathbf{Q}}_{3,2,5,4}\ /.\ (\mathbf{p}\mid\mathbf{x})_3\to 0\ /.\ p\_\to p\right]$$

Out[∘]=

$$\frac{2\,p\,x_1}{t}-\frac{2\,p\,x_5}{t}$$

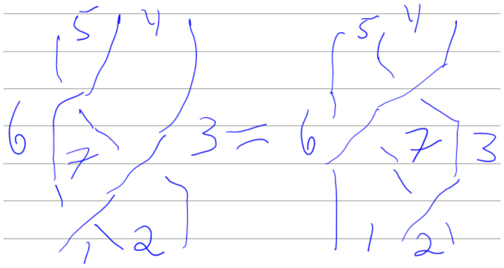
In[∘]:=

$$\mathbf{CF}\left[\mathbf{Q}_{1,2,3,4}+\overline{\mathbf{Q}}_{3,2,5,4}\ /.\ p\_\to p\right]$$

Out[∘]=

$$0$$

R3



In[•]:= $\mathbf{lhs} = \mathbf{CF}[Q_{1,2,7,6} + Q_{2,3,4,7} + Q_{7,4,5,6}];$

$\mathbf{rhs} = \mathbf{CF}[Q_{2,3,7,1} + Q_{1,7,5,6} + Q_{7,3,4,5}]$

Out[•]=

$$-\frac{(-1+t)\ (1+t)\ p_1\,x_1}{t} - \frac{p_2\,x_1}{t} - \frac{2\,p_5\,x_1}{t} + t\,p_6\,x_1 + \frac{2\,p_7\,x_1}{t} + t\,p_1\,x_2 - \frac{(-1+t)\ (1+t)\ p_2\,x_2}{t} + \frac{p_3\,x_2}{t} - \frac{2\,p_7\,x_2}{t} - t\,p_2\,x_3 + t\,p_4\,x_3 - \frac{p_3\,x_4}{t} - \frac{(-1+t)\ (1+t)\ p_4\,x_4}{t} - t\,p_5\,x_4 + 2\,t\,p_7\,x_4 + 2\,t\,p_1\,x_5 + \frac{p_4\,x_5}{t} - \frac{(-1+t)\ (1+t)\ p_5\,x_5}{t} - t\,p_6\,x_5 - \frac{2\,p_7\,x_5}{t} - \frac{p_1\,x_6}{t} + \frac{p_5\,x_6}{t} - 2\,t\,p_1\,x_7 + 2\,t\,p_2\,x_7 - \frac{2\,p_4\,x_7}{t} + 2\,t\,p_5\,x_7 - \frac{2\ (-1+t)\ (1+t)\ p_7\,x_7}{t}$$

In[•]:= $\mathbf{Simplify}[\mathbf{lhs} == \mathbf{rhs}]$

Out[•]=

$$\frac{1}{t}\left(-2\,p_7\,x_1 - t^2\,p_1\,x_2 - p_4\,x_2 + p_7\,x_2 + t^2\,p_7\,x_2 - 2\,t^2\,p_7\,x_4 - t^2\,p_1\,x_5 - p_4\,x_5 + p_7\,x_5 + t^2\,p_7\,x_5 + 2\,t^2\,p_1\,x_7 + 2\,p_4\,x_7 + p_2\left(x_1 + t^2\,x_4 - \left(1+t^2\right)\,x_7\right) + p_5\left(x_1 + t^2\,x_4 - \left(1+t^2\right)\,x_7\right)\right) == 0$$

In[•]:= $\mathbf{CF}[\mathbf{lhs},\ (\mathbf{p}\mid\mathbf{x})_7]$

Out[•]=

$$\frac{2\,p_7\left(-x_1 + t^2\,x_2 - t^2\,x_4 + t^2\,x_5\right)}{t} - \frac{1}{t}\left(-p_1\,x_1 + t^2\,p_1\,x_1 - p_2\,x_1 - t^2\,p_6\,x_1 + t^2\,p_1\,x_2 - p_2\,x_2 + t^2\,p_2\,x_2 - p_3\,x_2 + 2\,p_4\,x_2 + t^2\,p_2\,x_3 - t^2\,p_4\,x_3 - 2\,t^2\,p_2\,x_4 + p_3\,x_4 - p_4\,x_4 + t^2\,p_4\,x_4 - t^2\,p_5\,x_4 + p_4\,x_5 - p_5\,x_5 + t^2\,p_5\,x_5 + t^2\,p_6\,x_5 + p_1\,x_6 - p_5\,x_6\right) + \frac{2\left(t^2\,p_1 - p_2 + p_4 - p_5\right)\,x_7}{t} - \frac{2\ (-1+t)\ (1+t)\ p_7\,x_7}{t}$$

In[∘]:= $\int \mathbb{E}[\mathbf{lhs}]\, \mathbb{d}\{p_7, x_7\}$

Out[∘]=

$$-\frac{1}{2\left(-1+t^2\right)}$$

$$\mathbb{i}\, t\, \mathbb{E}\Big[-\frac{\left(1+t^4\right) p_1 x_1}{(-1+t)\, t\, (1+t)}+\frac{\left(1+t^2\right) p_2 x_1}{(-1+t)\, t\, (1+t)}-\frac{2 p_4 x_1}{(-1+t)\, t\, (1+t)}+\frac{2 p_5 x_1}{(-1+t)\, t\, (1+t)}+t p_6 x_1+\frac{t\left(1+t^2\right) p_1 x_2}{(-1+t)(1+t)}-\frac{\left(1+t^4\right) p_2 x_2}{(-1+t)\, t\, (1+t)}+\frac{p_3 x_2}{t}+\frac{2 p_4 x_2}{(-1+t)\, t\, (1+t)}-\frac{2 t p_5 x_2}{(-1+t)(1+t)}-t p_2 x_3+t p_4 x_3-\frac{2 t^3 p_1 x_4}{(-1+t)(1+t)}+\frac{2 t^3 p_2 x_4}{(-1+t)(1+t)}-\frac{p_3 x_4}{t}-\frac{\left(1+t^4\right) p_4 x_4}{(-1+t)\, t\, (1+t)}+\frac{t\left(1+t^2\right) p_5 x_4}{(-1+t)(1+t)}+\frac{2 t^3 p_1 x_5}{(-1+t)(1+t)}-\frac{2 t p_2 x_5}{(-1+t)(1+t)}+\frac{\left(1+t^2\right) p_4 x_5}{(-1+t)\, t\, (1+t)}-\frac{\left(1+t^4\right) p_5 x_5}{(-1+t)\, t\, (1+t)}-t p_6 x_5-\frac{p_1 x_6}{t}+\frac{p_5 x_6}{t}\Big]$$

In[∘]:= $\mathbf{CF}[\mathbf{rhs}, (p \mid x)_7]$

Out[∘]=

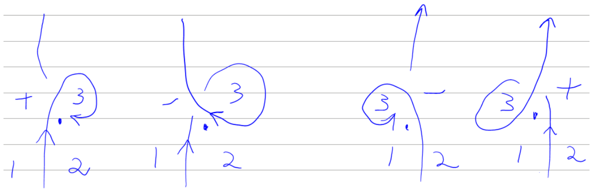
$$\frac{2 p_7\left(x_1-x_2+t^2 x_4-x_5\right)}{t}-\frac{1}{t}\left(-p_1 x_1+t^2 p_1 x_1+p_2 x_1+2 p_5 x_1-t^2 p_6 x_1-t^2 p_1 x_2-p_2 x_2+t^2 p_2 x_2-p_3 x_2+t^2 p_2 x_3-t^2 p_4 x_3+p_3 x_4-p_4 x_4+t^2 p_4 x_4+t^2 p_5 x_4-2 t^2 p_1 x_5-p_4 x_5-p_5 x_5+t^2 p_5 x_5+t^2 p_6 x_5+p_1 x_6-p_5 x_6\right)-\frac{2\left(t^2 p_1-t^2 p_2+p_4-t^2 p_5\right) x_7}{t}-\frac{2(-1+t)(1+t) p_7 x_7}{t}$$

In[∘]:= $\left(\int \mathbb{E}[\mathbf{rhs}]\, \mathbb{d}\{p_7, x_7\}\right) == \left(\int \mathbb{E}[\mathbf{lhs}]\, \mathbb{d}\{p_7, x_7\}\right)$

Out[∘]=

True

R1s

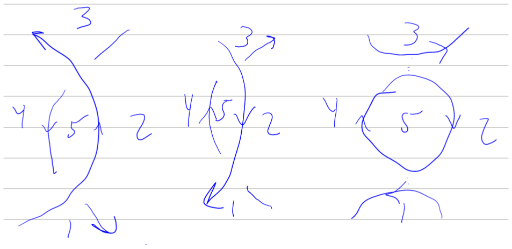


In[∘]:= $\mathbf{R1s} = \mathbf{Simplify}@\left\{Q_{2,3,2,1},\ \overline{Q}_{2,3,2,1},\ \overline{Q}_{1,2,1,3},\ Q_{1,2,1,3}\right\}$

Out[∘]=

{0, 0, 0, 0}

R2C$_+$ and R2C$_-$



In[∘]:=

$$\left\{\text{Collect}\left[\overline{Q}_{4,1,2,5}+Q_{2,3,4,5},\ t,\ \text{Simplify}\right],\ \text{Collect}\left[\overline{Q}_{2,5,4,1}+Q_{4,5,2,3},\ t,\ \text{Simplify}\right]\right\}$$

Out[∘]=

$$\left\{t\,(p_2-p_4)\,(x_1-x_3)-\frac{(p_1-p_3)\,(x_2-x_4)}{t},\ -\frac{(p_2-p_4)\,(x_1-x_3)}{t}+t\,(p_1-p_3)\,(x_2-x_4)\right\}$$

In[∘]:=

$$\left\{\text{Collect}\left[Q_{4,1,2,5}+\overline{Q}_{2,3,4,5},\ t,\ \text{Simplify}\right],\ \text{Collect}\left[Q_{2,5,4,1}+\overline{Q}_{4,5,2,3},\ t,\ \text{Simplify}\right]\right\}$$

Out[∘]=

$$\left\{t\,(p_2-p_4)\,(x_1-x_3)-\frac{(p_1-p_3)\,(x_2-x_4)}{t},\ -\frac{(p_2-p_4)\,(x_1-x_3)}{t}+t\,(p_1-p_3)\,(x_2-x_4)\right\}$$

In[∘]:=

$$\mathbb{E}\left[\overline{Q}_{4,1,2,5}+Q_{2,3,4,5}+i\left(\sum_{k=1}^{4}(\pi_k\,p_k+\xi_k\,x_k)\right)\right]$$

Out[∘]=

$$\mathbb{E}\Big[(t\,p_2-t\,p_4)\,x_1+\left(-\frac{p_1}{t}+\left(-\frac{1}{t}+t\right)p_2+\frac{2\,p_4}{t}-t\,p_5\right)x_2+\left(\left(\frac{1}{t}-t\right)p_2+\frac{p_3}{t}-\frac{2\,p_4}{t}+t\,p_5\right)x_2+$$
$$(-t\,p_2+t\,p_4)\,x_3+\left(2\,t\,p_2-\frac{p_3}{t}+\left(\frac{1}{t}-t\right)p_4-t\,p_5\right)x_4+\left(\frac{p_1}{t}-2\,t\,p_2+\left(-\frac{1}{t}+t\right)p_4+t\,p_5\right)x_4+$$
$$\left(\frac{p_2}{t}-\frac{p_4}{t}\right)x_5+\left(-\frac{p_2}{t}+\frac{p_4}{t}\right)x_5+i\,(p_1\,\pi_1+p_2\,\pi_2+p_3\,\pi_3+p_4\,\pi_4+x_1\,\xi_1+x_2\,\xi_2+x_3\,\xi_3+x_4\,\xi_4)\Big]$$

In[∘]:=

$$\text{px}_{i\_}:=\text{Sequence}[p_i,\ x_i];$$

In[∘]:=

$$\int\mathbb{E}\left[\overline{Q}_{4,1,2,5}+Q_{2,3,4,5}+i\left(\sum_{k=1}^{4}(\pi_k\,p_k+\xi_k\,x_k)\right)\right]d\{\text{px}_4\}$$

Out[∘]=

Degenerate Q!

In[∘]:=

$$\text{CF}\left[\mathbb{E}\left[\overline{Q}_{4,1,2,5}+Q_{2,3,4,5}+i\left(\sum_{k=1}^{4}(\pi_k\,p_k+\xi_k\,x_k)\right)\right],\ (p\,|\,x)_4\right]$$

Out[∘]=

$$\mathbb{E}\Big[p_4\,(i\,\pi_4-t\,x_1+t\,x_3)+$$
$$\frac{i\,t\,p_1\,\pi_1+i\,t\,p_2\,\pi_2+i\,t\,p_3\,\pi_3+t^2\,p_2\,x_1-p_1\,x_2+p_3\,x_2-t^2\,p_2\,x_3+i\,t\,x_1\,\xi_1+i\,t\,x_2\,\xi_2+i\,t\,x_3\,\xi_3}{t}+$$
$$\frac{x_4\,(p_1-p_3+i\,t\,\xi_4)}{t}\Big]$$

In[∘]:= $$\int \mathbb{E}\left[\overline{Q}_{4,1,2,5} + Q_{2,3,4,5} + i\left(\sum_{k=1}^{4} (\pi_k\, p_k + \xi_k\, x_k)\right)\right] d\{px_3, px_4\}$$

Out[∘]= $$\mathbb{E}\left[i\, p_1\, \pi_1 + i\, p_2\, \pi_2 + i\, p_1\, \pi_3 + i\, p_2\, \pi_4 + i\, x_1\, \xi_1 + i\, x_2\, \xi_2 + \frac{\pi_4\, \xi_3}{t} + i\, x_1\, \xi_3 - t\, \pi_3\, \xi_4 + i\, x_2\, \xi_4\right]$$

In[∘]:= $$\left(\int \mathbb{E}\left[\overline{Q}_{4,1,2,5} + Q_{2,3,4,5} + i\left(\sum_{k=1}^{4} (\pi_k\, p_k + \xi_k\, x_k)\right)\right] d\{px_3, px_4\}\right) /. \{\pi_1 \to -\pi_3,\ \xi_1 \to -\xi_3,\ \pi_2 \to -\pi_4,\ \xi_2 \to -\xi_4\}$$

Out[∘]= $$\mathbb{E}\left[\frac{\pi_4\, \xi_3}{t} - t\, \pi_3\, \xi_4\right]$$

In[∘]:= $$\mathrm{CF}\left[\overline{Q}_{4,1,2,5} + Q_{2,3,4,5} + i\,(\pi_v\,(p_1 - p_3) + \xi_v\,(x_1 - x_3) + \pi_h\,(p_2 - p_4) + \xi_h\,(x_2 - x_4))\right]$$

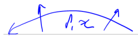
Out[∘]= $$i\, p_2\, \pi_h - i\, p_4\, \pi_h + i\, p_1\, \pi_v - i\, p_3\, \pi_v + t\, p_2\, x_1 - t\, p_4\, x_1 - \frac{p_1\, x_2}{t} + \frac{p_3\, x_2}{t} - t\, p_2\, x_3 + t\, p_4\, x_3 + \frac{p_1\, x_4}{t} - \frac{p_3\, x_4}{t} + i\, x_2\, \xi_h - i\, x_4\, \xi_h + i\, x_1\, \xi_v - i\, x_3\, \xi_v$$

In[∘]:= $$\int \mathbb{E}\left[\overline{Q}_{4,1,2,5} + Q_{2,3,4,5} + i\,(\pi v\,(p_1 - p_3) + \xi v\,(x_1 - x_3) + \pi h\,(p_2 - p_4) + \xi h\,(x_2 - x_4))\right] d\{px_3, px_4\}$$

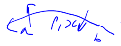
Out[∘]= $$\mathbb{E}\left[-\frac{-\xi v\, \pi h + t^2\, \xi h\, \pi v}{t}\right]$$

The potentials

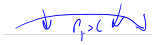
The potential conjecture:



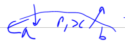
$\longrightarrow (\partial_p \to (t+t^{-1})x,\ \partial_x \to -(t+t^{-1})p)$



$\to (\partial_p \to t^{-1}(x_b - x_a),\ \partial_x \to t(p_a - p_b))$



$\longrightarrow (\partial_p \to -(t+t^{-1})x,\ \partial_x \to (t+t^{-1})p)$



$\to (\partial_p \to t(x_b - x_a),\ \partial_x \to t^{-1}(p_b - p_a))$

(False: The left two seem to have side terms)

For the crossings:

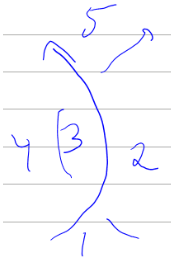
In[∘]:= $$L = Q_{i,j,k,l};\ \mathrm{CF}@\left\{(\partial_{p_i} L) + (\partial_{p_k} L),\ (\partial_{x_i} L) + (\partial_{x_k} L),\ \partial_{p_j} L,\ \partial_{x_j} L,\ \partial_{p_l} L,\ \partial_{x_l} L\right\}$$

Out[∘]= $$\left\{-\frac{(1+t^2)\, x_i}{t} + \frac{(1+t^2)\, x_k}{t},\ \frac{(1+t^2)\, p_i}{t} - \frac{(1+t^2)\, p_k}{t},\ \frac{x_i}{t} - \frac{x_k}{t},\ -t\, p_i + t\, p_k,\ t\, x_i - t\, x_k,\ -\frac{p_i}{t} + \frac{p_k}{t}\right\}$$

In[∘]:= $$L = \overline{Q}_{i,j,k,l};\ \mathrm{CF}@\left\{(\partial_{p_i} L) + (\partial_{p_k} L),\ (\partial_{x_i} L) + (\partial_{x_k} L),\ \partial_{p_j} L,\ \partial_{x_j} L,\ \partial_{p_l} L,\ \partial_{x_l} L\right\}$$

Out[∘]= $$\left\{-\frac{(1+t^2)\, x_i}{t} + \frac{(1+t^2)\, x_k}{t},\ \frac{(1+t^2)\, p_i}{t} - \frac{(1+t^2)\, p_k}{t},\ \frac{x_i}{t} - \frac{x_k}{t},\ -t\, p_i + t\, p_k,\ t\, x_i - t\, x_k,\ -\frac{p_i}{t} + \frac{p_k}{t}\right\}$$

For R2b:



In[∘]:=

$$L = Q_{1,2,3,4} + \overline{Q}_{3,2,5,4};$$
$$CF@\{(\partial_{p_1}L) + (\partial_{p_3}L) + (\partial_{p_5}L),\ (\partial_{x_1}L) + (\partial_{x_3}L) + (\partial_{x_5}L),\ \partial_{p_2}L,\ \partial_{x_2}L,\ \partial_{p_4}L,\ \partial_{x_4}L\}$$

Out[∘]=

$$\left\{-\frac{(1+t^2)\,x_1}{t} + \frac{(1+t^2)\,x_5}{t},\ \frac{(1+t^2)\,p_1}{t} - \frac{(1+t^2)\,p_5}{t},\ \frac{x_1}{t} - \frac{x_5}{t},\ -t\,p_1 + t\,p_5,\ t\,x_1 - t\,x_5,\ -\frac{p_1}{t} + \frac{p_5}{t}\right\}$$

In[∘]:=

$$L = Q_{1,2,3,4} + \overline{Q}_{3,2,5,4}\ /.\ (p\,|\,x)_3 \to 0;$$
$$CF@\{(\partial_{p_1}L) + (\partial_{p_5}L),\ (\partial_{x_1}L) + (\partial_{x_5}L),\ \partial_{p_2}L,\ \partial_{x_2}L,\ \partial_{p_4}L,\ \partial_{x_4}L\}$$

Out[∘]=

$$\left\{-\frac{(-1+t)\,(1+t)\,x_1}{t} + \frac{(-1+t)\,(1+t)\,x_5}{t},\right.$$
$$\left.-\frac{(-1+t)\,(1+t)\,p_1}{t} + \frac{(-1+t)\,(1+t)\,p_5}{t},\ \frac{x_1}{t} - \frac{x_5}{t},\ -t\,p_1 + t\,p_5,\ t\,x_1 - t\,x_5,\ -\frac{p_1}{t} + \frac{p_5}{t}\right\}$$

For σ_1^2:

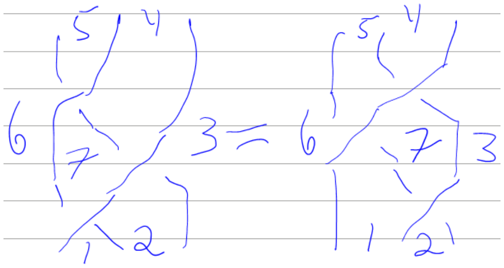
In[∘]:=

$$L = Q_{1,2,3,4} + Q_{3,2,5,4};$$
$$CF@\{(\partial_{p_1}L) + (\partial_{p_3}L) + (\partial_{p_5}L),\ (\partial_{x_1}L) + (\partial_{x_3}L) + (\partial_{x_5}L),\ \partial_{p_2}L,\ \partial_{x_2}L,\ \partial_{p_4}L,\ \partial_{x_4}L\}$$

Out[∘]=

$$\left\{-\frac{(1+t^2)\,x_1}{t} + \frac{(1+t^2)\,x_5}{t},\ \frac{(1+t^2)\,p_1}{t} - \frac{(1+t^2)\,p_5}{t},\ \frac{x_1}{t} - \frac{x_5}{t},\ -t\,p_1 + t\,p_5,\ t\,x_1 - t\,x_5,\ -\frac{p_1}{t} + \frac{p_5}{t}\right\}$$

For R3:



```
In[∘]:= lhs = CF[Q_{1,2,7,6} + Q_{2,3,4,7} + Q_{7,4,5,6}];
        rhs = CF[Q_{2,3,7,1} + Q_{1,7,5,6} + Q_{7,3,4,5}];
        L = lhs; CF@{(∂_{p_1} L) + (∂_{p_7} L) + (∂_{p_5} L), (∂_{x_1} L) + (∂_{x_7} L) + (∂_{x_5} L),
           (∂_{p_2} L) + (∂_{p_4} L), (∂_{x_2} L) + (∂_{x_4} L), ∂_{p_3} L, ∂_{x_3} L, ∂_{p_6} L, ∂_{x_6} L}
        L = rhs; CF@{(∂_{p_1} L) + (∂_{p_5} L), (∂_{x_1} L) + (∂_{x_5} L),
           (∂_{p_2} L) + (∂_{p_7} L) + (∂_{p_4} L), (∂_{x_2} L) + (∂_{x_7} L) + (∂_{x_4} L), ∂_{p_3} L, ∂_{x_3} L, ∂_{p_6} L, ∂_{x_6} L}
        L = Cases[(∫𝔼[lhs] ⅆ{p_7, x_7}), 𝔼[L_] :> L][[1]];
        CF@{(∂_{p_1} L) + (∂_{p_5} L), (∂_{x_1} L) + (∂_{x_5} L), (∂_{p_2} L) + (∂_{p_4} L), (∂_{x_2} L) + (∂_{x_4} L), ∂_{p_3} L, ∂_{x_3} L, ∂_{p_6} L, ∂_{x_6} L}
        L = Cases[(∫𝔼[rhs] ⅆ{p_7, x_7}), 𝔼[L_] :> L][[1]];
        CF@{(∂_{p_1} L) + (∂_{p_5} L), (∂_{x_1} L) + (∂_{x_5} L), (∂_{p_2} L) + (∂_{p_4} L), (∂_{x_2} L) + (∂_{x_4} L), ∂_{p_3} L, ∂_{x_3} L, ∂_{p_6} L, ∂_{x_6} L}
```

Out[∘]=

$$\left\{-\frac{\left(1+t^2\right) x_1}{t}+t\,x_2-t\,x_4+\frac{\left(1+t^2\right) x_5}{t},\ \frac{\left(1+t^2\right) p_1}{t}-\frac{p_2}{t}+\frac{p_4}{t}-\frac{\left(1+t^2\right) p_5}{t},\ \frac{x_1}{t}-\frac{\left(1+t^2\right) x_2}{t}+\frac{\left(1+t^2\right) x_4}{t}-\frac{x_5}{t},\ -t\,p_1+\frac{\left(1+t^2\right) p_2}{t}-\frac{\left(1+t^2\right) p_4}{t}+t\,p_5,\ \frac{x_2}{t}-\frac{x_4}{t},\ -t\,p_2+t\,p_4,\ t\,x_1-t\,x_5,\ -\frac{p_1}{t}+\frac{p_5}{t}\right\}$$

Out[∘]=

$$\left\{-\frac{\left(1+t^2\right) x_1}{t}+t\,x_2-t\,x_4+\frac{\left(1+t^2\right) x_5}{t},\ \frac{\left(1+t^2\right) p_1}{t}-\frac{p_2}{t}+\frac{p_4}{t}-\frac{\left(1+t^2\right) p_5}{t},\ \frac{x_1}{t}-\frac{\left(1+t^2\right) x_2}{t}+\frac{\left(1+t^2\right) x_4}{t}-\frac{x_5}{t},\ -t\,p_1+\frac{\left(1+t^2\right) p_2}{t}-\frac{\left(1+t^2\right) p_4}{t}+t\,p_5,\ \frac{x_2}{t}-\frac{x_4}{t},\ -t\,p_2+t\,p_4,\ t\,x_1-t\,x_5,\ -\frac{p_1}{t}+\frac{p_5}{t}\right\}$$

Out[∘]=

$$\left\{-\frac{\left(1+t^2\right) x_1}{t}+t\,x_2-t\,x_4+\frac{\left(1+t^2\right) x_5}{t},\ \frac{\left(1+t^2\right) p_1}{t}-\frac{p_2}{t}+\frac{p_4}{t}-\frac{\left(1+t^2\right) p_5}{t},\ \frac{x_1}{t}-\frac{\left(1+t^2\right) x_2}{t}+\frac{\left(1+t^2\right) x_4}{t}-\frac{x_5}{t},\ -t\,p_1+\frac{\left(1+t^2\right) p_2}{t}-\frac{\left(1+t^2\right) p_4}{t}+t\,p_5,\ \frac{x_2}{t}-\frac{x_4}{t},\ -t\,p_2+t\,p_4,\ t\,x_1-t\,x_5,\ -\frac{p_1}{t}+\frac{p_5}{t}\right\}$$

Out[∘]=

$$\left\{-\frac{\left(1+t^2\right) x_1}{t}+t\,x_2-t\,x_4+\frac{\left(1+t^2\right) x_5}{t},\ \frac{\left(1+t^2\right) p_1}{t}-\frac{p_2}{t}+\frac{p_4}{t}-\frac{\left(1+t^2\right) p_5}{t},\ \frac{x_1}{t}-\frac{\left(1+t^2\right) x_2}{t}+\frac{\left(1+t^2\right) x_4}{t}-\frac{x_5}{t},\ -t\,p_1+\frac{\left(1+t^2\right) p_2}{t}-\frac{\left(1+t^2\right) p_4}{t}+t\,p_5,\ \frac{x_2}{t}-\frac{x_4}{t},\ -t\,p_2+t\,p_4,\ t\,x_1-t\,x_5,\ -\frac{p_1}{t}+\frac{p_5}{t}\right\}$$

R3 with flipped middle crossing:

In[∘]:=
```
lhs = CF[Q_{1,2,7,6} + Q̄_{2,3,4,7} + Q_{7,4,5,6}];
rhs = CF[Q_{2,3,7,1} + Q̄_{1,7,5,6} + Q_{7,3,4,5}];
L = lhs; CF@{(∂_{p_1} L) + (∂_{p_7} L) + (∂_{p_5} L), (∂_{x_1} L) + (∂_{x_7} L) + (∂_{x_5} L),
   (∂_{p_2} L) + (∂_{p_4} L), (∂_{x_2} L) + (∂_{x_4} L), ∂_{p_3} L, ∂_{x_3} L, ∂_{p_6} L, ∂_{x_6} L}
L = rhs; CF@{(∂_{p_1} L) + (∂_{p_5} L), (∂_{x_1} L) + (∂_{x_5} L),
   (∂_{p_2} L) + (∂_{p_7} L) + (∂_{p_4} L), (∂_{x_2} L) + (∂_{x_7} L) + (∂_{x_4} L), ∂_{p_3} L, ∂_{x_3} L, ∂_{p_6} L, ∂_{x_6} L}
L = Cases[(∫𝔼[lhs] ⅆ{p_7, x_7}), 𝔼[L_] :→ L]⟦1⟧;
CF@{(∂_{p_1} L) + (∂_{p_5} L), (∂_{x_1} L) + (∂_{x_5} L), (∂_{p_2} L) + (∂_{p_4} L), (∂_{x_2} L) + (∂_{x_4} L), ∂_{p_3} L, ∂_{x_3} L, ∂_{p_6} L, ∂_{x_6} L}
L = Cases[(∫𝔼[rhs] ⅆ{p_7, x_7}), 𝔼[L_] :→ L]⟦1⟧;
CF@{(∂_{p_1} L) + (∂_{p_5} L), (∂_{x_1} L) + (∂_{x_5} L), (∂_{p_2} L) + (∂_{p_4} L), (∂_{x_2} L) + (∂_{x_4} L), ∂_{p_3} L, ∂_{x_3} L, ∂_{p_6} L, ∂_{x_6} L}
```

Out[∘]=

$$\left\{-\frac{\left(1+t^2\right) x_1}{t}+t\,x_2-t\,x_4+\frac{\left(1+t^2\right) x_5}{t},\ \frac{\left(1+t^2\right) p_1}{t}-\frac{p_2}{t}+\frac{p_4}{t}-\frac{\left(1+t^2\right) p_5}{t},\ \frac{x_1}{t}-\frac{\left(1+t^2\right) x_2}{t}+\frac{\left(1+t^2\right) x_4}{t}-\frac{x_5}{t},\ -t\,p_1+\frac{\left(1+t^2\right) p_2}{t}-\frac{\left(1+t^2\right) p_4}{t}+t\,p_5,\ \frac{x_2}{t}-\frac{x_4}{t},\ -t\,p_2+t\,p_4,\ t\,x_1-t\,x_5,\ -\frac{p_1}{t}+\frac{p_5}{t}\right\}$$

Out[∘]=

$$\left\{-\frac{\left(1+t^2\right) x_1}{t}+t\,x_2-t\,x_4+\frac{\left(1+t^2\right) x_5}{t},\ \frac{\left(1+t^2\right) p_1}{t}-\frac{p_2}{t}+\frac{p_4}{t}-\frac{\left(1+t^2\right) p_5}{t},\ \frac{x_1}{t}-\frac{\left(1+t^2\right) x_2}{t}+\frac{\left(1+t^2\right) x_4}{t}-\frac{x_5}{t},\ -t\,p_1+\frac{\left(1+t^2\right) p_2}{t}-\frac{\left(1+t^2\right) p_4}{t}+t\,p_5,\ \frac{x_2}{t}-\frac{x_4}{t},\ -t\,p_2+t\,p_4,\ t\,x_1-t\,x_5,\ -\frac{p_1}{t}+\frac{p_5}{t}\right\}$$

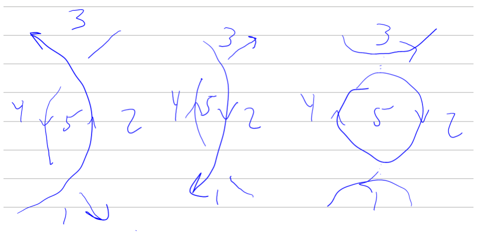
Out[∘]=

$$\left\{-\frac{\left(1+t^2\right) x_1}{t}+t\,x_2-t\,x_4+\frac{\left(1+t^2\right) x_5}{t},\ \frac{\left(1+t^2\right) p_1}{t}-\frac{p_2}{t}+\frac{p_4}{t}-\frac{\left(1+t^2\right) p_5}{t},\ \frac{x_1}{t}-\frac{\left(1+t^2\right) x_2}{t}+\frac{\left(1+t^2\right) x_4}{t}-\frac{x_5}{t},\ -t\,p_1+\frac{\left(1+t^2\right) p_2}{t}-\frac{\left(1+t^2\right) p_4}{t}+t\,p_5,\ \frac{x_2}{t}-\frac{x_4}{t},\ -t\,p_2+t\,p_4,\ t\,x_1-t\,x_5,\ -\frac{p_1}{t}+\frac{p_5}{t}\right\}$$

Out[∘]=

$$\left\{-\frac{\left(1+t^2\right) x_1}{t}+t\,x_2-t\,x_4+\frac{\left(1+t^2\right) x_5}{t},\ \frac{\left(1+t^2\right) p_1}{t}-\frac{p_2}{t}+\frac{p_4}{t}-\frac{\left(1+t^2\right) p_5}{t},\ \frac{x_1}{t}-\frac{\left(1+t^2\right) x_2}{t}+\frac{\left(1+t^2\right) x_4}{t}-\frac{x_5}{t},\ -t\,p_1+\frac{\left(1+t^2\right) p_2}{t}-\frac{\left(1+t^2\right) p_4}{t}+t\,p_5,\ \frac{x_2}{t}-\frac{x_4}{t},\ -t\,p_2+t\,p_4,\ t\,x_1-t\,x_5,\ -\frac{p_1}{t}+\frac{p_5}{t}\right\}$$

For R2C$_+$:



$$L = CF\left[\overline{Q}_{4,1,2,5} + Q_{2,3,4,5}\right];$$
$$CF@\{(\partial_{p_2} L) + (\partial_{p_4} L),\ (\partial_{x_2} L) + (\partial_{x_4} L),\ \partial_{p_1} L,\ \partial_{x_1} L,\ \partial_{p_3} L,\ \partial_{x_3} L\}$$

Out[•]=

$$\left\{0,\ 0,\ -\frac{x_2}{t} + \frac{x_4}{t},\ t\,p_2 - t\,p_4,\ \frac{x_2}{t} - \frac{x_4}{t},\ -t\,p_2 + t\,p_4\right\}$$

R2C$_+$ with one crossing flipped:

In[•]:=

$$L = CF[Q_{4,1,2,5} + Q_{2,3,4,5}];$$
$$CF@\{(\partial_{p_2} L) + (\partial_{p_4} L),\ (\partial_{x_2} L) + (\partial_{x_4} L),\ \partial_{p_1} L,\ \partial_{x_1} L,\ \partial_{p_3} L,\ \partial_{x_3} L\}$$

Out[•]=

$$\left\{0,\ 0,\ -\frac{x_2}{t} + \frac{x_4}{t},\ t\,p_2 - t\,p_4,\ \frac{x_2}{t} - \frac{x_4}{t},\ -t\,p_2 + t\,p_4\right\}$$

In[•]:=

$$\text{Factor}\left[Q_{4,1,2,5} - \overline{Q}_{4,1,2,5}\right]$$

Out[•]=

$$-\frac{2\,(-1+t)\,(1+t)\,(p_2 - p_4)\,(x_2 - x_4)}{t}$$