Pensieve header: A unified verification notebook for the sl_2-portfolio project; continues pensieve://Projects/PPSA/nb/Verification.pdf.

Continues pensieve://2017-06/ and pensieve://2017-08/.

Prolog

```
Go;

wdir = SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\SL2Portfolio"];
NotebookOpen[wdir <> "\\MakeSnips.nb"];
```

In[∘]:=
```
HL[ℰ_] := Style[ℰ, Background → Yellow];
```

DocileQ

DocileQ

In[∘]:=
```
DQ[ℰ_] := (Exponent[Normal@ℰ /.
     {a → a / ϵ, a_i_ → a_i / ϵ, (u : x | y) ⧴ ϵ^-1/2 u, (u : x | y)_i_ ⧴ ϵ^-1/2 u_i}, ϵ, Min] ≥ 0);
```

In[∘]:=
```
DQ /@ {ϵ^2 x y a_2, ϵ^2 x^2 y^3}
```

Out[∘]=
```
{True, False}
```

Initialization / Utilities

It is verification-risky to work with low $E!

TD

In[∘]:=
```
$p = 2; $k = 1; $E := {$k, $p};
$trim := {ħ^p_· /; p > $p → 0, ϵ^k_· /; k > $k → 0};
SetAttributes[{SS, SST}, HoldAll];
TRule = {T_i_ → e^(ħ t_i), T → e^(ħ t)}; q_ħ = e^(γ ϵ ħ);
SS[ℰ_, op_] := Collect[
   Normal@Series[If[$p > 0, ℰ, ℰ /. TRule], {ħ, 0, $p}],
   ħ, op];
SS[ℰ_] := SS[ℰ, Together];
SST[ℰ_, op___] := SS[ℰ /. TRule, op];
Simp[ℰ_, op_] := Collect[ℰ, _CU | _QU, op];
Simp[ℰ_] := Simp[ℰ, SS[#, Expand] &];
SimpT[ℰ_] := Collect[ℰ, _CU | _QU, SST[#, Expand] &];
```

Differential polynomials (DP):

Utils

In[∘]:=
```
DP_{α_→D_x_, β_→D_y_}[P_][λ_] :=
 Total[CoefficientRules[Normal@P, {α, β}] /. ({m_, n_} → c_) ⧴ c ∂_{{x,m},{y,n}} λ]
```

```
HL[DP_{x→D_ξ, y→D_η}[x^2 y^3][e^{δ ξ η}] == 6 e^{δ η ξ} δ^3 ξ + 6 e^{δ η ξ} δ^4 η ξ^2 + e^{δ η ξ} δ^5 η^2 ξ^3]
```

True

CF

```
In[∘]:= CF[ℰ_] := ExpandDenominator@
           ExpandNumerator@Together[Expand[ℰ] //. e^{x_} e^{y_} ⧴ e^{x+y} /. e^{x_} ⧴ e^{CF[x]}];
```

SeriesData

```
In[∘]:= Unprotect[SeriesData];
        SeriesData /: CF[sd_SeriesData] := MapAt[CF, sd, 3];
        SeriesData /: Expand[sd_SeriesData] := MapAt[Expand, sd, 3];
        SeriesData /: Simplify[sd_SeriesData] := MapAt[Simplify, sd, 3];
        SeriesData /: Together[sd_SeriesData] := MapAt[Together, sd, 3];
        SeriesData /: Collect[sd_SeriesData, specs__] := MapAt[Collect[#, specs] &, sd, 3];
        Protect[SeriesData];
```

Self-Pair (SP):

SP

```
In[∘]:= SP_{}[P_] := P; SP_{ξ_→x_, ps___}[P_] := Expand[P // SP_{ps}] /. f_. ξ^{d_.} ⧴ ∂_{x,d} f
```

```
SP_{ξ→x}[(ξ^2 + ξ + 3) (x^5 e^x + 7 x) + 99 a]
```

$7 + 99\, a + 21\, x + 20\, e^x x^3 + 15\, e^x x^4 + 5\, e^x x^5$

```
SP_{ξ→x, η→y}[(ξ^2 + ξ + 3 + 2 ξ η) (x^5 e^x + 7 x) + 99 a + e^{δ x y} ξ η]
```

$7 + 99\, a + 21\, x + 20\, e^x x^3 + 15\, e^x x^4 + 5\, e^x x^5 + e^{x y \delta} \delta + e^{x y \delta} x\, y\, \delta^2$

DeclareAlgebra

QLImplementation

```
In[∘]:= Unprotect[NonCommutativeMultiply]; Attributes[NonCommutativeMultiply] = {};
        (NCM = NonCommutativeMultiply)[x_] := x;
        NCM[x_, y_, z__] := (x ** y) ** z;
        0 ** _ = _ ** 0 = 0;
        (x_Plus) ** y_ := (# ** y) & /@ x; x_ ** (y_Plus) := (x ** #) & /@ y;
        B[x_, x_] = 0; B[x_, y_] := x ** y - y ** x;
        B[x_, y_, e_] := B[x, y, e] = B[x, y];
```

QLImplementation

```
In[•]:= DeclareAlgebra[U_Symbol, opts__Rule] := Module[{gp, sr, g, cp, M, CE, pow, k = 0,
     gs = Generators /. {opts},
     cs = Centrals /. {opts} /. Centrals → {} },
   (#_U = U@#) & /@ gs;
   gp = Alternatives @@ gs; gp = gp | gp_; (* gens *)
   sr = Flatten@Table[{g → ++k, g_i_ → {i, k}}, {g, gs}]; (* sorting → *)
   cp = Alternatives @@ cs; (* cents *)
   SetAttributes[M, HoldRest]; M[0, _] = 0; M[a_, x_] := a x;
   CE[ℰ_] := Collect[ℰ, _U, Expand] /. $trim;
   U_i_[ℰ_] := ℰ /. {t : cp :> t_i, u_U :> (#_i &) /@ u};
   U_i_[NCM[]] = pow[ℰ_, 0] = U@{} = 1_U = U[];
   B[U@(x_)_i_, U@(y_)_i_] := U_i@B[U@x, U@y];
   B[U@(x_)_i_, U@(y_)_j_] /; i =!= j := 0;
   B[U@y_, U@x_] := CE[-B[U@x, U@y]];
   x_ ** (c_. 1_U) := CE[c x]; (c_. 1_U) ** x_ := CE[c x];
   (a_. U[xx___, x_]) ** (b_. U[y_, yy___]) := If[OrderedQ[{x, y} /. sr],
     CE@M[a b /. $trim, U[xx, x, y, yy]],
     U@xx ** CE@M[a b /. $trim, U@y ** U@x + B[U@x, U@y, $E]] ** U@yy ];
   U@{c_. * (l : gp)^n_, r___} /; FreeQ[c, gp] := CE[c U@Table[l, {n}] ** U@{r}];
   U@{c_. * l : gp, r___} := CE[c U[l] ** U@{r}];
   U@{c_, r___} /; FreeQ[c, gp] := CE[c U@{r}];
   U@{l_Plus, r___} := CE[U@{#, r} & /@ l];
   U@{l_, r___} := U@{Expand[l], r};
   U[ℰ_NonCommutativeMultiply] := U /@ ℰ;
   𝕆_U[specs___, poly_] := Module[{sp, null, vs, us},
     sp = Replace[{specs}, l_List :> l_null, {1}];
     vs = Join @@ (First /@ sp);
     us = Join @@ (sp /. l_s_ :> (l /. x_i_ :> x_s));
     CE[Total[
        CoefficientRules[poly, vs] /. (p_ → c_) :> c U@(us^p)
       ]] /. x_null :> x];
   pow[ℰ_, n_] := pow[ℰ, n - 1] ** ℰ;
   𝕊_U[ℰ_, ss___Rule] := CE@Total[
      CoefficientRules[ℰ, First /@ {ss}] /.
       (p_ → c_) :> c NCM @@ MapThread[pow, {Last /@ {ss}, p}]];
   m_j_→k_[c_. * u_U] := CE[((c /. (t : cp)_j → t_k) DeleteCases[u, _j|k]) **
      U @@ Cases[u, w_j :> w_k] ** U @@ Cases[u, _k]];
   U /: c_. * u_U * v_U := CE[c u ** v];
   S_i_[c_. * u_U] := CE[((c /. S_i[U, Centrals]) DeleteCases[u, _i]) **
      U_i[NCM @@ Reverse@Cases[u, x_i :> S@U@x]] ]
```

DeclareMorphism

QLImplementation

```
In[∘]:= DeclareMorphism[m_, U_ → V_, ongs_List, oncs_List: {}] := (
          Replace[ongs,
            {(g_ → img_) ⧴ (m[U[g]] = img), (g_ ⧴ img_) ⧴ (m[U[g]] := img /. $trim)}, {1}];
          m[1_U] = 1_V;
          m[U[g_i_]] := V_i[m[U@g]];
          m[U[vs__]] := NCM @@ (m /@ U /@ {vs});
          m[ℰ_] := Simp[ℰ /. oncs /. u_U ⧴ m[u]] /. $trim; )
```

Meta-Operations

QLImplementation

```
In[∘]:= m_{j_→j_} = Identity;
        m_{j_→k_}[ℰ_Plus] := Simp[m_{j→k} /@ ℰ];
        m_{is___,i_,j_→k_}[ℰ_] := m_{j→k}@m_{is,i→j}@ℰ;
        S_{i_}[ℰ_Plus] := Simp[S_i /@ ℰ];
```

Implementing CU = $\mathcal{U}(sl_2^{\gamma\epsilon})$

CU

```
In[∘]:= DeclareAlgebra[CU, Generators → {y, a, x}, Centrals → {t}];
        B[a_CU, y_CU] = -γ y_CU; B[x_CU, a_CU] = -γ x_CU;
        B[x_CU, y_CU] = 2 ϵ a_CU - t 1_CU;
        (S@y_CU = -y_CU; S@a_CU = -a_CU; S@x_CU = -x_CU;)
        S_{i_}[CU, Centrals] = {t_i → -t_i};
        Δ@y_CU = CU@y_1 + T_1 CU@y_2; Δ@a_CU = CU@a_1 + CU@a_2; Δ@x_CU = CU@x_1 + CU@x_2;
```

Verifying associativity on triples of generators:

```
With[{bas = CU /@ {y, a, x}},
 Table[z1 ** (z2 ** z3) - (z1 ** z2) ** z3 // Simp // HL,
  {z1, bas}, {z2, bas}, {z3, bas}]]
```

```
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying associativity on a “random” triple:

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
   (rhs = (z1 ** z2) ** z3 // Simp) // Short,
   z1 ** (z2 ** z3) - rhs // Simp // HL
  }] // Timing
```

```
{1.32813,
 {(28 t^2 γ^4 + 116 t γ^5 ϵ) CU[y, y, y, x, x] + <<21>> + CU[y, y, y, y, y, a, a, a, a, x, x, x, x], 0}}
```

Verifying that *S* is an anti-homomorphism on CU:

In[∘]:=
```
With[{bas = CU /@ {y₁, a₁, x₁}},
 Table[HL@Simp[S₁[z1 ** z2] - S₁[z2] ** S₁[z1]],
  {z1, bas}, {z2, bas} ] ]
```
Out[∘]= {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

Verifying the involutivity of S on products of triples:

In[∘]:=
```
With[{bas = CU /@ {y₁, a₁, x₁}},
 Table[HL@Simp[z1 ** z2 ** z3 - S₁@S₁[z1 ** z2 ** z3]],
  {z1, bas}, {z2, bas}, {z3, bas} ] ]
```
Out[∘]= {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}

Verifying relabeling:

```
t₁ t₃ CU[y₁, a₁, x₂] + t₁ t₁ CU[y₁, a₂, x₂] // m₁→₃
```

$CU[a_2, x_2, y_3]\, t_3^2 + CU[x_2, y_3, a_3]\, t_3^2$

Verifying meta-associativity:

```
Module[{z, u},
 Table[u = CU[z[[1]]₁, z[[2]]₂, z[[3]]₃]; z → HL[m₁,₃→₃@m₂,₃→₃@u == m₂,₃→₃@m₁,₂→₂@u],
  {z, Tuples[{y, a, x}, 3]}]]
```

{{y, y, y} → True, {y, y, a} → True, {y, y, x} → True, {y, a, y} → True,
 {y, a, a} → True, {y, a, x} → True, {y, x, y} → True, {y, x, a} → True,
 {y, x, x} → True, {a, y, y} → True, {a, y, a} → True, {a, y, x} → True, {a, a, y} → True,
 {a, a, a} → True, {a, a, x} → True, {a, x, y} → True, {a, x, a} → True, {a, x, x} → True,
 {x, y, y} → True, {x, y, a} → True, {x, y, x} → True, {x, a, y} → True, {x, a, a} → True,
 {x, a, x} → True, {x, x, y} → True, {x, x, a} → True, {x, x, x} → True}

Implementing QU = $\mathcal{U}_q(sl_2^{\gamma\epsilon})$

Aside

$\text{Series}\left[\left(1 - T\, e^{-2\epsilon a \hbar}\right)/\hbar,\ \{a, 0, 3\}\right]$

Aside

$$\frac{1-T}{\hbar} + 2\,T\,\epsilon\,a - 2\left(T\,\epsilon^2\,\hbar\right)a^2 + \frac{4}{3}\,T\,\epsilon^3\,\hbar^2\,a^3 + O[a]^4$$

In[∘]:= $\text{HL /@ DQ /@ Series}\left[\left\{\left(1 - T\, e^{-2\epsilon a \hbar}\right)/\hbar,\ e^{\hbar\epsilon a}\right\},\ \{\epsilon, 0, 5\}\right]$

Out[∘]= {True, True}

QU

In[∘]:=
```
DeclareAlgebra[QU, Generators → {y, a, x}, Centrals → {t, T}];
B[a_QU, y_QU] = -γ y_QU; B[x_QU, a_QU] = -γ QU@x;
B[x_QU, y_QU] := SS[q_ħ - 1] QU@{y, x} + O_QU[{a}, SS[(1 - T e^(-2 ϵ a ħ)) / ħ]];
(S@y_QU := O_QU[{a, y}, SS[-T^-1 e^(ħ ϵ a) y]]; S@a_QU = -a_QU; S@x_QU := O_QU[{a, x}, SS[-e^(ħ ϵ a) x]];)
S_i_[QU, Centrals] = {t_i → -t_i, T_i → T_i^-1};
Δ@y_QU := O_QU[{y₁, a₁}₁, {y₂}₂, SS[y₁ + T₁ e^(-ħ ϵ a₁) y₂]];
Δ@a_QU = QU@a₁ + QU@a₂; Δ@x_QU := O_QU[{a₁, x₁}₁, {x₂}₂, SS[x₁ + e^(-ħ ϵ a₁) x₂]];
```

```
With[{bas = QU /@ {y, a, x}}, Table[{z1, z2} → Simp[z1 ** z2 - z2 ** z1], {z1, bas}, {z2, bas}] ]
```

$\{\{\{QU[y], QU[y]\} \to 0, \{QU[y], QU[a]\} \to \gamma\, QU[y], \{QU[y], QU[x]\} \to \frac{(-1+T)\, QU[]}{\hbar} - 2\, T\, \epsilon\, QU[a] - \gamma\, \epsilon\, \hbar\, QU[y, x]\},$
$\{\{QU[a], QU[y]\} \to -\gamma\, QU[y], \{QU[a], QU[a]\} \to 0, \{QU[a], QU[x]\} \to \gamma\, QU[x]\},$
$\{\{QU[x], QU[y]\} \to \frac{(1-T)\, QU[]}{\hbar} + 2\, T\, \epsilon\, QU[a] + \gamma\, \epsilon\, \hbar\, QU[y, x], \{QU[x], QU[a]\} \to -\gamma\, QU[x], \{QU[x], QU[x]\} \to 0\}\}$

Verifying associativity on triples of generators:

```
With[{bas = QU /@ {y, a, x}},
 Table[HL[z1 ** (z2 ** z3) - (z1 ** z2) ** z3] // Simp,
  {z1, bas}, {z2, bas}, {z3, bas} ] ]
```

{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}

Verifying associativity on a “random” triple (~34 secs @ $p=5, $k=2):

```
With[{z1 = QU[y, y, a, a, x, x], z2 = QU[y, a, x], z3 = QU[y, y, a, x]}, {
   (rhs = (z1 ** z2) ** z3 // Simp) // Short,
   HL[z1 ** (z2 ** z3) - rhs // Simp]
  }] // Timing
```

$\{3.78125, \{\left(\frac{28\gamma^4 - 56\, T\gamma^4 + 28\, T^2\gamma^4}{\hbar^2} + \frac{82\gamma^5\epsilon - 280\, T \ll 1 \gg \epsilon + 198\, T^2\gamma^5\epsilon}{\hbar}\right) QU[y, y, y, x, x] + \ll 18 \gg + (1 + 8\gamma\epsilon\hbar)\, QU[y, \ll 11 \gg, x], 0\}\}$

Verifying that S is an anti-homomorphism on QU:

```
With[{bas = QU /@ {y1, a1, x1}},
 Table[{z1, z2} → HL@Simp[S1[z1 ** z2] - S1[z2] ** S1[z1]],
  {z1, bas}, {z2, bas} ] ]
```

$\{\{\{QU[y_1], QU[y_1]\} \to 0, \{QU[y_1], QU[a_1]\} \to 0, \{QU[y_1], QU[x_1]\} \to 0\},$
$\{\{QU[a_1], QU[y_1]\} \to 0, \{QU[a_1], QU[a_1]\} \to 0, \{QU[a_1], QU[x_1]\} \to 0\},$
$\{\{QU[x_1], QU[y_1]\} \to 0, \{QU[x_1], QU[a_1]\} \to 0, \{QU[x_1], QU[x_1]\} \to 0\}\}$

Verifying that $\mathrm{Lim}_{\hbar\to 0}$ QU = CU using a “random” product (~23 secs @ $p=5, $k=2):

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
   Short[lhs = z1 ** (z2 ** z3)],
   Short[rhs = (QU @@ z1) ** ((QU @@ z2) ** (QU @@ z3))],
   Expand[Limit[rhs /. TRule ⋃ {QU → CU}, ℏ → 0] - lhs] // HL
  }] // Timing
```

$\{10.125, \{28\, t^2\gamma^4\, CU[y, y, y, x, x] + 116\, t\,\gamma^5\epsilon\, CU[y, y, y, x, x] + \ll 44 \gg + CU[y, y, y, y, y, a, a, a, a, x, x, x, x],$
$2\left(\frac{\gamma^4}{\hbar^2} - \frac{2\, T\gamma^4}{\hbar^2} + \frac{T^2\gamma^4}{\hbar^2} + \frac{\gamma^5\epsilon}{\hbar} - \frac{2\, T\gamma^5\epsilon}{\hbar} + \frac{T^2\gamma^5\epsilon}{\hbar}\right) QU[y, y, y, x, x] + \ll 209 \gg + (1 + 8\gamma\epsilon\hbar)\, QU[y, y, y, \ll 7 \gg, x, x, x], 0\}\}$

Implementing θ

theta

In[ ]:=
```
DeclareMorphism[Cθ, CU → CU, {y → -x_CU, a → -a_CU, x → -y_CU}, {t → -t, T → T^-1}];
DeclareMorphism[Qθ, QU → QU, {y :→ 𝕆_QU[{a, x}, SS[-T^(-1/2) 𝕖^(ℏ ϵ a) x]],
   a → -a_QU, x :→ 𝕆_QU[{a, y}, SS[-T^(-1/2) 𝕖^(ℏ ϵ a) y]]}, {t → -t, T → T^-1}]
```

Verifying involutivity on CU:

```
With[{bas = CU /@ {y, a, x}},
 Table[z → Cθ[z] → HL[Cθ[Cθ[z]]], {z, bas}] ]
```

{CU[y] → -CU[x] → CU[y], CU[a] → -CU[a] → CU[a], CU[x] → -CU[y] → CU[x]}

Verifying that θ is a multiplicative homomorphism on CU:

```
With[{bas = CU /@ {y, a, x}},
 Table[Cθ[z1 ** z2] - Cθ[z1] ** Cθ[z2] // HL, {z1, bas}, {z2, bas}] ]
```

{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

Verifying involutivity on QU:

```
With[{bas = QU /@ {y, a, x}},
 Table[z → Qθ[z] → HL[Simp[Qθ[Qθ[z]], PowerExpand]], {z, bas}] ]
```

$\{QU[y] \to -\frac{QU[x]}{\sqrt{T}} - \frac{\epsilon\,\hbar\,QU[a, x]}{\sqrt{T}} \to QU[y],\ QU[a] \to -QU[a] \to QU[a],$

$QU[x] \to \left(-\frac{1}{\sqrt{T}} + \frac{\gamma\,\epsilon\,\hbar}{\sqrt{T}}\right) QU[y] - \frac{\epsilon\,\hbar\,QU[y, a]}{\sqrt{T}} \to QU[x]\}$

Verifying that θ is a multiplicative homomorphism on QU:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL[Simp[Qθ[z1 ** z2] - Qθ[z1] ** Qθ[z2]]], {z1, bas}, {z2, bas}] ]
```

{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}

The Asymmetric Dequantizator

Following pensieve://People/VanDerVeen/Dequant1.pdf.

ADeq

In[ ]:=

$$A\mathbb{D}\$f = \gamma\left(\left(\mathrm{Cosh}\left[\hbar\left(a\,\epsilon + \frac{\gamma\,\epsilon}{2} - \frac{t}{2}\right)\right] - \mathrm{Cosh}\left[\hbar\sqrt{\left(\frac{t-\gamma\,\epsilon}{2}\right)^2 + \epsilon\,\omega}\right]\right)\Big/\left(\hbar\,e^{\hbar\,((a+\gamma)\,\epsilon - t/2)}\,\mathrm{Sinh}\left[\frac{\gamma\,\epsilon\,\hbar}{2}\right]\left(a^2\,\epsilon + a\,\gamma\,\epsilon - a\,t - \omega\right)\right)\right);$$

Docility of A𝔻$f:

```
In[∘]:= HL@DQ@Block[{$p = 4}, Collect[SS@A𝔻$f /. ω → a1, ϵ]]
Out[∘]= True
```

Scaling behaviour of A𝔻$f:

```
HL@Simplify[A𝔻$f == ((A𝔻$f /. γ → 1) /. {ϵ → γ ϵ, a → γ^-1 a, ω → γ^-1 ω})]
True
```

```
HL@FullSimplify[
  A𝔻$f == ((A𝔻$f /. γ → 1) /. {ℏ → γ^2 ℏ, ϵ → ϵ / γ, a → a / γ, t → γ^-2 t, ω → γ^-3 ω})]
True
```

ADeq

```
In[∘]:= A𝔻$ω = γ CU[y, x] + ϵ CU[a, a] - (t - γ ϵ) CU[a];
```

ADeq

```
In[∘]:= DeclareMorphism[A𝔻, QU → CU,
  {a → aCU, x → CU@x, y :→ 𝕊CU[SS[A𝔻$f], a → aCU, ω → A𝔻$ω] ** yCU}]
```

Verifying that the asymmetric dequantizator is a homomorphism:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL[SimpT[A𝔻[z1 ** z2] - A𝔻[z1] ** A𝔻[z2]]], {z1, bas}, {z2, bas}] ]
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

The Symmetric Dequantizator

Following pensieve://People/VanDerVeen/Dequant1.pdf.

SDeq

In[∘]:=

$$S\mathbb{D}\$g = \sqrt{\left(\left(2\,\gamma\left(\mathrm{Cosh}\left[\frac{\hbar}{2}\sqrt{t^2+\gamma^2\,\epsilon^2+4\,\epsilon\,\varpi}\,\right]-\mathrm{Cosh}\left[\frac{t-\epsilon\,\gamma-2\,\epsilon\,a}{2/\hbar}\right]\right)\right)\Big/\left(\mathrm{Sinh}\left[\frac{\gamma\,\epsilon\,\hbar}{2}\right]\left(t\,(2\,a+\gamma)-2\,a\,(a+\gamma)\,\epsilon+2\,\varpi\right)\hbar\right)\right)};$$

SDeq

```
In[∘]:= S𝔻$f = Simplify[e^(ℏ (t/2-ϵ a)) (S𝔻$g /. {a → -a, t → -t})];
```

Verify agreement with the formulas in pensieve://People/VanDerVeen/Dequant1.pdf:

In[∘]:=
```
{SD$P = (Cosh[ħ ((ϵ-t)/2 + ϵ a)] - Cosh[ħ Sqrt[(t^2+ϵ^2)/4 + ϵ ϖ]]) / (ħ Sinh[(-ϵ ħ)/2] (ϖ - ϵ a^2 + (t - ϵ) a + t/2)),
 Simplify[SD$P == (SD$P /. {a → -a - 1, t → -t})] // HL,
 PowerExpand@Simplify[(SD$P /. {ħ → γ^2 ħ, ϵ → ϵ/γ, a → a/γ, t → γ^-2 t, ϖ → γ^-3 ϖ}) ==
   SD$g (SD$g /. {a → -a - γ, t → -t})] // HL,
 SD$Q = Simplify[SD$P /. {a → c - 1/2}],
 Simplify[SD$Q == (SD$Q /. {c → -c, t → -t})] // HL,
 FullSimplify[SD$g == FullSimplify[
   Sqrt[SD$Q] /. c → a + 1/2 /. {ħ → γ^2 ħ, ϵ → ϵ/γ, a → a/γ, t → γ^-2 t, ϖ → γ^-3 ϖ}]] // HL,
 HL@DQ@Block[{$p = 4}, Collect[SS@SD$g /. ω → a1, ϵ]],
 HL@DQ@Block[{$p = 4}, Collect[SS@SD$f /. ω → a1, ϵ]]
}
```

Out[∘]=

$$\left\{-\left(\left(\left(\mathrm{Cosh}\left[\left(a\,\epsilon+\frac{1}{2}(-t+\epsilon)\right)\hbar\right]-\mathrm{Cosh}\left[\sqrt{\frac{1}{4}(t^2+\epsilon^2)+\epsilon\varpi}\;\hbar\right]\right)\mathrm{Csch}\left[\frac{\epsilon\hbar}{2}\right]\right)\Big/\left(\left(\frac{t}{2}+a(t-\epsilon)-a^2\epsilon+\varpi\right)\hbar\right)\right),\ \mathrm{True},\ \mathrm{True},\right.$$

$$\left.-\left(\left(4\left(\mathrm{Cosh}\left[\frac{1}{2}(t-2c\epsilon)\hbar\right]-\mathrm{Cosh}\left[\frac{1}{2}\sqrt{t^2+\epsilon^2+4\epsilon\varpi}\;\hbar\right]\right)\mathrm{Csch}\left[\frac{\epsilon\hbar}{2}\right]\right)\Big/\left((4ct+\epsilon-4c^2\epsilon+4\varpi)\hbar\right)\right),\ \mathrm{True},\ \mathrm{True},\ \mathrm{True},\ \mathrm{True}\right\}$$

SDeq

In[∘]:=
```
SD$ϖ = γ CU[y, x] + ϵ CU[a, a] - (t - γ ϵ) CU[a] - t γ 1_CU / 2;
```

SDeq

In[∘]:=
```
DeclareMorphism[SD, QU → CU, {a → a_CU,
  x :→ S_CU[SS[SD$f], a → a_CU, ϖ → SD$ϖ] ** x_CU,
  y :→ S_CU[SS[SD$g], a → a_CU, ϖ → SD$ϖ] ** y_CU }]
```

Verifying the θ-symmetry:

```
Table[HL@SimpT[Cθ[SD[z]] == SD[Qθ[z]]], {z, QU /@ {y, a, x}}]
```

{True, True, True}

Verifying that the symmetric dequantizator is a homomorphism:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL@SimpT[SD[z1 ** z2] - SD[z1] ** SD[z2]], {z1, bas}, {z2, bas}] ]
```

```
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

The representation ρ

rho

In[ ]:=
```
ρ@y_CU = ρ@y_QU = (0 0; ϵ 0); ρ@a_CU = ρ@a_QU = (γ 0; 0 0);
ρ@x_CU = (0 γ; 0 0); ρ@x_QU = (0 (1 - e^(-γ ϵ ħ)) / (ϵ ħ); 0 0);
ρ[e^𝓔_] := MatrixExp[ρ[𝓔]];
ρ[𝓔_] := (𝓔 /. TRule /. t → γ ϵ /. (U : CU | QU) [u___] ⧴ Fold[Dot, (1 0; 0 1), ρ /@ U /@ {u}])
```

Verifying that ρ represents CU and QU:

```
Table[HL[SS[ρ[z1 ** z2] == ρ[z1].ρ[z2]] /. e^(k_·) /; k > $k → 0],
 {U, {CU, QU}}, {z1, U /@ {y, a, x}}, {z2, U /@ {y, a, x}} ]
```

```
{{{True, True, True}, {True, True, True}, {True, True, True}},
 {{True, True, True}, {True, True, True}, {True, True, True}}}
```

Commuting $e^{\alpha a}$ with $e^{\xi x}$:

```
Table[HL[ρ[e^(ξ U@x)].ρ[e^(α U@a)] == ρ[e^(α U@a)].ρ[e^(e^(-γ α) ξ U@x)]], {U, {CU, QU}}]
```

```
{True, True}
```

Œ and the logoi Λ

Logoi from Pensieve://Talks/Toulouse-1705/DogmaDemo.nb and from Pensieve://Talks/Sydney-1708/ExtraDetails@@.nb.

MultiplyingOEs

In[ ]:=
```
Œ_U_ [s1__, Q1_, P1_] Œ_U_ [s2__, Q2_, P2_] ^:= Œ_U[s1, s2, Q1 + Q2, P1 P2];
```

CdsO

In[ ]:=
```
CU@Œ_CU[specs___, Q_, P_] := 𝕆_CU[specs, SS[e^Q P]];
QU@Œ_QU[specs___, Q_, P_] := 𝕆_QU[specs, SS[e^Q P]];
```

Logos

In[ ]:=
```
c_Integer_(k_Integer) := c + O[ϵ]^(k+1);
Λ_(U_,k_) [{α_, β_}, {x_, x_}] := Œ_U[{x}, (α + β) x, 1_k];
Λ_(U_,k_) [{ξ_, α_}, {x, a}] := Œ_U[{a, x}, α a + e^(-γ α) ξ x, 1_k];
Λ_(U_,k_) [{α_, η_}, {a, y}] := Œ_U[{y, a}, α a + e^(-γ α) η y, 1_k];
```

```
Table[
 {Λ_{U,1}[{α, β}, {u, u}],
  lhs = U@Œ_U[{u_1, u_2}, ℏ (α u_1 + β u_2), 1], HL[lhs == U@Λ_{U,1}[ℏ {α, β}, {u, u}]]},
 {U, {CU, QU}}, {u, {y, a, x}}]
```

$\{\{\{Œ_{CU}[\{y\}, y\,(\alpha+\beta), 1+O[\epsilon]^2],$
$CU[\,] + (\alpha\hbar+\beta\hbar)\,CU[y] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right) CU[y, y], \text{True}\},$
$\{Œ_{CU}[\{a\}, a\,(\alpha+\beta), 1+O[\epsilon]^2], CU[\,] + (\alpha\hbar+\beta\hbar)\,CU[a] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right) CU[a, a],$
$\text{True}\}, \{Œ_{CU}[\{x\}, x\,(\alpha+\beta), 1+O[\epsilon]^2],$
$CU[\,] + (\alpha\hbar+\beta\hbar)\,CU[x] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right) CU[x, x], \text{True}\}\},$
$\{\{Œ_{QU}[\{y\}, y\,(\alpha+\beta), 1+O[\epsilon]^2], QU[\,] + (\alpha\hbar+\beta\hbar)\,QU[y] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right) QU[y, y],$
$\text{True}\}, \{Œ_{QU}[\{a\}, a\,(\alpha+\beta), 1+O[\epsilon]^2], QU[\,] + (\alpha\hbar+\beta\hbar)\,QU[a] +$
$\left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right) QU[a, a], \text{True}\}, \{Œ_{QU}[\{x\}, x\,(\alpha+\beta), 1+O[\epsilon]^2],$
$QU[\,] + (\alpha\hbar+\beta\hbar)\,QU[x] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right) QU[x, x], \text{True}\}\}\}$

```
{Λ_{#,1}[{ξ, α}, {x, a}], lhs = #@Œ_#[{x, a}, ℏ (ξ x + α a), 1],
   HL[lhs == #@Λ_{#,1}[ℏ {ξ, α}, {x, a}]]} & /@ {CU, QU}
```

$\{\{Œ_{CU}[\{a, x\}, a\,\alpha + e^{-\alpha\gamma}\,x\,\xi, 1+O[\epsilon]^2],$
$CU[\,] + \alpha\hbar\,CU[a] + (\xi\hbar - \alpha\gamma\xi\hbar^2)\,CU[x] + \frac{1}{2}\alpha^2\hbar^2\,CU[a, a] + \alpha\xi\hbar^2\,CU[a, x] + \frac{1}{2}\xi^2\hbar^2\,CU[x, x],$
$\text{True}\}, \{Œ_{QU}[\{a, x\}, a\,\alpha + e^{-\alpha\gamma}\,x\,\xi, 1+O[\epsilon]^2], QU[\,] + \alpha\hbar\,QU[a] +$
$(\xi\hbar - \alpha\gamma\xi\hbar^2)\,QU[x] + \frac{1}{2}\alpha^2\hbar^2\,QU[a, a] + \alpha\xi\hbar^2\,QU[a, x] + \frac{1}{2}\xi^2\hbar^2\,QU[x, x], \text{True}\}\}$

```
{Λ_{#,2}[{α, η}, {a, y}], lhs = #@Œ_#[{a, y}, ℏ (η y + α a), 1],
   HL[lhs == #@Λ_{#,2}[ℏ {α, η}, {a, y}]]} & /@ {CU, QU}
```

$\{\{Œ_{CU}[\{y, a\}, a\,\alpha + e^{-\alpha\gamma}\,y\,\eta, 1+O[\epsilon]^3],$
$CU[\,] + \alpha\hbar\,CU[a] + (\eta\hbar - \alpha\gamma\eta\hbar^2)\,CU[y] + \frac{1}{2}\alpha^2\hbar^2\,CU[a, a] + \alpha\eta\hbar^2\,CU[y, a] + \frac{1}{2}\eta^2\hbar^2\,CU[y, y],$
$\text{True}\}, \{Œ_{QU}[\{y, a\}, a\,\alpha + e^{-\alpha\gamma}\,y\,\eta, 1+O[\epsilon]^3], QU[\,] + \alpha\hbar\,QU[a] +$
$(\eta\hbar - \alpha\gamma\eta\hbar^2)\,QU[y] + \frac{1}{2}\alpha^2\hbar^2\,QU[a, a] + \alpha\eta\hbar^2\,QU[y, a] + \frac{1}{2}\eta^2\hbar^2\,QU[y, y], \text{True}\}\}$

Goal. In either U, compute $F = e^{-\eta y} e^{\xi x} e^{\eta y} e^{-\xi x}$. First compute $G = e^{\xi x}\, y\, e^{-\xi x}$, a finite sum. Now F satisfies the ODE $\partial_\eta F = \partial_\eta\left(e^{-\eta y} e^{\eta G}\right) = -yF + FG$ with initial conditions $F(\eta = 0) = 1$. So we set it up and solve:

```
If[$k > 0, With[{U = CU},
  Module[{G, F, fs, bs, e, b, es, sol},
   G = Echo@Simp[Table[ξ^k / k!, {k, 0, $k + 1}].NestList[Simp[B[x_U , #]] &, y_U, $k + 1]];
   fs = Echo@Flatten@Table[f_{l,i,j,k}[η], {l, 0, $k}, {i, 0, 1}, {j, 0, 1}, {k, 0, 1}];
   F = Echo[fs.(bs = fs /. f_{l_,i_,j_,k_}[η] :→ ϵ^l U@{y^i, a^j, x^k})];
   es = Flatten[
     Table[Coefficient[e, b] == 0, {e, {F - 1_U /. η → 0, F ** G - y_U ** F - ∂_η F}}, {b, bs}]];
   sol = Echo@First[F /. DSolve[es, fs, η]];
   Echo[sol /. {e- → 1, U → Times}];
   Collect[sol /. {e- → 1, U → Times}, ϵ, Simplify]
  ]]]
```

$-t\,\xi\,CU[\,] + 2\,\epsilon\,\xi\,CU[a] - \gamma\,\epsilon\,\xi^2\,CU[x] + CU[y]$

$\{f_{0,0,0,0}[\eta],\ f_{1,0,0,0}[\eta],\ f_{1,0,0,1}[\eta],\ f_{1,0,1,0}[\eta],\ f_{1,0,1,1}[\eta],\ f_{1,1,0,0}[\eta],\ f_{1,1,0,1}[\eta],\ f_{1,1,1,0}[\eta],\ f_{1,1,1,1}[\eta]\}$

$CU[\,]\,f_{0,0,0,0}[\eta] + \epsilon\,CU[\,]\,f_{1,0,0,0}[\eta] + \epsilon\,CU[x]\,f_{1,0,0,1}[\eta] + \epsilon\,CU[a]\,f_{1,0,1,0}[\eta] + \epsilon\,CU[a, x]\,f_{1,0,1,1}[\eta] + \epsilon\,CU[y]\,f_{1,1,0,0}[\eta] + \epsilon\,CU[y, x]\,f_{1,1,0,1}[\eta] + \epsilon\,CU[y, a]\,f_{1,1,1,0}[\eta] + \epsilon\,CU[y, a, x]\,f_{1,1,1,1}[\eta]$

$e^{-t\eta\xi}\,CU[\,] + \frac{1}{2}\,e^{-t\eta\xi}\,t\,\gamma\,\epsilon\,\eta^2\,\xi^2\,CU[\,] + 2\,e^{-t\eta\xi}\,\epsilon\,\eta\,\xi\,CU[a] - e^{-t\eta\xi}\,\gamma\,\epsilon\,\eta\,\xi^2\,CU[x] - e^{-t\eta\xi}\,\gamma\,\epsilon\,\eta^2\,\xi\,CU[y]$

$1 + 2\,a\,\epsilon\,\eta\,\xi - y\,\gamma\,\epsilon\,\eta^2\,\xi - x\,\gamma\,\epsilon\,\eta\,\xi^2 + \frac{1}{2}\,t\,\gamma\,\epsilon\,\eta^2\,\xi^2$

$1 + \frac{1}{2}\,\epsilon\,\eta\,\xi\,\left(4\,a + \gamma\,\left(-2\,y\,\eta - 2\,x\,\xi + t\,\eta\,\xi\right)\right)$

Logos

In[∘]:=
```
Λ_{U_,kk_}[{ξ1_, η1_}, {x, y}] := Λ_{U,kk}[{ξ1, η1}, {x, y}] =
  Block[{$k = kk, $p = kk}, Module[{ξ, η, G, F, fs, f, bs, e, b, es},
    G = Simp[Table[ξ^k / k!, {k, 0, $k + 1}].NestList[Simp[B[x_U , #]] &, y_U, $k + 1]];
    fs = Flatten@Table[f_{l,i,j,k}[η], {l, 0, $k}, {i, 0, 1}, {j, 0, 1}, {k, 0, 1}];
    F = fs.(bs = fs /. f_{l_,i_,j_,k_}[η] :→ ϵ^l U@{y^i, a^j, x^k});
    es = Flatten[
      Table[Coefficient[e, b] == 0, {e, {F - 1_U /. η → 0, F ** G - y_U ** F - ∂_η F}}, {b, bs}]];
    F = F /. DSolve[es, fs, η][[1]];
    Œ_U[{y, a, x},
      ξ x + η y + (U /. {CU → -t η ξ, QU → η ξ (1 - T) / ℏ}),
      F + O_{$k} /. {e- → 1, U → Times}
     ] /. {ξ → ξ1, η → η1}]];
```

In[∘]:= **Timing@$\Lambda_{QU,2}$[{ξ, η}, {x, y}]**

Out[∘]= $\Big\{1.64063,\ \mathbb{E}_{QU}\Big[\{y, a, x\},\ y\,\eta + x\,\xi + \frac{(1-T)\,\eta\,\xi}{\hbar},\ 1 + \frac{1}{4\,\hbar}\,\eta\,\xi\,\big(\gamma\,\eta\,\xi - 4\,T\,\gamma\,\eta\,\xi + 3\,T^2\,\gamma\,\eta\,\xi + 8\,a\,T\,\hbar + 2\,y\,\gamma\,\eta\,\hbar - 6\,T\,y\,\gamma\,\eta\,\hbar + 2\,x\,\gamma\,\xi\,\hbar - 6\,T\,x\,\gamma\,\xi\,\hbar + 4\,x\,y\,\gamma\,\hbar^2\big)\,\epsilon + \Big(-a\,T\,y\,\gamma\,\eta^2\,\xi\,(-\eta\,\xi + 3\,T\,\eta\,\xi - 3\,\hbar) - a\,T\,x\,\gamma\,\eta\,\xi^2\,(-\eta\,\xi + 3\,T\,\eta\,\xi - 3\,\hbar) + 2\,a^2\,T\,\eta\,\xi\,(T\,\eta\,\xi - \hbar) + 2\,a\,T\,x\,y\,\gamma\,\eta^2\,\xi^2\,\hbar - \frac{1}{2}\,x\,y^2\,\gamma^2\,\eta^2\,\xi\,(-\eta\,\xi + 3\,T\,\eta\,\xi - \hbar)\,\hbar - \frac{1}{2}\,x^2\,y\,\gamma^2\,\eta\,\xi^2\,(-\eta\,\xi + 3\,T\,\eta\,\xi - \hbar)\,\hbar + \frac{1}{2}\,x^2\,y^2\,\gamma^2\,\eta^2\,\xi^2\,\hbar^2 + \frac{1}{24}\,y^2\,\gamma^2\,\eta^3\,\xi\,(3\,\eta\,\xi - 18\,T\,\eta\,\xi + 27\,T^2\,\eta\,\xi + 4\,\hbar - 28\,T\,\hbar) + \frac{1}{24}\,x^2\,\gamma^2\,\eta\,\xi^3\,(3\,\eta\,\xi - 18\,T\,\eta\,\xi + 27\,T^2\,\eta\,\xi + 4\,\hbar - 28\,T\,\hbar) + \frac{1}{2\,\hbar}\,a\,T\,\gamma\,\eta^2\,\xi^2\,(\eta\,\xi - 4\,T\,\eta\,\xi + 3\,T^2\,\eta\,\xi + 4\,\hbar - 6\,T\,\hbar) + \frac{1}{4}\,x\,y\,\gamma^2\,\eta\,\xi\,(2\,\eta^2\,\xi^2 - 10\,T\,\eta^2\,\xi^2 + 12\,T^2\,\eta^2\,\xi^2 + 5\,\eta\,\xi\,\hbar - 21\,T\,\eta\,\xi\,\hbar + 2\,\hbar^2) - \frac{1}{24\,\hbar}\,y\,\gamma^2\,\eta^2\,\xi\,(-3\,\eta^2\,\xi^2 + 21\,T\,\eta^2\,\xi^2 - 45\,T^2\,\eta^2\,\xi^2 + 27\,T^3\,\eta^2\,\xi^2 - 10\,\eta\,\xi\,\hbar + 68\,T\,\eta\,\xi\,\hbar - 82\,T^2\,\eta\,\xi\,\hbar - 6\,\hbar^2 + 30\,T\,\hbar^2) - \frac{1}{24\,\hbar}\,x\,\gamma^2\,\eta\,\xi^2\,(-3\,\eta^2\,\xi^2 + 21\,T\,\eta^2\,\xi^2 - 45\,T^2\,\eta^2\,\xi^2 + 27\,T^3\,\eta^2\,\xi^2 - 10\,\eta\,\xi\,\hbar + 68\,T\,\eta\,\xi\,\hbar - 82\,T^2\,\eta\,\xi\,\hbar - 6\,\hbar^2 + 30\,T\,\hbar^2) + \frac{1}{288\,\hbar^2}\,(-1 + T)\,\gamma^2\,\eta^2\,\xi^2\,(-9\,\eta^2\,\xi^2 + 63\,T\,\eta^2\,\xi^2 - 135\,T^2\,\eta^2\,\xi^2 + 81\,T^3\,\eta^2\,\xi^2 - 40\,\eta\,\xi\,\hbar + 272\,T\,\eta\,\xi\,\hbar - 328\,T^2\,\eta\,\xi\,\hbar - 36\,\hbar^2 + 180\,T\,\hbar^2)\Big)\,\epsilon^2 + O[\epsilon]^3\Big]\Big\}$

**{$\Lambda_{CU,1}$[{ξ, η}, {x, y}], lhs = CU@$\mathbb{E}_{CU}$[{x, y}, ℏ (ξ x + η y), 1],
HL[lhs == CU@$\Lambda_{CU,1}$[ℏ {ξ, η}, {x, y}]]}**

$\Big\{\mathbb{E}_{CU}\Big[\{y, a, x\},\ y\,\eta + x\,\xi - t\,\eta\,\xi,\ 1 + \frac{1}{2}\,\eta\,\xi\,(4\,a - 2\,y\,\gamma\,\eta - 2\,x\,\gamma\,\xi + t\,\gamma\,\eta\,\xi)\,\epsilon + O[\epsilon]^2\Big],\ (1 - t\,\eta\,\xi\,\hbar^2)\,CU[\,] + 2\,\epsilon\,\eta\,\xi\,\hbar^2\,CU[a] + \xi\,\hbar\,CU[x] + \eta\,\hbar\,CU[y] + \frac{1}{2}\,\xi^2\,\hbar^2\,CU[x, x] + \eta\,\xi\,\hbar^2\,CU[y, x] + \frac{1}{2}\,\eta^2\,\hbar^2\,CU[y, y],\ \text{True}\Big\}$

In[∘]:= **{$\Lambda_{QU,1}$[{ξ, η}, {x, y}], lhs = QU@$\mathbb{E}_{QU}$[{x, y}, ℏ (ξ x + η y), 1],
HL@SimpT[lhs == QU@$\Lambda_{QU,1}$[ℏ {ξ, η}, {x, y}]]}**

Out[∘]= $\Big\{\mathbb{E}_{QU}\Big[\{y, a, x\},\ y\,\eta + x\,\xi + \frac{(1-T)\,\eta\,\xi}{\hbar},\ 1 + \frac{1}{4\,\hbar}\,\eta\,\xi\,(\gamma\,\eta\,\xi - 4\,T\,\gamma\,\eta\,\xi + 3\,T^2\,\gamma\,\eta\,\xi + 8\,a\,T\,\hbar + 2\,y\,\gamma\,\eta\,\hbar - 6\,T\,y\,\gamma\,\eta\,\hbar + 2\,x\,\gamma\,\xi\,\hbar - 6\,T\,x\,\gamma\,\xi\,\hbar + 4\,x\,y\,\gamma\,\hbar^2)\,\epsilon + O[\epsilon]^2\Big],\ (1 + \eta\,\xi\,\hbar - T\,\eta\,\xi\,\hbar)\,QU[\,] + 2\,T\,\epsilon\,\eta\,\xi\,\hbar^2\,QU[a] + \xi\,\hbar\,QU[x] + \eta\,\hbar\,QU[y] + \frac{1}{2}\,\xi^2\,\hbar^2\,QU[x, x] + \eta\,\xi\,\hbar^2\,QU[y, x] + \frac{1}{2}\,\eta^2\,\hbar^2\,QU[y, y],\ \text{True}\Big\}$

```
{tt = Last[Λ_{CU,2}[{ξ, η}, {x, y}]], Log[tt],
  Exponent[Normal@Log[tt] /. {ξ → ℏ ξ, η → ℏ η, x → ℏ x, y → ℏ y}, ℏ]} // Expand
```

$$\Big\{1+\left(2\,a\,\eta\,\xi-y\,\gamma\,\eta^2\,\xi-x\,\gamma\,\eta\,\xi^2+\frac{1}{2}\,t\,\gamma\,\eta^2\,\xi^2\right)\epsilon+\left(2\,a^2\,\eta^2\,\xi^2-a\,\gamma\,\eta^2\,\xi^2-2\,a\,y\,\gamma\,\eta^3\,\xi^2+y\,\gamma^2\,\eta^3\,\xi^2+\frac{1}{2}\,y^2\,\gamma^2\,\eta^4\,\xi^2-2\,a\,x\,\gamma\,\eta^2\,\xi^3+x\,\gamma^2\,\eta^2\,\xi^3+a\,t\,\gamma\,\eta^3\,\xi^3-\frac{1}{3}\,t\,\gamma^2\,\eta^3\,\xi^3+x\,y\,\gamma^2\,\eta^3\,\xi^3-\frac{1}{2}\,t\,y\,\gamma^2\,\eta^4\,\xi^3+\frac{1}{2}\,x^2\,\gamma^2\,\eta^2\,\xi^4-\frac{1}{2}\,t\,x\,\gamma^2\,\eta^3\,\xi^4+\frac{1}{8}\,t^2\,\gamma^2\,\eta^4\,\xi^4\right)\epsilon^2+O[\epsilon]^3,$$

$$\left(2\,a\,\eta\,\xi-y\,\gamma\,\eta^2\,\xi-x\,\gamma\,\eta\,\xi^2+\frac{1}{2}\,t\,\gamma\,\eta^2\,\xi^2\right)\epsilon+\left(-a\,\gamma\,\eta^2\,\xi^2+y\,\gamma^2\,\eta^3\,\xi^2+x\,\gamma^2\,\eta^2\,\xi^3-\frac{1}{3}\,t\,\gamma^2\,\eta^3\,\xi^3\right)\epsilon^2+O[\epsilon]^3,\ 6\Big\}$$

```
{tt = Last[Λ_{QU,2}[{ξ, η}, {x, y}]], Log[tt],
  Exponent[Normal@Log[tt] /. {ξ → d ξ, η → d η, x → d x, y → d y}, d]} // Expand
```

$$\Big\{1+\Big(2\,a\,T\,\eta\,\xi+\frac{1}{2}\,y\,\gamma\,\eta^2\,\xi-\frac{3}{2}\,T\,y\,\gamma\,\eta^2\,\xi+$$
$$\frac{1}{2}\,x\,\gamma\,\eta\,\xi^2-\frac{3}{2}\,T\,x\,\gamma\,\eta\,\xi^2+\frac{\gamma\,\eta^2\,\xi^2}{4\,\hbar}-\frac{T\,\gamma\,\eta^2\,\xi^2}{\hbar}+\frac{3\,T^2\,\gamma\,\eta^2\,\xi^2}{4\,\hbar}+x\,y\,\gamma\,\eta\,\xi\,\hbar\Big)\,\epsilon+$$
$$\Big(2\,a^2\,T^2\,\eta^2\,\xi^2+2\,a\,T\,\gamma\,\eta^2\,\xi^2-3\,a\,T^2\,\gamma\,\eta^2\,\xi^2+\frac{1}{8}\,\gamma^2\,\eta^2\,\xi^2-\frac{3}{4}\,T\,\gamma^2\,\eta^2\,\xi^2+\frac{5}{8}\,T^2\,\gamma^2\,\eta^2\,\xi^2+$$
$$a\,T\,y\,\gamma\,\eta^3\,\xi^2-3\,a\,T^2\,y\,\gamma\,\eta^3\,\xi^2+\frac{5}{12}\,y\,\gamma^2\,\eta^3\,\xi^2-\frac{17}{6}\,T\,y\,\gamma^2\,\eta^3\,\xi^2+\frac{41}{12}\,T^2\,y\,\gamma^2\,\eta^3\,\xi^2+\frac{1}{8}\,y^2\,\gamma^2\,\eta^4\,\xi^2-$$
$$\frac{3}{4}\,T\,y^2\,\gamma^2\,\eta^4\,\xi^2+\frac{9}{8}\,T^2\,y^2\,\gamma^2\,\eta^4\,\xi^2+a\,T\,x\,\gamma\,\eta^2\,\xi^3-3\,a\,T^2\,x\,\gamma\,\eta^2\,\xi^3+\frac{5}{12}\,x\,\gamma^2\,\eta^2\,\xi^3-\frac{17}{6}\,T\,x\,\gamma^2\,\eta^2\,\xi^3+$$
$$\frac{41}{12}\,T^2\,x\,\gamma^2\,\eta^2\,\xi^3+\frac{1}{2}\,x\,y\,\gamma^2\,\eta^3\,\xi^3-\frac{5}{2}\,T\,x\,y\,\gamma^2\,\eta^3\,\xi^3+3\,T^2\,x\,y\,\gamma^2\,\eta^3\,\xi^3+\frac{1}{8}\,x^2\,\gamma^2\,\eta^2\,\xi^4-$$
$$\frac{3}{4}\,T\,x^2\,\gamma^2\,\eta^2\,\xi^4+\frac{9}{8}\,T^2\,x^2\,\gamma^2\,\eta^2\,\xi^4+\frac{\gamma^2\,\eta^4\,\xi^4}{32\,\hbar^2}-\frac{T\,\gamma^2\,\eta^4\,\xi^4}{4\,\hbar^2}+\frac{11\,T^2\,\gamma^2\,\eta^4\,\xi^4}{16\,\hbar^2}-\frac{3\,T^3\,\gamma^2\,\eta^4\,\xi^4}{4\,\hbar^2}+$$
$$\frac{9\,T^4\,\gamma^2\,\eta^4\,\xi^4}{32\,\hbar^2}+\frac{a\,T\,\gamma\,\eta^3\,\xi^3}{2\,\hbar}-\frac{2\,a\,T^2\,\gamma\,\eta^3\,\xi^3}{\hbar}+\frac{3\,a\,T^3\,\gamma\,\eta^3\,\xi^3}{2\,\hbar}+\frac{5\,\gamma^2\,\eta^3\,\xi^3}{36\,\hbar}-\frac{13\,T\,\gamma^2\,\eta^3\,\xi^3}{12\,\hbar}+$$
$$\frac{25\,T^2\,\gamma^2\,\eta^3\,\xi^3}{12\,\hbar}-\frac{41\,T^3\,\gamma^2\,\eta^3\,\xi^3}{36\,\hbar}+\frac{y\,\gamma^2\,\eta^4\,\xi^3}{8\,\hbar}-\frac{7\,T\,y\,\gamma^2\,\eta^4\,\xi^3}{8\,\hbar}+\frac{15\,T^2\,y\,\gamma^2\,\eta^4\,\xi^3}{8\,\hbar}-\frac{9\,T^3\,y\,\gamma^2\,\eta^4\,\xi^3}{8\,\hbar}+$$
$$\frac{x\,\gamma^2\,\eta^3\,\xi^4}{8\,\hbar}-\frac{7\,T\,x\,\gamma^2\,\eta^3\,\xi^4}{8\,\hbar}+\frac{15\,T^2\,x\,\gamma^2\,\eta^3\,\xi^4}{8\,\hbar}-\frac{9\,T^3\,x\,\gamma^2\,\eta^3\,\xi^4}{8\,\hbar}-2\,a^2\,T\,\eta\,\xi\,\hbar+3\,a\,T\,y\,\gamma\,\eta^2\,\xi\,\hbar+$$
$$\frac{1}{4}\,y\,\gamma^2\,\eta^2\,\xi\,\hbar-\frac{5}{4}\,T\,y\,\gamma^2\,\eta^2\,\xi\,\hbar+\frac{1}{6}\,y^2\,\gamma^2\,\eta^3\,\xi\,\hbar-\frac{7}{6}\,T\,y^2\,\gamma^2\,\eta^3\,\xi\,\hbar+3\,a\,T\,x\,\gamma\,\eta\,\xi^2\,\hbar+\frac{1}{4}\,x\,\gamma^2\,\eta\,\xi^2\,\hbar-$$
$$\frac{5}{4}\,T\,x\,\gamma^2\,\eta\,\xi^2\,\hbar+2\,a\,T\,x\,y\,\gamma\,\eta^2\,\xi^2\,\hbar+\frac{5}{4}\,x\,y\,\gamma^2\,\eta^2\,\xi^2\,\hbar-\frac{21}{4}\,T\,x\,y\,\gamma^2\,\eta^2\,\xi^2\,\hbar+\frac{1}{2}\,x\,y^2\,\gamma^2\,\eta^3\,\xi^2\,\hbar-$$
$$\frac{3}{2}\,T\,x\,y^2\,\gamma^2\,\eta^3\,\xi^2\,\hbar+\frac{1}{6}\,x^2\,\gamma^2\,\eta\,\xi^3\,\hbar-\frac{7}{6}\,T\,x^2\,\gamma^2\,\eta\,\xi^3\,\hbar+\frac{1}{2}\,x^2\,y\,\gamma^2\,\eta^2\,\xi^3\,\hbar-\frac{3}{2}\,T\,x^2\,y\,\gamma^2\,\eta^2\,\xi^3\,\hbar+$$
$$\frac{1}{2}\,x\,y\,\gamma^2\,\eta\,\xi\,\hbar^2+\frac{1}{2}\,x\,y^2\,\gamma^2\,\eta^2\,\xi\,\hbar^2+\frac{1}{2}\,x^2\,y\,\gamma^2\,\eta\,\xi^2\,\hbar^2+\frac{1}{2}\,x^2\,y^2\,\gamma^2\,\eta^2\,\xi^2\,\hbar^2\Big)\,\epsilon^2+O[\epsilon]^3,$$
$$\Big(2\,a\,T\,\eta\,\xi+\frac{1}{2}\,y\,\gamma\,\eta^2\,\xi-\frac{3}{2}\,T\,y\,\gamma\,\eta^2\,\xi+\frac{1}{2}\,x\,\gamma\,\eta\,\xi^2-\frac{3}{2}\,T\,x\,\gamma\,\eta\,\xi^2+\frac{\gamma\,\eta^2\,\xi^2}{4\,\hbar}-$$
$$\frac{T\,\gamma\,\eta^2\,\xi^2}{\hbar}+\frac{3\,T^2\,\gamma\,\eta^2\,\xi^2}{4\,\hbar}+x\,y\,\gamma\,\eta\,\xi\,\hbar\Big)\,\epsilon+$$
$$\Big(2\,a\,T\,\gamma\,\eta^2\,\xi^2-3\,a\,T^2\,\gamma\,\eta^2\,\xi^2+\frac{1}{8}\,\gamma^2\,\eta^2\,\xi^2-\frac{3}{4}\,T\,\gamma^2\,\eta^2\,\xi^2+\frac{5}{8}\,T^2\,\gamma^2\,\eta^2\,\xi^2+\frac{5}{12}\,y\,\gamma^2\,\eta^3\,\xi^2-$$
$$\frac{17}{6}\,T\,y\,\gamma^2\,\eta^3\,\xi^2+\frac{41}{12}\,T^2\,y\,\gamma^2\,\eta^3\,\xi^2+\frac{5}{12}\,x\,\gamma^2\,\eta^2\,\xi^3-\frac{17}{6}\,T\,x\,\gamma^2\,\eta^2\,\xi^3+\frac{41}{12}\,T^2\,x\,\gamma^2\,\eta^2\,\xi^3+$$
$$\frac{5\,\gamma^2\,\eta^3\,\xi^3}{36\,\hbar}-\frac{13\,T\,\gamma^2\,\eta^3\,\xi^3}{12\,\hbar}+\frac{25\,T^2\,\gamma^2\,\eta^3\,\xi^3}{12\,\hbar}-\frac{41\,T^3\,\gamma^2\,\eta^3\,\xi^3}{36\,\hbar}-2\,a^2\,T\,\eta\,\xi\,\hbar+3\,a\,T\,y\,\gamma\,\eta^2\,\xi\,\hbar+$$
$$\frac{1}{4}\,y\,\gamma^2\,\eta^2\,\xi\,\hbar-\frac{5}{4}\,T\,y\,\gamma^2\,\eta^2\,\xi\,\hbar+\frac{1}{6}\,y^2\,\gamma^2\,\eta^3\,\xi\,\hbar-\frac{7}{6}\,T\,y^2\,\gamma^2\,\eta^3\,\xi\,\hbar+3\,a\,T\,x\,\gamma\,\eta\,\xi^2\,\hbar+$$
$$\frac{1}{4}\,x\,\gamma^2\,\eta\,\xi^2\,\hbar-\frac{5}{4}\,T\,x\,\gamma^2\,\eta\,\xi^2\,\hbar+\frac{5}{4}\,x\,y\,\gamma^2\,\eta^2\,\xi^2\,\hbar-\frac{21}{4}\,T\,x\,y\,\gamma^2\,\eta^2\,\xi^2\,\hbar+\frac{1}{6}\,x^2\,\gamma^2\,\eta\,\xi^3\,\hbar-$$
$$\frac{7}{6}\,T\,x^2\,\gamma^2\,\eta\,\xi^3\,\hbar+\frac{1}{2}\,x\,y\,\gamma^2\,\eta\,\xi\,\hbar^2+\frac{1}{2}\,x\,y^2\,\gamma^2\,\eta^2\,\xi\,\hbar^2+\frac{1}{2}\,x^2\,y\,\gamma^2\,\eta\,\xi^2\,\hbar^2\Big)\,\epsilon^2+O[\epsilon]^3,\ 6\Big\}$$

Logos

```
In[•]:= Simp[Œ_U_[specs___, Q_, P_]] := Œ_U[specs, CF[Q], CF[P]];
```

Logos

```
In[•]:= Λ_U_,k_[{𝓋1_, ω1_, 𝛿_}, {u_, w_}] := Simp@Module[{𝓋, ω, yax, q, p, Q, d},
    {yax, q, p} = List @@ Λ_U,k[{𝓋, ω}, {u, w}];
    Œ_U[yax, Q = (𝓋 u + ω w + 𝛿 u w + d 𝓋 ω) / (1 - d 𝛿),
      Expand[(1 - d 𝛿)^-1 e^-Q DP_{𝓋→D_u, ω→D_w}[p][e^Q]] + 0_k] /. {d → ∂_{𝓋,ω} q} /. {𝓋 → 𝓋1, ω → ω1}];
```

```
Block[{$p = 4, $k = 1},
 {Λ_CU,$k[ℏ {ξ, η, δ}, {x, y}],
  Short[lhs = CU@Œ_CU[{x, y}, ℏ (ξ x + η y + δ x y), 1_$k], 5],
  HL@Simp[lhs - CU@Λ_CU,$k[ℏ {ξ, η, δ}, {x, y}]]}
]
```

$$\Big\{Œ_{CU}\big[\{y, a, x\}, \frac{x\,y\,\delta\,\hbar + y\,\eta\,\hbar + x\,\xi\,\hbar - t\,\eta\,\xi\,\hbar^2}{1 + t\,\delta\,\hbar},$$

$$\frac{1}{1+t\,\delta\,\hbar} + \Big(\big(4\,a\,\delta\,\hbar + 12\,a\,t\,\delta^2\,\hbar^2 + 4\,a\,x\,y\,\delta^2\,\hbar^2 + 2\,t\,\gamma\,\delta^2\,\hbar^2 - 8\,x\,y\,\gamma\,\delta^2\,\hbar^2 + 4\,a\,y\,\delta\,\eta\,\hbar^2 -$$
$$4\,y\,\gamma\,\delta\,\eta\,\hbar^2 + 4\,a\,x\,\delta\,\xi\,\hbar^2 - 4\,x\,\gamma\,\delta\,\xi\,\hbar^2 + 4\,a\,\eta\,\xi\,\hbar^2 + 12\,a\,t^2\,\delta^3\,\hbar^3 + 8\,a\,t\,x\,y\,\delta^3\,\hbar^3 +$$
$$4\,t^2\,\gamma\,\delta^3\,\hbar^3 - 12\,t\,x\,y\,\gamma\,\delta^3\,\hbar^3 - 4\,x^2\,y^2\,\gamma\,\delta^3\,\hbar^3 + 8\,a\,t\,y\,\delta^2\,\eta\,\hbar^3 - 4\,t\,y\,\gamma\,\delta^2\,\eta\,\hbar^3 -$$
$$6\,x\,y^2\,\gamma\,\delta^2\,\eta\,\hbar^3 - 2\,y^2\,\gamma\,\delta\,\eta^2\,\hbar^3 + 8\,a\,t\,x\,\delta^2\,\xi\,\hbar^3 - 4\,t\,x\,\gamma\,\delta^2\,\xi\,\hbar^3 - 6\,x^2\,y\,\gamma\,\delta^2\,\xi\,\hbar^3 +$$
$$8\,a\,t\,\delta\,\eta\,\xi\,\hbar^3 + 4\,t\,\gamma\,\delta\,\eta\,\xi\,\hbar^3 - 8\,x\,y\,\gamma\,\delta\,\eta\,\xi\,\hbar^3 - 2\,y\,\gamma\,\eta^2\,\xi\,\hbar^3 - 2\,x^2\,\gamma\,\delta\,\xi^2\,\hbar^3 - 2\,x\,\gamma\,\eta\,\xi^2\,\hbar^3 +$$
$$4\,a\,t^3\,\delta^4\,\hbar^4 + 4\,a\,t^2\,x\,y\,\delta^4\,\hbar^4 + 2\,t^3\,\gamma\,\delta^4\,\hbar^4 - 4\,t^2\,x\,y\,\gamma\,\delta^4\,\hbar^4 - 3\,t\,x^2\,y^2\,\gamma\,\delta^4\,\hbar^4 +$$
$$4\,a\,t^2\,y\,\delta^3\,\eta\,\hbar^4 - 4\,t\,x\,y^2\,\gamma\,\delta^3\,\eta\,\hbar^4 - t\,y^2\,\gamma\,\delta^2\,\eta^2\,\hbar^4 + 4\,a\,t^2\,x\,\delta^3\,\xi\,\hbar^4 - 4\,t\,x^2\,y\,\gamma\,\delta^3\,\xi\,\hbar^4 +$$
$$4\,a\,t^2\,\delta^2\,\eta\,\xi\,\hbar^4 + 4\,t^2\,\gamma\,\delta^2\,\eta\,\xi\,\hbar^4 - 4\,t\,x\,y\,\gamma\,\delta^2\,\eta\,\xi\,\hbar^4 - t\,x^2\,\gamma\,\delta^2\,\xi^2\,\hbar^4 + t\,\gamma\,\eta^2\,\xi^2\,\hbar^4\big)\,\epsilon\Big) \Big/$$
$$\big(2 + 10\,t\,\delta\,\hbar + 20\,t^2\,\delta^2\,\hbar^2 + 20\,t^3\,\delta^3\,\hbar^3 + 10\,t^4\,\delta^4\,\hbar^4 + 2\,t^5\,\delta^5\,\hbar^5\big) + O[\epsilon]^2\big],$$

$$\Big(1 - t\,\delta\,\hbar + t^2\,\delta^2\,\hbar^2 + t\,\gamma\,\delta^2\,\epsilon\,\hbar^2 - t\,\eta\,\xi\,\hbar^2 - t^3\,\delta^3\,\hbar^3 - 3\,t^2\,\gamma\,\delta^3\,\epsilon\,\hbar^3 + 2\,t^2\,\delta\,\eta\,\xi\,\hbar^3 +$$
$$2\,t\,\gamma\,\delta\,\epsilon\,\eta\,\xi\,\hbar^3 + t^4\,\delta^4\,\hbar^4 + 6\,t^3\,\gamma\,\delta^4\,\epsilon\,\hbar^4 - 3\,t^3\,\delta^2\,\eta\,\xi\,\hbar^4 -$$
$$9\,t^2\,\gamma\,\delta^2\,\epsilon\,\eta\,\xi\,\hbar^4 + \frac{1}{2}\,t^2\,\eta^2\,\xi^2\,\hbar^4 + \frac{1}{2}\,t\,\gamma\,\epsilon\,\eta^2\,\xi^2\,\hbar^4\Big)\,\mathrm{CU}[\,] +$$

$$\big(2\,\delta\,\epsilon\,\hbar - 4\,t\,\delta^2\,\epsilon\,\hbar^2 + 2\,\epsilon\,\eta\,\xi\,\hbar^2 + 6\,t^2\,\delta^3\,\epsilon\,\hbar^3 - 8\,t\,\delta\,\epsilon\,\eta\,\xi\,\hbar^3 - 8\,t^3\,\delta^4\,\epsilon\,\hbar^4 +$$
$$18\,t^2\,\delta^2\,\epsilon\,\eta\,\xi\,\hbar^4 - 2\,t\,\epsilon\,\eta^2\,\xi^2\,\hbar^4\big)\,\mathrm{CU}[a] +$$

$$\langle\langle 37\rangle\rangle + \frac{1}{6}\,\delta^3\,\eta\,\hbar^4\,\mathrm{CU}[y, y, y, y, x, x, x] +$$

$$\frac{1}{24}\,\delta^4\,\hbar^4\,\mathrm{CU}[y, y, y, y, x, x, x, x],\ 0\Big\}$$

```
{Λ_{QU,2}[{ξ, η, δ}, {x, y}], lhs = QU@ℭ_{QU}[{x, y}, ℏ (ξ x + η y + δ x y), 1],
 HL@SimpT[lhs == QU@Λ_{QU,1}[ℏ {ξ, η, δ}, {x, y}]]}
```

$\{\mathbb{C}_{QU}[\{y, a, x\}, \frac{\cdots 1 \cdots}{\cdots 1 \cdots},$
$\frac{\hbar}{-\delta+T\,\delta+\hbar} + ((-8\,a\,T\,\delta^4\,\hbar^2 + 24\,a\,T^2\,\delta^4\,\hbar^2 - 24\,a\,T^3\,\delta^4\,\hbar^2 + 8\,a\,T^4\,\delta^4\,\hbar^2 + \cdots 149 \cdots + 4\,x^2\,y^2\,\gamma\,\delta^2\,\hbar^6 + 4\,x\,y^2\,\gamma\,\delta\,\eta\,\hbar^6 + 4\,x^2\,y\,\gamma\,\delta\,\xi\,\hbar^6 + 4\,x\,y\,\gamma\,\eta\,\xi\,\hbar^6)\,\epsilon) / (-4\,\delta^5 + 20\,T\,\delta^5 - 40\,T^2\,\delta^5 + 40\,T^3\,\delta^5 - 20\,T^4\,\delta^5 + 4\,T^5\,\delta^5 + \cdots 12 \cdots + 40\,T^3\,\delta^3\,\hbar^2 + 40\,\delta^2\,\hbar^3 - 80\,T\,\delta^2\,\hbar^3 + 40\,T^2\,\delta^2\,\hbar^3 - 20\,\delta\,\hbar^4 + 20\,T\,\delta\,\hbar^4 + 4\,\hbar^5) + \frac{(\cdots 1 \cdots)\,\cdots 1 \cdots}{\cdots 1 \cdots} + O[\epsilon]^3], \cdots 1 \cdots, \text{True}\}$

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```
{tt = ComposeSeries[(1 + t δ) Last[Λ_{CU,2}[{ξ, η, δ}, {x, y}]], (1 + t δ)^4 ϵ + O[ϵ]^18];
  Together@Log[tt],
  Exponent[Normal@Together@Log[tt] /. {ξ → d ξ, η → d η, x → d x, y → d y}, d],
  Exponent[Normal@Together@Log[tt] /. {x → d x, y → d y}, d]
 } // Expand
```

$\{(2\,a\,\delta + 6\,a\,t\,\delta^2 + 2\,a\,x\,y\,\delta^2 + t\,\gamma\,\delta^2 - 4\,x\,y\,\gamma\,\delta^2 + 6\,a\,t^2\,\delta^3 + 4\,a\,t\,x\,y\,\delta^3 + 2\,t^2\,\gamma\,\delta^3 - 6\,t\,x\,y\,\gamma\,\delta^3 - 2\,x^2\,y^2\,\gamma\,\delta^3 + 2\,a\,t^3\,\delta^4 + 2\,a\,t^2\,x\,y\,\delta^4 + t^3\,\gamma\,\delta^4 - 2\,t^2\,x\,y\,\gamma\,\delta^4 - \frac{3}{2}\,t\,x^2\,y^2\,\gamma\,\delta^4 + 2\,a\,y\,\delta\,\eta - 2\,y\,\gamma\,\delta\,\eta + 4\,a\,t\,y\,\delta^2\,\eta - 2\,t\,y\,\gamma\,\delta^2\,\eta - 3\,x\,y^2\,\gamma\,\delta^2\,\eta + 2\,a\,t^2\,y\,\delta^3\,\eta - 2\,t\,x\,y^2\,\gamma\,\delta^3\,\eta - y^2\,\gamma\,\delta\,\eta^2 - \frac{1}{2}\,t\,y^2\,\gamma\,\delta^2\,\eta^2 + 2\,a\,x\,\delta\,\xi - 2\,x\,\gamma\,\delta\,\xi + 4\,a\,t\,x\,\delta^2\,\xi - 2\,t\,x\,\gamma\,\delta^2\,\xi - 3\,x^2\,y\,\gamma\,\delta^2\,\xi + 2\,a\,t^2\,x\,\delta^3\,\xi - 2\,t\,x^2\,y\,\gamma\,\delta^3\,\xi + 2\,a\,\eta\,\xi + 4\,a\,t\,\delta\,\eta\,\xi + 2\,t\,\gamma\,\delta\,\eta\,\xi - 4\,x\,y\,\gamma\,\delta\,\eta\,\xi + 2\,a\,t^2\,\delta^2\,\eta\,\xi + 2\,t^2\,\gamma\,\delta^2\,\eta\,\xi - 2\,t\,x\,y\,\gamma\,\delta^2\,\eta\,\xi - y\,\gamma\,\eta^2\,\xi - x^2\,\gamma\,\delta\,\xi^2 - \frac{1}{2}\,t\,x^2\,\gamma\,\delta^2\,\xi^2 - x\,\gamma\,\eta\,\xi^2 + \frac{1}{2}\,t\,\gamma\,\eta^2\,\xi^2)\,\epsilon +$

$(2\,a^2\,\delta^2 - 2\,a\,\gamma\,\delta^2 + 12\,a^2\,t\,\delta^3 + 4\,a^2\,x\,y\,\delta^3 - 8\,a\,t\,\gamma\,\delta^3 - 20\,a\,x\,y\,\gamma\,\delta^3 - 2\,t\,\gamma^2\,\delta^3 + 18\,x\,y\,\gamma^2\,\delta^3 + 30\,a^2\,t^2\,\delta^4 + 20\,a^2\,t\,x\,y\,\delta^4 - 10\,a\,t^2\,\gamma\,\delta^4 - 88\,a\,t\,x\,y\,\gamma\,\delta^4 - 13\,a\,x^2\,y^2\,\gamma\,\delta^4 - \frac{15}{2}\,t^2\,\gamma^2\,\delta^4 + 64\,t\,x\,y\,\gamma^2\,\delta^4 + 34\,x^2\,y^2\,\gamma^2\,\delta^4 + 40\,a^2\,t^3\,\delta^5 + 40\,a^2\,t^2\,x\,y\,\delta^5 - 152\,a\,t^2\,x\,y\,\gamma\,\delta^5 - 48\,a\,t\,x^2\,y^2\,\gamma\,\delta^5 - 10\,t^3\,\gamma^2\,\delta^5 + 86\,t^2\,x\,y\,\gamma^2\,\delta^5 + 107\,t\,x^2\,y^2\,\gamma^2\,\delta^5 + 11\,x^3\,y^3\,\gamma^2\,\delta^5 + 30\,a^2\,t^4\,\delta^6 + 40\,a^2\,t^3\,x\,y\,\delta^6 + 10\,a\,t^4\,\gamma\,\delta^6 - 128\,a\,t^3\,x\,y\,\gamma\,\delta^6 - 66\,a\,t^2\,x^2\,y^2\,\gamma\,\delta^6 - 5\,t^4\,\gamma^2\,\delta^6 + 54\,t^3\,x\,y\,\gamma^2\,\delta^6 + \frac{247}{2}\,t^2\,x^2\,y^2\,\gamma^2\,\delta^6 + \frac{80}{3}\,t\,x^3\,y^3\,\gamma^2\,\delta^6 + 12\,a^2\,t^5\,\delta^7 + 20\,a^2\,t^4\,x\,y\,\delta^7 + 8\,a\,t^5\,\gamma\,\delta^7 - 52\,a\,t^4\,x\,y\,\gamma\,\delta^7 - 40\,a\,t^3\,x^2\,y^2\,\gamma\,\delta^7 + 16\,t^4\,x\,y\,\gamma^2\,\delta^7 + 62\,t^3\,x^2\,y^2\,\gamma^2\,\delta^7 + \frac{64}{3}\,t^2\,x^3\,y^3\,\gamma^2\,\delta^7 + 2\,a^2\,t^6\,\delta^8 + 4\,a^2\,t^5\,x\,y\,\delta^8 + 2\,a\,t^6\,\gamma\,\delta^8 - 8\,a\,t^5\,x\,y\,\gamma\,\delta^8 - 9\,a\,t^4\,x^2\,y^2\,\gamma\,\delta^8 + \frac{1}{2}\,t^6\,\gamma^2\,\delta^8 + 2\,t^5\,x\,y\,\gamma^2\,\delta^8 + \frac{23}{2}\,t^4\,x^2\,y^2\,\gamma^2\,\delta^8 + \frac{17}{3}\,t^3\,x^3\,y^3\,\gamma^2\,\delta^8 + 4\,a^2\,y\,\delta^2\,\eta - 12\,a\,y\,\gamma\,\delta^2\,\eta + 6\,y\,\gamma^2\,\delta^2\,\eta + 20\,a^2\,t\,y\,\delta^3\,\eta - 48\,a\,t\,y\,\gamma\,\delta^3\,\eta - 20\,a\,x\,y^2\,\gamma\,\delta^3\,\eta + 14\,t\,y\,\gamma^2\,\delta^3\,\eta + 40\,x\,y^2\,\gamma^2\,\delta^3\,\eta + 40\,a^2\,t^2\,y\,\delta^4\,\eta - 72\,a\,t^2\,y\,\gamma\,\delta^4\,\eta - 72\,a\,t\,x\,y^2\,\gamma\,\delta^4\,\eta + 6\,t^2\,y\,\gamma^2\,\delta^4\,\eta + 115\,t\,x\,y^2\,\gamma^2\,\delta^4\,\eta + 23\,x^2\,y^3\,\gamma^2\,\delta^4\,\eta + 40\,a^2\,t^3\,y\,\delta^5\,\eta - 48\,a\,t^3\,y\,\gamma\,\delta^5\,\eta - 96\,a\,t^2\,x\,y^2\,\gamma\,\delta^5\,\eta - 6\,t^3\,y\,\gamma^2\,\delta^5\,\eta + 118\,t^2\,x\,y^2\,\gamma^2\,\delta^5\,\eta + 53\,t\,x^2\,y^3\,\gamma^2\,\delta^5\,\eta + 20\,a^2\,t^4\,y\,\delta^6\,\eta - 12\,a\,t^4\,y\,\gamma\,\delta^6\,\eta - 56\,a\,t^3\,x\,y^2\,\gamma\,\delta^6\,\eta - 4\,t^4\,y\,\gamma^2\,\delta^6\,\eta + 51\,t^3\,x\,y^2\,\gamma^2\,\delta^6\,\eta + 40\,t^2\,x^2\,y^3\,\gamma^2\,\delta^6\,\eta + 4\,a^2\,t^5\,y\,\delta^7\,\eta -$

$$12\,a\,t^4\,x\,y^2\,\gamma\,\delta^7\,\eta + 8\,t^4\,x\,y^2\,\gamma^2\,\delta^7\,\eta + 10\,t^3\,x^2\,y^3\,\gamma^2\,\delta^7\,\eta - 7\,a\,y^2\,\gamma\,\delta^2\,\eta^2 + 10\,y^2\,\gamma^2\,\delta^2\,\eta^2 -$$
$$24\,a\,t\,y^2\,\gamma\,\delta^3\,\eta^2 + 24\,t\,y^2\,\gamma^2\,\delta^3\,\eta^2 + 15\,x\,y^3\,\gamma^2\,\delta^3\,\eta^2 - 30\,a\,t^2\,y^2\,\gamma\,\delta^4\,\eta^2 + \frac{37}{2}\,t^2\,y^2\,\gamma^2\,\delta^4\,\eta^2 +$$
$$32\,t\,x\,y^3\,\gamma^2\,\delta^4\,\eta^2 - 16\,a\,t^3\,y^2\,\gamma\,\delta^5\,\eta^2 + 5\,t^3\,y^2\,\gamma^2\,\delta^5\,\eta^2 + 22\,t^2\,x\,y^3\,\gamma^2\,\delta^5\,\eta^2 - 3\,a\,t^4\,y^2\,\gamma\,\delta^6\,\eta^2 +$$
$$\frac{1}{2}\,t^4\,y^2\,\gamma^2\,\delta^6\,\eta^2 + 5\,t^3\,x\,y^3\,\gamma^2\,\delta^6\,\eta^2 + 3\,y^3\,\gamma^2\,\delta^2\,\eta^3 + \frac{17}{3}\,t\,y^3\,\gamma^2\,\delta^3\,\eta^3 + \frac{10}{3}\,t^2\,y^3\,\gamma^2\,\delta^4\,\eta^3 +$$
$$\frac{2}{3}\,t^3\,y^3\,\gamma^2\,\delta^5\,\eta^3 + 4\,a^2\,x\,\delta^2\,\xi - 12\,a\,x\,\gamma\,\delta^2\,\xi + 6\,x\,\gamma^2\,\delta^2\,\xi + 20\,a^2\,t\,x\,\delta^3\,\xi - 48\,a\,t\,x\,\gamma\,\delta^3\,\xi -$$
$$20\,a\,x^2\,y\,\gamma\,\delta^3\,\xi + 14\,t\,x\,\gamma^2\,\delta^3\,\xi + 40\,x^2\,y\,\gamma^2\,\delta^3\,\xi + 40\,a^2\,t^2\,x\,\delta^4\,\xi - 72\,a\,t^2\,x\,\gamma\,\delta^4\,\xi - 72\,a\,t\,x^2\,y\,\gamma\,\delta^4\,\xi +$$
$$6\,t^2\,x\,\gamma^2\,\delta^4\,\xi + 115\,t\,x^2\,y\,\gamma^2\,\delta^4\,\xi + 23\,x^3\,y^2\,\gamma^2\,\delta^4\,\xi + 40\,a^2\,t^3\,x\,\delta^5\,\xi - 48\,a\,t^3\,x\,\gamma\,\delta^5\,\xi -$$
$$96\,a\,t^2\,x^2\,y\,\gamma\,\delta^5\,\xi - 6\,t^3\,x\,\gamma^2\,\delta^5\,\xi + 118\,t^2\,x^2\,y\,\gamma^2\,\delta^5\,\xi + 53\,t\,x^3\,y^2\,\gamma^2\,\delta^5\,\xi + 20\,a^2\,t^4\,x\,\delta^6\,\xi -$$
$$12\,a\,t^4\,x\,\gamma\,\delta^6\,\xi - 56\,a\,t^3\,x^2\,y\,\gamma\,\delta^6\,\xi - 4\,t^4\,x\,\gamma^2\,\delta^6\,\xi + 51\,t^3\,x^2\,y\,\gamma^2\,\delta^6\,\xi + 40\,t^2\,x^3\,y^2\,\gamma^2\,\delta^6\,\xi +$$
$$4\,a^2\,t^5\,x\,\delta^7\,\xi - 12\,a\,t^4\,x^2\,y\,\gamma\,\delta^7\,\xi + 8\,t^4\,x^2\,y\,\gamma^2\,\delta^7\,\xi + 10\,t^3\,x^3\,y^2\,\gamma^2\,\delta^7\,\xi + 4\,a^2\,\delta\,\eta\,\xi - 4\,a\,\gamma\,\delta\,\eta\,\xi +$$
$$20\,a^2\,t\,\delta^2\,\eta\,\xi - 8\,a\,t\,\gamma\,\delta^2\,\eta\,\xi - 28\,a\,x\,y\,\gamma\,\delta^2\,\eta\,\xi - 6\,t\,\gamma^2\,\delta^2\,\eta\,\xi + 38\,x\,y\,\gamma^2\,\delta^2\,\eta\,\xi + 40\,a^2\,t^2\,\delta^3\,\eta\,\xi +$$
$$8\,a\,t^2\,\gamma\,\delta^3\,\eta\,\xi - 96\,a\,t\,x\,y\,\gamma\,\delta^3\,\eta\,\xi - 14\,t^2\,\gamma^2\,\delta^3\,\eta\,\xi + 88\,t\,x\,y\,\gamma^2\,\delta^3\,\eta\,\xi + 44\,x^2\,y^2\,\gamma^2\,\delta^3\,\eta\,\xi +$$
$$40\,a^2\,t^3\,\delta^4\,\eta\,\xi + 32\,a\,t^3\,\gamma\,\delta^4\,\eta\,\xi - 120\,a\,t^2\,x\,y\,\gamma\,\delta^4\,\eta\,\xi - 6\,t^3\,\gamma^2\,\delta^4\,\eta\,\xi + 62\,t^2\,x\,y\,\gamma^2\,\delta^4\,\eta\,\xi +$$
$$93\,t\,x^2\,y^2\,\gamma^2\,\delta^4\,\eta\,\xi + 20\,a^2\,t^4\,\delta^5\,\eta\,\xi + 28\,a\,t^4\,\gamma\,\delta^5\,\eta\,\xi - 64\,a\,t^3\,x\,y\,\gamma\,\delta^5\,\eta\,\xi + 6\,t^4\,\gamma^2\,\delta^5\,\eta\,\xi +$$
$$12\,t^3\,x\,y\,\gamma^2\,\delta^5\,\eta\,\xi + 63\,t^2\,x^2\,y^2\,\gamma^2\,\delta^5\,\eta\,\xi + 4\,a^2\,t^5\,\delta^6\,\eta\,\xi + 8\,a\,t^5\,\gamma\,\delta^6\,\eta\,\xi - 12\,a\,t^4\,x\,y\,\gamma\,\delta^6\,\eta\,\xi +$$
$$4\,t^5\,\gamma^2\,\delta^6\,\eta\,\xi + 14\,t^3\,x^2\,y^2\,\gamma^2\,\delta^6\,\eta\,\xi - 8\,a\,y\,\gamma\,\delta\,\eta^2\,\xi + 6\,y\,\gamma^2\,\delta\,\eta^2\,\xi - 24\,a\,t\,y\,\gamma\,\delta^2\,\eta^2\,\xi +$$
$$5\,t\,y\,\gamma^2\,\delta^2\,\eta^2\,\xi + 25\,x\,y^2\,\gamma^2\,\delta^2\,\eta^2\,\xi - 24\,a\,t^2\,y\,\gamma\,\delta^3\,\eta^2\,\xi - 8\,t^2\,y\,\gamma^2\,\delta^3\,\eta^2\,\xi + 45\,t\,x\,y^2\,\gamma^2\,\delta^3\,\eta^2\,\xi -$$
$$8\,a\,t^3\,y\,\gamma\,\delta^4\,\eta^2\,\xi - 7\,t^3\,y\,\gamma^2\,\delta^4\,\eta^2\,\xi + 24\,t^2\,x\,y^2\,\gamma^2\,\delta^4\,\eta^2\,\xi + 4\,t^3\,x\,y^2\,\gamma^2\,\delta^5\,\eta^2\,\xi + 4\,y^2\,\gamma^2\,\delta\,\eta^3\,\xi +$$
$$5\,t\,y^2\,\gamma^2\,\delta^2\,\eta^3\,\xi + t^2\,y^2\,\gamma^2\,\delta^3\,\eta^3\,\xi - 7\,a\,x^2\,\gamma\,\delta^2\,\xi^2 + 10\,x^2\,\gamma^2\,\delta^2\,\xi^2 - 24\,a\,t\,x^2\,\gamma\,\delta^3\,\xi^2 +$$
$$24\,t\,x^2\,\gamma^2\,\delta^3\,\xi^2 + 15\,x^3\,y\,\gamma^2\,\delta^3\,\xi^2 - 30\,a\,t^2\,x^2\,\gamma\,\delta^4\,\xi^2 + \frac{37}{2}\,t^2\,x^2\,\gamma^2\,\delta^4\,\xi^2 + 32\,t\,x^3\,y\,\gamma^2\,\delta^4\,\xi^2 -$$
$$16\,a\,t^3\,x^2\,\gamma\,\delta^5\,\xi^2 + 5\,t^3\,x^2\,\gamma^2\,\delta^5\,\xi^2 + 22\,t^2\,x^3\,y\,\gamma^2\,\delta^5\,\xi^2 - 3\,a\,t^4\,x^2\,\gamma\,\delta^6\,\xi^2 + \frac{1}{2}\,t^4\,x^2\,\gamma^2\,\delta^6\,\xi^2 +$$
$$5\,t^3\,x^3\,y\,\gamma^2\,\delta^6\,\xi^2 - 8\,a\,x\,\gamma\,\delta\,\eta\,\xi^2 + 6\,x\,\gamma^2\,\delta\,\eta\,\xi^2 - 24\,a\,t\,x\,\gamma\,\delta^2\,\eta\,\xi^2 + 5\,t\,x\,\gamma^2\,\delta^2\,\eta\,\xi^2 +$$
$$25\,x^2\,y\,\gamma^2\,\delta^2\,\eta\,\xi^2 - 24\,a\,t^2\,x\,\gamma\,\delta^3\,\eta\,\xi^2 - 8\,t^2\,x\,\gamma^2\,\delta^3\,\eta\,\xi^2 + 45\,t\,x^2\,y\,\gamma^2\,\delta^3\,\eta\,\xi^2 - 8\,a\,t^3\,x\,\gamma\,\delta^4\,\eta\,\xi^2 -$$
$$7\,t^3\,x\,\gamma^2\,\delta^4\,\eta\,\xi^2 + 24\,t^2\,x^2\,y\,\gamma^2\,\delta^4\,\eta\,\xi^2 + 4\,t^3\,x^2\,y\,\gamma^2\,\delta^5\,\eta\,\xi^2 - a\,\gamma\,\eta^2\,\xi^2 - 3\,t\,\gamma^2\,\delta\,\eta^2\,\xi^2 +$$
$$11\,x\,y\,\gamma^2\,\delta\,\eta^2\,\xi^2 + 6\,a\,t^2\,\gamma\,\delta^2\,\eta^2\,\xi^2 - \frac{5}{2}\,t^2\,\gamma^2\,\delta^2\,\eta^2\,\xi^2 + 12\,t\,x\,y\,\gamma^2\,\delta^2\,\eta^2\,\xi^2 + 8\,a\,t^3\,\gamma\,\delta^3\,\eta^2\,\xi^2 +$$
$$4\,t^3\,\gamma^2\,\delta^3\,\eta^2\,\xi^2 + 3\,a\,t^4\,\gamma\,\delta^4\,\eta^2\,\xi^2 + \frac{7}{2}\,t^4\,\gamma^2\,\delta^4\,\eta^2\,\xi^2 - t^3\,x\,y\,\gamma^2\,\delta^4\,\eta^2\,\xi^2 + y\,\gamma^2\,\eta^3\,\xi^2 -$$
$$t\,y\,\gamma^2\,\delta\,\eta^3\,\xi^2 - 2\,t^2\,y\,\gamma^2\,\delta^2\,\eta^3\,\xi^2 + 3\,x^3\,\gamma^2\,\delta^2\,\xi^3 + \frac{17}{3}\,t\,x^3\,\gamma^2\,\delta^3\,\xi^3 + \frac{10}{3}\,t^2\,x^3\,\gamma^2\,\delta^4\,\xi^3 +$$
$$\frac{2}{3}\,t^3\,x^3\,\gamma^2\,\delta^5\,\xi^3 + 4\,x^2\,\gamma^2\,\delta\,\eta\,\xi^3 + 5\,t\,x^2\,\gamma^2\,\delta^2\,\eta\,\xi^3 + t^2\,x^2\,\gamma^2\,\delta^3\,\eta\,\xi^3 + x\,\gamma^2\,\eta^2\,\xi^3 - t\,x\,\gamma^2\,\delta\,\eta^2\,\xi^3 -$$
$$2\,t^2\,x\,\gamma^2\,\delta^2\,\eta^2\,\xi^3 - \frac{1}{3}\,t\,\gamma^2\,\eta^3\,\xi^3 + \frac{1}{3}\,t^2\,\gamma^2\,\delta\,\eta^3\,\xi^3 + \frac{2}{3}\,t^3\,\gamma^2\,\delta^2\,\eta^3\,\xi^3\Big)\,\epsilon^2 + O[\epsilon]^3,\ 6,\ 6\Big\}$$

```
{tt = Last[Λ_QU,2[{ξ, η, δ}, {x, y}]];
  Log[tt],
  Exponent[Normal@Together@Log[tt] /. {ξ → d ξ, η → d η, x → d x, y → d y}, d]} // Expand
```

$$\left\{\text{Log}\left[\frac{\hbar}{-\delta+T\,\delta+\hbar}\right]+\left(\frac{2\,a\,T\,\delta^5\,\hbar}{(-\delta+T\,\delta+\hbar)^5}-\frac{8\,a\,T^2\,\delta^5\,\hbar}{(-\delta+T\,\delta+\hbar)^5}+\frac{12\,a\,T^3\,\delta^5\,\hbar}{(-\delta+T\,\delta+\hbar)^5}-\frac{8\,a\,T^4\,\delta^5\,\hbar}{(-\delta+T\,\delta+\hbar)^5}+\langle\langle 267\rangle\rangle+\frac{x^2\,y^2\,\gamma\,\delta^2\,\hbar^6}{(-\delta+T\,\delta+\hbar)^5}+\frac{x\,y^2\,\gamma\,\delta\,\eta\,\hbar^6}{(-\delta+T\,\delta+\hbar)^5}+\frac{x^2\,y\,\gamma\,\delta\,\xi\,\hbar^6}{(-\delta+T\,\delta+\hbar)^5}+\frac{x\,y\,\gamma\,\eta\,\xi\,\hbar^6}{(-\delta+T\,\delta+\hbar)^5}\right)\epsilon+\left(-\frac{32\,a^2\,T^2\,\delta^{10}\,\hbar^2}{(\langle\langle 1\rangle\rangle)^2}+\langle\langle 8307\rangle\rangle+\langle\langle 1\rangle\rangle+\frac{144\,x^2\,y^2\,\gamma^2\,\eta^2\,\xi^2\,\hbar^{11}}{\langle\langle 1\rangle\rangle}\right)\epsilon^2+O[\epsilon]^3,\ 6\right\}$$

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Reorderings with Rord

Rord

In[ ]:=
```
Rord_{u_i_,w_j_→k_}[𝔼_U_[L___, {l___, u_i_, w_j_, r___}_s_, R___, Q_, P_]] :=
  Simp@Module[{υ, ω, δ, Λ1, yax, q, p, kk = P⟦5⟧, δ1 = ∂_{u_i,w_j} Q},
    {yax, q, p} = Echo[List @@ If[δ1 === 0, Λ_{U,kk}[{υ, ω}, {u, w}],
        Λ_{U,kk}[{υ, ω, δ}, {u, w}]] /. {y → y_k, a → a_k, x → x_k, t → t_s, T → T_s}];
    𝔼_U[L, {l, Sequence @@ yax, r}_s, R, q + (Q /. u_i | w_j → 0), e^{-q} DP_{u_i→D_υ, w_j→D_ω}[P][p e^q]] /.
     {υ → ∂_{u_i} Q /. w_j → 0, ω → ∂_{w_j} Q /. u_i → 0, δ → δ1}];
```

Rord

In[ ]:=
```
Rord_{u_i_,w_j_→k_}[𝔼_U_[L___, {l___, u_i_, w_j_, r___}_s_, R___, Q_, P_]] :=
  Simp@Module[{υ, ω, δ, Λ1, yax, q, p, n, kk = P⟦5⟧, δ1 = ∂_{u_i,w_j} Q},
    {yax, q, p} = List @@ If[δ1 === 0, Λ_{U,kk}[{υ, ω}, {u, w}], Λ_{U,kk}[{υ, ω, δ}, {u, w}]] /.
      {y → y_n, a → a_n, x → x_n, t → t_s, T → T_s};
    (*Echo@{{{u_i,υ},{w_j,ω}},P,p e^q};*)
    𝔼_U[L, {l, Sequence @@ yax, r}_s, R, q + (Q /. u_i | w_j → 0), e^{-q} SP_{u_i→υ, w_j→ω}[P p e^q]] /.
     {n → k, υ → ∂_{u_i} Q /. w_j → 0, ω → ∂_{w_j} Q /. u_i → 0, δ → δ1}];
```

```
With[{c𝟘 = 𝔼_CU[{y_1, x_1}_1, {x_2, a_2, y_2}_2, ℏ t_1 a_2 + ℏ t_1^{-1} (e^{t_1} - 1) y_1 x_2, 1_2 + ϵ x_1 y_2]},
 {Short[rhs = c𝟘 // Rord_{x_2,a_2→3}, 3], HL[CU[c𝟘] == CU[rhs]]}]
```

$$\left\{\mathbb{E}_{CU}\left[\{y_1,\,x_1\}_1,\,\{a_3,\,x_3,\,y_2\}_2,\,\frac{e^{-\gamma\,\hbar\,t_1}\left(e^{\gamma\,\hbar\,t_1}\,\hbar\,a_3\,t_1^2-\hbar\,x_3\,y_1+e^{t_1}\,\hbar\,x_3\,y_1\right)}{t_1},\,1+x_1\,y_2\,\epsilon+O[\epsilon]^3\right],\,\text{True}\right\}$$

```
With[{c𝟘 = 𝔼_CU[{y_1, a_1, a_2}_1, {x_2, x_1, y_2}_2,
     ℏ (l_11 t_1 a_1 + l_12 t_1 a_2 + l_21 t_2 a_1 + l_22 t_2 a_2 + γ_11 x_1 y_1 + γ_12 x_1 y_2 + γ_21 x_2 y_1 + γ_22 x_2 y_2),
     1_2 + ϵ (l_1 a_1 + l_2 a_2 + p_11 x_1 y_1 + p_12 x_1 y_2 + p_21 x_2 y_1 + p_22 x_2 y_2)]},
 {Short[rhs = c𝟘 // Rord_{a_1,a_2→3} // Rord_{x_2,x_1→4}, 3], HL[CU[c𝟘] == CU[rhs]]}]
```

$$\{\mathbb{E}_{CU}[\{y_1,\,a_3\}_1,\,\{x_4,\,y_2\}_2,\,\hbar\,a_3\,l_{11}\,t_1+\hbar\,a_3\,l_{12}\,t_1+\hbar\,a_3\,l_{21}\,t_2+\hbar\,a_3\,l_{22}\,t_2+\hbar\,x_4\,y_1\,\gamma_{11}+\hbar\,x_4\,y_2\,\gamma_{12}+\hbar\,x_4\,y_1\,\gamma_{21}+\hbar\,x_4\,y_2\,\gamma_{22},\,1+(a_3\,l_1+a_3\,l_2+p_{11}\,x_4\,y_1+p_{21}\,x_4\,y_1+p_{12}\,x_4\,y_2+p_{22}\,x_4\,y_2)\,\epsilon+O[\epsilon]^3],\,\text{True}\}$$

```
ℏ a_3 l_11 t_1 + ℏ a_3 l_12 t_1 + ℏ a_3 l_21 t_2 + ℏ a_3 l_22 t_2 +
  ℏ x_4 y_1 γ_11 + ℏ x_4 y_2 γ_12 + ℏ x_4 y_1 γ_21 + ℏ x_4 y_2 γ_22 // Simplify
```

$$\hbar\,(a_3\,(l_{11}\,t_1+l_{12}\,t_1+(l_{21}+l_{22})\,t_2)+x_4\,(y_1\,(\gamma_{11}+\gamma_{21})+y_2\,(\gamma_{12}+\gamma_{22})))$$

$$\mathbf{With}\Big[\Big\{c\mathbb{O} = \mathbb{E}_{CU}\big[\{y_1, a_1, x_1\}_1,\ \{x_2, a_2, y_2\}_2,\ \hbar\,(l_{11}\,t_1\,a_1 + l_{12}\,t_1\,a_2 + l_{21}\,t_2\,a_1 + l_{22}\,t_2\,a_2 + \gamma_{11}\,x_1\,y_1 + \gamma_{12}\,x_1\,y_2 + \gamma_{21}\,x_2\,y_1 + \gamma_{22}\,x_2\,y_2),\ 1_2 + \epsilon\,(l_1\,a_1 + l_2\,a_2 + p_{11}\,x_1\,y_1 + p_{12}\,x_1\,y_2 + p_{21}\,x_2\,y_1 + p_{22}\,x_2\,y_2)\big]\Big\},\ \{\mathbf{Short}[rhs = c\mathbb{O}\ //\ \mathbf{Rord}_{x_2,a_2\to 3},\ 3],\ \mathbf{HL}[\mathbf{CU}[c\mathbb{O}] == \mathbf{CU}[rhs]]\}\Big]$$

$$\Big\{\mathbb{E}_{CU}\big[\{y_1, a_1, x_1\}_1,\ \ll 1\gg_2,\ \ll 1\gg\ \ll 1\gg,\ 1 + e^{-\gamma\hbar l_{12} t_1 - \gamma\hbar l_{22} t_2}\,\big(e^{\gamma\hbar l_{12} t_1+\gamma\hbar l_{22} t_2}\,a_1\,l_1 + e^{\gamma\hbar l_{12} t_1+\gamma\hbar l_{22} t_2}\,a_3\,l_2 + e^{\gamma\hbar l_{12} t_1+\gamma\hbar l_{22} t_2}\,p_{11}\,x_1\,y_1 + p_{21}\,x_3\,y_1 + e^{\ll 1\gg + \ll 1\gg}\,p_{12}\,x_1\,y_2 + p_{22}\,x_3\,y_2 - \gamma\,\hbar\,l_2\,x_3\,y_1\,\gamma_{21} - \gamma\,\hbar\,l_2\,x_3\,y_2\,\gamma_{22}\big)\,\epsilon + O[\epsilon]^3\big],\ \text{True}\Big\}$$

$$\mathbf{With}\Big[\Big\{q\mathbb{O} = \mathbb{E}_{QU}\big[\{y_1, a_1, x_1\}_1,\ \{x_2, a_2, y_2\}_2,\ \hbar\,(l_{11}\,t_1\,a_1 + l_{12}\,t_1\,a_2 + l_{21}\,t_2\,a_1 + l_{22}\,t_2\,a_2 + \gamma_{11}\,x_1\,y_1 + \gamma_{12}\,x_1\,y_2 + \gamma_{21}\,x_2\,y_1 + \gamma_{22}\,x_2\,y_2),\ 1_2 + \epsilon\,(l_1\,a_1 + l_2\,a_2 + p_{11}\,x_1\,y_1 + p_{12}\,x_1\,y_2 + p_{21}\,x_2\,y_1 + p_{22}\,x_2\,y_2)\big]\Big\},\ \{\mathbf{Short}[rhs = q\mathbb{O}\ //\ \mathbf{Rord}_{x_2,a_2\to 3},\ 3],\ \mathbf{HL}[\mathbf{QU}[q\mathbb{O}] == \mathbf{QU}[rhs]]\}\Big]$$

$$\Big\{\mathbb{E}_{QU}\big[\{y_1, a_1, x_1\}_1,\ \ll 1\gg_2,\ \ll 1\gg\ \ll 1\gg,\ 1 + e^{-\gamma\hbar l_{12} t_1 - \gamma\hbar l_{22} t_2}\,\big(e^{\gamma\hbar l_{12} t_1+\gamma\hbar l_{22} t_2}\,a_1\,l_1 + e^{\gamma\hbar l_{12} t_1+\gamma\hbar l_{22} t_2}\,a_3\,l_2 + e^{\gamma\hbar l_{12} t_1+\gamma\hbar l_{22} t_2}\,p_{11}\,x_1\,y_1 + p_{21}\,x_3\,y_1 + e^{\ll 1\gg + \ll 1\gg}\,p_{12}\,x_1\,y_2 + p_{22}\,x_3\,y_2 - \gamma\,\hbar\,l_2\,x_3\,y_1\,\gamma_{21} - \gamma\,\hbar\,l_2\,x_3\,y_2\,\gamma_{22}\big)\,\epsilon + O[\epsilon]^3\big],\ \text{True}\Big\}$$

$$\mathbf{With}\Big[\Big\{q\mathbb{O} = \mathbb{E}_{QU}\big[\{y_1, a_1, x_1\}_1,\ \{x_2, a_2, y_2\}_2,\ \hbar\,(l_{11}\,t_1\,a_1 + l_{12}\,t_1\,a_2 + l_{21}\,t_2\,a_1 + l_{22}\,t_2\,a_2 + \gamma_{11}\,x_1\,y_1 + \gamma_{12}\,x_1\,y_2 + \gamma_{21}\,x_2\,y_1 + \gamma_{22}\,x_2\,y_2),\ 1_2 + \epsilon\,(l_1\,a_1 + l_2\,a_2 + p_{11}\,x_1\,y_1 + p_{12}\,x_1\,y_2 + p_{21}\,x_2\,y_1 + p_{22}\,x_2\,y_2)\big]\Big\},\ \{\mathbf{Short}[rhs = q\mathbb{O}\ //\ \mathbf{Rord}_{a_2,y_2\to 3},\ 3],\ \mathbf{HL}[\mathbf{QU}[q\mathbb{O}] == \mathbf{QU}[rhs]]\}\Big]$$

$$\{\ll 1\gg,\ \text{True}\}$$

$$\mathbf{Timing@With}\Big[\Big\{q\mathbb{O} = \mathbb{E}_{QU}\big[\{x_1, y_1\}_1,\ \{x_2, a_2, y_2\}_2,\ \hbar\,(l_{12}\,t_1\,a_2 + l_{22}\,t_2\,a_2 + \gamma_{11}\,x_1\,y_1 + \gamma_{12}\,x_1\,y_2 + \gamma_{21}\,x_2\,y_1 + \gamma_{22}\,x_2\,y_2),\ 0_2 + \epsilon\,(l_2\,a_2 + p_{11}\,x_1\,y_1 + p_{12}\,x_1\,y_2 + p_{21}\,x_2\,y_1 + p_{22}\,x_2\,y_2)\big]\Big\},\ \{\mathbf{Short}[rhs = q\mathbb{O}\ //\ \mathbf{Rord}_{x_1,y_1\to 3},\ 5]\}\Big]$$

$$\Big\{116.156,\ \Big\{\mathbb{E}_{QU}\Big[\{y_3, a_3, x_3\}_1,\ \ll 1\gg_2,\ \frac{\ll 1\gg}{1 - \ll 1\gg + \ll 1\gg},\ \Big(\big(\hbar\,a_2\,l_2 + p_{11} - p_{11}\,T_1 + \hbar\,p_{22}\,x_2\,y_2 + \hbar\,p_{12}\,x_3\,y_2 + \ll 46\gg + 2\,\hbar\,p_{12}\,T_1\,x_2\,y_2\,\gamma_{11}\,\gamma_{21} - \hbar\,p_{12}\,T_1^2\,x_2\,y_2\,\gamma_{11}\,\gamma_{21} + \hbar\,p_{11}\,x_2\,y_2\,\gamma_{12}\,\gamma_{21} - 2\,\hbar\,p_{11}\,T_1\,x_2\,y_2\,\gamma_{12}\,\gamma_{21} + \hbar\,p_{11}\,T_1^2\,x_2\,y_2\,\gamma_{12}\,\gamma_{21}\big)\,\epsilon\Big)\Big/\big(\hbar - 3\,\hbar\,\gamma_{11} + 3\,\hbar\,T_1\,\gamma_{11} + 3\,\hbar\,\gamma_{11}^2 - 6\,\hbar\,T_1\,\gamma_{11}^2 + 3\,\hbar\,T_1^2\,\gamma_{11}^2 - \hbar\,\gamma_{11}^3 + 3\,\hbar\,T_1\,\gamma_{11}^3 - 3\,\hbar\,T_1^2\,\gamma_{11}^3 + \hbar\,T_1^3\,\gamma_{11}^3\big) + \Big(\big(8\,a_3\,p_{11}\,T_1 + \ll 1\gg + \ll 2726\gg + 3\,\gamma\ \ll 6\gg\ \gamma_{21}^3\big)\ \ll 1\gg\Big)\Big/\big(4 - 28\,\gamma_{11} + \ll 48\gg + 4\,T_1^7\,\gamma_{11}^7\big) + O[\epsilon]^3\Big]\Big\}\Big\}$$

```
Timing@With[{qΦ = ŒQU[{x1, y1}1, {x2, a2, y2}2,
      ℏ (l12 t1 a2 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
      12 + ϵ (l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
  {Short[rhs = qΦ // Rord_{x1,y1→3}, 5], HL@SimpT[QU[qΦ] == QU[rhs]]}]
```

$\{388.922,$
$\{Œ_{QU}[\{y_3, a_3, x_3\}_1, \{\ll 1\gg\}_2, \frac{\ll 1\gg}{\ll 1\gg}, \frac{1}{1-\gamma_{11}+T_1\gamma_{11}} + ((4\hbar a_2 l_2 + 4 p_{11} - 4 p_{11} T_1 + 4\hbar p_{22} x_2 y_2 + \ll 339\gg + \gamma\hbar^4 x_2^2 y_2^2 \gamma_{12}^2 \gamma_{21}^2 - 4\gamma\hbar^4 T_1 x_2^2 y_2^2 \gamma_{12}^2 \gamma_{21}^2 + 3\gamma\hbar^4 T_1^2 x_2^2 y_2^2 \gamma_{12}^2 \gamma_{21}^2)\,\epsilon) /$
$(4\hbar - 20\hbar\gamma_{11} + 20\hbar T_1\gamma_{11} + 40\hbar\gamma_{11}^2 - 80\hbar T_1\gamma_{11}^2 + 40\hbar T_1^2\gamma_{11}^2 - 40\hbar\gamma_{11}^3 + \ll 13\gg + 20\hbar T_1\gamma_{11}^5 - 40\hbar T_1^2\gamma_{11}^5 + 40\hbar T_1^3\gamma_{11}^5 - 20\hbar T_1^4\gamma_{11}^5 + 4\hbar T_1^5\gamma_{11}^5) +$
$\frac{(576 a_3 p_{11} T_1 + \ll 8073\gg + \ll 1\gg)\ \ll 1\gg}{\ll 79\gg + 288 T_1^9 \gamma_{11}^9} + O[\epsilon]^3], \text{True}\}\}$

```
Timing@With[{qΦ = ŒQU[{x1, y1}1, {x2, a2, y2}2,
      ℏ (l12 t1 a2 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
      12 + ϵ (l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
  {Short[rhs = qΦ // Rord_{x1,y1→1}, 5], HL@SimpT[QU[qΦ] == QU[rhs]]}]
```

$\{336.781,$
$\{Œ_{QU}[\{y_1, a_1, x_1\}_1, \{\ll 1\gg\}_2, \frac{\ll 1\gg}{\ll 1\gg}, \frac{1}{1-\gamma_{11}+T_1\gamma_{11}} + ((4\hbar a_2 l_2 + 4 p_{11} - 4 p_{11} T_1 + 4\hbar p_{11} x_1 y_1 + 4\hbar p_{21} x_2 y_1 + \ll 338\gg + \gamma\hbar^4 x_2^2 y_2^2 \gamma_{12}^2 \gamma_{21}^2 - 4\gamma\hbar^4 T_1 x_2^2 y_2^2 \gamma_{12}^2 \gamma_{21}^2 + 3\gamma\hbar^4 T_1^2 x_2^2 y_2^2 \gamma_{12}^2 \gamma_{21}^2)\,\epsilon) /$
$(4\hbar - 20\hbar\gamma_{11} + 20\hbar T_1\gamma_{11} + 40\hbar\gamma_{11}^2 - 80\hbar T_1\gamma_{11}^2 + 40\hbar T_1^2\gamma_{11}^2 - 40\hbar\gamma_{11}^3 + \ll 10\gg + 20\hbar T_1^4\gamma_{11}^4 - 4\hbar\gamma_{11}^5 + 20\hbar T_1\gamma_{11}^5 - 40\hbar T_1^2\gamma_{11}^5 + 40\hbar T_1^3\gamma_{11}^5 - 20\hbar T_1^4\gamma_{11}^5 + 4\hbar T_1^5\gamma_{11}^5) +$
$\frac{(576 a_1 p_{11} T_1 + \ll 8073\gg + \ll 1\gg)\ \ll 1\gg}{\ll 79\gg + 288 T_1^9 \gamma_{11}^9} + O[\epsilon]^3], \text{True}\}\}$

Canonical ordering with Cord

Cord

In[∘]:=
```
Cord[ŒU_[L___, {l___, u_i_, w_j_, r___}s_, R___, Q_, P_]] /;
   OrderedQ[{w, u} /. {y → 1, a → 2, x → 3}] :=
  ((*Echo@{ui,wj};*)Cord[Rord_{ui,wj→Unique[]}[ŒU[L, {l, ui, wj, r}s, R, Q, P]]]);
Cord[ŒU_[specs__, Q_, P_]] := ŒU[Sequence @@ Sort@{specs}, Q, P] /.
  Flatten[{specs} /. {yax__}s_ :> ({yax} /. u_i_ :> (ui → us))]
```

```
Cord@ŒCU[{x1, y1}1, 0, 01 + x1 y1]
```

$Œ_{CU}[\{y_1, a_1, x_1\}_1, 0, (-t_1 + x_1 y_1) + 2 a_1 \epsilon + O[\epsilon]^2]$

```
Block[{$p = 4, $k = 0, cⲟ = ℭCU[{y1, a1, x1, x2, a2, y2}1,
    ħ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    1θ + ∈ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
  Timing@{Short[Cord[cⲟ], 8], HL@Simp[CU[cⲟ] - CU[Cord[cⲟ]]]}]
```

$$\Big\{4.53125, \Big\{\mathfrak{C}_{CU}\big[\{y_1, a_1, x_1\}_1, \big(e^{\gamma\hbar l_{11} t_1+2\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+2\gamma\hbar l_{22} t_2}\hbar\, a_1 l_{11} t_1 + e^{\gamma\hbar l_{11} t_1+2\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+2\gamma\hbar l_{22} t_2}\hbar\, a_1 l_{12} t_1 + \ll 12\gg + e^{\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2}\hbar^2 a_1 l_{22} t_1 t_2 \gamma_{22} + \hbar x_1 y_1 \gamma_{22}\big) \Big/ \big(e^{\gamma\hbar l_{11} t_1+2\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+2\gamma\hbar l_{22} t_2} + e^{\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2}\hbar\, t_1 \gamma_{12} + e^{\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2}\hbar\, t_1 \gamma_{22}\big), \frac{e^{\gamma\hbar l_{12} t_1+\gamma\hbar l_{22} t_2}}{e^{\gamma\hbar l_{12} t_1+\gamma\hbar l_{22} t_2} + \hbar t_1 \gamma_{12} + \hbar t_1 \gamma_{22}} + O[\epsilon]^1\big], 0\Big\}\Big\}$$

```
Block[{$p = 4, $k = 1, cⲟ = ℭCU[{y1, a1, x1, x2, a2, y2}1,
    ħ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    11 + ∈ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
  Timing@{Short[Cord[cⲟ], 8], HL@Simp[CU[cⲟ] - CU[Cord[cⲟ]]]}]
```

$$\Big\{81.2656, \Big\{\mathfrak{C}_{CU}\big[\{y_1, a_1, x_1\}_1, \big(e^{\gamma\hbar l_{11} t_1+2\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+2\gamma\hbar l_{22} t_2}\hbar\, a_1 l_{11} t_1 + \ll 14\gg + \hbar x_1 y_1 \gamma_{22}\big) \Big/ \big(e^{\gamma\hbar l_{11} t_1+2\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+2\gamma\hbar l_{22} t_2} + e^{\gamma\hbar l_{11} t_1+\ll 2\gg+\gamma\hbar \ll 1\gg t_2}\hbar\, t_1 \gamma_{12} + e^{\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2}\hbar\, t_1 \gamma_{22}\big), \frac{e^{\gamma\hbar l_{12} t_1+\gamma\hbar l_{22} t_2}}{e^{\gamma\hbar l_{12} t_1+\gamma\hbar l_{22} t_2} + \hbar t_1 \gamma_{12} + \hbar t_1 \gamma_{22}} + \Big(\big(2\, e^{2\gamma\hbar l_{11} t_1+6\gamma\hbar l_{12} t_1+2\gamma\hbar l_{21} t_2+6\gamma\hbar l_{22} t_2} a_1 l_1 + \ll 419\gg\big)\epsilon\Big) \Big/ \big(2\, e^{2\gamma\hbar l_{11} t_1+6\gamma\hbar l_{12} t_1+2\gamma\hbar l_{21} t_2+6\gamma\hbar l_{22} t_2} + 10\, e^{2\gamma\hbar l_{11} t_1+5\gamma\hbar l_{12} t_1+2\gamma\hbar l_{21} t_2+5\gamma\hbar l_{22} t_2}\hbar\, t_1 \gamma_{12} + \ll 18\gg + 2\, e^{2\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+2\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2}\hbar^5 t_1^5 \gamma_{22}^5\big) + O[\epsilon]^2\big], 0\Big\}\Big\}$$

```
With[{qⲟ = ℭQU[{y1, a1, x1, x2, a2, y2}1,
    ħ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    1θ + ∈ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
  Cord[
   qⲟ]]
```

Writing $E_1 = \gamma\hbar l_{11} t_1+2\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+2\gamma\hbar l_{22} t_2$ and $E_2 = \gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2$ for the exponents as printed:

$$\mathfrak{C}_{QU}\big[\{y_1, a_1, x_1\}_1, \big(e^{E_1}\hbar\, a_1 l_{11} t_1 + e^{E_1}\hbar\, a_1 l_{12} t_1 + e^{E_1}\hbar\, a_1 l_{21} t_2 + e^{E_1}\hbar\, a_1 l_{22} t_2 + e^{E_2}\hbar\, x_1 y_1 \gamma_{11} - e^{E_2}\hbar\, a_1 l_{11} t_1 \gamma_{12} - e^{E_2}\hbar\, a_1 l_{12} t_1 \gamma_{12} - e^{E_2}\hbar\, a_1 l_{21} t_2 \gamma_{12} - e^{E_2}\hbar\, a_1 l_{22} t_2 \gamma_{12} + e^{E_2}\hbar\, a_1 l_{11} t_1 T_1 \gamma_{12} + e^{E_2}\hbar\, a_1 l_{12} t_1 T_1 \gamma_{12} + e^{E_2}\hbar\, a_1 l_{21} t_2 T_1 \gamma_{12} + e^{E_2}\hbar\, a_1 l_{22} t_2 T_1 \gamma_{12} + \hbar x_1 y_1 \gamma_{12} + e^{E_2}\hbar\, x_1 y_1 \gamma_{21} - e^{E_2}\hbar\, a_1 l_{11} t_1 \gamma_{22} - e^{E_2}\hbar\, a_1 l_{12} t_1 \gamma_{22} - e^{E_2}\hbar\, a_1 l_{21} t_2 \gamma_{22} - e^{E_2}\hbar\, a_1 l_{22} t_2 \gamma_{22} + e^{E_2}\hbar\, a_1 l_{11} t_1 T_1 \gamma_{22} + e^{E_2}\hbar\, a_1 l_{12} t_1 T_1 \gamma_{22} + e^{E_2}\hbar\, a_1 l_{21} t_2 T_1 \gamma_{22} + e^{E_2}\hbar\, a_1 l_{22} t_2 T_1 \gamma_{22} + \hbar x_1 y_1 \gamma_{22}\big) \Big/ \big(e^{E_1} - e^{E_2}\gamma_{12} + e^{E_2} T_1 \gamma_{12} - e^{E_2}\gamma_{22} + e^{E_2} T_1 \gamma_{22}\big), \frac{e^{\gamma\hbar l_{12} t_1+\gamma\hbar l_{22} t_2}}{e^{\gamma\hbar l_{12} t_1+\gamma\hbar l_{22} t_2} - \gamma_{12} + T_1\gamma_{12} - \gamma_{22} + T_1\gamma_{22}} + O[\epsilon]^1\big]$$

Stitching Œ's.

StitchingOEs

In[ ]:=
```
m_{j_→k_}[Œ_U_[specs__, Q_, P_]] := Cord[Œ_U[Sequence @@ Append[DeleteCases[{specs}, {__}_{j|k}],
      Flatten[{Cases[{specs}, {us__}_j ⧴ {us}], Cases[{specs}, {us__}_k ⧴ {us}]}]_k],
    Q, P] /. {t_j → t_k, T_j → T_k}]
```

```
cœ = Œ_CU[{y1, a1, x1}_1, {y2, a2, x2}_2,
   {y3, a3, x3}_3, ℏ Sum[l_{10 i+j} t_i a_j + γ_{10 i+j} y_i x_j, {i, 3}, {j, 3}], 1_2];
{cœ // m_{3→4}, HL@Simp[CU[m_{3→4}[cœ]] - m_{3→4}[CU[cœ]]]}
```

$\{Œ_{CU}[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, \{y_4, a_4, x_4\}_4,$
$\hbar\,(a_1 l_{11} t_1 + a_2 l_{12} t_1 + a_4 l_{13} t_1 + a_1 l_{21} t_2 + a_2 l_{22} t_2 + a_4 l_{23} t_2 +$
$a_1 l_{31} t_4 + a_2 l_{32} t_4 + a_4 l_{33} t_4 + x_1 y_1 \gamma_{11} + x_2 y_1 \gamma_{12} + x_4 y_1 \gamma_{13} + x_1 y_2 \gamma_{21} +$
$x_2 y_2 \gamma_{22} + x_4 y_2 \gamma_{23} + x_1 y_4 \gamma_{31} + x_2 y_4 \gamma_{32} + x_4 y_4 \gamma_{33}), 1 + O[\epsilon]^3], 0\}$

Verifying that *m* commutes with evaluation, in CU:

```
cœ = Œ_CU[{y1, a1, x1}_1, {y2, a2, x2}_2,
   {y3, a3, x3}_3, ℏ Sum[l_{10 i+j} t_i a_j + γ_{10 i+j} y_i x_j, {i, 3}, {j, 3}], 1_2];
Timing@{cœ // m_{2→3}, HL@Simp[CU[m_{2→3}[cœ]] - m_{2→3}[CU[cœ]]]}
```

$\{513.453, \{Œ_{CU}[\{y_1, a_1, x_1\}_1, \{y_3, a_3, x_3\}_3, \frac{\cdots 1 \cdots}{e^{\cdots 1 \cdots}\,\cdots 1 \cdots\,\cdots 1 \cdots},$
$\frac{1}{1+\hbar t_3 \gamma_{32}} + \Big(\big(4\, e^{2\gamma\hbar l_{12} t_1 + \cdots 4 \cdots + 2\gamma\hbar l_{33} t_3}\hbar^2 a_3 x_1 y_1 \gamma_{12}\gamma_{31} - 2 \cdots 7 \cdots \gamma_{31} + \cdots 154 \cdots\big)\epsilon\Big) \Big/$
$\Big(2\, e^{2\gamma\hbar l_{12} t_1 + 2\gamma\hbar l_{13} t_1 + \cdots 3 \cdots + 2\gamma\hbar l_{33} t_3} + 10\, e^{\cdots 1 \cdots}\hbar t_3 \gamma_{32} + \cdots 2 \cdots +$
$\cdots 1 \cdots + 2\, e^{2\gamma\hbar l_{12} t_1 + \cdots 4 \cdots + \cdots 1 \cdots}\hbar^5 t_3^5 \gamma_{32}^5\Big) + \frac{(\cdots 1 \cdots)\,\epsilon^2}{\cdots 1 \cdots} + O[\epsilon]^3], 0\}\}$

large output | show less | show more | show all | set size limit...

Verifying that *m* commutes with evaluation, in QU:

```
qœ = Œ_QU[{y1, a1, x1}_1, {y2, a2, x2}_2,
   {y3, a3, x3}_3, ℏ Sum[l_{10 i+j} t_i a_j + γ_{10 i+j} y_i x_j, {i, 3}, {j, 3}], 1_2];
Timing@{qœ // m_{2→3}, HL@SimpT[QU[m_{2→3}[qœ]] - m_{2→3}[QU[qœ]]]}
```

$\{7831.47, \{Œ_{QU}[\{y_1, a_1, x_1\}_1, \{y_3, a_3, x_3\}_3, \frac{\cdots 1 \cdots}{\cdots 1 \cdots}, \frac{1}{1-\gamma_{32}+T_3\gamma_{32}} +$
$\Big(\big(8\, e^{2\gamma\hbar l_{12} t_1 + \cdots 4 \cdots + 2\gamma\hbar l_{33} t_3}\hbar^2 a_3 T_3 x_1 y_1 \gamma_{12}\gamma_{31} + 4 \cdots 8 \cdots \gamma_{31} + \cdots 371 \cdots\big)\epsilon\Big) \Big/$
$\Big(4\, e^{2\gamma\hbar l_{12} t_1 + 2\gamma\hbar l_{13} t_1 + \cdots 3 \cdots + 2\gamma\hbar l_{33} t_3} - 20\, e^{\cdots 1 \cdots}\gamma_{32} + \cdots 26 \cdots +$
$4\, e^{2\gamma\hbar l_{12} t_1 + \cdots 4 \cdots + \cdots 1 \cdots} T_3^5 \gamma_{32}^5\Big) + \frac{(\cdots 1 \cdots)\,\epsilon^2}{\cdots 79 \cdots + \cdots 1 \cdots} + O[\epsilon]^3], 0\}\}$

large output | show less | show more | show all | set size limit...

In[∘]:=
```
Œ_U_[sp1__, Q1_, P1_] ≡ Œ_U_[sp2__, Q2_, P2_] :=
  Sort[{sp1}] == Sort[{sp2}] ∧ Simplify[Q1 == Q2] ∧ Simplify[Normal[P1 - P2] == 0]
```

Verifying meta-associativity in CU:

```
co = Œ_CU[{y1, a1, x1}_1, {y2, a2, x2}_2,
    {y3, a3, x3}_3, ℏ Sum[λ_{10 i+j} t_i a_j + γ_{10 i+j} y_i x_j, {i, 3}, {j, 3}], 1_0];
Timing@HL[(lhs = co // m_{1,2→1} // m_{1,3→1}) ≡ (rhs = co // m_{2,3→2} // m_{1,2→1})]
```

{41.9219, True}

```
co = Œ_CU[{y1, a1, x1}_1, {y2, a2, x2}_2,
    {y3, a3, x3}_3, ℏ Sum[λ_{10 i+j} t_i a_j + γ_{10 i+j} y_i x_j, {i, 3}, {j, 3}], 1_1];
Timing@HL[(lhs = co // m_{1,2→1} // m_{1,3→1}) ≡ (rhs = co // m_{2,3→2} // m_{1,2→1})]
```

{30119.8, True}

mexamples

```
co = Œ_CU[{y1, a1, x1}_1, {y2, a2, x2}_2, ℏ Sum[l_{10 i+j} t_i a_j + γ_{10 i+j} y_i x_j, {i, 2}, {j, 2}], 1_1];
Short[Simplify /@ (cexample = co // m_{1→2}), 12]
Short[Simplify /@ (qexample = (qo = co /. CU → QU) // m_{1→2}), 12]
```

mexamples

$$\text{Œ}_{CU}\Big[\{y_2, a_2, x_2\}_2,\ \hbar\, a_2\,(l_{11}+l_{12}+l_{21}+l_{22})\,t_2 + \frac{1}{1+\hbar t_2 \gamma_{21}} e^{-\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar x_2 y_2 \Big(\gamma_{21} + e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\gamma_{12}(1+\hbar t_2\gamma_{21}) + e^{\gamma\hbar(l_{12}+l_{22})t_2}\gamma_{22} + \gamma_{11}\big(e^{\gamma\hbar(l_{11}+l_{21})t_2} - e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar t_2\gamma_{22}\big)\Big),$$
$$\frac{1}{1+\hbar t_2\gamma_{21}} + \frac{1}{2(1+\hbar t_2\gamma_{21})^5} e^{-2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar\Big(4 a_2 (1+\hbar t_2\gamma_{21})^2 \big(e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar\,(e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}t_2 + x_2 y_2)\gamma_{21}^2 + e^{2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar x_2 y_2\gamma_{11}\gamma_{22} + \gamma_{21}\big(e^{2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2} + \hbar x_2 y_2 (e^{\gamma\hbar(2l_{11}+l_{12}+2l_{21}+l_{22})t_2}\gamma_{11} + e^{\gamma\hbar(l_{11}+2l_{12}+l_{21}+2l_{22})t_2}\gamma_{22})\big)\big) - \gamma\hbar\Big(-2e^{2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}t_2\gamma_{21}^2(1+\hbar t_2\gamma_{21})^2 + 4 \ll 5 \gg (\ll 1 \gg) + \hbar \ll 4 \gg \big(3\hbar t_2\gamma_{21}^2 + 2e^{\gamma\hbar(l_{12}+l_{\ll 2 \gg})t_2}\gamma_{22} + \gamma_{21}(4 + e^{\gamma \ll 3 \gg}\hbar t_2\gamma_{22}) + e^{\gamma\hbar(l_{11}+l_{21})t_2}\gamma_{11}(2+\hbar t_2(\gamma_{21} - e^{\gamma\hbar(l_{12}+l_{22})t_2}\gamma_{22}))\big)\Big)\Big)\epsilon + O[\epsilon]^2\Big]$$

mexamples

$$\text{Œ}_{QU}\Big[\{y_2, a_2, x_2\}_2,\ \hbar\, a_2\,(l_{11}+l_{12}+l_{21}+l_{22})\,t_2 + \frac{1}{1+(-1+T_2)\gamma_{21}} e^{-\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar x_2 y_2\Big(\gamma_{21} + e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\gamma_{12}(1+(-1+T_2)\gamma_{21}) + e^{\gamma\hbar(l_{12}+l_{22})t_2}\gamma_{22} + \gamma_{11}\big(e^{\gamma\hbar(l_{11}+l_{21})t_2} - e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}(-1+T_2)\gamma_{22}\big)\Big),$$
$$\frac{1}{1+(-1+T_2)\gamma_{21}} + \frac{1}{4(1+(-1+T_2)\gamma_{21})^5} e^{-2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar\Big(8 a_2 T_2 (1+(-1+T_2)\gamma_{21})^2\big(e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}(-e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2} + e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}T_2 + \hbar x_2 y_2)\gamma_{21}^2 + e^{2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar x_2 y_2\gamma_{11}\gamma_{22} + \gamma_{21}\big(e^{2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2} + \hbar x_2 y_2(e^{\gamma\hbar(2l_{11}+l_{12}+2l_{21}+l_{22})t_2}\gamma_{11} + e^{\gamma\hbar(l_{11}+2l_{12}+l_{21}+2l_{22})t_2}\gamma_{22})\big)\big) + \gamma\Big(2e^{2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}(1-4T_2+3T_2^2)\gamma_{21}^2(1+(-1+T_2)\gamma_{21})^2 + 4e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar x_2 y_2\gamma_{21}(1+(-1+T_2)\gamma_{21})(\ll 1 \gg) - \ll 1 \gg\Big)\Big)\epsilon + O[\epsilon]^2\Big]$$

R in QU.

The Faddeev-Quesne formula:

Faddeev

In[∘]:=
```
𝕖_{q_,k_}[x_] := 𝕖^(Sum_{j=1}^{k+1} (1 - q)^j x^j / (j (1 - q^j))); 𝕖_{q_}[x_] := 𝕖_{q,$k}[x]
```

```
Table[Series[𝕖_{q_ℏ,k}[x], {ϵ, 0, 4}], {k, 0, 5}] // Column
```

$\mathbb{e}^x$

$\mathbb{e}^x - \frac{1}{4}\left(\mathbb{e}^x x^2 \gamma \hbar\right) \epsilon + \frac{1}{32}\mathbb{e}^x x^4 \gamma^2 \hbar^2 \epsilon^2 - \frac{1}{384}\left(\mathbb{e}^x x^2 \left(-8 + x^4\right) \gamma^3 \hbar^3\right) \epsilon^3 + \frac{\mathbb{e}^x x^4 \left(-32 + x^4\right) \gamma^4 \hbar^4 \epsilon^4}{6144} + O[\epsilon]^5$

$\mathbb{e}^x - \frac{1}{4}\left(\mathbb{e}^x x^2 \gamma \hbar\right) \epsilon + \frac{1}{288}\mathbb{e}^x x^3 \left(32 + 9x\right) \gamma^2 \hbar^2 \epsilon^2 - \frac{\left(\mathbb{e}^x x^2 \left(-24 + 32x^3 + 3x^4\right) \gamma^3 \hbar^3\right) \epsilon^3}{1152} + \frac{\mathbb{e}^x x^3 \left(-4608 - 864x + 1024x^3 + 576x^4 + 27x^5\right) \gamma^4 \hbar^4 \epsilon^4}{165888} + O[\epsilon]^5$

$\mathbb{e}^x - \frac{1}{4}\left(\mathbb{e}^x x^2 \gamma \hbar\right) \epsilon + \frac{1}{288}\mathbb{e}^x x^3 \left(32 + 9x\right) \gamma^2 \hbar^2 \epsilon^2 - \frac{\left(\mathbb{e}^x x^2 \left(-24 + 72x^2 + 32x^3 + 3x^4\right) \gamma^3 \hbar^3\right) \epsilon^3}{1152} + \frac{\mathbb{e}^x x^3 \left(-4608 - 864x + 3616x^3 + 576x^4 + 27x^5\right) \gamma^4 \hbar^4 \epsilon^4}{165888} + O[\epsilon]^5$

$\mathbb{e}^x - \frac{1}{4}\left(\mathbb{e}^x x^2 \gamma \hbar\right) \epsilon + \frac{1}{288}\mathbb{e}^x x^3 \left(32 + 9x\right) \gamma^2 \hbar^2 \epsilon^2 - \frac{\left(\mathbb{e}^x x^2 \left(-24 + 72x^2 + 32x^3 + 3x^4\right) \gamma^3 \hbar^3\right) \epsilon^3}{1152} + \frac{1}{4147200} \mathbb{e}^x x^3 \left(-115200 - 21600x + 165888x^2 + 90400x^3 + 14400x^4 + 675x^5\right) \gamma^4 \hbar^4 \epsilon^4 + O[\epsilon]^5$

$\mathbb{e}^x - \frac{1}{4}\left(\mathbb{e}^x x^2 \gamma \hbar\right) \epsilon + \frac{1}{288}\mathbb{e}^x x^3 \left(32 + 9x\right) \gamma^2 \hbar^2 \epsilon^2 - \frac{\left(\mathbb{e}^x x^2 \left(-24 + 72x^2 + 32x^3 + 3x^4\right) \gamma^3 \hbar^3\right) \epsilon^3}{1152} + \frac{1}{4147200} \mathbb{e}^x x^3 \left(-115200 - 21600x + 165888x^2 + 90400x^3 + 14400x^4 + 675x^5\right) \gamma^4 \hbar^4 \epsilon^4 + O[\epsilon]^5$

```
Table[Together@SeriesCoefficient[𝕖_{q,5}[x], {x, 0, n}], {n, 0, 5}]
```

$\left\{1, 1, \frac{1}{1+q}, \frac{1}{(1+q)\left(1+q+q^2\right)}, \frac{1}{(1+q)^2\left(1+q^2\right)\left(1+q+q^2\right)}, 1\Big/\left((1+q)^2\left(1+q^2\right)\left(1+q+q^2\right)\left(1+q+q^2+q^3+q^4\right)\right)\right\}$

```
Table[HL@FunctionExpand[QFactorial[n, q] SeriesCoefficient[𝕖_{q,5}[x], {x, 0, n}]], {n, 0, 5}]
```

{1, 1, 1, 1, 1, 1}

R

In[∘]:=
```
QU[R_{i_,j_}] := 𝕆_QU[{y_1, a_1}_i, {a_2, x_2}_j, SS[𝕖^{ℏ b_1 a_2} 𝕖_{q_ℏ}[ℏ y_1 x_2] /. b_1 → γ^-1 (ϵ a_1 - t_i)]];
QU[R^-1_{i_,j_}] := S_j@QU[R_{i,j}];
```

```
QU[R_{3,4}] // Short
```

$\text{QU}[\,] + \frac{\epsilon \hbar\, \text{QU}[a_3, a_4]}{\gamma} + \hbar\, \text{QU}[y_3, x_4] + \frac{\epsilon \ll 1 \gg \ll 1 \gg}{\gamma} + \frac{1}{2} \ll 1 \gg \ll 1 \gg - \frac{\ll 1 \gg}{\gamma} - \frac{\epsilon \hbar^2 \ll 1 \gg t_3}{\gamma^2} - \frac{\hbar^2\, \text{QU}[y_3, a_4, x_4]\, t_3}{\gamma} + \frac{\hbar^2\, \text{QU}[a_4, a_4]\, t_3^2}{2\gamma^2}$

Verifying R2 (~2 secs @ $p=4, $k=2):

```
QU[R_{1,2} ** R^-1_{1,2}] // Simp // HL // Timing
```

{0.078125, QU[]}

Verifying R3 (~156 secs @ $p=4, $k=2):

```
{Short[lhs = QU[R_{1,2} ** R_{1,3} ** R_{2,3}]], HL@SimpT[lhs - QU[R_{2,3} ** R_{1,3} ** R_{1,2}]]} // Timing
```

$$\left\{0.203125,\ \left\{QU[\,] + \frac{\epsilon\,\hbar\,QU[a_1, a_2]}{\gamma} + \frac{\epsilon\,\hbar\,QU[a_1, a_3]}{\gamma} + \ll 73 \gg + 2\,\epsilon\,\hbar^2\,QU[y_1, a_2, x_3]\,T_2 + QU[y_1, x_3]\,(\hbar - \hbar\,T_2),\ 0\right\}\right\}$$

R in Œ$_{QU}$.

RinOE

In[∘]:=

```
Œ_{QU,k_}[R_{i_,j_}] := Œ_{QU}[{y_i, a_i, x_i}_i, {y_j, a_j, x_j}_j, -ħ γ^{-1} t_i a_j + ħ y_i x_j,
  Series[e^{ħ γ^{-1} t_i a_j - ħ y_i x_j} (e^{ħ b_i a_j} e_{q_ħ,k}[ħ y_i x_j] /. b_i → γ^{-1} (ϵ a_i - t_i)), {ϵ, 0, k}]]
```

```
{Œ_{QU,1}[R_{1,2}], Œ_{QU,2}[R_{1,2}]}
```

$$\left\{Œ_{QU}\left[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, -\frac{\hbar\,a_2\,t_1}{\gamma} + \hbar\,x_2\,y_1, 1 + \left(\frac{\hbar\,a_1\,a_2}{\gamma} - \frac{1}{4}\,\gamma\,\hbar^3\,x_2^2\,y_1^2\right)\epsilon + O[\epsilon]^2\right],\right.$$

$$Œ_{QU}\left[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, -\frac{\hbar\,a_2\,t_1}{\gamma} + \hbar\,x_2\,y_1, 1 + \left(\frac{\hbar\,a_1\,a_2}{\gamma} - \frac{1}{4}\,\gamma\,\hbar^3\,x_2^2\,y_1^2\right)\epsilon + \right.$$

$$\left.\left.\frac{1}{288\,\gamma^2}\left(144\,\hbar^2\,a_1^2\,a_2^2 - 72\,\gamma^2\,\hbar^4\,a_1\,a_2\,x_2^2\,y_1^2 + 32\,\gamma^4\,\hbar^5\,x_2^3\,y_1^3 + 9\,\gamma^4\,\hbar^6\,x_2^4\,y_1^4\right)\epsilon^2 + O[\epsilon]^3\right]\right\}$$

The morphism Œ$_{U,k}$.

MorphismOE

In[∘]:=

```
Œ_{U_,k_}[a_ * b_] := Œ_{U,k}[a] Œ_{U,k}[b];
Œ_{U_,k_}[m_{is__}[a_]] := m_{is}[Œ_{U,k}[a]];
```

```
Œ_{QU,1}[R_{1,2} R_{3,4} R_{5,6}]
```

$$Œ_{QU}\left[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, \{y_3, a_3, x_3\}_3, \{y_4, a_4, x_4\}_4,\right.$$
$$\{y_5, a_5, x_5\}_5, \{y_6, a_6, x_6\}_6, -\frac{\hbar\,a_2\,t_1}{\gamma} - \frac{\hbar\,a_4\,t_3}{\gamma} - \frac{\hbar\,a_6\,t_5}{\gamma} + \hbar\,x_2\,y_1 + \hbar\,x_4\,y_3 + \hbar\,x_6\,y_5,$$
$$\left.1 + \left(\frac{\hbar\,a_1\,a_2}{\gamma} + \frac{\hbar\,a_3\,a_4}{\gamma} + \frac{\hbar\,a_5\,a_6}{\gamma} - \frac{1}{4}\,\gamma\,\hbar^3\,x_2^2\,y_1^2 - \frac{1}{4}\,\gamma\,\hbar^3\,x_4^2\,y_3^2 - \frac{1}{4}\,\gamma\,\hbar^3\,x_6^2\,y_5^2\right)\epsilon + O[\epsilon]^2\right]$$

```
Œ_{QU,1}[R_{1,2} R_{3,4} R_{5,6} // m_{1,3→1} // m_{2,5→2} // m_{4,6→4}]
```

$$Œ_{QU}\left[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, \{y_4, a_4, x_4\}_4, \frac{1}{\gamma}\right.$$
$$\left(-\hbar\,a_2\,t_1 - \hbar\,a_4\,t_1 - \hbar\,a_4\,t_2 + \gamma\,\hbar\,x_2\,y_1 + \gamma\,\hbar\,x_4\,y_1 + e^{\hbar\,t_2}\,\gamma\,\hbar\,x_4\,y_1 - \gamma\,\hbar\,T_2\,x_4\,y_1 + e^{\hbar\,t_1}\,\gamma\,\hbar\,x_4\,y_2\right),$$
$$1 + \frac{1}{4\,\gamma}\left(4\,\hbar\,a_1\,a_2 + 4\,\hbar\,a_1\,a_4 + 4\,\hbar\,a_2\,a_4 - 4\,\gamma\,\hbar^2\,a_4\,x_2\,y_1 - 8\,e^{\hbar\,t_2}\,\gamma\,\hbar^2\,a_2\,x_4\,y_1 +\right.$$
$$8\,\gamma\,\hbar^2\,a_2\,T_2\,x_4\,y_1 - \gamma^2\,\hbar^3\,x_2^2\,y_1^2 + 4\,e^{\hbar\,t_2}\,\gamma^2\,\hbar^3\,x_2\,x_4\,y_1^2 - 4\,\gamma^2\,\hbar^3\,T_2\,x_2\,x_4\,y_1^2 - \gamma^2\,\hbar^3\,x_4^2\,y_1^2 -$$
$$e^{2\,\hbar\,t_2}\,\gamma^2\,\hbar^3\,x_4^2\,y_1^2 + \gamma^2\,\hbar^3\,T_2^2\,x_4^2\,y_1^2 - 4\,e^{\hbar\,t_1}\,\gamma\,\hbar^2\,a_1\,x_4\,y_2 + 4\,e^{\hbar\,t_1}\,\gamma^2\,\hbar^3\,x_2\,x_4\,y_1\,y_2 +$$
$$\left.\left.4\,e^{\hbar\,t_1+\hbar\,t_2}\,\gamma^2\,\hbar^3\,x_4^2\,y_1\,y_2 - 4\,e^{\hbar\,t_1}\,\gamma^2\,\hbar^3\,T_2\,x_4^2\,y_1\,y_2 - e^{2\,\hbar\,t_1}\,\gamma^2\,\hbar^3\,x_4^2\,y_2^2\right)\epsilon + O[\epsilon]^2\right]$$

$\text{Œ}_{QU,1}[R_{1,2}\, R_{3,4}\, R_{5,6}\ //\ m_{3,1\to 1}\ //\ m_{5,2\to 2}\ //\ m_{6,4\to 4}]$

$$\text{Œ}_{QU}\Big[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, \{y_4, a_4, x_4\}_4, \frac{1}{\gamma}\left(-\hbar\, a_2\, t_1 - \hbar\, a_4\, t_1 - \hbar\, a_4\, t_2 + \gamma\,\hbar\, x_2\, y_1 + \gamma\,\hbar\, x_4\, y_1 + e^{\hbar\, t_1}\,\gamma\,\hbar\, x_4\, y_2\right),$$
$$1 + \frac{1}{4\gamma}\Big(4\,\hbar\, a_1\, a_2 + 4\,\hbar\, a_1\, a_4 + 4\,\hbar\, a_2\, a_4 - 4\,\gamma\,\hbar^2\, a_4\, x_2\, y_1 - \gamma^2\,\hbar^3\, x_2^2\, y_1^2 - \gamma^2\,\hbar^3\, x_4^2\, y_1^2 -$$
$$4\, e^{\hbar\, t_1}\,\gamma\,\hbar^2\, a_1\, x_4\, y_2 + 4\, e^{\hbar\, t_1}\,\gamma^2\,\hbar^3\, x_2\, x_4\, y_1\, y_2 - e^{2\hbar\, t_1}\,\gamma^2\,\hbar^3\, x_4^2\, y_2^2\Big)\,\epsilon + O[\epsilon]^2\Big]$$

$\text{Œ}_{QU,1}[R_{1,2}\, R_{3,4}\, R_{5,6}\ //\ m_{1,3\to 1}\ //\ m_{2,5\to 2}\ //\ m_{4,6\to 4}] \equiv \text{Œ}_{QU,1}[R_{1,2}\, R_{3,4}\, R_{5,6}\ //\ m_{3,1\to 1}\ //\ m_{5,2\to 2}\ //\ m_{6,4\to 4}]$

$$\hbar\left(e^{\hbar\, t_2} - T_2\right) x_4\, y_1 == 0\ \&\&\ \epsilon\,\hbar\left(e^{\hbar\, t_2} - T_2\right) x_4\, y_1\left(8\, a_2 + \gamma\,\hbar\left(-4\, x_2\, y_1 + x_4\left(\left(e^{\hbar\, t_2} + T_2\right) y_1 - 4\, e^{\hbar\, t_1}\, y_2\right)\right)\right) == 0$$

Exponentials as needed.

ReqExp

In[•]:=
```
Block[{$p = 2, $k = 2}, TableForm[StringSplit[
    "y|a|x|Cθ@y_CU|Cθ@a_CU|Cθ@x_CU|Qθ@y_QU|Qθ@a_QU|Qθ@x_QU|A𝔻@y_QU|A𝔻@a_QU|A𝔻@x_QU|S𝔻@y_QU|S𝔻@a_QU|S𝔻
      @x_QU|S@y_CU|S@a_CU|S@x_CU|S@y_QU|S@a_QU|S@x_QU|Δ@y_CU|Δ@a_CU|Δ@x_CU|Δ@y_QU|Δ@a_QU|Δ@x_QU",
    "|"] /. s_String ⧴
    {s, Normal@Simplify@Series[ToExpression[s] /. CU | QU → Times, {ϵ, 0, $k}]}]]
```

Out[•]//TableForm=
ReqExp

y	y
a	a
x	x
Cθ@y_{CU}	$-x$
Cθ@a_{CU}	$-a$
Cθ@x_{CU}	$-y$
Qθ@y_{QU}	$-\frac{x}{\sqrt{T}}-\frac{a\,x\,\epsilon\,\hbar}{\sqrt{T}}-\frac{a^2\,x\,\epsilon^2\,\hbar^2}{2\sqrt{T}}$
Qθ@a_{QU}	$-a$
Qθ@x_{QU}	$-\frac{y}{\sqrt{T}}+\frac{y\,(-a+\gamma)\,\epsilon\,\hbar}{\sqrt{T}}-\frac{y\,(a-\gamma)^2\,\epsilon^2\,\hbar^2}{2\sqrt{T}}$
A𝔻@y_{QU}	$\frac{2}{3}a^2\,y\,\epsilon^2\,\hbar^2+\frac{1}{6}y\,(6+3\,t\,\hbar+t^2\,\hbar^2)+\frac{1}{12}y\,\epsilon\,\hbar\,(x\,y\,\gamma\,\hbar-4\,a\,(3+2\,t\,\hbar))$
A𝔻@a_{QU}	a
A𝔻@x_{QU}	x
S𝔻@y_{QU}	$y+\frac{1}{48}t^2\,y\,\hbar^2+\frac{1}{24}y\,(-2\,a\,t+x\,y\,\gamma)\,\epsilon\,\hbar^2+\frac{1}{12}a^2\,y\,\epsilon^2\,\hbar^2$
S𝔻@a_{QU}	a
S𝔻@x_{QU}	$\frac{7}{12}a^2\,x\,\epsilon^2\,\hbar^2+x\left(1+\frac{t\,\hbar}{2}+\frac{7\,t^2\,\hbar^2}{48}\right)+\frac{1}{24}x\,\epsilon\,\hbar\,(x\,y\,\gamma\,\hbar-2\,a\,(12+7\,t\,\hbar))$
S@y_{CU}	$-y$
S@a_{CU}	$-a$
S@x_{CU}	$-x$
S@y_{QU}	$-\frac{y}{T}+\frac{y\,(-a+\gamma)\,\epsilon\,\hbar}{T}-\frac{y\,(a-\gamma)^2\,\epsilon^2\,\hbar^2}{2\,T}$
S@a_{QU}	$-a$
S@x_{QU}	$-x-a\,x\,\epsilon\,\hbar-\frac{1}{2}a^2\,x\,\epsilon^2\,\hbar^2$
Δ@y_{CU}	$y_1+T_1\,y_2$
Δ@a_{CU}	a_1+a_2
Δ@x_{CU}	x_1+x_2
Δ@y_{QU}	$y_1+T_1\,y_2-\epsilon\,\hbar\,a_1\,T_1\,y_2+\frac{1}{2}\epsilon^2\,\hbar^2\,a_1^2\,T_1\,y_2$
Δ@a_{QU}	a_1+a_2
Δ@x_{QU}	$x_1+x_2-\epsilon\,\hbar\,a_1\,x_2+\frac{1}{2}\epsilon^2\,\hbar^2\,a_1^2\,x_2$

Exp

Task. Define $\mathrm{Exp}_{U_i,k}[\xi, P]$ which computes $e^{\xi\,\mathbb{O}(P)}$ to ϵ^k in the algebra U_i, where ξ is a scalar, X is x_i or y_i, and P is an ϵ-dependent near-docile element, giving the answer in Œ-form. Should satisfy $U@\mathrm{Exp}_{U_i,k}[\xi, P] == \mathbb{S}_U\left[e^{\xi x},\ x \to \mathbb{O}(P)\right]$.

Methodology. If $P_0 := P_{\epsilon=0}$ and $e^{\xi\,\mathbb{O}(P)} = \mathbb{O}\left(e^{\xi P_0} F(\xi)\right)$, then $F(\xi = 0) = 1$ and we have:
$\mathbb{O}\left(e^{\xi P_0}(P_0 F(\xi) + \partial_\xi F\right) = \mathbb{O}\left(\partial_\xi e^{\xi P_0} F(\xi)\right) = \partial_\xi\,\mathbb{O}\left(e^{\xi P_0} F(\xi)\right) = \partial_\xi e^{\xi\,\mathbb{O}(P)} = e^{\xi\,\mathbb{O}(P)}\,\mathbb{O}(P) = \mathbb{O}\left(e^{\xi P_0} F(\xi)\right)\mathbb{O}(P)$.
This is an ODE for F. Setting inductively $F_k = F_{k-1} + \epsilon^k \varphi$ we find that $F_0 = 1$ and solve for φ.

Exp

```
(* Bug: The first line is valid only if 𝕆(e^P0) == e^𝕆(P0). *)
(* Bug: ξ must be a symbol. *)
Exp_{U_i_,0}[ξ_, P_] := 𝔼_U[{y_i, a_i, x_i}_i, Normal@P /. ϵ → 0, 1 + O_0];
Exp_{U_i_,k_}[ξ_, P_] := Module[{yax = {y_i, a_i, x_i}, P0, φ, φs, F, j, rhs, at0, atξ},
  P0 = Normal@P /. ϵ → 0;
  φs =
   Flatten@Table[φ_{j1,j2,j3}[ξ], {j2, 0, k}, {j1, 0, 2 k + 1 - j2}, {j3, 0, 2 k + 1 - j2 - j1}];
  F = Normal@Last@Exp_{U_i,k-1}[ξ, P] + ϵ^k φs.(φs /. φ_{js__}[ξ] :→ Times @@ yax^{js});
  rhs = Normal@Last@m_{i,j→i}[𝔼_U[yax_i, ξ P0, F + O_k] m_{i→j}@𝔼_U[{y_i, a_i, x_i}_i, 0, P + O_k]];
  at0 = (# == 0) & /@ Flatten@CoefficientList[F - 1 /. ξ → 0, yax];
  atξ = (# == 0) & /@ Flatten@CoefficientList[(∂_ξ F) + P0 F - rhs, yax];
  𝔼_U[yax_i, ξ P0, F + O_k] /. DSolve[And @@ (at0 ⋃ atξ), φs, ξ][[1]] ]
```

In[∘]:=
```
Timing@Block[{$p = 4, $k = 2}, {
   s = S_1[QU[y_1]] /. QU → Times,
   exps = Exp_{QU_1,$k}[η, s], (* Warning: wrong unless $p≥$k+1! *)
   HL@Simp[S_1@𝕆_QU[{y_1}_1, SS[e^{ħ η y_1}]] - QU@(exps /. η → ħ η)]
  }]
```

Out[∘]=
$$\left\{35.8281,\ \left\{a_1\left(-\frac{\epsilon\hbar}{T_1}+\frac{\gamma\epsilon^2\hbar^2}{T_1}\right)y_1+\left(-\frac{1}{T_1}+\frac{\gamma\epsilon\hbar}{T_1}-\frac{\gamma^2\epsilon^2\hbar^2}{2T_1}\right)y_1-\frac{\epsilon^2\hbar^2a_1^2y_1}{2T_1},\right.\right.$$
$$\mathbb{E}_{QU}\left[\{y_1,a_1,x_1\}_1,-\frac{\eta y_1}{T_1},1+\frac{\left(2\gamma\eta\hbar T_1y_1-2\eta\hbar a_1T_1y_1-\gamma\eta^2\hbar y_1^2\right)\epsilon}{2T_1^2}+\right.$$
$$\left(-\frac{\gamma^2\eta\hbar^2y_1}{2T_1}+\frac{\gamma\eta\hbar^2a_1y_1}{T_1}-\frac{\eta\hbar^2a_1^2y_1}{2T_1}+\frac{7\gamma^2\eta^2\hbar^2y_1^2}{4T_1^2}-\frac{2\gamma\eta^2\hbar^2a_1y_1^2}{T_1^2}+\right.$$
$$\left.\left.\left.\frac{\eta^2\hbar^2a_1^2y_1^2}{2T_1^2}-\frac{\gamma^2\eta^3\hbar^2y_1^3}{T_1^3}+\frac{\gamma\eta^3\hbar^2a_1y_1^3}{2T_1^3}+\frac{\gamma^2\eta^4\hbar^2y_1^4}{8T_1^4}\right)\epsilon^2+O[\epsilon]^3\right],\ 0\right\}\right\}$$

In[∘]:=
```
Timing@Block[{$p = 4, $k = 2}, {
   s = S_1[QU[a_1]] /. QU → Times,
   exps = Exp_{QU_1,$k}[α, s], (* Warning: wrong unless $p≥$k+1! *)
   HL@Simp[S_1@𝕆_QU[{a_1}_1, SS[e^{ħ α a_1}]] - QU@(exps /. α → ħ α)]
  }]
```

Out[∘]=
$$\left\{33.5938,\ \left\{-a_1,\ \mathbb{E}_{QU}\left[\{y_1,a_1,x_1\}_1,-\alpha a_1,1+O[\epsilon]^3\right],\ 0\right\}\right\}$$

```
In[•]:= Timing@Block[{$p = 4, $k = 2}, {
    s = S₁[QU[x₁]] /. QU → Times,
    exps = Exp_{QU₁,$k}[ξ, s], (* Warning: wrong unless $p≥$k+1! *)
    HL@Simp[S₁@𝕆_QU[{x₁}₁, SS[e^{ħ ξ x₁}]] - QU@(exps /. ξ → ħ ξ)]
  }]
```

Out[•]= $\{34.0625, \{-x_1 - \epsilon\hbar a_1 x_1 - \frac{1}{2}\epsilon^2\hbar^2 a_1^2 x_1,$

$\mathbb{E}_{QU}[\{y_1, a_1, x_1\}_1, -\xi x_1, 1 + \left(-\xi\hbar a_1 x_1 - \frac{1}{2}\gamma\xi^2\hbar x_1^2\right)\epsilon + \left(-\frac{1}{2}\xi\hbar^2 a_1^2 x_1 + \frac{1}{4}\gamma^2\xi^2\hbar^2 x_1^2 - \gamma\xi^2\hbar^2 a_1 x_1^2 + \frac{1}{2}\xi^2\hbar^2 a_1^2 x_1^2 - \frac{1}{2}\gamma^2\xi^3\hbar^2 x_1^3 + \frac{1}{2}\gamma\xi^3\hbar^2 a_1 x_1^3 + \frac{1}{8}\gamma^2\xi^4\hbar^2 x_1^4\right)\epsilon^2 + O[\epsilon]^3], 0\}\}$

$S(e^{\eta y} e^{\alpha a} e^{\xi x})$

```
In[•]:= Timing@Block[{$p = 3, $k = 1}, {
    sexp = m_{3,2,1→1}[Exp_{QU₁,$k}[η, S₁[QU[y₁]] /. QU → Times] Exp_{QU₂,$k}[α, S₂[QU[a₂]] /. QU → Times]
        Exp_{QU₃,$k}[ξ, S₃[QU[x₃]] /. QU → Times]] /. {η → ħ η, α → ħ α, ξ → ħ ξ},
    HL@SimpT[QU@sexp - S₁@𝕆_QU[{y₁, a₁, x₁}₁, SS[e^{ħ (η y₁+α a₁+ξ x₁)}]]]
  }]
```

Out[•]= $\{9.34375,$

$\{\mathbb{E}_{QU}[\{y_1, a_1, x_1\}_1, \frac{1}{\hbar T_1}\left(e^{\alpha\gamma\hbar}\eta\xi\hbar^2 - e^{\alpha\gamma\hbar}\eta\xi\hbar^2 T_1 - \alpha\hbar^2 a_1 T_1 - e^{\alpha\gamma\hbar}\xi\hbar^2 T_1 x_1 - e^{\alpha\gamma\hbar}\eta\hbar^2 y_1\right),$

$1 + \frac{1}{4\hbar T_1^2}\left(-3 e^{2\alpha\gamma\hbar}\gamma\eta^2\xi^2\hbar^4 - 4 e^{\alpha\gamma\hbar}\gamma\eta\xi\hbar^3 T_1 + 4 e^{2\alpha\gamma\hbar}\gamma\eta^2\xi^2\hbar^4 T_1 + 8 e^{\alpha\gamma\hbar}\eta\xi\hbar^3 a_1 T_1 + 4 e^{\alpha\gamma\hbar}\gamma\eta\xi\hbar^3 T_1^2 - e^{2\alpha\gamma\hbar}\gamma\eta^2\xi^2\hbar^4 T_1^2 + 6 e^{2\alpha\gamma\hbar}\gamma\eta\xi^2\hbar^4 T_1 x_1 - 2 e^{2\alpha\gamma\hbar}\gamma\eta\xi^2\hbar^4 T_1^2 x_1 - 4 e^{\alpha\gamma\hbar}\xi\hbar^3 a_1 T_1^2 x_1 - 2 e^{2\alpha\gamma\hbar}\gamma\xi^2\hbar^4 T_1^2 x_1^2 + 6 e^{2\alpha\gamma\hbar}\gamma\eta^2\xi\hbar^4 y_1 + 4 e^{\alpha\gamma\hbar}\gamma\eta\hbar^3 T_1 y_1 - 2 e^{2\alpha\gamma\hbar}\gamma\eta^2\xi\hbar^4 T_1 y_1 - 4 e^{\alpha\gamma\hbar}\eta\hbar^3 a_1 T_1 y_1 - 4 e^{2\alpha\gamma\hbar}\gamma\eta\xi\hbar^4 T_1 x_1 y_1 - 2 e^{2\alpha\gamma\hbar}\gamma\eta^2\hbar^4 y_1^2\right)\epsilon + O[\epsilon]^2], 0\}\}$

```
In[•]:= Timing@Block[{$p = 4, $k = 2}, {
    sexp = m_{3,2,1→1}[Exp_{QU₁,$k}[η, S₁[QU[y₁]] /. QU → Times] Exp_{QU₂,$k}[α, S₂[QU[a₂]] /. QU → Times]
        Exp_{QU₃,$k}[ξ, S₃[QU[x₃]] /. QU → Times]] /. {η → ħ η, α → ħ α, ξ → ħ ξ},
    HL@SimpT[QU@sexp - S₁@𝕆_QU[{y₁, a₁, x₁}₁, SS[e^{ħ (η y₁+α a₁+ξ x₁)}]]]
  }]
```

Out[∘]= $\{169.703,$

$$\Big\{\mathbb{E}_{QU}[\{y_1, a_1, x_1\}_1, \frac{1}{\hbar T_1}\left(\mathrm{e}^{\alpha\gamma\hbar}\eta\xi\hbar^2 - \mathrm{e}^{\alpha\gamma\hbar}\eta\xi\hbar^2 T_1 - \alpha\hbar^2 a_1 T_1 - \mathrm{e}^{\alpha\gamma\hbar}\xi\hbar^2 T_1 x_1 - \mathrm{e}^{\alpha\gamma\hbar}\eta\hbar^2 y_1\right),$$

$$1 + \frac{1}{4\hbar T_1^2}\Big(-3\,\mathrm{e}^{2\alpha\gamma\hbar}\gamma\eta^2\xi^2\hbar^4 - 4\,\mathrm{e}^{\alpha\gamma\hbar}\gamma\eta\xi\hbar^3 T_1 + 4\,\mathrm{e}^{2\alpha\gamma\hbar}\gamma\eta^2\xi^2\hbar^4 T_1 + 8\,\mathrm{e}^{\alpha\gamma\hbar}\eta\xi\hbar^3 a_1 T_1 +$$
$$4\,\mathrm{e}^{\alpha\gamma\hbar}\gamma\eta\xi\hbar^3 T_1^2 - \mathrm{e}^{2\alpha\gamma\hbar}\gamma\eta^2\xi^2\hbar^4 T_1^2 + 6\,\mathrm{e}^{2\alpha\gamma\hbar}\gamma\eta\xi^2\hbar^4 T_1 x_1 - 2\,\mathrm{e}^{2\alpha\gamma\hbar}\gamma\eta\xi^2\hbar^4 T_1^2 x_1 -$$
$$4\,\mathrm{e}^{\alpha\gamma\hbar}\xi\hbar^3 a_1 T_1^2 x_1 - 2\,\mathrm{e}^{2\alpha\gamma\hbar}\gamma\xi^2\hbar^4 T_1^2 x_1^2 + 6\,\mathrm{e}^{2\alpha\gamma\hbar}\gamma\eta^2\xi\hbar^4 y_1 + 4\,\mathrm{e}^{\alpha\gamma\hbar}\gamma\eta\hbar^3 T_1 y_1 -$$
$$2\,\mathrm{e}^{2\alpha\gamma\hbar}\gamma\eta^2\xi\hbar^4 T_1 y_1 - 4\,\mathrm{e}^{\alpha\gamma\hbar}\eta\hbar^3 a_1 T_1 y_1 - 4\,\mathrm{e}^{2\alpha\gamma\hbar}\gamma\eta\xi\hbar^4 T_1 x_1 y_1 - 2\,\mathrm{e}^{2\alpha\gamma\hbar}\gamma\eta^2\hbar^4 y_1^2\Big)\,\epsilon +$$

$$\frac{1}{288\hbar^2 T_1^4}\Big(81\,\mathrm{e}^{4\alpha\gamma\hbar}\gamma^2\eta^4\xi^4\hbar^8 + 544\,\mathrm{e}^{3\alpha\gamma\hbar}\gamma^2\eta^3\xi^3\hbar^7 T_1 - 216\,\mathrm{e}^{4\alpha\gamma\hbar}\gamma^2\eta^4\xi^4\hbar^8 T_1 -$$
$$432\,\mathrm{e}^{3\alpha\gamma\hbar}\gamma\eta^3\xi^3\hbar^7 a_1 T_1 + 756\,\mathrm{e}^{2\alpha\gamma\hbar}\gamma^2\eta^2\xi^2\hbar^6 T_1^2 - 1104\,\mathrm{e}^{3\alpha\gamma\hbar}\gamma^2\eta^3\xi^3\hbar^7 T_1^2 +$$
$$198\,\mathrm{e}^{4\alpha\gamma\hbar}\gamma^2\eta^4\xi^4\hbar^8 T_1^2 - 1440\,\mathrm{e}^{2\alpha\gamma\hbar}\gamma\eta^2\xi^2\hbar^6 a_1 T_1^2 + 576\,\mathrm{e}^{3\alpha\gamma\hbar}\gamma\eta^3\xi^3\hbar^7 a_1 T_1^2 +$$
$$576\,\mathrm{e}^{2\alpha\gamma\hbar}\eta^2\xi^2\hbar^6 a_1^2 T_1^2 + 144\,\mathrm{e}^{\alpha\gamma\hbar}\gamma^2\eta\xi\hbar^5 T_1^3 - 1080\,\mathrm{e}^{2\alpha\gamma\hbar}\gamma^2\eta^2\xi^2\hbar^6 T_1^3 +$$
$$672\,\mathrm{e}^{3\alpha\gamma\hbar}\gamma^2\eta^3\xi^3\hbar^7 T_1^3 - 72\,\mathrm{e}^{4\alpha\gamma\hbar}\gamma^2\eta^4\xi^4\hbar^8 T_1^3 - 576\,\mathrm{e}^{\alpha\gamma\hbar}\gamma\eta\xi\hbar^5 a_1 T_1^3 +$$
$$1152\,\mathrm{e}^{2\alpha\gamma\hbar}\gamma\eta^2\xi^2\hbar^6 a_1 T_1^3 - 144\,\mathrm{e}^{3\alpha\gamma\hbar}\gamma\eta^3\xi^3\hbar^7 a_1 T_1^3 + 576\,\mathrm{e}^{\alpha\gamma\hbar}\eta\xi\hbar^5 a_1^2 T_1^3 -$$
$$144\,\mathrm{e}^{\alpha\gamma\hbar}\gamma^2\eta\xi\hbar^5 T_1^4 + 324\,\mathrm{e}^{2\alpha\gamma\hbar}\gamma^2\eta^2\xi^2\hbar^6 T_1^4 - 112\,\mathrm{e}^{3\alpha\gamma\hbar}\gamma^2\eta^3\xi^3\hbar^7 T_1^4 +$$
$$9\,\mathrm{e}^{4\alpha\gamma\hbar}\gamma^2\eta^4\xi^4\hbar^8 T_1^4 - 324\,\mathrm{e}^{4\alpha\gamma\hbar}\gamma^2\eta^3\xi^4\hbar^8 T_1 x_1 - 1416\,\mathrm{e}^{3\alpha\gamma\hbar}\gamma^2\eta^2\xi^3\hbar^7 T_1^2 x_1 +$$
$$540\,\mathrm{e}^{4\alpha\gamma\hbar}\gamma^2\eta^3\xi^4\hbar^8 T_1^2 x_1 + 1080\,\mathrm{e}^{3\alpha\gamma\hbar}\gamma\eta^2\xi^3\hbar^7 a_1 T_1^2 x_1 - 792\,\mathrm{e}^{2\alpha\gamma\hbar}\gamma^2\eta\xi^2\hbar^6 T_1^3 x_1 +$$
$$1392\,\mathrm{e}^{3\alpha\gamma\hbar}\gamma^2\eta^2\xi^3\hbar^7 T_1^3 x_1 - 252\,\mathrm{e}^{4\alpha\gamma\hbar}\gamma^2\eta^3\xi^4\hbar^8 T_1^3 x_1 + 1584\,\mathrm{e}^{2\alpha\gamma\hbar}\gamma\eta\xi^2\hbar^6 a_1 T_1^3 x_1 -$$
$$576\,\mathrm{e}^{3\alpha\gamma\hbar}\gamma\eta^2\xi^3\hbar^7 a_1 T_1^3 x_1 - 576\,\mathrm{e}^{2\alpha\gamma\hbar}\eta\xi^2\hbar^6 a_1^2 T_1^3 x_1 + 216\,\mathrm{e}^{2\alpha\gamma\hbar}\gamma^2\eta\xi^2\hbar^6 T_1^4 x_1 -$$
$$264\,\mathrm{e}^{3\alpha\gamma\hbar}\gamma^2\eta^2\xi^3\hbar^7 T_1^4 x_1 + 36\,\mathrm{e}^{4\alpha\gamma\hbar}\gamma^2\eta^3\xi^4\hbar^8 T_1^4 x_1 - 432\,\mathrm{e}^{2\alpha\gamma\hbar}\gamma\eta\xi^2\hbar^6 a_1 T_1^4 x_1 +$$
$$72\,\mathrm{e}^{3\alpha\gamma\hbar}\gamma\eta^2\xi^3\hbar^7 a_1 T_1^4 x_1 - 144\,\mathrm{e}^{\alpha\gamma\hbar}\xi\hbar^5 a_1^2 T_1^4 x_1 + 432\,\mathrm{e}^{4\alpha\gamma\hbar}\gamma^2\eta^2\xi^4\hbar^8 T_1^2 x_1^2 +$$
$$912\,\mathrm{e}^{3\alpha\gamma\hbar}\gamma^2\eta\xi^3\hbar^7 T_1^3 x_1^2 - 360\,\mathrm{e}^{4\alpha\gamma\hbar}\gamma^2\eta^2\xi^4\hbar^8 T_1^3 x_1^2 - 720\,\mathrm{e}^{3\alpha\gamma\hbar}\gamma\eta\xi^3\hbar^7 a_1 T_1^3 x_1^2 +$$
$$72\,\mathrm{e}^{2\alpha\gamma\hbar}\gamma^2\xi^2\hbar^6 T_1^4 x_1^2 - 336\,\mathrm{e}^{3\alpha\gamma\hbar}\gamma^2\eta\xi^3\hbar^7 T_1^4 x_1^2 + 72\,\mathrm{e}^{4\alpha\gamma\hbar}\gamma^2\eta^2\xi^4\hbar^8 T_1^4 x_1^2 -$$
$$288\,\mathrm{e}^{2\alpha\gamma\hbar}\gamma\xi^2\hbar^6 a_1 T_1^4 x_1^2 + 144\,\mathrm{e}^{3\alpha\gamma\hbar}\gamma\eta\xi^3\hbar^7 a_1 T_1^4 x_1^2 + 144\,\mathrm{e}^{2\alpha\gamma\hbar}\xi^2\hbar^6 a_1^2 T_1^4 x_1^2 -$$
$$216\,\mathrm{e}^{4\alpha\gamma\hbar}\gamma^2\eta\xi^4\hbar^8 T_1^3 x_1^3 - 144\,\mathrm{e}^{3\alpha\gamma\hbar}\gamma^2\xi^3\hbar^7 T_1^4 x_1^3 + 72\,\mathrm{e}^{4\alpha\gamma\hbar}\gamma^2\eta\xi^4\hbar^8 T_1^4 x_1^3 +$$
$$144\,\mathrm{e}^{3\alpha\gamma\hbar}\gamma\xi^3\hbar^7 a_1 T_1^4 x_1^3 + 36\,\mathrm{e}^{4\alpha\gamma\hbar}\gamma^2\xi^4\hbar^8 T_1^4 x_1^4 - 324\,\mathrm{e}^{4\alpha\gamma\hbar}\gamma^2\eta^4\xi^3\hbar^8 y_1 -$$
$$1632\,\mathrm{e}^{3\alpha\gamma\hbar}\gamma^2\eta^3\xi^2\hbar^7 T_1 y_1 + 540\,\mathrm{e}^{4\alpha\gamma\hbar}\gamma^2\eta^4\xi^3\hbar^8 T_1 y_1 + 1080\,\mathrm{e}^{3\alpha\gamma\hbar}\gamma\eta^3\xi^2\hbar^7 a_1 T_1 y_1 -$$
$$1512\,\mathrm{e}^{2\alpha\gamma\hbar}\gamma^2\eta^2\xi\hbar^6 T_1^2 y_1 + 1680\,\mathrm{e}^{3\alpha\gamma\hbar}\gamma^2\eta^3\xi^2\hbar^7 T_1^2 y_1 - 252\,\mathrm{e}^{4\alpha\gamma\hbar}\gamma^2\eta^4\xi^3\hbar^8 T_1^2 y_1 +$$
$$2160\,\mathrm{e}^{2\alpha\gamma\hbar}\gamma\eta^2\xi\hbar^6 a_1 T_1^2 y_1 - 576\,\mathrm{e}^{3\alpha\gamma\hbar}\gamma\eta^3\xi^2\hbar^7 a_1 T_1^2 y_1 - 576\,\mathrm{e}^{2\alpha\gamma\hbar}\eta^2\xi\hbar^6 a_1^2 T_1^2 y_1 -$$
$$144\,\mathrm{e}^{\alpha\gamma\hbar}\gamma^2\eta\hbar^5 T_1^3 y_1 + 648\,\mathrm{e}^{2\alpha\gamma\hbar}\gamma^2\eta^2\xi\hbar^6 T_1^3 y_1 - 336\,\mathrm{e}^{3\alpha\gamma\hbar}\gamma^2\eta^3\xi^2\hbar^7 T_1^3 y_1 +$$
$$36\,\mathrm{e}^{4\alpha\gamma\hbar}\gamma^2\eta^4\xi^3\hbar^8 T_1^3 y_1 + 288\,\mathrm{e}^{\alpha\gamma\hbar}\gamma\eta\hbar^5 a_1 T_1^3 y_1 - 432\,\mathrm{e}^{2\alpha\gamma\hbar}\gamma\eta^2\xi\hbar^6 a_1 T_1^3 y_1 +$$
$$72\,\mathrm{e}^{3\alpha\gamma\hbar}\gamma\eta^3\xi^2\hbar^7 a_1 T_1^3 y_1 - 144\,\mathrm{e}^{\alpha\gamma\hbar}\eta\hbar^5 a_1^2 T_1^3 y_1 + 864\,\mathrm{e}^{4\alpha\gamma\hbar}\gamma^2\eta^3\xi^3\hbar^8 T_1 x_1 y_1 + 2232$$
$$\mathrm{e}^{3\alpha\gamma\hbar}\gamma^2\eta^2\xi^2\hbar^7 T_1^2 x_1 y_1 - 720\,\mathrm{e}^{4\alpha\gamma\hbar}\gamma^2\eta^3\xi^3\hbar^8 T_1^2 x_1 y_1 - 1440\,\mathrm{e}^{3\alpha\gamma\hbar}\gamma\eta^2\xi^2\hbar^7 a_1 T_1^2 x_1 y_1 +$$
$$432\,\mathrm{e}^{2\alpha\gamma\hbar}\gamma^2\eta\xi\hbar^6 T_1^3 x_1 y_1 - 792\,\mathrm{e}^{3\alpha\gamma\hbar}\gamma^2\eta^2\xi^2\hbar^7 T_1^3 x_1 y_1 + 144\,\mathrm{e}^{4\alpha\gamma\hbar}\gamma^2\eta^3\xi^3\hbar^8 T_1^3 x_1 y_1 -$$
$$864\,\mathrm{e}^{2\alpha\gamma\hbar}\gamma\eta\xi\hbar^6 a_1 T_1^3 x_1 y_1 + 288\,\mathrm{e}^{3\alpha\gamma\hbar}\gamma\eta^2\xi^2\hbar^7 a_1 T_1^3 x_1 y_1 + 288\,\mathrm{e}^{2\alpha\gamma\hbar}\eta\xi\hbar^6 a_1^2 T_1^3 x_1 y_1 -$$
$$648\,\mathrm{e}^{4\alpha\gamma\hbar}\gamma^2\eta^2\xi^3\hbar^8 T_1^2 x_1^2 y_1 - 576\,\mathrm{e}^{3\alpha\gamma\hbar}\gamma^2\eta\xi^2\hbar^7 T_1^3 x_1^2 y_1 + 216\,\mathrm{e}^{4\alpha\gamma\hbar}\gamma^2\eta^2\xi^3\hbar^8 T_1^3 x_1^2 y_1 +$$
$$432\,\mathrm{e}^{3\alpha\gamma\hbar}\gamma\eta\xi^2\hbar^7 a_1 T_1^3 x_1^2 y_1 + 144\,\mathrm{e}^{4\alpha\gamma\hbar}\gamma^2\eta\xi^3\hbar^8 T_1^3 x_1^3 y_1 + 432\,\mathrm{e}^{4\alpha\gamma\hbar}\gamma^2\eta^4\xi^2\hbar^8 y_1^2 +$$
$$1344\,\mathrm{e}^{3\alpha\gamma\hbar}\gamma^2\eta^3\xi\hbar^7 T_1 y_1^2 - 360\,\mathrm{e}^{4\alpha\gamma\hbar}\gamma^2\eta^4\xi^2\hbar^8 T_1 y_1^2 - 720\,\mathrm{e}^{3\alpha\gamma\hbar}\gamma\eta^3\xi\hbar^7 a_1 T_1 y_1^2 +$$
$$504\,\mathrm{e}^{2\alpha\gamma\hbar}\gamma^2\eta^2\hbar^6 T_1^2 y_1^2 - 480\,\mathrm{e}^{3\alpha\gamma\hbar}\gamma^2\eta^3\xi\hbar^7 T_1^2 y_1^2 + 72\,\mathrm{e}^{4\alpha\gamma\hbar}\gamma^2\eta^4\xi^2\hbar^8 T_1^2 y_1^2 -$$
$$576\,\mathrm{e}^{2\alpha\gamma\hbar}\gamma\eta^2\hbar^6 a_1 T_1^2 y_1^2 + 144\,\mathrm{e}^{3\alpha\gamma\hbar}\gamma\eta^3\xi\hbar^7 a_1 T_1^2 y_1^2 + 144\,\mathrm{e}^{2\alpha\gamma\hbar}\eta^2\hbar^6 a_1^2 T_1^2 y_1^2 -$$
$$648\,\mathrm{e}^{4\alpha\gamma\hbar}\gamma^2\eta^3\xi^2\hbar^8 T_1 x_1 y_1^2 - 720\,\mathrm{e}^{3\alpha\gamma\hbar}\gamma^2\eta^2\xi\hbar^7 T_1^2 x_1 y_1^2 + 216\,\mathrm{e}^{4\alpha\gamma\hbar}\gamma^2\eta^3\xi^2\hbar^8 T_1^2 x_1 y_1^2 +$$
$$432\,\mathrm{e}^{3\alpha\gamma\hbar}\gamma\eta^2\xi\hbar^7 a_1 T_1^2 x_1 y_1^2 + 216\,\mathrm{e}^{4\alpha\gamma\hbar}\gamma^2\eta^2\xi^2\hbar^8 T_1^2 x_1^2 y_1^2 - 216\,\mathrm{e}^{4\alpha\gamma\hbar}\gamma^2\eta^4\xi\hbar^8 y_1^3 -$$
$$288\,\mathrm{e}^{3\alpha\gamma\hbar}\gamma^2\eta^3\hbar^7 T_1 y_1^3 + 72\,\mathrm{e}^{4\alpha\gamma\hbar}\gamma^2\eta^4\xi\hbar^8 T_1 y_1^3 + 144\,\mathrm{e}^{3\alpha\gamma\hbar}\gamma\eta^3\hbar^7 a_1 T_1 y_1^3 +$$
$$144\,\mathrm{e}^{4\alpha\gamma\hbar}\gamma^2\eta^3\xi\hbar^8 T_1 x_1 y_1^3 + 36\,\mathrm{e}^{4\alpha\gamma\hbar}\gamma^2\eta^4\hbar^8 y_1^4\Big)\,\epsilon^2 + O[\epsilon]^3],\ 0\Big\}\Big\}$$

Alternative Algorithms

AltLogos

In[ ]:=

```
λ_{alt,k_}[CU] := If[k == 0, 1, Module[{eq, d, b, c, so},
    eq = ρ@𝕖^(ξ x_CU).ρ@𝕖^(η y_CU) == ρ@𝕖^(d y_CU).ρ@𝕖^(c (t 1_CU - 2 ϵ a_CU)).ρ@𝕖^(b x_CU);
    {so} = Solve[Thread[Flatten /@ eq], {d, b, c}] /. C@1 → 0;
    Series[𝕖^(-η y - ξ x + η ξ t + c t + d y - 2 ϵ c a + b x) /. so, {ϵ, 0, k}]]];
```

```
{λ_{alt,2}[CU], HL@Simplify@Normal[λ_{alt,2}[CU] == Last[Λ_{CU,2}[{ξ, η}, {x, y}]]]}
```

$$\left\{1+\left(2\,a\,\eta\,\xi-y\,\gamma\,\eta^2\,\xi-x\,\gamma\,\eta\,\xi^2+\frac{1}{2}\,t\,\gamma\,\eta^2\,\xi^2\right)\epsilon+\frac{1}{2}\left(\left(2\,a\,\eta\,\xi-y\,\gamma\,\eta^2\,\xi-x\,\gamma\,\eta\,\xi^2+\frac{1}{2}\,t\,\gamma\,\eta^2\,\xi^2\right)^2+2\left(-a\,\gamma\,\eta^2\,\xi^2+y\,\gamma^2\,\eta^3\,\xi^2+x\,\gamma^2\,\eta^2\,\xi^3-\frac{1}{3}\,t\,\gamma^2\,\eta^3\,\xi^3\right)\right)\epsilon^2+O[\epsilon]^3,\ \text{True}\right\}$$