Pensieve header: A unified verification notebook for the sl_2-portfolio project, Uxi version. Continues pensieve://Projects/PPSA/nb/Verification.pdf.

Continues pensieve://2017-06/ and pensieve://2017-08/.

Prolog

```
wdir = SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\SL2Portfolio"];
NotebookOpen[wdir <> "\\MakeSnips.nb"];
```

In[∘]:=
```
HL[ℰ_] := Style[ℰ, Background → Yellow];
```

DocileQ

DocileQ

In[∘]:=
```
DQ[ℰ_] := (Exponent[Normal@ℰ /.
     {a → a / ϵ, a_i_ → a_i / ϵ, (u : x | y) :→ ϵ^(-1/2) u, (u : x | y)_i_ :→ ϵ^(-1/2) u_i}, ϵ, Min] ≥ 0);
```

In[∘]:=
```
DQ /@ {ϵ^2 x y a_2, ϵ^2 x^2 y^3}
```

Out[∘]=
```
{True, False}
```

Initialization / Utilities

It is verification-risky to work with low $E!

TD

In[∘]:=
```
$p = 2; $k = 1; $U = QU; $E := {$k, $p};
$trim := {ħ^(p_·) /; p > $p → 0, ϵ^(k_·) /; k > $k → 0};
SetAttributes[{SS, SST}, HoldAll];
T2t = {T_i_ → 𝕖^(ħ t_i), T → 𝕖^(ħ t)}; q_ħ = 𝕖^(γ ϵ ħ);
t2T = {𝕖^(c_· t_i_+b_·) :→ T_i^(c/ħ) 𝕖^b, 𝕖^(c_· t+b_·) :→ T^(c/ħ) 𝕖^b, 𝕖^ℰ_ :→ 𝕖^(Expand@ℰ)};
SS[ℰ_, op_] := Collect[
   Normal@Series[If[$p > 0, ℰ, ℰ /. T2t], {ħ, 0, $p}],
   ħ, op];
SS[ℰ_] := SS[ℰ, Together];
SST[ℰ_, op___] := SS[ℰ /. T2t, op];
Simp[ℰ_, op_] := Collect[ℰ, _CU | _QU, op];
Simp[ℰ_] := Simp[ℰ, SS[#, Expand] &];
SimpT[ℰ_] := Collect[ℰ, _CU | _QU, SST[#, Expand] &];
Kδ /: Kδ_(i_,j_) := If[i === j, 1, 0];
```

Differential polynomials (DP):

Utils

In[∘]:=
```
DP_(α_→D_x_, β_→D_y_)[P_][λ_] :=
 Total[CoefficientRules[Normal@P, {α, β}] /. ({m_, n_} → c_) :→ c ∂_({x,m},{y,n}) λ]
```

```
HL[DP_{x→D_ξ,y→D_η}[x^2 y^3][e^(δ ξ η)] == 6 e^(δ η ξ) δ^3 ξ + 6 e^(δ η ξ) δ^4 η ξ^2 + e^(δ η ξ) δ^5 η^2 ξ^3]
True
```

CF

```
In[∘]:= CF[ℰ_] := ExpandDenominator@
          ExpandNumerator@Together[Expand[ℰ] //. e^x_ e^y_ ⧴ e^(x+y) /. e^x_ ⧴ e^(CF[x])];
```

SeriesData

```
In[∘]:= Unprotect[SeriesData];
        SeriesData /: CF[sd_SeriesData] := MapAt[CF, sd, 3];
        SeriesData /: Expand[sd_SeriesData] := MapAt[Expand, sd, 3];
        SeriesData /: Simplify[sd_SeriesData] := MapAt[Simplify, sd, 3];
        SeriesData /: Together[sd_SeriesData] := MapAt[Together, sd, 3];
        SeriesData /: Collect[sd_SeriesData, specs__] := MapAt[Collect[#, specs] &, sd, 3];
        Protect[SeriesData];
```

Self-Pair (SP):

SP

```
In[∘]:= SP_{}[P_] := P; SP_{ξ_→x_,ps___}[P_] := Expand[P // SP_{ps}] /. f_. ξ^d_. ⧴ ∂_{x,d} f
```

```
SP_{ξ→x}[(ξ^2 + ξ + 3) (x^5 e^x + 7 x) + 99 a]
7 + 99 a + 21 x + 20 e^x x^3 + 15 e^x x^4 + 5 e^x x^5

SP_{ξ→x,η→y}[(ξ^2 + ξ + 3 + 2 ξ η) (x^5 e^x + 7 x) + 99 a + e^(δ x y) ξ η]
7 + 99 a + 21 x + 20 e^x x^3 + 15 e^x x^4 + 5 e^x x^5 + e^(x y δ) δ + e^(x y δ) x y δ^2
```

DeclareAlgebra

QLImplementation

```
In[∘]:= Unprotect[NonCommutativeMultiply]; Attributes[NonCommutativeMultiply] = {};
        (NCM = NonCommutativeMultiply)[x_] := x;
        NCM[x_, y_, z__] := (x ** y) ** z;
        0 ** _ = _ ** 0 = 0;
        (x_Plus) ** y_ := (# ** y) & /@ x; x_ ** (y_Plus) := (x ** #) & /@ y;
        B[x_, x_] = 0; B[x_, y_] := x ** y - y ** x;
        B[x_, y_, e_] := B[x, y, e] = B[x, y];
```

QLImplementation

In[ ]:=

```
DeclareAlgebra[U_Symbol, opts__Rule] := Module[{gp, sr, g, cp, M, CE, pow, k = 0,
    gs = Generators /. {opts},
    cs = Centrals /. {opts} /. Centrals → {} },
   (#_{U} = U@#) & /@ gs;
   gp = Alternatives @@ gs; gp = gp | gp_; (* gens *)
   sr = Flatten@Table[{g → ++k, g_{i_} → {i, k}}, {g, gs}]; (* sorting → *)
   cp = Alternatives @@ cs; (* cents *)
   SetAttributes[M, HoldRest]; M[0, _] = 0; M[a_, x_] := a x;
   CE[𝓔_] := Collect[𝓔, _U, Expand] /. $trim;
   U_{i_} [𝓔_] := 𝓔 /. {t : cp ⧴ t_{i}, u_U ⧴ (#_{i} &) /@ u};
   U_{i_} [NCM[]] = pow[𝓔_, 0] = U@{} = 1_{U} = U[];
   B[U@(x_)_{i_}, U@(y_)_{i_}] := U_{i}@B[U@x, U@y];
   B[U@(x_)_{i_}, U@(y_)_{j_}] /; i =!= j := 0;
   B[U@y_, U@x_] := CE[-B[U@x, U@y]];
   x_ ** (c_. 1_{U}) := CE[c x]; (c_. 1_{U}) ** x_ := CE[c x];
   (a_. U[xx___, x_]) ** (b_. U[y_, yy___]) := If[OrderedQ[{x, y} /. sr],
     CE@M[a b /. $trim, U[xx, x, y, yy]],
     U@xx ** CE@M[a b /. $trim, U@y ** U@x + B[U@x, U@y, $E]] ** U@yy ];
   U@{c_. * (l : gp)^{n_}, r___} /; FreeQ[c, gp] := CE[c U@Table[l, {n}] ** U@{r}];
   U@{c_. * l : gp, r___} := CE[c U[l] ** U@{r}];
   U@{c_, r___} /; FreeQ[c, gp] := CE[c U@{r}];
   U@{l_Plus, r___} := CE[U@{#, r} & /@ l];
   U@{l_, r___} := U@{Expand[l], r};
   U[𝓔_NonCommutativeMultiply] := U /@ 𝓔;
   𝕆_{U}[specs___, poly_] := Module[{sp, null, vs, us},
     sp = Replace[{specs}, l_List ⧴ l_{null}, {1}];
     vs = Join @@ (First /@ sp);
     us = Join @@ (sp /. l_{s_} ⧴ (l /. x_{i_} ⧴ x_{s}));
     CE[Total[
        CoefficientRules[poly, vs] /. (p_ → c_) ⧴ c U@(us^{p})
       ]] /. x_{null} ⧴ x];
   pow[𝓔_, n_] := pow[𝓔, n - 1] ** 𝓔;
   𝕊_{U}[𝓔_, ss___Rule] := CE@Total[
      CoefficientRules[𝓔, First /@ {ss}] /.
       (p_ → c_) ⧴ c NCM @@ MapThread[pow, {Last /@ {ss}, p}]];
   σ_{rs___} [c_. * u_U] := (c /. (t : cp)_{j_} ⧴ t_{j /. {rs}}) U[List @@ (u /. v_{j_} ⧴ v_{j /. {rs}})];
   m_{j_→k_} [c_. * u_U] := CE[((c /. (t : cp)_{j} → t_{k}) DeleteCases[u, _{j|k}]) **
      U @@ Cases[u, w_{j} ⧴ w_{k}] ** U @@ Cases[u, _{k}]];
   U /: c_. * u_U * v_U := CE[c u ** v];
   S_{i_} [c_. * u_U] := CE[((c /. S_{i}[U, Centrals]) DeleteCases[u, _{i}]) **
      U_{i}[NCM @@ Reverse@Cases[u, x_{i} ⧴ S@U@x]]];
   Δ_{i_→j_,k_} [c_. * u_U] := CE[((c /. Δ_{i→j,k}[U, Centrals]) DeleteCases[u, _{i}]) **
      (NCM @@ Cases[u, x_{i} ⧴ σ_{1→j,2→k}@Δ@U@x] /. NCM[] → U[])]; ]
```

DeclareMorphism

QLImplementation

```
In[∘]:= DeclareMorphism[m_, U_ → V_, ongs_List, oncs_List: {}] := (
  Replace[ongs,
   {(g_ → img_) :→ (m[U[g]] = img), (g_ :→ img_) :→ (m[U[g]] := img /. $trim)}, {1}];
  m[1_U] = 1_V;
  m[U[g_i_]] := V_i[m[U@g]];
  m[U[vs__]] := NCM @@ (m /@ U /@ {vs});
  m[ℰ_] := Simp[ℰ /. oncs /.  u_U :→ m[u]] /. $trim; )
```

Meta-Operations

QLImplementation

```
In[∘]:= σ_rs___[ℰ_Plus] := σ_rs /@ ℰ;
m_j_→j_ = Identity; m_j_→k_[0] = 0;
m_j_→k_[ℰ_Plus] := Simp[m_j→k /@ ℰ];
m_is___,i_,j_→k_[ℰ_] := m_j→k@m_is,i→j@ℰ;
S_i_[ℰ_Plus] := Simp[S_i /@ ℰ];
Δ_is___[ℰ_Plus] := Simp[Δ_is /@ ℰ];
```

Implementing CU = $\mathcal{U}(sl_2^{\gamma\epsilon})$

Verify σ and Δ! Also Generalize Δ to $\Delta_{i,j_1,j_2,\ldots}$.

CU

```
In[∘]:= DeclareAlgebra[CU, Generators → {y, a, x}, Centrals → {t}];
B[a_CU, y_CU] = -γ y_CU; B[x_CU, a_CU] = -γ x_CU;
B[x_CU, y_CU] = 2 ϵ a_CU - t 1_CU;
(S@y_CU = -y_CU; S@a_CU = -a_CU; S@x_CU = -x_CU;)
S_i_[CU, Centrals] = {t_i → -t_i};
Δ@y_CU = CU@y_1 + CU@y_2; Δ@a_CU = CU@a_1 + CU@a_2; Δ@x_CU = CU@x_1 + CU@x_2;
Δ_i_→j_,k_[CU, Centrals] = {t_i → t_j + t_k};
```

Verifying associativity on triples of generators:

```
With[{bas = CU /@ {y, a, x}},
 Table[z1 ** (z2 ** z3) - (z1 ** z2) ** z3 // Simp // HL,
  {z1, bas}, {z2, bas}, {z3, bas} ] ]
```

```
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying associativity on a “random” triple:

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
   (rhs = (z1 ** z2) ** z3 // Simp) // Short,
   z1 ** (z2 ** z3) - rhs // Simp // HL
  }] // Timing
```

{1.32813,
 {(28 t^2 γ^4 + 116 t γ^5 ϵ) CU[y, y, y, x, x] + ≪21≫ + CU[y, y, y, y, y, a, a, a, a, x, x, x, x], 0}}

Implementing QU = $\mathcal{U}_q(sl_2^{\gamma\epsilon})$

Aside

```
Series[(1 - T e^(-2 ϵ a ħ)) / ħ, {a, 0, 3}]
```

Aside

$$\frac{1-T}{\hbar} + 2 T \epsilon a - 2 (T \epsilon^2 \hbar) a^2 + \frac{4}{3} T \epsilon^3 \hbar^2 a^3 + O[a]^4$$

In[◦]:=
```
HL /@ DQ /@ Series[{(1 - T e^(-2 ϵ a ħ)) / ħ, e^(ħ ϵ a)}, {ϵ, 0, 5}]
```

Out[◦]= {True, True}

QU

In[◦]:=
```
DeclareAlgebra[QU, Generators → {y, a, x}, Centrals → {t, T}];
B[a_QU, y_QU] = -γ y_QU; B[x_QU, a_QU] = -γ QU@x;
B[x_QU, y_QU] := SS[q_ħ - 1] QU@{y, x} + O_QU[{a}, SS[(1 - T e^(-2 ϵ a ħ)) / ħ]];
(S@y_QU := O_QU[{a, y}, SS[-T^-1 e^(ħ ϵ a) y]]; S@a_QU = -a_QU; S@x_QU := O_QU[{a, x}, SS[-e^(ħ ϵ a) x]];)
S_i_[QU, Centrals] = {t_i → -t_i, T_i → T_i^-1};
Δ@y_QU := O_QU[{y_1, a_1}_1, {y_2}_2, SS[y_1 + T_1 e^(-ħ ϵ a_1) y_2]];
Δ@a_QU = QU@a_1 + QU@a_2; Δ@x_QU := O_QU[{a_1, x_1}_1, {x_2}_2, SS[x_1 + e^(-ħ ϵ a_1) x_2]];
Δ_(i_→j_,k_)[QU, Centrals] = {t_i → t_j + t_k, T_i → T_j T_k};
```

```
With[{bas = QU /@ {y, a, x}}, Table[{z1, z2} → Simp[z1 ** z2 - z2 ** z1], {z1, bas}, {z2, bas}] ]
```

$\{\{\{QU[y], QU[y]\} \to 0,\ \{QU[y], QU[a]\} \to \gamma\, QU[y],$
$\ \{QU[y], QU[x]\} \to \frac{(-1+T)\, QU[]}{\hbar} - 2 T \epsilon\, QU[a] - \gamma \epsilon \hbar\, QU[y, x]\},$
$\{\{QU[a], QU[y]\} \to -\gamma\, QU[y],\ \{QU[a], QU[a]\} \to 0,\ \{QU[a], QU[x]\} \to \gamma\, QU[x]\},$
$\{\{QU[x], QU[y]\} \to \frac{(1-T)\, QU[]}{\hbar} + 2 T \epsilon\, QU[a] + \gamma \epsilon \hbar\, QU[y, x],$
$\ \{QU[x], QU[a]\} \to -\gamma\, QU[x],\ \{QU[x], QU[x]\} \to 0\}\}$

Verifying associativity on triples of generators:

```
With[{bas = QU /@ {y, a, x}},
 Table[HL[z1 ** (z2 ** z3) - (z1 ** z2) ** z3] // Simp,
  {z1, bas}, {z2, bas}, {z3, bas}] ]
```

{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}

Verifying associativity on a “random” triple (~34 secs @ $p=5, $k=2):

```
With[{z1 = QU[y, y, a, a, x, x], z2 = QU[y, a, x], z3 = QU[y, y, a, x]}, {
  (rhs = (z1 ** z2) ** z3 // Simp) // Short,
  HL[z1 ** (z2 ** z3) - rhs // Simp]
 }] // Timing
```

$\{3.78125, \{\left(\frac{28\gamma^4 - 56 T\gamma^4 + 28 T^2\gamma^4}{\hbar^2} + \frac{82\gamma^5\epsilon - 280 T \ll 1 \gg \epsilon + 198 T^2\gamma^5\epsilon}{\hbar}\right)$ QU[y, y, y, x, x] + $\ll 18 \gg + (1 + 8\gamma\epsilon\hbar)$ QU[y, $\ll 11 \gg$, x], 0}}

Verifying that $\text{Lim}_{\hbar\to 0}$ QU = CU using a "random" product (~23 secs @ $p=5, $k=2):

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
  Short[lhs = z1 ** (z2 ** z3)],
  Short[rhs = (QU @@ z1) ** ((QU @@ z2) ** (QU @@ z3))],
  Expand[Limit[rhs /. T2t ⋃ {QU → CU}, ℏ → 0] - lhs] // HL
 }] // Timing
```

$\{10.125, \{28 t^2\gamma^4$ CU[y, y, y, x, x] + $116 t\gamma^5\epsilon$ CU[y, y, y, x, x] + $\ll 44 \gg$ + CU[y, y, y, y, y, a, a, a, a, x, x, x, x],
$2\left(\frac{\gamma^4}{\hbar^2} - \frac{2T\gamma^4}{\hbar^2} + \frac{T^2\gamma^4}{\hbar^2} + \frac{\gamma^5\epsilon}{\hbar} - \frac{2T\gamma^5\epsilon}{\hbar} + \frac{T^2\gamma^5\epsilon}{\hbar}\right)$ QU[y, y, y, x, x] +
$\ll 209 \gg + (1 + 8\gamma\epsilon\hbar)$ QU[y, y, y, $\ll 7 \gg$, x, x, x], 0}}

Verifying σ, m, S, and Δ.

Verifying $\sigma_{i\to j,k\to l}$:

In[∘]:= $\text{CU@}x_1 + \text{CU@}x_2$ // $\sigma_{1\to 3,2\to 4}$

Out[∘]= $\text{CU}[x_3] + \text{CU}[x_4]$

Verifying relabeling using m:

In[∘]:= $t_1 t_3 \text{CU}[y_1, a_1, x_2] + t_1 t_1 \text{CU}[y_1, a_2, x_2]$ // $m_{1\to 3}$

Out[∘]= $\text{CU}[a_2, x_2, y_3]\, t_3^2 + \text{CU}[x_2, y_3, a_3]\, t_3^2$

Verifying the meta-associativity of m:

In[∘]:=
```
Module[{z, u},
 Table[u = CU[z⟦1⟧₁, z⟦2⟧₂, z⟦3⟧₃]; HL[m₁,₃→₃@m₂,₃→₃@u == m₂,₃→₃@m₁,₂→₂@u],
  {z, Tuples[{y, a, x}, 3]}, {U, {CU, QU}}]]
```

Out[∘]= {{True, True}, {True, True}, {True, True}, {True, True}, {True, True},
{True, True}, {True, True}, {True, True}, {True, True}, {True, True},
{True, True}, {True, True}, {True, True}, {True, True}, {True, True}, {True, True},
{True, True}, {True, True}, {True, True}, {True, True}, {True, True}, {True, True},
{True, True}, {True, True}, {True, True}, {True, True}, {True, True}}

Verifying the involutivity of S on CU on products of triples:

```
In[∘]:= With[{bas = CU /@ {y₁, a₁, x₁}},
          Table[HL@Simp[z1 ** z2 ** z3 - S₁@S₁[z1 ** z2 ** z3]],
           {z1, bas}, {z2, bas}, {z3, bas} ] ]
Out[∘]= {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
         {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying that S is an anti-homomorphism on CU/QU:

```
In[∘]:= With[{bas = U /@ {y₁, a₁, x₁}},
          Table[HL@Simp[S₁[z1 ** z2] - S₁[z2] ** S₁[z1]],
           {z1, bas}, {z2, bas} , {U, {CU, QU}}] ]
Out[∘]= {{{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}}
```

Verifying the co-associativity of Δ:

```
In[∘]:= Block[{bas = U /@ {y₁, a₁, x₁}},
          Table[(z1 ** z2 ** z3 // Δ1→1,2 // Δ2→2,3) - (z1 ** z2 ** z3 // Δ1→1,3 // Δ1→1,2) // Simp // HL,
           {z1, bas}, {z2, bas}, {z3, bas} , {U, {CU, QU}}] ]
Out[∘]= {{{{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}},
         {{{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}},
         {{{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}}}
```

Verifying S-Δ compatibility:

```
In[∘]:= Block[{bas = U /@ {y₁, a₁, x₁}},
          Table[z1 ** z2 ** z3 // Δ1→1,2 // Si // m1,2→1 // Simp // HL,
           {U, {CU, QU}}, {i, 2}, {z1, bas}, {z2, bas}, {z3, bas}] ]
Out[∘]= {{{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
            {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}, {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
            {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}},
          {{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
            {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}, {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
            {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}}}
```

Verifying S-Δ compatibility for opposite m, only for CU:

```
In[∘]:= Block[{bas = CU /@ {y₁, a₁, x₁}},
          Table[z1 ** z2 ** z3 // Δ1→1,2 // Si // m2,1→1 // Simp // HL,
           {i, 2}, {z1, bas}, {z2, bas}, {z3, bas} ]]
Out[∘]= {{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
           {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}, {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
           {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}}
```

Verifying m-Δ compatibility:

```
In[∘]:= Block[{bas1 = U /@ {y1, a1, x1}, bas2 = U /@ {y2, a2, x2}},
          Table[(z1 ** z2 ** z3 ** z4 // m1,2→1 // Δ1→1,2) -
              (z1 ** z2 ** z3 ** z4 // Δ1→3,4 // Δ2→5,6 // m3,5→1 // m4,6→2) // Simp // HL,
           {U, {CU, QU}}, {z1, bas1}, {z2, bas1}, {z3, bas2}, {z4, bas2}] ]

Out[∘]= {{{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
          {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}},
         {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
          {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}, {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
          {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}},
        {{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
          {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}, {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
          {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}},
         {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
          {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}}}
```

Implementing θ

theta

```
In[∘]:= DeclareMorphism[Cθ, CU → CU, {y → -x_CU, a → -a_CU, x → -y_CU}, {t → -t, T → T^-1}];
        DeclareMorphism[Qθ, QU → QU, {y :→ 𝕆_QU[{a, x}, SS[-T^(-1/2) e^(ħ ϵ a) x]],
          a → -a_QU, x :→ 𝕆_QU[{a, y}, SS[-T^(-1/2) e^(ħ ϵ a) y]]}, {t → -t, T → T^-1}]
```

Verifying involutivity on CU:

```
With[{bas = CU /@ {y, a, x}},
 Table[z → Cθ[z] → HL[Cθ[Cθ[z]]], {z, bas}] ]
```

{CU[y] → -CU[x] → CU[y], CU[a] → -CU[a] → CU[a], CU[x] → -CU[y] → CU[x]}

Verifying that θ is a multiplicative homomorphism on CU:

```
With[{bas = CU /@ {y, a, x}},
 Table[Cθ[z1 ** z2] - Cθ[z1] ** Cθ[z2] // HL, {z1, bas}, {z2, bas}] ]
```

{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

Verifying involutivity on QU:

```
With[{bas = QU /@ {y, a, x}},
 Table[z → Qθ[z] → HL[Simp[Qθ[Qθ[z]], PowerExpand]], {z, bas}] ]
```

$\{QU[y] \to -\frac{QU[x]}{\sqrt{T}} - \frac{\epsilon\hbar\, QU[a, x]}{\sqrt{T}} \to QU[y],\ QU[a] \to -QU[a] \to QU[a],$

$QU[x] \to \left(-\frac{1}{\sqrt{T}} + \frac{\gamma\epsilon\hbar}{\sqrt{T}}\right) QU[y] - \frac{\epsilon\hbar\, QU[y, a]}{\sqrt{T}} \to QU[x]\}$

Verifying that θ is a multiplicative homomorphism on QU:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL[Simp[Qθ[z1 ** z2] - Qθ[z1] ** Qθ[z2]]], {z1, bas}, {z2, bas}]]
```

```
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

The Asymmetric Dequantizator

Following pensieve://People/VanDerVeen/Dequant1.pdf.

ADeq

In[∘]:=
$$A\mathbb{D}\$f = \gamma \left(\left(\text{Cosh}\left[\hbar \left(a\,\epsilon + \frac{\gamma\,\epsilon}{2} - \frac{t}{2} \right) \right] - \text{Cosh}\left[\hbar \sqrt{\left(\frac{t - \gamma\,\epsilon}{2} \right)^2 + \epsilon\,\omega} \right] \right) \Big/ \left(\hbar\, e^{\hbar\,((a+\gamma)\,\epsilon - t/2)}\, \text{Sinh}\left[\frac{\gamma\,\epsilon\,\hbar}{2} \right] \left(a^2\,\epsilon + a\,\gamma\,\epsilon - a\,t - \omega \right) \right) \right);$$

Docility of A𝔻$f:

In[∘]:= `HL@DQ@Block[{$p = 4}, Collect[SS@A𝔻$f /. ω → a₁, ϵ]]`

Out[∘]= True

Scaling behaviour of A𝔻$f:

$$\text{HL@Simplify}\left[A\mathbb{D}\$f == \left(\left(A\mathbb{D}\$f\ /.\ \gamma \to 1 \right)\ /.\ \left\{ \epsilon \to \gamma\,\epsilon,\ a \to \gamma^{-1}\,a,\ \omega \to \gamma^{-1}\,\omega \right\} \right) \right]$$

True

$$\text{HL@FullSimplify}\left[A\mathbb{D}\$f == \left(\left(A\mathbb{D}\$f\ /.\ \gamma \to 1 \right)\ /.\ \left\{ \hbar \to \gamma^2\,\hbar,\ \epsilon \to \epsilon/\gamma,\ a \to a/\gamma,\ t \to \gamma^{-2}\,t,\ \omega \to \gamma^{-3}\,\omega \right\} \right) \right]$$

True

ADeq

In[∘]:=
$$A\mathbb{D}\$\omega = \gamma\,\text{CU}[y, x] + \epsilon\,\text{CU}[a, a] - (t - \gamma\,\epsilon)\,\text{CU}[a];$$

ADeq

In[∘]:=
```
DeclareMorphism[A𝔻, QU → CU,
 {a → a_CU, x → CU@x, y :→ 𝕊_CU[SS[A𝔻$f], a → a_CU, ω → A𝔻$ω] ** y_CU}]
```

Verifying that the asymmetric dequantizator is a homomorphism:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL[SimpT[A𝔻[z1 ** z2] - A𝔻[z1] ** A𝔻[z2]]], {z1, bas}, {z2, bas}]]
```

```
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

The Symmetric Dequantizator

Following pensieve://People/VanDerVeen/Dequant1.pdf.

SDeq

In[∘]:=
```
SD$g = Sqrt[((2 γ (Cosh[ħ/2 Sqrt[t^2 + γ^2 ϵ^2 + 4 ϵ ϖ]] - Cosh[(t - ϵ γ - 2 ϵ a)/(2/ħ)]))/
    (Sinh[γ ϵ ħ/2] (t (2 a + γ) - 2 a (a + γ) ϵ + 2 ϖ) ħ))];
```

SDeq

In[∘]:=
```
SD$f = Simplify[e^(ħ (t/2-ϵ a)) (SD$g /. {a → -a, t → -t})];
```

Verify agreement with the formulas in pensieve://People/VanDerVeen/Dequant1.pdf:

In[∘]:=
```
{SD$P = (Cosh[ħ ((ϵ-t)/2 + ϵ a)] - Cosh[ħ Sqrt[(t^2+ϵ^2)/4 + ϵ ϖ]])/
   (ħ Sinh[-ϵ ħ/2] (ϖ - ϵ a^2 + (t - ϵ) a + t/2)),
 Simplify[SD$P == (SD$P /. {a → -a - 1, t → -t})] // HL,
 PowerExpand@Simplify[(SD$P /. {ħ → γ^2 ħ, ϵ → ϵ/γ, a → a/γ, t → γ^-2 t, ϖ → γ^-3 ϖ}) ==
    SD$g (SD$g /. {a → -a - γ, t → -t})] // HL,
 SD$Q = Simplify[SD$P /. {a → c - 1/2}],
 Simplify[SD$Q == (SD$Q /. {c → -c, t → -t})] // HL,
 FullSimplify[SD$g == FullSimplify[
    Sqrt[SD$Q] /. c → a + 1/2 /. {ħ → γ^2 ħ, ϵ → ϵ/γ, a → a/γ, t → γ^-2 t, ϖ → γ^-3 ϖ}]] // HL,
 HL@DQ@Block[{$p = 4}, Collect[SS@SD$g /. ω → a1, ϵ]],
 HL@DQ@Block[{$p = 4}, Collect[SS@SD$f /. ω → a1, ϵ]]
}
```

Out[∘]=

$$\left\{-\left(\left(\left(\mathrm{Cosh}\left[\left(a\,\epsilon+\frac{1}{2}(-t+\epsilon)\right)\hbar\right]-\mathrm{Cosh}\left[\sqrt{\frac{1}{4}(t^2+\epsilon^2)+\epsilon\,\varpi}\;\hbar\right]\right)\mathrm{Csch}\left[\frac{\epsilon\,\hbar}{2}\right]\right)\Big/\left(\left(\frac{t}{2}+a\,(t-\epsilon)-a^2\epsilon+\varpi\right)\hbar\right)\right),\ \mathrm{True},\ \mathrm{True},\right.$$

$$\left.-\left(\left(4\left(\mathrm{Cosh}\left[\frac{1}{2}(t-2\,c\,\epsilon)\,\hbar\right]-\mathrm{Cosh}\left[\frac{1}{2}\sqrt{t^2+\epsilon^2+4\,\epsilon\,\varpi}\;\hbar\right]\right)\mathrm{Csch}\left[\frac{\epsilon\,\hbar}{2}\right]\right)\Big/\left(\left(4\,c\,t+\epsilon-4\,c^2\epsilon+4\,\varpi\right)\hbar\right)\right),\ \mathrm{True},\ \mathrm{True},\ \mathrm{True},\ \mathrm{True}\right\}$$

SDeq

In[∘]:=
```
SD$ϖ = γ CU[y, x] + ϵ CU[a, a] - (t - γ ϵ) CU[a] - t γ 1_CU/2;
```

SDeq

In[∘]:=
```
DeclareMorphism[SD, QU → CU, {a → a_CU,
   x :→ S_CU[SS[SD$f], a → a_CU, ϖ → SD$ϖ] ** x_CU,
   y :→ S_CU[SS[SD$g], a → a_CU, ϖ → SD$ϖ] ** y_CU }]
```

Verifying the θ-symmetry:

```
Table[HL@SimpT[Cθ[SD[z]] == SD[Qθ[z]]], {z, QU /@ {y, a, x}}]
{True, True, True}
```

Verifying that the symmetric dequantizator is a homomorphism:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL@SimpT[SD[z1 ** z2] - SD[z1] ** SD[z2]], {z1, bas}, {z2, bas}] ]
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

The representation ρ

rho

In[∘]:=
```
ρ@y_CU = ρ@y_QU = (0 0; ϵ 0); ρ@a_CU = ρ@a_QU = (γ 0; 0 0);
ρ@x_CU = (0 γ; 0 0); ρ@x_QU = (0 (1 - e^(-γ ϵ ħ)) / (ϵ ħ); 0 0);
ρ[e^𝓔_] := MatrixExp[ρ[𝓔]];
ρ[𝓔_] := (𝓔 /. T2t /. t → γ ϵ /. (U : CU | QU)[u___] :→ Fold[Dot, (1 0; 0 1), ρ /@ U /@ {u}])
```

Verifying that ρ represents CU and QU:

```
Table[HL[SS[ρ[z1 ** z2] == ρ[z1].ρ[z2]] /. ϵ^k_· /; k > $k → 0],
 {U, {CU, QU}}, {z1, U /@ {y, a, x}}, {z2, U /@ {y, a, x}} ]
{{{True, True, True}, {True, True, True}, {True, True, True}},
 {{True, True, True}, {True, True, True}, {True, True, True}}}
```

Commuting $e^{\alpha a}$ with $e^{\xi x}$:

```
Table[HL[ρ[e^(ξ U@x)].ρ[e^(α U@a)] == ρ[e^(α U@a)].ρ[e^(e^(-γ α) ξ U@x)]], {U, {CU, QU}}]
{True, True}
```

Œ and the logoi Λ

Logoi from Pensieve://Talks/Toulouse-1705/DogmaDemo.nb and from Pensieve://Talks/Sydney-1708/ExtraDetails@@.nb.

MultiplyingOEs

In[∘]:=
```
Œ_U_[s1__, Q1_, P1_] Œ_U_[s2__, Q2_, P2_] ^:= Œ_U[s1, s2, Q1 + Q2, P1 P2];
```

CdsO

```
In[∘]:= CU@𝔼CU[specs___, Q_, P_] := 𝕆CU[specs, SS[e^Q P]];
        QU@𝔼QU[specs___, Q_, P_] := 𝕆QU[specs, SS[e^Q P]];
```

Logos

```
In[∘]:= c_Integer_(k_Integer) := c + O[ϵ]^(k+1);
        Λ_(U_,k_)[{α_, β_}, {x_, x_}] := 𝔼_U[{x}, (α + β) x, 1_k];
        Λ_(U_,k_)[{ξ_, α_}, {x, a}] := 𝔼_U[{a, x}, α a + e^(-γ α) ξ x, 1_k];
        Λ_(U_,k_)[{α_, η_}, {a, y}] := 𝔼_U[{y, a}, α a + e^(-γ α) η y, 1_k];
```

```
Table[
  {Λ_(U,1)[{α, β}, {u, u}],
   lhs = U@𝔼_U[{u1, u2}, ℏ (α u1 + β u2), 1], HL[lhs == U@Λ_(U,1)[ℏ {α, β}, {u, u}]]},
  {U, {CU, QU}}, {u, {y, a, x}}]
```

$\{\{\{\mathbb{E}_{CU}[\{y\}, y\,(\alpha+\beta), 1+O[\epsilon]^2],\ CU[\,] + (\alpha\hbar+\beta\hbar)\,CU[y] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right) CU[y,y], \text{True}\},$

$\{\mathbb{E}_{CU}[\{a\}, a\,(\alpha+\beta), 1+O[\epsilon]^2],\ CU[\,] + (\alpha\hbar+\beta\hbar)\,CU[a] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right) CU[a,a], \text{True}\},\ \{\mathbb{E}_{CU}[\{x\}, x\,(\alpha+\beta), 1+O[\epsilon]^2],$

$CU[\,] + (\alpha\hbar+\beta\hbar)\,CU[x] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right) CU[x,x], \text{True}\}\},$

$\{\{\mathbb{E}_{QU}[\{y\}, y\,(\alpha+\beta), 1+O[\epsilon]^2],\ QU[\,] + (\alpha\hbar+\beta\hbar)\,QU[y] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right) QU[y,y],$

$\text{True}\},\ \{\mathbb{E}_{QU}[\{a\}, a\,(\alpha+\beta), 1+O[\epsilon]^2],\ QU[\,] + (\alpha\hbar+\beta\hbar)\,QU[a] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right) QU[a,a], \text{True}\},\ \{\mathbb{E}_{QU}[\{x\}, x\,(\alpha+\beta), 1+O[\epsilon]^2],$

$QU[\,] + (\alpha\hbar+\beta\hbar)\,QU[x] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right) QU[x,x], \text{True}\}\}\}$

```
{Λ_(#,1)[{ξ, α}, {x, a}], lhs = #@𝔼_#[{x, a}, ℏ (ξ x + α a), 1],
   HL[lhs == #@Λ_(#,1)[ℏ {ξ, α}, {x, a}]]} & /@ {CU, QU}
```

$\{\{\mathbb{E}_{CU}[\{a, x\}, a\,\alpha + e^{-\alpha\gamma} x\,\xi, 1+O[\epsilon]^2],$

$CU[\,] + \alpha\hbar\,CU[a] + (\xi\hbar - \alpha\gamma\xi\hbar^2)\,CU[x] + \frac{1}{2}\alpha^2\hbar^2\,CU[a,a] + \alpha\xi\hbar^2\,CU[a,x] + \frac{1}{2}\xi^2\hbar^2\,CU[x,x],$

$\text{True}\},\ \{\mathbb{E}_{QU}[\{a, x\}, a\,\alpha + e^{-\alpha\gamma} x\,\xi, 1+O[\epsilon]^2],\ QU[\,] + \alpha\hbar\,QU[a] +$

$(\xi\hbar - \alpha\gamma\xi\hbar^2)\,QU[x] + \frac{1}{2}\alpha^2\hbar^2\,QU[a,a] + \alpha\xi\hbar^2\,QU[a,x] + \frac{1}{2}\xi^2\hbar^2\,QU[x,x], \text{True}\}\}$

```
{Λ#,2[{α, η}, {a, y}], lhs = #@Œ#[{a, y}, ℏ (η y + α a), 1],
   HL[lhs ⩵ #@Λ#,2[ℏ {α, η}, {a, y}]]} & /@ {CU, QU}
```

$\{\{\mathbb{E}_{CU}[\{y, a\}, a\alpha + e^{-\alpha\gamma} y\eta, 1 + O[\epsilon]^3],$
$CU[] + \alpha\hbar\, CU[a] + (\eta\hbar - \alpha\gamma\eta\hbar^2)\, CU[y] + \frac{1}{2}\alpha^2\hbar^2\, CU[a, a] + \alpha\eta\hbar^2\, CU[y, a] + \frac{1}{2}\eta^2\hbar^2\, CU[y, y],$
$\text{True}\}, \{\mathbb{E}_{QU}[\{y, a\}, a\alpha + e^{-\alpha\gamma} y\eta, 1 + O[\epsilon]^3], QU[] + \alpha\hbar\, QU[a] +$
$(\eta\hbar - \alpha\gamma\eta\hbar^2)\, QU[y] + \frac{1}{2}\alpha^2\hbar^2\, QU[a, a] + \alpha\eta\hbar^2\, QU[y, a] + \frac{1}{2}\eta^2\hbar^2\, QU[y, y], \text{True}\}\}$

Goal. In either U, compute $F = e^{-\eta y} e^{\xi x} e^{\eta y} e^{-\xi x}$. First compute $G = e^{\xi x} y e^{-\xi x}$, a finite sum. Now F satisfies the ODE $\partial_\eta F = \partial_\eta(e^{-\eta y} e^{\eta G}) = -yF + FG$ with initial conditions $F(\eta = 0) = 1$. So we set it up and solve:

```
If[$k > 0, With[{U = CU},
  Module[{G, F, fs, bs, e, b, es, sol},
   G = Echo@Simp[Table[ξ^k / k!, {k, 0, $k + 1}].NestList[Simp[B[x_U , #]] &, y_U, $k + 1]];
   fs = Echo@Flatten@Table[f_{l,i,j,k}[η], {l, 0, $k}, {i, 0, 1}, {j, 0, 1}, {k, 0, 1}];
   F = Echo[fs.(bs = fs /. f_{l_,i_,j_,k_}[η] :→ ϵ^l U@{y^i, a^j, x^k})];
   es = Flatten[
     Table[Coefficient[e, b] == 0, {e, {F - 1_U /. η → 0, F ** G - y_U ** F - ∂_η F}}, {b, bs}]];
   sol = Echo@First[F /. DSolve[es, fs, η]];
   Echo[sol /. {e- → 1, U → Times}];
   Collect[sol /. {e- → 1, U → Times}, ϵ, Simplify]
  ]]]
```

» $-t\,\xi\, CU[] + 2\epsilon\,\xi\, CU[a] - \gamma\epsilon\,\xi^2\, CU[x] + CU[y]$

» $\{f_{0,0,0,0}[\eta], f_{1,0,0,0}[\eta], f_{1,0,0,1}[\eta], f_{1,0,1,0}[\eta],$
$f_{1,0,1,1}[\eta], f_{1,1,0,0}[\eta], f_{1,1,0,1}[\eta], f_{1,1,1,0}[\eta], f_{1,1,1,1}[\eta]\}$

» $CU[]\, f_{0,0,0,0}[\eta] + \epsilon\, CU[]\, f_{1,0,0,0}[\eta] + \epsilon\, CU[x]\, f_{1,0,0,1}[\eta] + \epsilon\, CU[a]\, f_{1,0,1,0}[\eta] + \epsilon\, CU[a, x]\, f_{1,0,1,1}[\eta] +$
$\epsilon\, CU[y]\, f_{1,1,0,0}[\eta] + \epsilon\, CU[y, x]\, f_{1,1,0,1}[\eta] + \epsilon\, CU[y, a]\, f_{1,1,1,0}[\eta] + \epsilon\, CU[y, a, x]\, f_{1,1,1,1}[\eta]$

» $e^{-t\eta\xi}\, CU[] + \frac{1}{2} e^{-t\eta\xi}\, t\gamma\epsilon\eta^2\xi^2\, CU[] + 2 e^{-t\eta\xi}\epsilon\eta\xi\, CU[a] - e^{-t\eta\xi}\gamma\epsilon\eta\xi^2\, CU[x] - e^{-t\eta\xi}\gamma\epsilon\eta^2\xi\, CU[y]$

» $1 + 2 a\epsilon\eta\xi - y\gamma\epsilon\eta^2\xi - x\gamma\epsilon\eta\xi^2 + \frac{1}{2} t\gamma\epsilon\eta^2\xi^2$

$1 + \frac{1}{2}\epsilon\eta\xi\,(4a + \gamma(-2y\eta - 2x\xi + t\eta\xi))$

Logos

In[∘]:=
```
ΛU_,kk_[{ξ1_, η1_}, {x, y}] := ΛU,kk[{ξ1, η1}, {x, y}] =
  Block[{$k = kk, $p = kk}, Module[{ξ, η, G, F, fs, f, bs, e, b, es},
    G = Simp[Table[ξ^k / k!, {k, 0, $k + 1}].NestList[Simp[B[xU, #]] &, yU, $k + 1]];
    fs = Flatten@Table[f_{l,i,j,k}[η], {l, 0, $k}, {i, 0, 1}, {j, 0, 1}, {k, 0, 1}];
    F = fs.(bs = fs /. f_{l_,i_,j_,k_}[η] :> ϵ^l U@{y^i, a^j, x^k});
    es = Flatten[
      Table[Coefficient[e, b] == 0, {e, {F - 1U /. η → 0, F ** G - yU ** F - ∂η F}}, {b, bs}]];
    F = F /. DSolve[es, fs, η][[1]];
    ŒU[{y, a, x},
      ξ x + η y + (U /. {CU → -t η ξ, QU → η ξ (1 - T) / ħ}),
      F + O$k /. {e- → 1, U → Times}
    ] /. {ξ → ξ1, η → η1}]];
```

In[∘]:=
```
Timing@ΛQU,2[{ξ, η}, {x, y}]
```

Out[∘]=

$$\Big\{1.64063,\ \text{Œ}_{QU}\Big[\{y, a, x\},\ y\,\eta + x\,\xi + \frac{(1-T)\,\eta\,\xi}{\hbar},\ 1 + \frac{1}{4\,\hbar}\,\eta\,\xi\,\big(\gamma\,\eta\,\xi - 4\,T\,\gamma\,\eta\,\xi + 3\,T^2\,\gamma\,\eta\,\xi + 8\,a\,T\,\hbar + 2\,y\,\gamma\,\eta\,\hbar - 6\,T\,y\,\gamma\,\eta\,\hbar + 2\,x\,\gamma\,\xi\,\hbar - 6\,T\,x\,\gamma\,\xi\,\hbar + 4\,x\,y\,\gamma\,\hbar^2\big)\,\epsilon +$$
$$\Big(-a\,T\,y\,\gamma\,\eta^2\,\xi\,(-\eta\,\xi + 3\,T\,\eta\,\xi - 3\,\hbar) - a\,T\,x\,\gamma\,\eta\,\xi^2\,(-\eta\,\xi + 3\,T\,\eta\,\xi - 3\,\hbar) + 2\,a^2\,T\,\eta\,\xi\,(T\,\eta\,\xi - \hbar) +$$
$$2\,a\,T\,x\,y\,\gamma\,\eta^2\,\xi^2\,\hbar - \frac{1}{2}\,x\,y^2\,\gamma^2\,\eta^2\,\xi\,(-\eta\,\xi + 3\,T\,\eta\,\xi - \hbar)\,\hbar - \frac{1}{2}\,x^2\,y\,\gamma^2\,\eta\,\xi^2\,(-\eta\,\xi + 3\,T\,\eta\,\xi - \hbar)\,\hbar +$$
$$\frac{1}{2}\,x^2\,y^2\,\gamma^2\,\eta^2\,\xi^2\,\hbar^2 + \frac{1}{24}\,y^2\,\gamma^2\,\eta^3\,\xi\,(3\,\eta\,\xi - 18\,T\,\eta\,\xi + 27\,T^2\,\eta\,\xi + 4\,\hbar - 28\,T\,\hbar) + \frac{1}{24}\,x^2\,\gamma^2\,\eta\,\xi^3$$
$$(3\,\eta\,\xi - 18\,T\,\eta\,\xi + 27\,T^2\,\eta\,\xi + 4\,\hbar - 28\,T\,\hbar) + \frac{1}{2\,\hbar}\,a\,T\,\gamma\,\eta^2\,\xi^2\,(\eta\,\xi - 4\,T\,\eta\,\xi + 3\,T^2\,\eta\,\xi + 4\,\hbar - 6\,T\,\hbar) +$$
$$\frac{1}{4}\,x\,y\,\gamma^2\,\eta\,\xi\,(2\,\eta^2\,\xi^2 - 10\,T\,\eta^2\,\xi^2 + 12\,T^2\,\eta^2\,\xi^2 + 5\,\eta\,\xi\,\hbar - 21\,T\,\eta\,\xi\,\hbar + 2\,\hbar^2) - \frac{1}{24\,\hbar}$$
$$y\,\gamma^2\,\eta^2\,\xi\,(-3\,\eta^2\,\xi^2 + 21\,T\,\eta^2\,\xi^2 - 45\,T^2\,\eta^2\,\xi^2 + 27\,T^3\,\eta^2\,\xi^2 - 10\,\eta\,\xi\,\hbar + 68\,T\,\eta\,\xi\,\hbar - 82\,T^2\,\eta\,\xi\,\hbar -$$
$$6\,\hbar^2 + 30\,T\,\hbar^2) - \frac{1}{24\,\hbar}\,x\,\gamma^2\,\eta\,\xi^2\,(-3\,\eta^2\,\xi^2 + 21\,T\,\eta^2\,\xi^2 - 45\,T^2\,\eta^2\,\xi^2 + 27\,T^3\,\eta^2\,\xi^2 - 10\,\eta\,\xi\,\hbar +$$
$$68\,T\,\eta\,\xi\,\hbar - 82\,T^2\,\eta\,\xi\,\hbar - 6\,\hbar^2 + 30\,T\,\hbar^2) + \frac{1}{288\,\hbar^2}\,(-1 + T)\,\gamma^2\,\eta^2\,\xi^2\,(-9\,\eta^2\,\xi^2 + 63\,T\,\eta^2\,\xi^2 -$$
$$135\,T^2\,\eta^2\,\xi^2 + 81\,T^3\,\eta^2\,\xi^2 - 40\,\eta\,\xi\,\hbar + 272\,T\,\eta\,\xi\,\hbar - 328\,T^2\,\eta\,\xi\,\hbar - 36\,\hbar^2 + 180\,T\,\hbar^2)\Big)\,\epsilon^2 + O[\epsilon]^3\Big]\Big\}$$

```
{ΛCU,1[{ξ, η}, {x, y}], lhs = CU@ŒCU[{x, y}, ħ (ξ x + η y), 1],
 HL[lhs == CU@ΛCU,1[ħ {ξ, η}, {x, y}]]}
```

$$\Big\{\text{Œ}_{CU}\Big[\{y, a, x\},\ y\,\eta + x\,\xi - t\,\eta\,\xi,\ 1 + \frac{1}{2}\,\eta\,\xi\,(4\,a - 2\,y\,\gamma\,\eta - 2\,x\,\gamma\,\xi + t\,\gamma\,\eta\,\xi)\,\epsilon + O[\epsilon]^2\Big],$$
$$(1 - t\,\eta\,\xi\,\hbar^2)\,CU[\,] + 2\,\epsilon\,\eta\,\xi\,\hbar^2\,CU[a] + \xi\,\hbar\,CU[x] + \eta\,\hbar\,CU[y] +$$
$$\frac{1}{2}\,\xi^2\,\hbar^2\,CU[x, x] + \eta\,\xi\,\hbar^2\,CU[y, x] + \frac{1}{2}\,\eta^2\,\hbar^2\,CU[y, y],\ \text{True}\Big\}$$

In[∘]:= **{$\Lambda_{QU,1}$[{ξ, η}, {x, y}], lhs = QU@$\mathbb{E}_{QU}$[{x, y}, ℏ (ξ x + η y), 1],
HL@SimpT[lhs == QU@$\Lambda_{QU,1}$[ℏ {ξ, η}, {x, y}]]}**

Out[∘]= $\left\{\mathbb{E}_{QU}\left[\{y, a, x\}, y\,\eta + x\,\xi + \frac{(1-T)\,\eta\,\xi}{\hbar}, 1 + \frac{1}{4\hbar}\eta\,\xi\left(\gamma\,\eta\,\xi - 4\,T\,\gamma\,\eta\,\xi + 3\,T^2\,\gamma\,\eta\,\xi + 8\,a\,T\,\hbar + 2\,y\,\gamma\,\eta\,\hbar - 6\,T\,y\,\gamma\,\eta\,\hbar + 2\,x\,\gamma\,\xi\,\hbar - 6\,T\,x\,\gamma\,\xi\,\hbar + 4\,x\,y\,\gamma\,\hbar^2\right)\epsilon + O[\epsilon]^2\right], (1 + \eta\,\xi\,\hbar - T\,\eta\,\xi\,\hbar)\,QU[\,] + 2\,T\,\epsilon\,\eta\,\xi\,\hbar^2\,QU[a] + \xi\,\hbar\,QU[x] + \eta\,\hbar\,QU[y] + \frac{1}{2}\xi^2\,\hbar^2\,QU[x, x] + \eta\,\xi\,\hbar^2\,QU[y, x] + \frac{1}{2}\eta^2\,\hbar^2\,QU[y, y], \text{True}\right\}$

**{tt = Last[$\Lambda_{CU,2}$[{ξ, η}, {x, y}]], Log[tt],
Exponent[Normal@Log[tt] /. {ξ → ℏ ξ, η → ℏ η, x → ℏ x, y → ℏ y}, ℏ]} // Expand**

$\left\{1 + \left(2\,a\,\eta\,\xi - y\,\gamma\,\eta^2\,\xi - x\,\gamma\,\eta\,\xi^2 + \frac{1}{2}t\,\gamma\,\eta^2\,\xi^2\right)\epsilon + \left(2\,a^2\,\eta^2\,\xi^2 - a\,\gamma\,\eta^2\,\xi^2 - 2\,a\,y\,\gamma\,\eta^3\,\xi^2 + y\,\gamma^2\,\eta^3\,\xi^2 + \frac{1}{2}y^2\,\gamma^2\,\eta^4\,\xi^2 - 2\,a\,x\,\gamma\,\eta^2\,\xi^3 + x\,\gamma^2\,\eta^2\,\xi^3 + a\,t\,\gamma\,\eta^3\,\xi^3 - \frac{1}{3}t\,\gamma^2\,\eta^3\,\xi^3 + x\,y\,\gamma^2\,\eta^3\,\xi^3 - \frac{1}{2}t\,y\,\gamma^2\,\eta^4\,\xi^3 + \frac{1}{2}x^2\,\gamma^2\,\eta^2\,\xi^4 - \frac{1}{2}t\,x\,\gamma^2\,\eta^3\,\xi^4 + \frac{1}{8}t^2\,\gamma^2\,\eta^4\,\xi^4\right)\epsilon^2 + O[\epsilon]^3, \left(2\,a\,\eta\,\xi - y\,\gamma\,\eta^2\,\xi - x\,\gamma\,\eta\,\xi^2 + \frac{1}{2}t\,\gamma\,\eta^2\,\xi^2\right)\epsilon + \left(-a\,\gamma\,\eta^2\,\xi^2 + y\,\gamma^2\,\eta^3\,\xi^2 + x\,\gamma^2\,\eta^2\,\xi^3 - \frac{1}{3}t\,\gamma^2\,\eta^3\,\xi^3\right)\epsilon^2 + O[\epsilon]^3, 6\right\}$

**{tt = Last[$\Lambda_{QU,2}$[{ξ, η}, {x, y}]], Log[tt],
Exponent[Normal@Log[tt] /. {ξ → d ξ, η → d η, x → d x, y → d y}, d]} // Expand**

$$\Big\{1+\Big(2\,a\,T\,\eta\,\xi+\frac{1}{2}\,y\,\gamma\,\eta^2\,\xi-\frac{3}{2}\,T\,y\,\gamma\,\eta^2\,\xi+$$

$$\frac{1}{2}\,x\,\gamma\,\eta\,\xi^2-\frac{3}{2}\,T\,x\,\gamma\,\eta\,\xi^2+\frac{\gamma\,\eta^2\,\xi^2}{4\,\hbar}-\frac{T\,\gamma\,\eta^2\,\xi^2}{\hbar}+\frac{3\,T^2\,\gamma\,\eta^2\,\xi^2}{4\,\hbar}+x\,y\,\gamma\,\eta\,\xi\,\hbar\Big)\,\epsilon+$$

$$\Big(2\,a^2\,T^2\,\eta^2\,\xi^2+2\,a\,T\,\gamma\,\eta^2\,\xi^2-3\,a\,T^2\,\gamma\,\eta^2\,\xi^2+\frac{1}{8}\,\gamma^2\,\eta^2\,\xi^2-\frac{3}{4}\,T\,\gamma^2\,\eta^2\,\xi^2+\frac{5}{8}\,T^2\,\gamma^2\,\eta^2\,\xi^2+$$

$$a\,T\,y\,\gamma\,\eta^3\,\xi^2-3\,a\,T^2\,y\,\gamma\,\eta^3\,\xi^2+\frac{5}{12}\,y\,\gamma^2\,\eta^3\,\xi^2-\frac{17}{6}\,T\,y\,\gamma^2\,\eta^3\,\xi^2+\frac{41}{12}\,T^2\,y\,\gamma^2\,\eta^3\,\xi^2+\frac{1}{8}\,y^2\,\gamma^2\,\eta^4\,\xi^2-$$

$$\frac{3}{4}\,T\,y^2\,\gamma^2\,\eta^4\,\xi^2+\frac{9}{8}\,T^2\,y^2\,\gamma^2\,\eta^4\,\xi^2+a\,T\,x\,\gamma\,\eta^2\,\xi^3-3\,a\,T^2\,x\,\gamma\,\eta^2\,\xi^3+\frac{5}{12}\,x\,\gamma^2\,\eta^2\,\xi^3-\frac{17}{6}\,T\,x\,\gamma^2\,\eta^2\,\xi^3+$$

$$\frac{41}{12}\,T^2\,x\,\gamma^2\,\eta^2\,\xi^3+\frac{1}{2}\,x\,y\,\gamma^2\,\eta^3\,\xi^3-\frac{5}{2}\,T\,x\,y\,\gamma^2\,\eta^3\,\xi^3+3\,T^2\,x\,y\,\gamma^2\,\eta^3\,\xi^3+\frac{1}{8}\,x^2\,\gamma^2\,\eta^2\,\xi^4-$$

$$\frac{3}{4}\,T\,x^2\,\gamma^2\,\eta^2\,\xi^4+\frac{9}{8}\,T^2\,x^2\,\gamma^2\,\eta^2\,\xi^4+\frac{\gamma^2\,\eta^4\,\xi^4}{32\,\hbar^2}-\frac{T\,\gamma^2\,\eta^4\,\xi^4}{4\,\hbar^2}+\frac{11\,T^2\,\gamma^2\,\eta^4\,\xi^4}{16\,\hbar^2}-\frac{3\,T^3\,\gamma^2\,\eta^4\,\xi^4}{4\,\hbar^2}+$$

$$\frac{9\,T^4\,\gamma^2\,\eta^4\,\xi^4}{32\,\hbar^2}+\frac{a\,T\,\gamma\,\eta^3\,\xi^3}{2\,\hbar}-\frac{2\,a\,T^2\,\gamma\,\eta^3\,\xi^3}{\hbar}+\frac{3\,a\,T^3\,\gamma\,\eta^3\,\xi^3}{2\,\hbar}+\frac{5\,\gamma^2\,\eta^3\,\xi^3}{36\,\hbar}-\frac{13\,T\,\gamma^2\,\eta^3\,\xi^3}{12\,\hbar}+$$

$$\frac{25\,T^2\,\gamma^2\,\eta^3\,\xi^3}{12\,\hbar}-\frac{41\,T^3\,\gamma^2\,\eta^3\,\xi^3}{36\,\hbar}+\frac{y\,\gamma^2\,\eta^4\,\xi^3}{8\,\hbar}-\frac{7\,T\,y\,\gamma^2\,\eta^4\,\xi^3}{8\,\hbar}+\frac{15\,T^2\,y\,\gamma^2\,\eta^4\,\xi^3}{8\,\hbar}-\frac{9\,T^3\,y\,\gamma^2\,\eta^4\,\xi^3}{8\,\hbar}+$$

$$\frac{x\,\gamma^2\,\eta^3\,\xi^4}{8\,\hbar}-\frac{7\,T\,x\,\gamma^2\,\eta^3\,\xi^4}{8\,\hbar}+\frac{15\,T^2\,x\,\gamma^2\,\eta^3\,\xi^4}{8\,\hbar}-\frac{9\,T^3\,x\,\gamma^2\,\eta^3\,\xi^4}{8\,\hbar}-2\,a^2\,T\,\eta\,\xi\,\hbar+3\,a\,T\,y\,\gamma\,\eta^2\,\xi\,\hbar+$$

$$\frac{1}{4}\,y\,\gamma^2\,\eta^2\,\xi\,\hbar-\frac{5}{4}\,T\,y\,\gamma^2\,\eta^2\,\xi\,\hbar+\frac{1}{6}\,y^2\,\gamma^2\,\eta^3\,\xi\,\hbar-\frac{7}{6}\,T\,y^2\,\gamma^2\,\eta^3\,\xi\,\hbar+3\,a\,T\,x\,\gamma\,\eta\,\xi^2\,\hbar+\frac{1}{4}\,x\,\gamma^2\,\eta\,\xi^2\,\hbar-$$

$$\frac{5}{4}\,T\,x\,\gamma^2\,\eta\,\xi^2\,\hbar+2\,a\,T\,x\,y\,\gamma\,\eta^2\,\xi^2\,\hbar+\frac{5}{4}\,x\,y\,\gamma^2\,\eta^2\,\xi^2\,\hbar-\frac{21}{4}\,T\,x\,y\,\gamma^2\,\eta^2\,\xi^2\,\hbar+\frac{1}{2}\,x\,y^2\,\gamma^2\,\eta^3\,\xi^2\,\hbar-$$

$$\frac{3}{2}\,T\,x\,y^2\,\gamma^2\,\eta^3\,\xi^2\,\hbar+\frac{1}{6}\,x^2\,\gamma^2\,\eta\,\xi^3\,\hbar-\frac{7}{6}\,T\,x^2\,\gamma^2\,\eta\,\xi^3\,\hbar+\frac{1}{2}\,x^2\,y\,\gamma^2\,\eta^2\,\xi^3\,\hbar-\frac{3}{2}\,T\,x^2\,y\,\gamma^2\,\eta^2\,\xi^3\,\hbar+$$

$$\frac{1}{2}\,x\,y\,\gamma^2\,\eta\,\xi\,\hbar^2+\frac{1}{2}\,x\,y^2\,\gamma^2\,\eta^2\,\xi\,\hbar^2+\frac{1}{2}\,x^2\,y\,\gamma^2\,\eta\,\xi^2\,\hbar^2+\frac{1}{2}\,x^2\,y^2\,\gamma^2\,\eta^2\,\xi^2\,\hbar^2\Big)\,\epsilon^2+O[\epsilon]^3,$$

$$\Big(2\,a\,T\,\eta\,\xi+\frac{1}{2}\,y\,\gamma\,\eta^2\,\xi-\frac{3}{2}\,T\,y\,\gamma\,\eta^2\,\xi+\frac{1}{2}\,x\,\gamma\,\eta\,\xi^2-\frac{3}{2}\,T\,x\,\gamma\,\eta\,\xi^2+\frac{\gamma\,\eta^2\,\xi^2}{4\,\hbar}-$$

$$\frac{T\,\gamma\,\eta^2\,\xi^2}{\hbar}+\frac{3\,T^2\,\gamma\,\eta^2\,\xi^2}{4\,\hbar}+x\,y\,\gamma\,\eta\,\xi\,\hbar\Big)\,\epsilon+$$

$$\Big(2\,a\,T\,\gamma\,\eta^2\,\xi^2-3\,a\,T^2\,\gamma\,\eta^2\,\xi^2+\frac{1}{8}\,\gamma^2\,\eta^2\,\xi^2-\frac{3}{4}\,T\,\gamma^2\,\eta^2\,\xi^2+\frac{5}{8}\,T^2\,\gamma^2\,\eta^2\,\xi^2+\frac{5}{12}\,y\,\gamma^2\,\eta^3\,\xi^2-$$

$$\frac{17}{6}\,T\,y\,\gamma^2\,\eta^3\,\xi^2+\frac{41}{12}\,T^2\,y\,\gamma^2\,\eta^3\,\xi^2+\frac{5}{12}\,x\,\gamma^2\,\eta^2\,\xi^3-\frac{17}{6}\,T\,x\,\gamma^2\,\eta^2\,\xi^3+\frac{41}{12}\,T^2\,x\,\gamma^2\,\eta^2\,\xi^3+$$

$$\frac{5\,\gamma^2\,\eta^3\,\xi^3}{36\,\hbar}-\frac{13\,T\,\gamma^2\,\eta^3\,\xi^3}{12\,\hbar}+\frac{25\,T^2\,\gamma^2\,\eta^3\,\xi^3}{12\,\hbar}-\frac{41\,T^3\,\gamma^2\,\eta^3\,\xi^3}{36\,\hbar}-2\,a^2\,T\,\eta\,\xi\,\hbar+3\,a\,T\,y\,\gamma\,\eta^2\,\xi\,\hbar+$$

$$\frac{1}{4}\,y\,\gamma^2\,\eta^2\,\xi\,\hbar-\frac{5}{4}\,T\,y\,\gamma^2\,\eta^2\,\xi\,\hbar+\frac{1}{6}\,y^2\,\gamma^2\,\eta^3\,\xi\,\hbar-\frac{7}{6}\,T\,y^2\,\gamma^2\,\eta^3\,\xi\,\hbar+3\,a\,T\,x\,\gamma\,\eta\,\xi^2\,\hbar+$$

$$\frac{1}{4}\,x\,\gamma^2\,\eta\,\xi^2\,\hbar-\frac{5}{4}\,T\,x\,\gamma^2\,\eta\,\xi^2\,\hbar+\frac{5}{4}\,x\,y\,\gamma^2\,\eta^2\,\xi^2\,\hbar-\frac{21}{4}\,T\,x\,y\,\gamma^2\,\eta^2\,\xi^2\,\hbar+\frac{1}{6}\,x^2\,\gamma^2\,\eta\,\xi^3\,\hbar-$$

$$\frac{7}{6}\,T\,x^2\,\gamma^2\,\eta\,\xi^3\,\hbar+\frac{1}{2}\,x\,y\,\gamma^2\,\eta\,\xi\,\hbar^2+\frac{1}{2}\,x\,y^2\,\gamma^2\,\eta^2\,\xi\,\hbar^2+\frac{1}{2}\,x^2\,y\,\gamma^2\,\eta\,\xi^2\,\hbar^2\Big)\,\epsilon^2+O[\epsilon]^3,\ 6\Big\}$$

Logos

```
In[∘]:= Simp[Œ_U_[specs___, Q_, P_]] := Œ_U[specs, CF[Q], CF[P]];
```

Logos

```
In[∘]:= Λ_U_,k_[{υ1_, ω1_, δ_}, {u_, w_}] := Simp@Module[{υ, ω, yax, q, p, Q, d},
          {yax, q, p} = List @@ Λ_U,k[{υ, ω}, {u, w}];
          Œ_U[yax, Q = (υ u + ω w + δ u w + d υ ω) / (1 - d δ),
            Expand[(1 - d δ)^-1 e^-Q DP_{υ→D_u, ω→D_w}[p][e^Q]] + 0_k] /. {d → ∂_{υ,ω} q} /. {υ → υ1, ω → ω1}];
```

```
Block[{$p = 4, $k = 1},
 {Λ_CU,$k[ħ {ξ, η, δ}, {x, y}],
  Short[lhs = CU@Œ_CU[{x, y}, ħ (ξ x + η y + δ x y), 1_$k], 5],
  HL@Simp[lhs - CU@Λ_CU,$k[ħ {ξ, η, δ}, {x, y}]]}
]
```

$\{Œ_{CU}[\{y, a, x\}, \frac{x y \delta \hbar + y \eta \hbar + x \xi \hbar - t \eta \xi \hbar^2}{1 + t \delta \hbar},$

$\frac{1}{1 + t \delta \hbar} + ((4 a \delta \hbar + 12 a t \delta^2 \hbar^2 + 4 a x y \delta^2 \hbar^2 + 2 t \gamma \delta^2 \hbar^2 - 8 x y \gamma \delta^2 \hbar^2 + 4 a y \delta \eta \hbar^2 -$
$4 y \gamma \delta \eta \hbar^2 + 4 a x \delta \xi \hbar^2 - 4 x \gamma \delta \xi \hbar^2 + 4 a \eta \xi \hbar^2 + 12 a t^2 \delta^3 \hbar^3 + 8 a t x y \delta^3 \hbar^3 +$
$4 t^2 \gamma \delta^3 \hbar^3 - 12 t x y \gamma \delta^3 \hbar^3 - 4 x^2 y^2 \gamma \delta^3 \hbar^3 + 8 a t y \delta^2 \eta \hbar^3 - 4 t y \gamma \delta^2 \eta \hbar^3 -$
$6 x y^2 \gamma \delta^2 \eta \hbar^3 - 2 y^2 \gamma \delta \eta^2 \hbar^3 + 8 a t x \delta^2 \xi \hbar^3 - 4 t x \gamma \delta^2 \xi \hbar^3 - 6 x^2 y \gamma \delta^2 \xi \hbar^3 +$
$8 a t \delta \eta \xi \hbar^3 + 4 t \gamma \delta \eta \xi \hbar^3 - 8 x y \gamma \delta \eta \xi \hbar^3 - 2 y \gamma \eta^2 \xi \hbar^3 - 2 x^2 \gamma \delta \xi^2 \hbar^3 - 2 x \gamma \eta \xi^2 \hbar^3 +$
$4 a t^3 \delta^4 \hbar^4 + 4 a t^2 x y \delta^4 \hbar^4 + 2 t^3 \gamma \delta^4 \hbar^4 - 4 t^2 x y \gamma \delta^4 \hbar^4 - 3 t x^2 y^2 \gamma \delta^4 \hbar^4 +$
$4 a t^2 y \delta^3 \eta \hbar^4 - 4 t x y^2 \gamma \delta^3 \eta \hbar^4 - t y^2 \gamma \delta^2 \eta^2 \hbar^4 + 4 a t^2 x \delta^3 \xi \hbar^4 - 4 t x^2 y \gamma \delta^3 \xi \hbar^4 +$
$4 a t^2 \delta^2 \eta \xi \hbar^4 + 4 t^2 \gamma \delta^2 \eta \xi \hbar^4 - 4 t x y \gamma \delta^2 \eta \xi \hbar^4 - t x^2 \gamma \delta^2 \xi^2 \hbar^4 + t \gamma \eta^2 \xi^2 \hbar^4) \epsilon) /$
$(2 + 10 t \delta \hbar + 20 t^2 \delta^2 \hbar^2 + 20 t^3 \delta^3 \hbar^3 + 10 t^4 \delta^4 \hbar^4 + 2 t^5 \delta^5 \hbar^5) + O[\epsilon]^2],$

$(1 - t \delta \hbar + t^2 \delta^2 \hbar^2 + t \gamma \delta^2 \epsilon \hbar^2 - t \eta \xi \hbar^2 - t^3 \delta^3 \hbar^3 - 3 t^2 \gamma \delta^3 \epsilon \hbar^3 + 2 t^2 \delta \eta \xi \hbar^3 +$
$2 t \gamma \delta \epsilon \eta \xi \hbar^3 + t^4 \delta^4 \hbar^4 + 6 t^3 \gamma \delta^4 \epsilon \hbar^4 - 3 t^3 \delta^2 \eta \xi \hbar^4 -$
$9 t^2 \gamma \delta^2 \epsilon \eta \xi \hbar^4 + \frac{1}{2} t^2 \eta^2 \xi^2 \hbar^4 + \frac{1}{2} t \gamma \epsilon \eta^2 \xi^2 \hbar^4)$ CU[] +
$(2 \delta \epsilon \hbar - 4 t \delta^2 \epsilon \hbar^2 + 2 \epsilon \eta \xi \hbar^2 + 6 t^2 \delta^3 \epsilon \hbar^3 - 8 t \delta \epsilon \eta \xi \hbar^3 - 8 t^3 \delta^4 \epsilon \hbar^4 +$
$18 t^2 \delta^2 \epsilon \eta \xi \hbar^4 - 2 t \epsilon \eta^2 \xi^2 \hbar^4)$ CU[a] +
<<37>> + $\frac{1}{6} \delta^3 \eta \hbar^4$ CU[y, y, y, y, x, x, x] +
$\frac{1}{24} \delta^4 \hbar^4$
CU[y, y, y, y, x, x, x, x], 0}

```
{Λ_QU,2[{ξ, η, δ}, {x, y}], lhs = QU@𝔈_QU[{x, y}, ℏ (ξ x + η y + δ x y), 1],
 HL@SimpT[lhs == QU@Λ_QU,1[ℏ {ξ, η, δ}, {x, y}]]}
```

$\{\mathbb{E}_{QU}[\{y, a, x\}, \frac{\cdots 1 \cdots}{\cdots 1 \cdots},$

$\frac{\hbar}{-\delta + T\,\delta + \hbar} + \Big(\big(-8\,a\,T\,\delta^4\,\hbar^2 + 24\,a\,T^2\,\delta^4\,\hbar^2 - 24\,a\,T^3\,\delta^4\,\hbar^2 + 8\,a\,T^4\,\delta^4\,\hbar^2 + \cdots 149 \cdots +$
$4\,x^2\,y^2\,\gamma\,\delta^2\,\hbar^6 + 4\,x\,y^2\,\gamma\,\delta\,\eta\,\hbar^6 + 4\,x^2\,y\,\gamma\,\delta\,\xi\,\hbar^6 + 4\,x\,y\,\gamma\,\eta\,\xi\,\hbar^6\big)\,\epsilon\Big) \Big/$
$\big(-4\,\delta^5 + 20\,T\,\delta^5 - 40\,T^2\,\delta^5 + 40\,T^3\,\delta^5 - 20\,T^4\,\delta^5 + 4\,T^5\,\delta^5 + \cdots 12 \cdots + 40\,T^3\,\delta^3\,\hbar^2 +$
$40\,\delta^2\,\hbar^3 - 80\,T\,\delta^2\,\hbar^3 + 40\,T^2\,\delta^2\,\hbar^3 - 20\,\delta\,\hbar^4 + 20\,T\,\delta\,\hbar^4 + 4\,\hbar^5\big) +$
$\frac{(\cdots 1 \cdots)\,\cdots 1 \cdots}{\cdots 1 \cdots} + O[\epsilon]^3],\ \cdots 1 \cdots,\ \text{True}\}$

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```
{tt = ComposeSeries[(1 + t δ) Last[Λ_CU,2[{ξ, η, δ}, {x, y}]], (1 + t δ)^4 ϵ + O[ϵ]^18];
  Together@Log[tt],
  Exponent[Normal@Together@Log[tt] /. {ξ → d ξ, η → d η, x → d x, y → d y}, d],
  Exponent[Normal@Together@Log[tt] /. {x → d x, y → d y}, d]
 } // Expand
```

$\{\Big(2\,a\,\delta + 6\,a\,t\,\delta^2 + 2\,a\,x\,y\,\delta^2 + t\,\gamma\,\delta^2 - 4\,x\,y\,\gamma\,\delta^2 + 6\,a\,t^2\,\delta^3 + 4\,a\,t\,x\,y\,\delta^3 + 2\,t^2\,\gamma\,\delta^3 - 6\,t\,x\,y\,\gamma\,\delta^3 -$
$2\,x^2\,y^2\,\gamma\,\delta^3 + 2\,a\,t^3\,\delta^4 + 2\,a\,t^2\,x\,y\,\delta^4 + t^3\,\gamma\,\delta^4 - 2\,t^2\,x\,y\,\gamma\,\delta^4 - \frac{3}{2}\,t\,x^2\,y^2\,\gamma\,\delta^4 + 2\,a\,y\,\delta\,\eta -$
$2\,y\,\gamma\,\delta\,\eta + 4\,a\,t\,y\,\delta^2\,\eta - 2\,t\,y\,\gamma\,\delta^2\,\eta - 3\,x\,y^2\,\gamma\,\delta^2\,\eta + 2\,a\,t^2\,y\,\delta^3\,\eta - 2\,t\,x\,y^2\,\gamma\,\delta^3\,\eta -$
$y^2\,\gamma\,\delta\,\eta^2 - \frac{1}{2}\,t\,y^2\,\gamma\,\delta^2\,\eta^2 + 2\,a\,x\,\delta\,\xi - 2\,x\,\gamma\,\delta\,\xi + 4\,a\,t\,x\,\delta^2\,\xi - 2\,t\,x\,\gamma\,\delta^2\,\xi - 3\,x^2\,y\,\gamma\,\delta^2\,\xi +$
$2\,a\,t^2\,x\,\delta^3\,\xi - 2\,t\,x^2\,y\,\gamma\,\delta^3\,\xi + 2\,a\,\eta\,\xi + 4\,a\,t\,\delta\,\eta\,\xi + 2\,t\,\gamma\,\delta\,\eta\,\xi - 4\,x\,y\,\gamma\,\delta\,\eta\,\xi + 2\,a\,t^2\,\delta^2\,\eta\,\xi +$
$2\,t^2\,\gamma\,\delta^2\,\eta\,\xi - 2\,t\,x\,y\,\gamma\,\delta^2\,\eta\,\xi - y\,\gamma\,\eta^2\,\xi - x^2\,\gamma\,\delta\,\xi^2 - \frac{1}{2}\,t\,x^2\,\gamma\,\delta^2\,\xi^2 - x\,\gamma\,\eta\,\xi^2 + \frac{1}{2}\,t\,\gamma\,\eta^2\,\xi^2\Big)\,\epsilon +$
$\Big(2\,a^2\,\delta^2 - 2\,a\,\gamma\,\delta^2 + 12\,a^2\,t\,\delta^3 + 4\,a^2\,x\,y\,\delta^3 - 8\,a\,t\,\gamma\,\delta^3 - 20\,a\,x\,y\,\gamma\,\delta^3 - 2\,t\,\gamma^2\,\delta^3 + 18\,x\,y\,\gamma^2\,\delta^3 +$
$30\,a^2\,t^2\,\delta^4 + 20\,a^2\,t\,x\,y\,\delta^4 - 10\,a\,t^2\,\gamma\,\delta^4 - 88\,a\,t\,x\,y\,\gamma\,\delta^4 - 13\,a\,x^2\,y^2\,\gamma\,\delta^4 - \frac{15}{2}\,t^2\,\gamma^2\,\delta^4 +$
$64\,t\,x\,y\,\gamma^2\,\delta^4 + 34\,x^2\,y^2\,\gamma^2\,\delta^4 + 40\,a^2\,t^3\,\delta^5 + 40\,a^2\,t^2\,x\,y\,\delta^5 - 152\,a\,t^2\,x\,y\,\gamma\,\delta^5 - 48\,a\,t\,x^2\,y^2\,\gamma\,\delta^5 -$
$10\,t^3\,\gamma^2\,\delta^5 + 86\,t^2\,x\,y\,\gamma^2\,\delta^5 + 107\,t\,x^2\,y^2\,\gamma^2\,\delta^5 + 11\,x^3\,y^3\,\gamma^2\,\delta^5 + 30\,a^2\,t^4\,\delta^6 + 40\,a^2\,t^3\,x\,y\,\delta^6 +$
$10\,a\,t^4\,\gamma\,\delta^6 - 128\,a\,t^3\,x\,y\,\gamma\,\delta^6 - 66\,a\,t^2\,x^2\,y^2\,\gamma\,\delta^6 - 5\,t^4\,\gamma^2\,\delta^6 + 54\,t^3\,x\,y\,\gamma^2\,\delta^6 + \frac{247}{2}\,t^2\,x^2\,y^2\,\gamma^2\,\delta^6 +$
$\frac{80}{3}\,t\,x^3\,y^3\,\gamma^2\,\delta^6 + 12\,a^2\,t^5\,\delta^7 + 20\,a^2\,t^4\,x\,y\,\delta^7 + 8\,a\,t^5\,\gamma\,\delta^7 - 52\,a\,t^4\,x\,y\,\gamma\,\delta^7 - 40\,a\,t^3\,x^2\,y^2\,\gamma\,\delta^7 +$
$16\,t^4\,x\,y\,\gamma^2\,\delta^7 + 62\,t^3\,x^2\,y^2\,\gamma^2\,\delta^7 + \frac{64}{3}\,t^2\,x^3\,y^3\,\gamma^2\,\delta^7 + 2\,a^2\,t^6\,\delta^8 + 4\,a^2\,t^5\,x\,y\,\delta^8 + 2\,a\,t^6\,\gamma\,\delta^8 -$
$8\,a\,t^5\,x\,y\,\gamma\,\delta^8 - 9\,a\,t^4\,x^2\,y^2\,\gamma\,\delta^8 + \frac{1}{2}\,t^6\,\gamma^2\,\delta^8 + 2\,t^5\,x\,y\,\gamma^2\,\delta^8 + \frac{23}{2}\,t^4\,x^2\,y^2\,\gamma^2\,\delta^8 + \frac{17}{3}\,t^3\,x^3\,y^3\,\gamma^2\,\delta^8 +$
$4\,a^2\,y\,\delta^2\,\eta - 12\,a\,y\,\gamma\,\delta^2\,\eta + 6\,y\,\gamma^2\,\delta^2\,\eta + 20\,a^2\,t\,y\,\delta^3\,\eta - 48\,a\,t\,y\,\gamma\,\delta^3\,\eta - 20\,a\,x\,y^2\,\gamma\,\delta^3\,\eta +$
$14\,t\,y\,\gamma^2\,\delta^3\,\eta + 40\,x\,y^2\,\gamma^2\,\delta^3\,\eta + 40\,a^2\,t^2\,y\,\delta^4\,\eta - 72\,a\,t^2\,y\,\gamma\,\delta^4\,\eta - 72\,a\,t\,x\,y^2\,\gamma\,\delta^4\,\eta + 6\,t^2\,y\,\gamma^2\,\delta^4\,\eta +$
$115\,t\,x\,y^2\,\gamma^2\,\delta^4\,\eta + 23\,x^2\,y^3\,\gamma^2\,\delta^4\,\eta + 40\,a^2\,t^3\,y\,\delta^5\,\eta - 48\,a\,t^3\,y\,\gamma\,\delta^5\,\eta - 96\,a\,t^2\,x\,y^2\,\gamma\,\delta^5\,\eta -$
$6\,t^3\,y\,\gamma^2\,\delta^5\,\eta + 118\,t^2\,x\,y^2\,\gamma^2\,\delta^5\,\eta + 53\,t\,x^2\,y^3\,\gamma^2\,\delta^5\,\eta + 20\,a^2\,t^4\,y\,\delta^6\,\eta - 12\,a\,t^4\,y\,\gamma\,\delta^6\,\eta -$
$56\,a\,t^3\,x\,y^2\,\gamma\,\delta^6\,\eta - 4\,t^4\,y\,\gamma^2\,\delta^6\,\eta + 51\,t^3\,x\,y^2\,\gamma^2\,\delta^6\,\eta + 40\,t^2\,x^2\,y^3\,\gamma^2\,\delta^6\,\eta + 4\,a^2\,t^5\,y\,\delta^7\,\eta -$

$$12\,a\,t^4\,x\,y^2\,\gamma\,\delta^7\,\eta + 8\,t^4\,x\,y^2\,\gamma^2\,\delta^7\,\eta + 10\,t^3\,x^2\,y^3\,\gamma^2\,\delta^7\,\eta - 7\,a\,y^2\,\gamma\,\delta^2\,\eta^2 + 10\,y^2\,\gamma^2\,\delta^2\,\eta^2 -$$
$$24\,a\,t\,y^2\,\gamma\,\delta^3\,\eta^2 + 24\,t\,y^2\,\gamma^2\,\delta^3\,\eta^2 + 15\,x\,y^3\,\gamma^2\,\delta^3\,\eta^2 - 30\,a\,t^2\,y^2\,\gamma\,\delta^4\,\eta^2 + \frac{37}{2}\,t^2\,y^2\,\gamma^2\,\delta^4\,\eta^2 +$$
$$32\,t\,x\,y^3\,\gamma^2\,\delta^4\,\eta^2 - 16\,a\,t^3\,y^2\,\gamma\,\delta^5\,\eta^2 + 5\,t^3\,y^2\,\gamma^2\,\delta^5\,\eta^2 + 22\,t^2\,x\,y^3\,\gamma^2\,\delta^5\,\eta^2 - 3\,a\,t^4\,y^2\,\gamma\,\delta^6\,\eta^2 +$$
$$\frac{1}{2}\,t^4\,y^2\,\gamma^2\,\delta^6\,\eta^2 + 5\,t^3\,x\,y^3\,\gamma^2\,\delta^6\,\eta^2 + 3\,y^3\,\gamma^2\,\delta^2\,\eta^3 + \frac{17}{3}\,t\,y^3\,\gamma^2\,\delta^3\,\eta^3 + \frac{10}{3}\,t^2\,y^3\,\gamma^2\,\delta^4\,\eta^3 +$$
$$\frac{2}{3}\,t^3\,y^3\,\gamma^2\,\delta^5\,\eta^3 + 4\,a^2\,x\,\delta^2\,\xi - 12\,a\,x\,\gamma\,\delta^2\,\xi + 6\,x\,\gamma^2\,\delta^2\,\xi + 20\,a^2\,t\,x\,\delta^3\,\xi - 48\,a\,t\,x\,\gamma\,\delta^3\,\xi -$$
$$20\,a\,x^2\,y\,\gamma\,\delta^3\,\xi + 14\,t\,x\,\gamma^2\,\delta^3\,\xi + 40\,x^2\,y\,\gamma^2\,\delta^3\,\xi + 40\,a^2\,t^2\,x\,\delta^4\,\xi - 72\,a\,t^2\,x\,\gamma\,\delta^4\,\xi - 72\,a\,t\,x^2\,y\,\gamma\,\delta^4\,\xi +$$
$$6\,t^2\,x\,\gamma^2\,\delta^4\,\xi + 115\,t\,x^2\,y\,\gamma^2\,\delta^4\,\xi + 23\,x^3\,y^2\,\gamma^2\,\delta^4\,\xi + 40\,a^2\,t^3\,x\,\delta^5\,\xi - 48\,a\,t^3\,x\,\gamma\,\delta^5\,\xi -$$
$$96\,a\,t^2\,x^2\,y\,\gamma\,\delta^5\,\xi - 6\,t^3\,x\,\gamma^2\,\delta^5\,\xi + 118\,t^2\,x^2\,y\,\gamma^2\,\delta^5\,\xi + 53\,t\,x^3\,y^2\,\gamma^2\,\delta^5\,\xi + 20\,a^2\,t^4\,x\,\delta^6\,\xi -$$
$$12\,a\,t^4\,x\,\gamma\,\delta^6\,\xi - 56\,a\,t^3\,x^2\,y\,\gamma\,\delta^6\,\xi - 4\,t^4\,x\,\gamma^2\,\delta^6\,\xi + 51\,t^3\,x^2\,y\,\gamma^2\,\delta^6\,\xi + 40\,t^2\,x^3\,y^2\,\gamma^2\,\delta^6\,\xi +$$
$$4\,a^2\,t^5\,x\,\delta^7\,\xi - 12\,a\,t^4\,x^2\,y\,\gamma\,\delta^7\,\xi + 8\,t^4\,x^2\,y\,\gamma^2\,\delta^7\,\xi + 10\,t^3\,x^3\,y^2\,\gamma^2\,\delta^7\,\xi + 4\,a^2\,\delta\,\eta\,\xi - 4\,a\,\gamma\,\delta\,\eta\,\xi +$$
$$20\,a^2\,t\,\delta^2\,\eta\,\xi - 8\,a\,t\,\gamma\,\delta^2\,\eta\,\xi - 28\,a\,x\,y\,\gamma\,\delta^2\,\eta\,\xi - 6\,t\,\gamma^2\,\delta^2\,\eta\,\xi + 38\,x\,y\,\gamma^2\,\delta^2\,\eta\,\xi + 40\,a^2\,t^2\,\delta^3\,\eta\,\xi +$$
$$8\,a\,t^2\,\gamma\,\delta^3\,\eta\,\xi - 96\,a\,t\,x\,y\,\gamma\,\delta^3\,\eta\,\xi - 14\,t^2\,\gamma^2\,\delta^3\,\eta\,\xi + 88\,t\,x\,y\,\gamma^2\,\delta^3\,\eta\,\xi + 44\,x^2\,y^2\,\gamma^2\,\delta^3\,\eta\,\xi +$$
$$40\,a^2\,t^3\,\delta^4\,\eta\,\xi + 32\,a\,t^3\,\gamma\,\delta^4\,\eta\,\xi - 120\,a\,t^2\,x\,y\,\gamma\,\delta^4\,\eta\,\xi - 6\,t^3\,\gamma^2\,\delta^4\,\eta\,\xi + 62\,t^2\,x\,y\,\gamma^2\,\delta^4\,\eta\,\xi +$$
$$93\,t\,x^2\,y^2\,\gamma^2\,\delta^4\,\eta\,\xi + 20\,a^2\,t^4\,\delta^5\,\eta\,\xi + 28\,a\,t^4\,\gamma\,\delta^5\,\eta\,\xi - 64\,a\,t^3\,x\,y\,\gamma\,\delta^5\,\eta\,\xi + 6\,t^4\,\gamma^2\,\delta^5\,\eta\,\xi +$$
$$12\,t^3\,x\,y\,\gamma^2\,\delta^5\,\eta\,\xi + 63\,t^2\,x^2\,y^2\,\gamma^2\,\delta^5\,\eta\,\xi + 4\,a^2\,t^5\,\delta^6\,\eta\,\xi + 8\,a\,t^5\,\gamma\,\delta^6\,\eta\,\xi - 12\,a\,t^4\,x\,y\,\gamma\,\delta^6\,\eta\,\xi +$$
$$4\,t^5\,\gamma^2\,\delta^6\,\eta\,\xi + 14\,t^3\,x^2\,y^2\,\gamma^2\,\delta^6\,\eta\,\xi - 8\,a\,y\,\gamma\,\delta\,\eta^2\,\xi + 6\,y\,\gamma^2\,\delta\,\eta^2\,\xi - 24\,a\,t\,y\,\gamma\,\delta^2\,\eta^2\,\xi +$$
$$5\,t\,y\,\gamma^2\,\delta^2\,\eta^2\,\xi + 25\,x\,y^2\,\gamma^2\,\delta^2\,\eta^2\,\xi - 24\,a\,t^2\,y\,\gamma\,\delta^3\,\eta^2\,\xi - 8\,t^2\,y\,\gamma^2\,\delta^3\,\eta^2\,\xi + 45\,t\,x\,y^2\,\gamma^2\,\delta^3\,\eta^2\,\xi -$$
$$8\,a\,t^3\,y\,\gamma\,\delta^4\,\eta^2\,\xi - 7\,t^3\,y\,\gamma^2\,\delta^4\,\eta^2\,\xi + 24\,t^2\,x\,y^2\,\gamma^2\,\delta^4\,\eta^2\,\xi + 4\,t^3\,x\,y^2\,\gamma^2\,\delta^5\,\eta^2\,\xi + 4\,y^2\,\gamma^2\,\delta\,\eta^3\,\xi +$$
$$5\,t\,y^2\,\gamma^2\,\delta^2\,\eta^3\,\xi + t^2\,y^2\,\gamma^2\,\delta^3\,\eta^3\,\xi - 7\,a\,x^2\,\gamma\,\delta^2\,\xi^2 + 10\,x^2\,\gamma^2\,\delta^2\,\xi^2 - 24\,a\,t\,x^2\,\gamma\,\delta^3\,\xi^2 +$$
$$24\,t\,x^2\,\gamma^2\,\delta^3\,\xi^2 + 15\,x^3\,y\,\gamma^2\,\delta^3\,\xi^2 - 30\,a\,t^2\,x^2\,\gamma\,\delta^4\,\xi^2 + \frac{37}{2}\,t^2\,x^2\,\gamma^2\,\delta^4\,\xi^2 + 32\,t\,x^3\,y\,\gamma^2\,\delta^4\,\xi^2 -$$
$$16\,a\,t^3\,x^2\,\gamma\,\delta^5\,\xi^2 + 5\,t^3\,x^2\,\gamma^2\,\delta^5\,\xi^2 + 22\,t^2\,x^3\,y\,\gamma^2\,\delta^5\,\xi^2 - 3\,a\,t^4\,x^2\,\gamma\,\delta^6\,\xi^2 + \frac{1}{2}\,t^4\,x^2\,\gamma^2\,\delta^6\,\xi^2 +$$
$$5\,t^3\,x^3\,y\,\gamma^2\,\delta^6\,\xi^2 - 8\,a\,x\,\gamma\,\delta\,\eta\,\xi^2 + 6\,x\,\gamma^2\,\delta\,\eta\,\xi^2 - 24\,a\,t\,x\,\gamma\,\delta^2\,\eta\,\xi^2 + 5\,t\,x\,\gamma^2\,\delta^2\,\eta\,\xi^2 +$$
$$25\,x^2\,y\,\gamma^2\,\delta^2\,\eta\,\xi^2 - 24\,a\,t^2\,x\,\gamma\,\delta^3\,\eta\,\xi^2 - 8\,t^2\,x\,\gamma^2\,\delta^3\,\eta\,\xi^2 + 45\,t\,x^2\,y\,\gamma^2\,\delta^3\,\eta\,\xi^2 - 8\,a\,t^3\,x\,\gamma\,\delta^4\,\eta\,\xi^2 -$$
$$7\,t^3\,x\,\gamma^2\,\delta^4\,\eta\,\xi^2 + 24\,t^2\,x^2\,y\,\gamma^2\,\delta^4\,\eta\,\xi^2 + 4\,t^3\,x^2\,y\,\gamma^2\,\delta^5\,\eta\,\xi^2 - a\,\gamma\,\eta^2\,\xi^2 - 3\,t\,\gamma^2\,\delta\,\eta^2\,\xi^2 +$$
$$11\,x\,y\,\gamma^2\,\delta\,\eta^2\,\xi^2 + 6\,a\,t^2\,\gamma\,\delta^2\,\eta^2\,\xi^2 - \frac{5}{2}\,t^2\,\gamma^2\,\delta^2\,\eta^2\,\xi^2 + 12\,t\,x\,y\,\gamma^2\,\delta^2\,\eta^2\,\xi^2 + 8\,a\,t^3\,\gamma\,\delta^3\,\eta^2\,\xi^2 +$$
$$4\,t^3\,\gamma^2\,\delta^3\,\eta^2\,\xi^2 + 3\,a\,t^4\,\gamma\,\delta^4\,\eta^2\,\xi^2 + \frac{7}{2}\,t^4\,\gamma^2\,\delta^4\,\eta^2\,\xi^2 - t^3\,x\,y\,\gamma^2\,\delta^4\,\eta^2\,\xi^2 + y\,\gamma^2\,\eta^3\,\xi^2 -$$
$$t\,y\,\gamma^2\,\delta\,\eta^3\,\xi^2 - 2\,t^2\,y\,\gamma^2\,\delta^2\,\eta^3\,\xi^2 + 3\,x^3\,\gamma^2\,\delta^2\,\xi^3 + \frac{17}{3}\,t\,x^3\,\gamma^2\,\delta^3\,\xi^3 + \frac{10}{3}\,t^2\,x^3\,\gamma^2\,\delta^4\,\xi^3 +$$
$$\frac{2}{3}\,t^3\,x^3\,\gamma^2\,\delta^5\,\xi^3 + 4\,x^2\,\gamma^2\,\delta\,\eta\,\xi^3 + 5\,t\,x^2\,\gamma^2\,\delta^2\,\eta\,\xi^3 + t^2\,x^2\,\gamma^2\,\delta^3\,\eta\,\xi^3 + x\,\gamma^2\,\eta^2\,\xi^3 - t\,x\,\gamma^2\,\delta\,\eta^2\,\xi^3 -$$
$$2\,t^2\,x\,\gamma^2\,\delta^2\,\eta^2\,\xi^3 - \frac{1}{3}\,t\,\gamma^2\,\eta^3\,\xi^3 + \frac{1}{3}\,t^2\,\gamma^2\,\delta\,\eta^3\,\xi^3 + \frac{2}{3}\,t^3\,\gamma^2\,\delta^2\,\eta^3\,\xi^3\Big)\,\epsilon^2 + O[\epsilon]^3,\ 6,\ 6\Big\}$$

```
{tt = Last[ΛQU,2[{ξ, η, δ}, {x, y}]];
  Log[tt],
  Exponent[Normal@Together@Log[tt] /. {ξ → d ξ, η → d η, x → d x, y → d y}, d]} // Expand
```

$\{\mathrm{Log}[\frac{\hbar}{-\delta+T\,\delta+\hbar}] + (\frac{2\,a\,T\,\delta^5\,\hbar}{(-\delta+T\,\delta+\hbar)^5} - \frac{8\,a\,T^2\,\delta^5\,\hbar}{(-\delta+T\,\delta+\hbar)^5} + \frac{12\,a\,T^3\,\delta^5\,\hbar}{(-\delta+T\,\delta+\hbar)^5} - \frac{8\,a\,T^4\,\delta^5\,\hbar}{(-\delta+T\,\delta+\hbar)^5} + \cdots 267 \cdots + \frac{x^2\,y^2\,\gamma\,\delta^2\,\hbar^6}{(-\delta+T\,\delta+\hbar)^5} + \frac{x\,y^2\,\gamma\,\delta\,\eta\,\hbar^6}{(-\delta+T\,\delta+\hbar)^5} + \frac{x^2\,y\,\gamma\,\delta\,\xi\,\hbar^6}{(-\delta+T\,\delta+\hbar)^5} + \frac{x\,y\,\gamma\,\eta\,\xi\,\hbar^6}{(-\delta+T\,\delta+\hbar)^5})\,\epsilon + (-\frac{32\,a^2\,T^2\,\delta^{10}\,\hbar^2}{(\cdots 1 \cdots)^2} + \cdots 8307 \cdots + \cdots 1 \cdots + \frac{144\,x^2\,y^2\,\gamma^2\,\eta^2\,\xi^2\,\hbar^{11}}{\cdots 1 \cdots})\,\epsilon^2 + O[\epsilon]^3,\ 6\}$

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Reorderings with Rord

Rord

In[∘]:=
```
Rord_{u_i_,w_j_→k_}[ℭ_U_[L___, {l___, u_i_, w_j_, r___}_s_, R___, Q_, P_]] :=
  Simp@Module[{υ, ω, δ, Λ1, yax, q, p, kk = P⟦5⟧, δ1 = ∂_{u_i,w_j} Q},
    {yax, q, p} = Echo[List @@ If[δ1 === 0, Λ_{U,kk}[{υ, ω}, {u, w}],
        Λ_{U,kk}[{υ, ω, δ}, {u, w}]] /. {y → y_k, a → a_k, x → x_k, t → t_s, T → T_s}];
    ℭ_U[L, {l, Sequence @@ yax, r}_s, R, q + (Q /. u_i | w_j → 0), e^{-q} DP_{u_i→D_υ, w_j→D_ω}[P][p e^q]] /.
     {υ → ∂_{u_i} Q /. w_j → 0, ω → ∂_{w_j} Q /. u_i → 0, δ → δ1}];
```

Rord

In[∘]:=
```
Rord_{u_i_,w_j_→k_}[ℭ_U_[L___, {l___, u_i_, w_j_, r___}_s_, R___, Q_, P_]] :=
  Simp@Module[{υ, ω, δ, Λ1, yax, q, p, n, kk = P⟦5⟧, δ1 = ∂_{u_i,w_j} Q},
    {yax, q, p} = List @@ If[δ1 === 0, Λ_{U,kk}[{υ, ω}, {u, w}], Λ_{U,kk}[{υ, ω, δ}, {u, w}]] /.
      {y → y_n, a → a_n, x → x_n, t → t_s, T → T_s};
    (*Echo@{{{u_i,υ},{w_j,ω}},P,p e^q};*)
    ℭ_U[L, {l, Sequence @@ yax, r}_s, R, q + (Q /. u_i | w_j → 0), e^{-q} SP_{{u_i→υ, w_j→ω}}[P p e^q]] /.
     {n → k, υ → ∂_{u_i} Q /. w_j → 0, ω → ∂_{w_j} Q /. u_i → 0, δ → δ1}];
```

```
With[{cω = ℭ_CU[{y_1, x_1}_1, {x_2, a_2, y_2}_2, ℏ t_1 a_2 + ℏ t_1^-1 (e^{t_1} - 1) y_1 x_2, 1_2 + ϵ x_1 y_2]},
 {Short[rhs = cω // Rord_{x_2,a_2→3}, 3], HL[CU[cω] == CU[rhs]]}]
```

$\{\mathbb{C}_{CU}[\{y_1,\ x_1\}_1,\ \{a_3,\ x_3,\ y_2\}_2,\ \frac{e^{-\gamma\,\hbar\,t_1}\,(e^{\gamma\,\hbar\,t_1}\,\hbar\,a_3\,t_1^2 - \hbar\,x_3\,y_1 + e^{t_1}\,\hbar\,x_3\,y_1)}{t_1},\ 1 + x_1\,y_2\,\epsilon + O[\epsilon]^3],\ \mathrm{True}\}$

```
With[{cω = ℭ_CU[{y_1, a_1, a_2}_1, {x_2, x_1, y_2}_2,
    ℏ (l_11 t_1 a_1 + l_12 t_1 a_2 + l_21 t_2 a_1 + l_22 t_2 a_2 + γ_11 x_1 y_1 + γ_12 x_1 y_2 + γ_21 x_2 y_1 + γ_22 x_2 y_2),
    1_2 + ϵ (l_1 a_1 + l_2 a_2 + p_11 x_1 y_1 + p_12 x_1 y_2 + p_21 x_2 y_1 + p_22 x_2 y_2)]},
 {Short[rhs = cω // Rord_{a_1,a_2→3} // Rord_{x_2,x_1→4}, 3], HL[CU[cω] == CU[rhs]]}]
```

$\{\mathbb{C}_{CU}[\{y_1,\ a_3\}_1,\ \{x_4,\ y_2\}_2,\ \hbar\,a_3\,l_{11}\,t_1 + \hbar\,a_3\,l_{12}\,t_1 + \hbar\,a_3\,l_{21}\,t_2 + \hbar\,a_3\,l_{22}\,t_2 + \hbar\,x_4\,y_1\,\gamma_{11} + \hbar\,x_4\,y_2\,\gamma_{12} + \hbar\,x_4\,y_1\,\gamma_{21} + \hbar\,x_4\,y_2\,\gamma_{22},\ 1 + (a_3\,l_1 + a_3\,l_2 + p_{11}\,x_4\,y_1 + p_{21}\,x_4\,y_1 + p_{12}\,x_4\,y_2 + p_{22}\,x_4\,y_2)\,\epsilon + O[\epsilon]^3],\ \mathrm{True}\}$

```
ℏ a_3 l_11 t_1 + ℏ a_3 l_12 t_1 + ℏ a_3 l_21 t_2 + ℏ a_3 l_22 t_2 +
  ℏ x_4 y_1 γ_11 + ℏ x_4 y_2 γ_12 + ℏ x_4 y_1 γ_21 + ℏ x_4 y_2 γ_22 // Simplify
```

$\hbar\,(a_3\,(l_{11}\,t_1 + l_{12}\,t_1 + (l_{21} + l_{22})\,t_2) + x_4\,(y_1\,(\gamma_{11} + \gamma_{21}) + y_2\,(\gamma_{12} + \gamma_{22})))$

$$\text{With}\Big[\Big\{c\odot = \mathbb{E}_{CU}\Big[\{y_1, a_1, x_1\}_1,\ \{x_2, a_2, y_2\}_2,\ \hbar\left(l_{11}\,t_1\,a_1 + l_{12}\,t_1\,a_2 + l_{21}\,t_2\,a_1 + l_{22}\,t_2\,a_2 + \gamma_{11}\,x_1\,y_1 + \gamma_{12}\,x_1\,y_2 + \gamma_{21}\,x_2\,y_1 + \gamma_{22}\,x_2\,y_2\right),\ 1_2 + \epsilon\left(l_1\,a_1 + l_2\,a_2 + p_{11}\,x_1\,y_1 + p_{12}\,x_1\,y_2 + p_{21}\,x_2\,y_1 + p_{22}\,x_2\,y_2\right)\Big]\Big\},\ \{\text{Short}[rhs = c\odot\ //\ \text{Rord}_{x_2,a_2\to 3},\ 3],\ \text{HL}[\text{CU}[c\odot] == \text{CU}[rhs]]\}\Big]$$

$$\Big\{\mathbb{E}_{CU}\Big[\{y_1, a_1, x_1\}_1,\ \ll 1\gg_2,\ \ll 1\gg\ \ll 1\gg,\ 1 + e^{-\gamma\hbar l_{12} t_1 - \gamma\hbar l_{22} t_2}\left(e^{\gamma\hbar l_{12} t_1 + \gamma\hbar l_{22} t_2}\,a_1\,l_1 + e^{\gamma\hbar l_{12} t_1 + \gamma\hbar l_{22} t_2}\,a_3\,l_2 + e^{\gamma\hbar l_{12} t_1 + \gamma\hbar l_{22} t_2}\,p_{11}\,x_1\,y_1 + p_{21}\,x_3\,y_1 + e^{\ll 1\gg + \ll 1\gg}\,p_{12}\,x_1\,y_2 + p_{22}\,x_3\,y_2 - \gamma\,\hbar\,l_2\,x_3\,y_1\,\gamma_{21} - \gamma\,\hbar\,l_2\,x_3\,y_2\,\gamma_{22}\right)\epsilon + O[\epsilon]^3\Big],\ \text{True}\Big\}$$

$$\text{With}\Big[\Big\{q\odot = \mathbb{E}_{QU}\Big[\{y_1, a_1, x_1\}_1,\ \{x_2, a_2, y_2\}_2,\ \hbar\left(l_{11}\,t_1\,a_1 + l_{12}\,t_1\,a_2 + l_{21}\,t_2\,a_1 + l_{22}\,t_2\,a_2 + \gamma_{11}\,x_1\,y_1 + \gamma_{12}\,x_1\,y_2 + \gamma_{21}\,x_2\,y_1 + \gamma_{22}\,x_2\,y_2\right),\ 1_2 + \epsilon\left(l_1\,a_1 + l_2\,a_2 + p_{11}\,x_1\,y_1 + p_{12}\,x_1\,y_2 + p_{21}\,x_2\,y_1 + p_{22}\,x_2\,y_2\right)\Big]\Big\},\ \{\text{Short}[rhs = q\odot\ //\ \text{Rord}_{x_2,a_2\to 3},\ 3],\ \text{HL}[\text{QU}[q\odot] == \text{QU}[rhs]]\}\Big]$$

$$\Big\{\mathbb{E}_{QU}\Big[\{y_1, a_1, x_1\}_1,\ \ll 1\gg_2,\ \ll 1\gg\ \ll 1\gg,\ 1 + e^{-\gamma\hbar l_{12} t_1 - \gamma\hbar l_{22} t_2}\left(e^{\gamma\hbar l_{12} t_1 + \gamma\hbar l_{22} t_2}\,a_1\,l_1 + e^{\gamma\hbar l_{12} t_1 + \gamma\hbar l_{22} t_2}\,a_3\,l_2 + e^{\gamma\hbar l_{12} t_1 + \gamma\hbar l_{22} t_2}\,p_{11}\,x_1\,y_1 + p_{21}\,x_3\,y_1 + e^{\ll 1\gg + \ll 1\gg}\,p_{12}\,x_1\,y_2 + p_{22}\,x_3\,y_2 - \gamma\,\hbar\,l_2\,x_3\,y_1\,\gamma_{21} - \gamma\,\hbar\,l_2\,x_3\,y_2\,\gamma_{22}\right)\epsilon + O[\epsilon]^3\Big],\ \text{True}\Big\}$$

$$\text{With}\Big[\Big\{q\odot = \mathbb{E}_{QU}\Big[\{y_1, a_1, x_1\}_1,\ \{x_2, a_2, y_2\}_2,\ \hbar\left(l_{11}\,t_1\,a_1 + l_{12}\,t_1\,a_2 + l_{21}\,t_2\,a_1 + l_{22}\,t_2\,a_2 + \gamma_{11}\,x_1\,y_1 + \gamma_{12}\,x_1\,y_2 + \gamma_{21}\,x_2\,y_1 + \gamma_{22}\,x_2\,y_2\right),\ 1_2 + \epsilon\left(l_1\,a_1 + l_2\,a_2 + p_{11}\,x_1\,y_1 + p_{12}\,x_1\,y_2 + p_{21}\,x_2\,y_1 + p_{22}\,x_2\,y_2\right)\Big]\Big\},\ \{\text{Short}[rhs = q\odot\ //\ \text{Rord}_{a_2,y_2\to 3},\ 3],\ \text{HL}[\text{QU}[q\odot] == \text{QU}[rhs]]\}\Big]$$

$$\{\ll 1\gg,\ \text{True}\}$$

$$\text{Timing@With}\Big[\Big\{q\odot = \mathbb{E}_{QU}\Big[\{x_1, y_1\}_1,\ \{x_2, a_2, y_2\}_2,\ \hbar\left(l_{12}\,t_1\,a_2 + l_{22}\,t_2\,a_2 + \gamma_{11}\,x_1\,y_1 + \gamma_{12}\,x_1\,y_2 + \gamma_{21}\,x_2\,y_1 + \gamma_{22}\,x_2\,y_2\right),\ 0_2 + \epsilon\left(l_2\,a_2 + p_{11}\,x_1\,y_1 + p_{12}\,x_1\,y_2 + p_{21}\,x_2\,y_1 + p_{22}\,x_2\,y_2\right)\Big]\Big\},\ \{\text{Short}[rhs = q\odot\ //\ \text{Rord}_{x_1,y_1\to 3},\ 5]\}\Big]$$

$$\Big\{116.156,\ \Big\{\mathbb{E}_{QU}\Big[\{y_3, a_3, x_3\}_1,\ \ll 1\gg_2,\ \frac{\ll 1\gg}{1 - \ll 1\gg + \ll 1\gg},\ \Big(\big(\hbar\,a_2\,l_2 + p_{11} - p_{11}\,T_1 + \hbar\,p_{22}\,x_2\,y_2 + \hbar\,p_{12}\,x_3\,y_2 + \ll 46\gg + 2\,\hbar\,p_{12}\,T_1\,x_2\,y_2\,\gamma_{11}\,\gamma_{21} - \hbar\,p_{12}\,T_1^2\,x_2\,y_2\,\gamma_{11}\,\gamma_{21} + \hbar\,p_{11}\,x_2\,y_2\,\gamma_{12}\,\gamma_{21} - 2\,\hbar\,p_{11}\,T_1\,x_2\,y_2\,\gamma_{12}\,\gamma_{21} + \hbar\,p_{11}\,T_1^2\,x_2\,y_2\,\gamma_{12}\,\gamma_{21}\big)\,\epsilon\Big)\Big/\left(\hbar - 3\,\hbar\,\gamma_{11} + 3\,\hbar\,T_1\,\gamma_{11} + 3\,\hbar\,\gamma_{11}^2 - 6\,\hbar\,T_1\,\gamma_{11}^2 + 3\,\hbar\,T_1^2\,\gamma_{11}^2 - \hbar\,\gamma_{11}^3 + 3\,\hbar\,T_1\,\gamma_{11}^3 - 3\,\hbar\,T_1^2\,\gamma_{11}^3 + \hbar\,T_1^3\,\gamma_{11}^3\right) + \Big(\left(8\,a_3\,p_{11}\,T_1 + \ll 1\gg + \ll 2726\gg + 3\,\gamma\,\ll 6\gg\,\gamma_{21}^3\right)\ \ll 1\gg\Big)\Big/\left(4 - 28\,\gamma_{11} + \ll 48\gg + 4\,T_1^7\,\gamma_{11}^7\right) + O[\epsilon]^3\Big]\Big\}\Big\}$$

```
Timing@With[{q⊙ = 𝔈QU[{x1, y1}1, {x2, a2, y2}2,
      ℏ (l12 t1 a2 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
      12 + ϵ (l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
  {Short[rhs = q⊙ // Rord_{x1,y1→3}, 5], HL@SimpT[QU[q⊙] == QU[rhs]]}]
```

{388.922,
{𝔈QU[{y3, a3, x3}1, {<<1>>}2, <<1>>/<<1>>, 1/(1 − γ11 + T1 γ11) + ((4 ℏ a2 l2 + 4 p11 − 4 p11 T1 + 4 ℏ p22 x2 y2 + <<339>> + γ ℏ^4 x2^2 y2^2 γ12^2 γ21^2 − 4 γ ℏ^4 T1 x2^2 y2^2 γ12^2 γ21^2 + 3 γ ℏ^4 T1^2 x2^2 y2^2 γ12^2 γ21^2) ϵ) / (4 ℏ − 20 ℏ γ11 + 20 ℏ T1 γ11 + 40 ℏ γ11^2 − 80 ℏ T1 γ11^2 + 40 ℏ T1^2 γ11^2 − 40 ℏ γ11^3 + <<13>> + 20 ℏ T1 γ11^5 − 40 ℏ T1^2 γ11^5 + 40 ℏ T1^3 γ11^5 − 20 ℏ T1^4 γ11^5 + 4 ℏ T1^5 γ11^5) + ((576 a3 p11 T1 + <<8073>> + <<1>>) <<1>>)/(<<79>> + 288 T1^9 γ11^9) + O[ϵ]^3], True}}

```
Timing@With[{q⊙ = 𝔈QU[{x1, y1}1, {x2, a2, y2}2,
      ℏ (l12 t1 a2 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
      12 + ϵ (l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
  {Short[rhs = q⊙ // Rord_{x1,y1→1}, 5], HL@SimpT[QU[q⊙] == QU[rhs]]}]
```

{336.781,
{𝔈QU[{y1, a1, x1}1, {<<1>>}2, <<1>>/<<1>>, 1/(1 − γ11 + T1 γ11) + ((4 ℏ a2 l2 + 4 p11 − 4 p11 T1 + 4 ℏ p11 x1 y1 + 4 ℏ p21 x2 y1 + <<338>> + γ ℏ^4 x2^2 y2^2 γ12^2 γ21^2 − 4 γ ℏ^4 T1 x2^2 y2^2 γ12^2 γ21^2 + 3 γ ℏ^4 T1^2 x2^2 y2^2 γ12^2 γ21^2) ϵ) / (4 ℏ − 20 ℏ γ11 + 20 ℏ T1 γ11 + 40 ℏ γ11^2 − 80 ℏ T1 γ11^2 + 40 ℏ T1^2 γ11^2 − 40 ℏ γ11^3 + <<10>> + 20 ℏ T1^4 γ11^4 − 4 ℏ γ11^5 + 20 ℏ T1 γ11^5 − 40 ℏ T1^2 γ11^5 + 40 ℏ T1^3 γ11^5 − 20 ℏ T1^4 γ11^5 + 4 ℏ T1^5 γ11^5) + ((576 a1 p11 T1 + <<8073>> + <<1>>) <<1>>)/(<<79>> + 288 T1^9 γ11^9) + O[ϵ]^3], True}}

Canonical ordering with Cord

Cord

In[∘]:=
```
Cord[𝔈U_[L___, {l___, u_i_, w_j_, r___}s_, R___, Q_, P_]] /;
    OrderedQ[{w, u} /. {y → 1, a → 2, x → 3}] :=
  ((*Echo@{ui,wj};*)Cord[Rord_{ui,wj→Unique[]}[𝔈U[L, {l, ui, wj, r}s, R, Q, P]]]);
Cord[𝔈U_[specs__, Q_, P_]] := 𝔈U[Sequence @@ Sort@{specs}, Q, P] /.
   Flatten[{specs} /. {yax__}s_ :> ({yax} /. u_i_ :> (ui → us))]
```

```
Cord@𝔈CU[{x1, y1}1, 0, 01 + x1 y1]
```

𝔈CU[{y1, a1, x1}1, 0, (−t1 + x1 y1) + 2 a1 ϵ + O[ϵ]^2]

```
Block[{$p = 4, $k = 0, c⊙ = ℭCU[{y1, a1, x1, x2, a2, y2}1,
    ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    1θ + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 Timing@{Short[Cord[c⊙], 8], HL@Simp[CU[c⊙] - CU[Cord[c⊙]]]}]
```

$$\{4.53125,\ \{\mathfrak{C}_{CU}[\{y_1, a_1, x_1\}_1,\ (e^{\gamma\hbar l_{11} t_1+2\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+2\gamma\hbar l_{22} t_2}\hbar a_1 l_{11} t_1 + e^{\gamma\hbar l_{11} t_1+2\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+2\gamma\hbar l_{22} t_2}\hbar a_1 l_{12} t_1 + \ll 12 \gg + e^{\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2}\hbar^2 a_1 l_{22} t_1 t_2 \gamma_{22} + \hbar x_1 y_1 \gamma_{22}) / (e^{\gamma\hbar l_{11} t_1+2\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+2\gamma\hbar l_{22} t_2} + e^{\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2}\hbar t_1 \gamma_{12} + e^{\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2}\hbar t_1 \gamma_{22}),\ \frac{e^{\gamma\hbar l_{12} t_1+\gamma\hbar l_{22} t_2}}{e^{\gamma\hbar l_{12} t_1+\gamma\hbar l_{22} t_2} + \hbar t_1 \gamma_{12} + \hbar t_1 \gamma_{22}} + O[\epsilon]^1],\ 0\}\}$$

```
Block[{$p = 4, $k = 1, c⊙ = ℭCU[{y1, a1, x1, x2, a2, y2}1,
    ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    11 + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 Timing@{Short[Cord[c⊙], 8], HL@Simp[CU[c⊙] - CU[Cord[c⊙]]]}]
```

$$\{81.2656,\ \{\mathfrak{C}_{CU}[\{y_1, a_1, x_1\}_1,\ (e^{\gamma\hbar l_{11} t_1+2\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+2\gamma\hbar l_{22} t_2}\hbar a_1 l_{11} t_1 + \ll 14 \gg + \hbar x_1 y_1 \gamma_{22}) / (e^{\gamma\hbar l_{11} t_1+2\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+2\gamma\hbar l_{22} t_2} + e^{\gamma\hbar l_{11} t_1+\ll 2 \gg+\gamma\hbar \ll 1 \gg t_2}\hbar t_1 \gamma_{12} + e^{\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2}\hbar t_1 \gamma_{22}),\ \frac{e^{\gamma\hbar l_{12} t_1+\gamma\hbar l_{22} t_2}}{e^{\gamma\hbar l_{12} t_1+\gamma\hbar l_{22} t_2} + \hbar t_1 \gamma_{12} + \hbar t_1 \gamma_{22}} + ((2 e^{2\gamma\hbar l_{11} t_1+6\gamma\hbar l_{12} t_1+2\gamma\hbar l_{21} t_2+6\gamma\hbar l_{22} t_2} a_1 l_1 + \ll 419 \gg)\epsilon) / (2 e^{2\gamma\hbar l_{11} t_1+6\gamma\hbar l_{12} t_1+2\gamma\hbar l_{21} t_2+6\gamma\hbar l_{22} t_2} + 10 e^{2\gamma\hbar l_{11} t_1+5\gamma\hbar l_{12} t_1+2\gamma\hbar l_{21} t_2+5\gamma\hbar l_{22} t_2}\hbar t_1 \gamma_{12} + \ll 18 \gg + 2 e^{2\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+2\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2}\hbar^5 t_1^5 \gamma_{22}^5) + O[\epsilon]^2],\ 0\}\}$$

```
With[{q⊙ = ℭQU[{y1, a1, x1, x2, a2, y2}1,
    ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    1θ + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 Cord[
  q⊙]]
```

Writing $E_2 = e^{\gamma\hbar l_{11} t_1+2\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+2\gamma\hbar l_{22} t_2}$ and $E_1 = e^{\gamma\hbar l_{11} t_1+\gamma\hbar l_{12} t_1+\gamma\hbar l_{21} t_2+\gamma\hbar l_{22} t_2}$ as printed in full on the page:

$$\mathfrak{C}_{QU}[\{y_1, a_1, x_1\}_1,\ (E_2\hbar a_1 l_{11} t_1 + E_2\hbar a_1 l_{12} t_1 + E_2\hbar a_1 l_{21} t_2 + E_2\hbar a_1 l_{22} t_2 + E_1\hbar x_1 y_1 \gamma_{11} - E_1\hbar a_1 l_{11} t_1 \gamma_{12} - E_1\hbar a_1 l_{12} t_1 \gamma_{12} - E_1\hbar a_1 l_{21} t_2 \gamma_{12} - E_1\hbar a_1 l_{22} t_2 \gamma_{12} + E_1\hbar a_1 l_{11} t_1 T_1 \gamma_{12} + E_1\hbar a_1 l_{12} t_1 T_1 \gamma_{12} + E_1\hbar a_1 l_{21} t_2 T_1 \gamma_{12} + E_1\hbar a_1 l_{22} t_2 T_1 \gamma_{12} + \hbar x_1 y_1 \gamma_{12} + E_1\hbar x_1 y_1 \gamma_{21} - E_1\hbar a_1 l_{11} t_1 \gamma_{22} - E_1\hbar a_1 l_{12} t_1 \gamma_{22} - E_1\hbar a_1 l_{21} t_2 \gamma_{22} + E_1\hbar a_1 l_{22} t_2 \gamma_{22} + E_1\hbar a_1 l_{11} t_1 T_1 \gamma_{22} + E_1\hbar a_1 l_{12} t_1 T_1 \gamma_{22} + E_1\hbar a_1 l_{21} t_2 T_1 \gamma_{22} + E_1\hbar a_1 l_{22} t_2 T_1 \gamma_{22} + \hbar x_1 y_1 \gamma_{22}) / (E_2 - E_1\gamma_{12} + E_1 T_1 \gamma_{12} - E_1\gamma_{22} + E_1 T_1 \gamma_{22}),\ \frac{e^{\gamma\hbar l_{12} t_1+\gamma\hbar l_{22} t_2}}{e^{\gamma\hbar l_{12} t_1+\gamma\hbar l_{22} t_2} - \gamma_{12} + T_1\gamma_{12} - \gamma_{22} + T_1\gamma_{22}} + O[\epsilon]^1]$$

Stitching Œ's.

StitchingOEs

In[∘]:=
```
m_{j_→k_}[Œ_{U_}[specs__, Q_, P_]] := Cord[Œ_U[Sequence @@ Append[DeleteCases[{specs}, {__}_{j|k}],
      Flatten[{Cases[{specs}, {us__}_j ⧴ {us}], Cases[{specs}, {us__}_k ⧴ {us}]}]_k],
    Q, P] /. {t_j → t_k, T_j → T_k}]
```

```
cœ = Œ_CU[{y_1, a_1, x_1}_1, {y_2, a_2, x_2}_2,
   {y_3, a_3, x_3}_3, ℏ Sum[l_{10 i+j} t_i a_j + γ_{10 i+j} y_i x_j, {i, 3}, {j, 3}], 1_2];
{cœ // m_{3→4}, HL@Simp[CU[m_{3→4}[cœ]] - m_{3→4}[CU[cœ]]]}
```

$\{Œ_{CU}[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, \{y_4, a_4, x_4\}_4, \hbar\,(a_1 l_{11} t_1 + a_2 l_{12} t_1 + a_4 l_{13} t_1 + a_1 l_{21} t_2 + a_2 l_{22} t_2 + a_4 l_{23} t_2 + a_1 l_{31} t_4 + a_2 l_{32} t_4 + a_4 l_{33} t_4 + x_1 y_1 \gamma_{11} + x_2 y_1 \gamma_{12} + x_4 y_1 \gamma_{13} + x_1 y_2 \gamma_{21} + x_2 y_2 \gamma_{22} + x_4 y_2 \gamma_{23} + x_1 y_4 \gamma_{31} + x_2 y_4 \gamma_{32} + x_4 y_4 \gamma_{33}), 1 + O[\epsilon]^3], 0\}$

Verifying that *m* commutes with evaluation, in CU:

```
cœ = Œ_CU[{y_1, a_1, x_1}_1, {y_2, a_2, x_2}_2,
   {y_3, a_3, x_3}_3, ℏ Sum[l_{10 i+j} t_i a_j + γ_{10 i+j} y_i x_j, {i, 3}, {j, 3}], 1_2];
Timing@{cœ // m_{2→3}, HL@Simp[CU[m_{2→3}[cœ]] - m_{2→3}[CU[cœ]]]}
```

$\{513.453, \{Œ_{CU}[\{y_1, a_1, x_1\}_1, \{y_3, a_3, x_3\}_3, \frac{\cdots 1 \cdots}{e^{\cdots 1 \cdots}\ \cdots 1 \cdots\ \cdots 1 \cdots}, \frac{1}{1+\hbar t_3 \gamma_{32}} + \left(\left(4\, e^{2\gamma\hbar l_{12} t_1 + \cdots 4 \cdots + 2\gamma\hbar l_{33} t_3}\, \hbar^2 a_3 x_1 y_1 \gamma_{12} \gamma_{31} - 2 \cdots 7 \cdots \gamma_{31} + \cdots 154 \cdots\right) \epsilon\right) \Big/ \left(2\, e^{2\gamma\hbar l_{12} t_1 + 2\gamma\hbar l_{13} t_1 + \cdots 3 \cdots + 2\gamma\hbar l_{33} t_3} + 10\, e^{\cdots 1 \cdots}\, \hbar t_3 \gamma_{32} + \cdots 2 \cdots + \cdots 1 \cdots + 2\, e^{2\gamma\hbar l_{12} t_1 + \cdots 4 \cdots + \cdots 1 \cdots}\, \hbar^5 t_3^5 \gamma_{32}^5\right) + \frac{(\cdots 1 \cdots)\, \epsilon^2}{\cdots 1 \cdots} + O[\epsilon]^3], 0\}\}$

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Verifying that *m* commutes with evaluation, in QU:

```
qœ = Œ_QU[{y_1, a_1, x_1}_1, {y_2, a_2, x_2}_2,
   {y_3, a_3, x_3}_3, ℏ Sum[l_{10 i+j} t_i a_j + γ_{10 i+j} y_i x_j, {i, 3}, {j, 3}], 1_2];
Timing@{qœ // m_{2→3}, HL@SimpT[QU[m_{2→3}[qœ]] - m_{2→3}[QU[qœ]]]}
```

$\{7831.47, \{Œ_{QU}[\{y_1, a_1, x_1\}_1, \{y_3, a_3, x_3\}_3, \frac{\cdots 1 \cdots}{\cdots 1 \cdots}, \frac{1}{1-\gamma_{32}+T_3 \gamma_{32}} + \left(\left(8\, e^{2\gamma\hbar l_{12} t_1 + \cdots 4 \cdots + 2\gamma\hbar l_{33} t_3}\, \hbar^2 a_3 T_3 x_1 y_1 \gamma_{12} \gamma_{31} + 4 \cdots 8 \cdots \gamma_{31} + \cdots 371 \cdots\right) \epsilon\right) \Big/ \left(4\, e^{2\gamma\hbar l_{12} t_1 + 2\gamma\hbar l_{13} t_1 + \cdots 3 \cdots + 2\gamma\hbar l_{33} t_3} - 20\, e^{\cdots 1 \cdots}\, \gamma_{32} + \cdots 26 \cdots + 4\, e^{2\gamma\hbar l_{12} t_1 + \cdots 4 \cdots + \cdots 1 \cdots}\, T_3^5 \gamma_{32}^5\right) + \frac{(\cdots 1 \cdots)\, \epsilon^2}{\cdots 79 \cdots + \cdots 1 \cdots} + O[\epsilon]^3], 0\}\}$

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In[ ]:=
```
𝔈U_[sp1__, Q1_, P1_] ≡ 𝔈U_[sp2__, Q2_, P2_] :=
  Sort[{sp1}] == Sort[{sp2}] ∧ Simplify[Q1 == Q2] ∧ Simplify[Normal[P1 - P2] == 0]
```

Verifying meta-associativity in CU:

```
co = 𝔈CU[{y1, a1, x1}1, {y2, a2, x2}2,
    {y3, a3, x3}3, ℏ Sum[λ10 i+j ti aj + γ10 i+j yi xj, {i, 3}, {j, 3}], 10];
Timing@HL[(lhs = co // m1,2→1 // m1,3→1) ≡ (rhs = co // m2,3→2 // m1,2→1)]
```

{41.9219, True}

```
co = 𝔈CU[{y1, a1, x1}1, {y2, a2, x2}2,
    {y3, a3, x3}3, ℏ Sum[λ10 i+j ti aj + γ10 i+j yi xj, {i, 3}, {j, 3}], 11];
Timing@HL[(lhs = co // m1,2→1 // m1,3→1) ≡ (rhs = co // m2,3→2 // m1,2→1)]
```

{30 119.8, True}

mexamples

```
co = 𝔈CU[{y1, a1, x1}1, {y2, a2, x2}2, ℏ Sum[l10 i+j ti aj + γ10 i+j yi xj, {i, 2}, {j, 2}], 11];
Short[Simplify /@ (cexample = co // m1→2), 12]
Short[Simplify /@ (qexample = (qo = co /. CU → QU) // m1→2), 12]
```

mexamples

$$\mathbb{E}_{CU}\Big[\{y_2,a_2,x_2\}_2,\ \hbar\, a_2\,(l_{11}+l_{12}+l_{21}+l_{22})\,t_2+\frac{1}{1+\hbar t_2\gamma_{21}}\,e^{-\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\,\hbar\,x_2y_2\,\Big(\gamma_{21}+e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\gamma_{12}\,(1+\hbar t_2\gamma_{21})+e^{\gamma\hbar(l_{12}+l_{22})t_2}\gamma_{22}+\gamma_{11}\big(e^{\gamma\hbar(l_{11}+l_{21})t_2}-e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar t_2\gamma_{22}\big)\Big),$$

$$\frac{1}{1+\hbar t_2\gamma_{21}}+\frac{1}{2\,(1+\hbar t_2\gamma_{21})^5}\,e^{-2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\,\hbar\,\Big(4a_2\,(1+\hbar t_2\gamma_{21})^2\Big(e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar\,\big(e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}t_2+x_2y_2\big)\gamma_{21}^2+e^{2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar\,x_2y_2\gamma_{11}\gamma_{22}+\gamma_{21}\big(e^{2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}+\hbar\,x_2y_2\,\big(e^{\gamma\hbar(2l_{11}+l_{12}+2l_{21}+l_{22})t_2}\gamma_{11}+e^{\gamma\hbar(l_{11}+2l_{12}+l_{21}+2l_{22})t_2}\gamma_{22}\big)\big)\Big)-\gamma\hbar\,\Big(-2e^{2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}t_2\gamma_{21}^2\,(1+\hbar t_2\gamma_{21})^2+4\ll5\gg(\ll1\gg)+\hbar\ll4\gg\Big(3\hbar t_2\gamma_{21}^2+2e^{\gamma\hbar(l_{12}+l_{\ll2\gg})t_2}\gamma_{22}+\gamma_{21}\big(4+e^{\gamma\ll3\gg}\hbar t_2\gamma_{22}\big)+e^{\gamma\hbar(l_{11}+l_{21})t_2}\gamma_{11}\big(2+\hbar t_2\,(\gamma_{21}-e^{\gamma\hbar(l_{12}+l_{22})t_2}\gamma_{22})\big)\Big)\Big)\Big)\,\epsilon+O[\epsilon]^2\Big]$$

mexamples

$$\mathbb{E}_{QU}\Big[\{y_2,a_2,x_2\}_2,\ \hbar\, a_2\,(l_{11}+l_{12}+l_{21}+l_{22})\,t_2+\frac{1}{1+(-1+T_2)\gamma_{21}}\,e^{-\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\,\hbar\,x_2y_2\,\Big(\gamma_{21}+e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\gamma_{12}\,(1+(-1+T_2)\gamma_{21})+e^{\gamma\hbar(l_{12}+l_{22})t_2}\gamma_{22}+\gamma_{11}\big(e^{\gamma\hbar(l_{11}+l_{21})t_2}-e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}(-1+T_2)\gamma_{22}\big)\Big),$$

$$\frac{1}{1+(-1+T_2)\gamma_{21}}+\frac{1}{4\,(1+(-1+T_2)\gamma_{21})^5}\,e^{-2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\,\hbar\,\Big(8a_2T_2\,(1+(-1+T_2)\gamma_{21})^2\Big(e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\big(-e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}+e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}T_2+\hbar\,x_2y_2\big)\gamma_{21}^2+e^{2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar\,x_2y_2\gamma_{11}\gamma_{22}+\gamma_{21}\big(e^{2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}+\hbar\,x_2y_2\,\big(e^{\gamma\hbar(2l_{11}+l_{12}+2l_{21}+l_{22})t_2}\gamma_{11}+e^{\gamma\hbar(l_{11}+2l_{12}+l_{21}+2l_{22})t_2}\gamma_{22}\big)\big)\Big)+\gamma\,\Big(2e^{2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\,(1-4T_2+3T_2^2)\,\gamma_{21}^2\,(1+(-1+T_2)\gamma_{21})^2+4e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar\,x_2y_2\gamma_{21}\,(1+(-1+T_2)\gamma_{21})\,\big((\ll1\gg)-\ll1\gg\big)\Big)\Big)\,\epsilon+O[\epsilon]^2\Big]$$

R in QU.

The Faddeev-Quesne formula:

Faddeev

In[∘]:=
```
e_{q_,k_}[x_] := e^(Sum_{j=1}^{k+1} (1 - q)^j x^j / (j (1 - q^j))); e_{q_}[x_] := e_{q,$k}[x]
```

```
Table[Series[e_{q_ħ,k}[x], {ϵ, 0, 4}], {k, 0, 5}] // Column
```

e^x

$e^x - \frac{1}{4}(e^x x^2 \gamma \hbar)\epsilon + \frac{1}{32} e^x x^4 \gamma^2 \hbar^2 \epsilon^2 - \frac{1}{384}(e^x x^2 (-8 + x^4)\gamma^3 \hbar^3)\epsilon^3 + \frac{e^x x^4 (-32+x^4)\gamma^4 \hbar^4 \epsilon^4}{6144} + O[\epsilon]^5$

$e^x - \frac{1}{4}(e^x x^2 \gamma \hbar)\epsilon + \frac{1}{288} e^x x^3 (32 + 9x)\gamma^2 \hbar^2 \epsilon^2 - \frac{(e^x x^2 (-24+32x^3+3x^4)\gamma^3\hbar^3)\epsilon^3}{1152} + \frac{e^x x^3 (-4608-864x+1024x^3+576x^4+27x^5)\gamma^4\hbar^4\epsilon^4}{165888} + O[\epsilon]^5$

$e^x - \frac{1}{4}(e^x x^2 \gamma \hbar)\epsilon + \frac{1}{288} e^x x^3 (32 + 9x)\gamma^2 \hbar^2 \epsilon^2 - \frac{(e^x x^2 (-24+72x^2+32x^3+3x^4)\gamma^3\hbar^3)\epsilon^3}{1152} + \frac{e^x x^3 (-4608-864x+3616x^3+576x^4+27x^5)\gamma^4\hbar^4\epsilon^4}{165888} + O[\epsilon]^5$

$e^x - \frac{1}{4}(e^x x^2 \gamma \hbar)\epsilon + \frac{1}{288} e^x x^3 (32 + 9x)\gamma^2 \hbar^2 \epsilon^2 - \frac{(e^x x^2 (-24+72x^2+32x^3+3x^4)\gamma^3\hbar^3)\epsilon^3}{1152} + \frac{1}{4147200} e^x x^3 (-115200 - 21600x + 165888x^2 + 90400x^3 + 14400x^4 + 675x^5)\gamma^4\hbar^4\epsilon^4 + O[\epsilon]^5$

$e^x - \frac{1}{4}(e^x x^2 \gamma \hbar)\epsilon + \frac{1}{288} e^x x^3 (32 + 9x)\gamma^2 \hbar^2 \epsilon^2 - \frac{(e^x x^2 (-24+72x^2+32x^3+3x^4)\gamma^3\hbar^3)\epsilon^3}{1152} + \frac{1}{4147200} e^x x^3 (-115200 - 21600x + 165888x^2 + 90400x^3 + 14400x^4 + 675x^5)\gamma^4\hbar^4\epsilon^4 + O[\epsilon]^5$

```
Table[Together@SeriesCoefficient[e_{q,5}[x], {x, 0, n}], {n, 0, 5}]
```

$\left\{1, 1, \frac{1}{1+q}, \frac{1}{(1+q)(1+q+q^2)}, \frac{1}{(1+q)^2(1+q^2)(1+q+q^2)}, 1\Big/\left((1+q)^2(1+q^2)(1+q+q^2)(1+q+q^2+q^3+q^4)\right)\right\}$

```
Table[HL@FunctionExpand[QFactorial[n, q] SeriesCoefficient[e_{q,5}[x], {x, 0, n}]], {n, 0, 5}]
```

{1, 1, 1, 1, 1, 1}

R

In[∘]:=
```
QU[R_{i_,j_}] := O_QU[{y_1, a_1}_i, {a_2, x_2}_j, SS[e^(ħ b_1 a_2) e_{q_ħ}[ħ y_1 x_2] /. b_1 → γ^-1 (ϵ a_1 - t_i)]];
QU[R^-1_{i_,j_}] := S_j@QU[R_{i,j}];
```

```
QU[R_{3,4}] // Short
```

$QU[] + \frac{\epsilon \hbar QU[a_3, a_4]}{\gamma} + \hbar QU[y_3, x_4] + \frac{\epsilon \ll1\gg \ll1\gg}{\gamma} + \frac{1}{2}\ll1\gg \ll1\gg - \frac{\ll1\gg}{\gamma} - \frac{\epsilon \hbar^2 \ll1\gg t_3}{\gamma^2} - \frac{\hbar^2 QU[y_3, a_4, x_4] t_3}{\gamma} + \frac{\hbar^2 QU[a_4, a_4] t_3^2}{2\gamma^2}$

Verifying R2 (~2 secs @ $p=4, $k=2):

```
QU[R_{1,2} ** R^-1_{1,2}] // Simp // HL // Timing
```

{0.078125, QU[]}

Verifying R3 (~156 secs @ $p=4, $k=2):

```
{Short[lhs = QU[R1,2 ** R1,3 ** R2,3]], HL@SimpT[lhs - QU[R2,3 ** R1,3 ** R1,2]]} // Timing
```

$$\left\{0.203125,\ \left\{QU[\,] + \frac{\epsilon\,\hbar\,QU[a_1, a_2]}{\gamma} + \frac{\epsilon\,\hbar\,QU[a_1, a_3]}{\gamma} + \ll 73 \gg + 2\,\epsilon\,\hbar^2\,QU[y_1, a_2, x_3]\,T_2 + QU[y_1, x_3]\,(\hbar - \hbar\,T_2),\ 0\right\}\right\}$$

R in $Œ_{QU}$.

RinOE

In[∘]:=

```
ŒQU,k_[Ri_,j_] := ŒQU[{yi, ai, xi}i, {yj, aj, xj}j, -ħ γ^-1 ti aj + ħ yi xj,
  Series[e^(ħ γ^-1 ti aj - ħ yi xj) (e^(ħ bi aj) eqħ,k[ħ yi xj] /. bi → γ^-1 (ϵ ai - ti)), {ϵ, 0, k}]]
```

```
{ŒQU,1[R1,2], ŒQU,2[R1,2]}
```

$$\left\{Œ_{QU}\left[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, -\frac{\hbar\,a_2\,t_1}{\gamma} + \hbar\,x_2\,y_1,\ 1 + \left(\frac{\hbar\,a_1\,a_2}{\gamma} - \frac{1}{4}\,\gamma\,\hbar^3\,x_2^2\,y_1^2\right)\epsilon + O[\epsilon]^2\right],\right.$$

$$Œ_{QU}\left[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, -\frac{\hbar\,a_2\,t_1}{\gamma} + \hbar\,x_2\,y_1,\ 1 + \left(\frac{\hbar\,a_1\,a_2}{\gamma} - \frac{1}{4}\,\gamma\,\hbar^3\,x_2^2\,y_1^2\right)\epsilon + \right.$$

$$\left.\left.\frac{1}{288\,\gamma^2}\left(144\,\hbar^2\,a_1^2\,a_2^2 - 72\,\gamma^2\,\hbar^4\,a_1\,a_2\,x_2^2\,y_1^2 + 32\,\gamma^4\,\hbar^5\,x_2^3\,y_1^3 + 9\,\gamma^4\,\hbar^6\,x_2^4\,y_1^4\right)\epsilon^2 + O[\epsilon]^3\right]\right\}$$

The morphism $Œ_{U,k}$.

MorphismOE

In[∘]:=

```
ŒU_,k_[a_ * b_] := ŒU,k[a] ŒU,k[b];
ŒU_,k_[mis__[a_]] := mis[ŒU,k[a]];
```

```
ŒQU,1[R1,2 R3,4 R5,6]
```

$$Œ_{QU}\left[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, \{y_3, a_3, x_3\}_3, \{y_4, a_4, x_4\}_4, \{y_5, a_5, x_5\}_5, \{y_6, a_6, x_6\}_6, -\frac{\hbar\,a_2\,t_1}{\gamma} - \frac{\hbar\,a_4\,t_3}{\gamma} - \frac{\hbar\,a_6\,t_5}{\gamma} + \hbar\,x_2\,y_1 + \hbar\,x_4\,y_3 + \hbar\,x_6\,y_5,\right.$$

$$\left.1 + \left(\frac{\hbar\,a_1\,a_2}{\gamma} + \frac{\hbar\,a_3\,a_4}{\gamma} + \frac{\hbar\,a_5\,a_6}{\gamma} - \frac{1}{4}\,\gamma\,\hbar^3\,x_2^2\,y_1^2 - \frac{1}{4}\,\gamma\,\hbar^3\,x_4^2\,y_3^2 - \frac{1}{4}\,\gamma\,\hbar^3\,x_6^2\,y_5^2\right)\epsilon + O[\epsilon]^2\right]$$

```
ŒQU,1[R1,2 R3,4 R5,6 // m1,3→1 // m2,5→2 // m4,6→4]
```

$$Œ_{QU}\left[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, \{y_4, a_4, x_4\}_4, \frac{1}{\gamma}\left(-\hbar\,a_2\,t_1 - \hbar\,a_4\,t_1 - \hbar\,a_4\,t_2 + \gamma\,\hbar\,x_2\,y_1 + \gamma\,\hbar\,x_4\,y_1 + e^{\hbar\,t_2}\,\gamma\,\hbar\,x_4\,y_1 - \gamma\,\hbar\,T_2\,x_4\,y_1 + e^{\hbar\,t_1}\,\gamma\,\hbar\,x_4\,y_2\right),\right.$$

$$1 + \frac{1}{4\,\gamma}\left(4\,\hbar\,a_1\,a_2 + 4\,\hbar\,a_1\,a_4 + 4\,\hbar\,a_2\,a_4 - 4\,\gamma\,\hbar^2\,a_4\,x_2\,y_1 - 8\,e^{\hbar\,t_2}\,\gamma\,\hbar^2\,a_2\,x_4\,y_1 + 8\,\gamma\,\hbar^2\,a_2\,T_2\,x_4\,y_1 - \gamma^2\,\hbar^3\,x_2^2\,y_1^2 + 4\,e^{\hbar\,t_2}\,\gamma^2\,\hbar^3\,x_2\,x_4\,y_1^2 - 4\,\gamma^2\,\hbar^3\,T_2\,x_2\,x_4\,y_1^2 - \gamma^2\,\hbar^3\,x_4^2\,y_1^2 - e^{2\,\hbar\,t_2}\,\gamma^2\,\hbar^3\,x_4^2\,y_1^2 + \gamma^2\,\hbar^3\,T_2^2\,x_4^2\,y_1^2 - 4\,e^{\hbar\,t_1}\,\gamma\,\hbar^2\,a_1\,x_4\,y_2 + 4\,e^{\hbar\,t_1}\,\gamma^2\,\hbar^3\,x_2\,x_4\,y_1\,y_2 + 4\,e^{\hbar\,t_1 + \hbar\,t_2}\,\gamma^2\,\hbar^3\,x_4^2\,y_1\,y_2 - 4\,e^{\hbar\,t_1}\,\gamma^2\,\hbar^3\,T_2\,x_4^2\,y_1\,y_2 - e^{2\,\hbar\,t_1}\,\gamma^2\,\hbar^3\,x_4^2\,y_2^2\right)\epsilon + O[\epsilon]^2\right]$$

$Œ_{QU,1}[R_{1,2}\,R_{3,4}\,R_{5,6}$ // $m_{3,1\to 1}$ // $m_{5,2\to 2}$ // $m_{6,4\to 4}]$

$$Œ_{QU}\Big[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, \{y_4, a_4, x_4\}_4, \frac{1}{\gamma}\left(-\hbar\, a_2\, t_1 - \hbar\, a_4\, t_1 - \hbar\, a_4\, t_2 + \gamma\,\hbar\, x_2\, y_1 + \gamma\,\hbar\, x_4\, y_1 + e^{\hbar\, t_1}\,\gamma\,\hbar\, x_4\, y_2\right),$$
$$1 + \frac{1}{4\,\gamma}\Big(4\,\hbar\, a_1\, a_2 + 4\,\hbar\, a_1\, a_4 + 4\,\hbar\, a_2\, a_4 - 4\,\gamma\,\hbar^2\, a_4\, x_2\, y_1 - \gamma^2\,\hbar^3\, x_2^2\, y_1^2 - \gamma^2\,\hbar^3\, x_4^2\, y_1^2 -$$
$$4\, e^{\hbar\, t_1}\,\gamma\,\hbar^2\, a_1\, x_4\, y_2 + 4\, e^{\hbar\, t_1}\,\gamma^2\,\hbar^3\, x_2\, x_4\, y_1\, y_2 - e^{2\,\hbar\, t_1}\,\gamma^2\,\hbar^3\, x_4^2\, y_2^2\Big)\,\epsilon + O[\epsilon]^2\Big]$$

$Œ_{QU,1}[R_{1,2}\,R_{3,4}\,R_{5,6}$ // $m_{1,3\to 1}$ // $m_{2,5\to 2}$ // $m_{4,6\to 4}] \equiv Œ_{QU,1}[R_{1,2}\,R_{3,4}\,R_{5,6}$ // $m_{3,1\to 1}$ // $m_{5,2\to 2}$ // $m_{6,4\to 4}]$

$$\hbar\,\left(e^{\hbar\, t_2} - T_2\right)\, x_4\, y_1 == 0\;\&\&\;\epsilon\,\hbar\,\left(e^{\hbar\, t_2} - T_2\right)\, x_4\, y_1\,\left(8\, a_2 + \gamma\,\hbar\,\left(-4\, x_2\, y_1 + x_4\,\left(\left(e^{\hbar\, t_2} + T_2\right)\, y_1 - 4\, e^{\hbar\, t_1}\, y_2\right)\right)\right) == 0$$

Exponentials as needed.

In[ ]:=
```
Block[{$p = 2, $k = 2}, TableForm[StringSplit[
  "y|a|x|CΘ@y_CU|CΘ@a_CU|CΘ@x_CU|QΘ@y_QU|QΘ@a_QU|QΘ@x_QU|A𝔻@y_QU|A𝔻@a_QU|A𝔻@x_QU|S𝔻@y_QU|S𝔻@a_QU|S𝔻@x_QU|S@y_CU|S@a_CU|S@x_CU|S@y_QU|S@a_QU|S@x_QU|Δ@y_CU|Δ@a_CU|Δ@x_CU|Δ@y_QU|Δ@a_QU|Δ@x_QU",
  "|"] /. s_String ↦
  {s, Normal@Simplify@Series[ToExpression[s] /. CU | QU → Times, {ϵ, 0, $k}]}]]
```

Out[ ]//TableForm=

y	y
a	a
x	x
$C\Theta@y_{CU}$	$-x$
$C\Theta@a_{CU}$	$-a$
$C\Theta@x_{CU}$	$-y$
$Q\Theta@y_{QU}$	$-\frac{x}{\sqrt{T}}-\frac{a\,x\,\epsilon\,\hbar}{\sqrt{T}}-\frac{a^2\,x\,\epsilon^2\,\hbar^2}{2\sqrt{T}}$
$Q\Theta@a_{QU}$	$-a$
$Q\Theta@x_{QU}$	$-\frac{y}{\sqrt{T}}+\frac{y\,(-a+\gamma)\,\epsilon\,\hbar}{\sqrt{T}}-\frac{y\,(a-\gamma)^2\,\epsilon^2\,\hbar^2}{2\sqrt{T}}$
$A\mathbb{D}@y_{QU}$	$\frac{2}{3}a^2\,y\,\epsilon^2\,\hbar^2+\frac{1}{6}y\left(6+3\,t\,\hbar+t^2\,\hbar^2\right)+\frac{1}{12}y\,\epsilon\,\hbar\left(x\,y\,\gamma\,\hbar-4\,a\left(3+2\,t\,\hbar\right)\right)$
$A\mathbb{D}@a_{QU}$	a
$A\mathbb{D}@x_{QU}$	x
$S\mathbb{D}@y_{QU}$	$y+\frac{1}{48}t^2\,y\,\hbar^2+\frac{1}{24}y\left(-2\,a\,t+x\,y\,\gamma\right)\epsilon\,\hbar^2+\frac{1}{12}a^2\,y\,\epsilon^2\,\hbar^2$
$S\mathbb{D}@a_{QU}$	a
$S\mathbb{D}@x_{QU}$	$\frac{7}{12}a^2\,x\,\epsilon^2\,\hbar^2+x\left(1+\frac{t\,\hbar}{2}+\frac{7\,t^2\,\hbar^2}{48}\right)+\frac{1}{24}x\,\epsilon\,\hbar\left(x\,y\,\gamma\,\hbar-2\,a\left(12+7\,t\,\hbar\right)\right)$
$S@y_{CU}$	$-y$
$S@a_{CU}$	$-a$
$S@x_{CU}$	$-x$
$S@y_{QU}$	$-\frac{y}{T}+\frac{y\,(-a+\gamma)\,\epsilon\,\hbar}{T}-\frac{y\,(a-\gamma)^2\,\epsilon^2\,\hbar^2}{2\,T}$
$S@a_{QU}$	$-a$
$S@x_{QU}$	$-x-a\,x\,\epsilon\,\hbar-\frac{1}{2}a^2\,x\,\epsilon^2\,\hbar^2$
$\Delta@y_{CU}$	y_1+y_2
$\Delta@a_{CU}$	a_1+a_2
$\Delta@x_{CU}$	x_1+x_2
$\Delta@y_{QU}$	$y_1+T_1\,y_2-\epsilon\,\hbar\,a_1\,T_1\,y_2+\frac{1}{2}\epsilon^2\,\hbar^2\,a_1^2\,T_1\,y_2$
$\Delta@a_{QU}$	a_1+a_2
$\Delta@x_{QU}$	$x_1+x_2-\epsilon\,\hbar\,a_1\,x_2+\frac{1}{2}\epsilon^2\,\hbar^2\,a_1^2\,x_2$

Exp

Task. Define $\text{Exp}_{U_i,k}[\xi, P]$ which computes $e^{\xi\,\mathbb{O}(P)}$ to ϵ^k in the algebra U_i, where ξ is a scalar, X is x_i or y_i, and P is an ϵ-dependent near-docile element, giving the answer in Œ-form. Should satisfy $U@\text{Exp}_{U_i,k}[\xi, P] == \mathbb{S}_U\left[e^{\xi x},\ x \to \mathbb{O}(P)\right]$.

Methodology. If $P_0 := P_{\epsilon=0}$ and $e^{\xi\,\mathbb{O}(P)} = \mathbb{O}\left(e^{\xi P_0} F(\xi)\right)$, then $F(\xi=0)=1$ and we have:

$\mathbb{O}\left(e^{\xi P_0}(P_0\,F(\xi) + \partial_\xi F\right) = \mathbb{O}\left(\partial_\xi e^{\xi P_0} F(\xi)\right) = \partial_\xi \mathbb{O}\left(e^{\xi P_0} F(\xi)\right) = \partial_\xi e^{\xi\,\mathbb{O}(P)} = e^{\xi\,\mathbb{O}(P)}\,\mathbb{O}(P) = \mathbb{O}\left(e^{\xi P_0} F(\xi)\right)\mathbb{O}(P)$.

This is an ODE for F. Setting inductively $F_k = F_{k-1} + \epsilon^k \varphi$ we find that $F_0 = 1$ and solve for φ.

Exp

In[∘]:=
```
(* Bug: The first line is valid only if 𝕆(𝕖^P0)==𝕖^𝕆(P0). *)
(* Bug: ξ must be a symbol. *)
Exp_{U_i_,0}[ξ_, P_] := 𝔼_U[{y_i, a_i, x_i}_i, Normal@P /. ϵ → 0, 1 + 0_0];
Exp_{U_i_,k_}[ξ_, P_] := Module[{yax = {y_i, a_i, x_i}, P0, φ, φs, F, j, rhs, at0, atξ},
  P0 = Normal@P /. ϵ → 0;
  φs =
   Flatten@Table[φ_{j1,j2,j3}[ξ], {j2, 0, k}, {j1, 0, 2 k + 1 - j2}, {j3, 0, 2 k + 1 - j2 - j1}];
  F = Normal@Last@Exp_{U_i,k-1}[ξ, P] + ϵ^k φs.(φs /. φ_{js__}[ξ] :→ Times @@ yax^{js});
  rhs = Normal@Last@m_{i,j→i}[𝔼_U[yax_i, ξ P0, F + 0_k] m_{i→j}@𝔼_U[{y_i, a_i, x_i}_i, 0, P + 0_k]];
  at0 = (# == 0) & /@ Flatten@CoefficientList[F - 1 /. ξ → 0, yax];
  atξ = (# == 0) & /@ Flatten@CoefficientList[(∂_ξ F) + P0 F - rhs, yax];
  𝔼_U[yax_i, ξ P0, F + 0_k] /. DSolve[And @@ (at0 ⋃ atξ), φs, ξ][[1]] ]
```

In[∘]:=
```
Timing@Block[{$p = 4, $k = 2}, {
   s = S_1[QU[y_1]] /. QU → Times,
   exps = Exp_{QU_1,$k}[η, s], (* Warning: wrong unless $p≥$k+1! *)
   HL@Simp[S_1@𝕆_QU[{y_1}_1, SS[𝕖^{ħ η y_1}]] - QU@(exps /. η → ħ η)]
  }]
```

Out[∘]=

$$\left\{35.8281,\ \left\{a_1\left(-\frac{\epsilon\,\hbar}{T_1}+\frac{\gamma\,\epsilon^2\,\hbar^2}{T_1}\right)y_1+\left(-\frac{1}{T_1}+\frac{\gamma\,\epsilon\,\hbar}{T_1}-\frac{\gamma^2\,\epsilon^2\,\hbar^2}{2\,T_1}\right)y_1-\frac{\epsilon^2\,\hbar^2\,a_1^2\,y_1}{2\,T_1},\right.\right.$$

$$\mathbb{E}_{QU}\left[\{y_1,a_1,x_1\}_1,\ -\frac{\eta\,y_1}{T_1},\ 1+\frac{\left(2\,\gamma\,\eta\,\hbar\,T_1\,y_1-2\,\eta\,\hbar\,a_1\,T_1\,y_1-\gamma\,\eta^2\,\hbar\,y_1^2\right)\epsilon}{2\,T_1^2}+\right.$$

$$\left(-\frac{\gamma^2\,\eta\,\hbar^2\,y_1}{2\,T_1}+\frac{\gamma\,\eta\,\hbar^2\,a_1\,y_1}{T_1}-\frac{\eta\,\hbar^2\,a_1^2\,y_1}{2\,T_1}+\frac{7\,\gamma^2\,\eta^2\,\hbar^2\,y_1^2}{4\,T_1^2}-\frac{2\,\gamma\,\eta^2\,\hbar^2\,a_1\,y_1^2}{T_1^2}+\right.$$

$$\left.\left.\left.\frac{\eta^2\,\hbar^2\,a_1^2\,y_1^2}{2\,T_1^2}-\frac{\gamma^2\,\eta^3\,\hbar^2\,y_1^3}{T_1^3}+\frac{\gamma\,\eta^3\,\hbar^2\,a_1\,y_1^3}{2\,T_1^3}+\frac{\gamma^2\,\eta^4\,\hbar^2\,y_1^4}{8\,T_1^4}\right)\epsilon^2+O[\epsilon]^3\right],\ 0\right\}\right\}$$

In[∘]:=
```
Timing@Block[{$p = 4, $k = 2}, {
   s = S_1[QU[a_1]] /. QU → Times,
   exps = Exp_{QU_1,$k}[α, s], (* Warning: wrong unless $p≥$k+1! *)
   HL@Simp[S_1@𝕆_QU[{a_1}_1, SS[𝕖^{ħ α a_1}]] - QU@(exps /. α → ħ α)]
  }]
```

Out[∘]=

$$\left\{33.5938,\ \left\{-a_1,\ \mathbb{E}_{QU}\left[\{y_1,a_1,x_1\}_1,\ -\alpha\,a_1,\ 1+O[\epsilon]^3\right],\ 0\right\}\right\}$$

```
In[∘]:= Timing@Block[{$p = 4, $k = 2}, {
    s = S₁[QU[x₁]] /. QU → Times,
    exps = Exp_{QU₁,$k}[ξ, s], (* Warning: wrong unless $p≥$k+1! *)
    HL@Simp[S₁@𝕆_QU[{x₁}₁, SS[e^{ħ ξ x₁}]] - QU@(exps /. ξ → ħ ξ)]
  }]
```

Out[∘]= $\{34.0625, \{-x_1 - \epsilon\,\hbar\,a_1\,x_1 - \frac{1}{2}\epsilon^2\,\hbar^2\,a_1^2\,x_1,$

$\mathbb{E}_{QU}[\{y_1, a_1, x_1\}_1, -\xi\,x_1, 1 + \left(-\xi\,\hbar\,a_1\,x_1 - \frac{1}{2}\gamma\,\xi^2\,\hbar\,x_1^2\right)\epsilon + \left(-\frac{1}{2}\xi\,\hbar^2\,a_1^2\,x_1 + \frac{1}{4}\gamma^2\,\xi^2\,\hbar^2\,x_1^2 - \gamma\,\xi^2\,\hbar^2\,a_1\,x_1^2 + \frac{1}{2}\xi^2\,\hbar^2\,a_1^2\,x_1^2 - \frac{1}{2}\gamma^2\,\xi^3\,\hbar^2\,x_1^3 + \frac{1}{2}\gamma\,\xi^3\,\hbar^2\,a_1\,x_1^3 + \frac{1}{8}\gamma^2\,\xi^4\,\hbar^2\,x_1^4\right)\epsilon^2 + O[\epsilon]^3], 0\}\}$

$S(\mathbb{e}^{\eta y}\,\mathbb{e}^{\alpha a}\,\mathbb{e}^{\xi x})$

LinearLambda

```
In[∘]:= Timing@Block[{$p = 3, $k = 1}, {
    (sexp = m_{3,2,1→1}[Exp_{QU₁,$k}[η, S₁[QU[y₁]] /. QU → Times] Exp_{QU₂,$k}[α, S₂[QU[a₂]] /. QU → Times]
         Exp_{QU₃,$k}[ξ, S₃[QU[x₃]] /. QU → Times]]) /. u_₁ :→ u,
    HL@SimpT[QU@(sexp /. {η → ħ η, α → ħ α, ξ → ħ ξ}) -
       S₁@𝕆_QU[{y₁, a₁, x₁}₁, SS[e^{ħ (η y₁+α a₁+ξ x₁)}]]]
  }]
```

LinearLambda

Out[∘]= $\{15.2969, \{\mathbb{E}_{QU}[\{y_1, a_1, x_1\}, \frac{1}{T\,\hbar}\left(e^{\alpha\gamma}\,\eta\,\xi - e^{\alpha\gamma}\,T\,\eta\,\xi - a\,T\,\alpha\,\hbar - e^{\alpha\gamma}\,y\,\eta\,\hbar - e^{\alpha\gamma}\,T\,x\,\xi\,\hbar\right),$

$1 + \frac{1}{4\,T^2\,\hbar}\left(-3\,e^{2\alpha\gamma}\,\gamma\,\eta^2\,\xi^2 + 4\,e^{2\alpha\gamma}\,T\,\gamma\,\eta^2\,\xi^2 - e^{2\alpha\gamma}\,T^2\,\gamma\,\eta^2\,\xi^2 + 8\,a\,e^{\alpha\gamma}\,T\,\eta\,\xi\,\hbar - 4\,e^{\alpha\gamma}\,T\,\gamma\,\eta\,\xi\,\hbar + 4\,e^{\alpha\gamma}\,T^2\,\gamma\,\eta\,\xi\,\hbar + 6\,e^{2\alpha\gamma}\,y\,\gamma\,\eta^2\,\xi\,\hbar - 2\,e^{2\alpha\gamma}\,T\,y\,\gamma\,\eta^2\,\xi\,\hbar + 6\,e^{2\alpha\gamma}\,T\,x\,\gamma\,\eta\,\xi^2\,\hbar - 2\,e^{2\alpha\gamma}\,T^2\,x\,\gamma\,\eta\,\xi^2\,\hbar - 4\,a\,e^{\alpha\gamma}\,T\,y\,\eta\,\hbar^2 + 4\,e^{\alpha\gamma}\,T\,y\,\gamma\,\eta\,\hbar^2 - 2\,e^{2\alpha\gamma}\,y^2\,\gamma\,\eta^2\,\hbar^2 - 4\,a\,e^{\alpha\gamma}\,T^2\,x\,\xi\,\hbar^2 - 4\,e^{2\alpha\gamma}\,T\,x\,y\,\gamma\,\eta\,\xi\,\hbar^2 - 2\,e^{2\alpha\gamma}\,T^2\,x^2\,\gamma\,\xi^2\,\hbar^2\right)\epsilon + O[\epsilon]^2], 0\}\}$

$\Delta_{1\to1,2}\left(e^{\eta y_1}\, e^{\alpha a_1}\, e^{\xi x_1}\right)$

LinearLambda

```
Timing@Block[{$p = 4, $k = 2}, {
   sexp = m1,3,5→1@m2,4,6→2@Times[ (* Warning: wrong unless $p≥$k+1! *)
        Prepend[{y2}2]@ExpQU1,$k[η, Δ1→1,2[QU[y1]] /. QU → Times],
        Prepend[{a4}4]@ExpQU3,$k[α, Δ3→3,4[QU[a3]] /. QU → Times],
        Prepend[{x6}6]@ExpQU5,$k[ξ, Δ5→5,6[QU[x5]] /. QU → Times]
      ] /. {η → ℏ η, α → ℏ α, ξ → ℏ ξ},
   HL@SimpT[QU@sexp - Δ1→1,2@OQU[{y1, a1, x1}1, SS[e^(ℏ (η y1+α a1+ξ x1))]]]
  }]
```

LinearLambda

Out[∘]= $\Big\{162.,\ \Big\{\mathbb{E}_{QU}[\{y_2, a_2, x_2\}_2, \{y_1, a_1, x_1\}_1,\ \alpha\hbar a_1 + \alpha\hbar a_2 + \xi\hbar x_1 + \xi\hbar x_2 + \eta\hbar y_1 + \eta\hbar T_1 y_2,$
$1 + \frac{1}{2}\left(-2\xi\hbar^2 a_1 x_2 + \gamma\xi^2\hbar^3 x_1 x_2 - 2\eta\hbar^2 a_1 T_1 y_2 + \gamma\eta^2\hbar^3 T_1 y_1 y_2\right)\epsilon +$
$\frac{1}{24}\left(12\xi\hbar^3 a_1^2 x_2 + 6\gamma^2\xi^2\hbar^4 x_1 x_2 - 12\gamma\xi^2\hbar^4 a_1 x_1 x_2 + 4\gamma^2\xi^3\hbar^5 x_1^2 x_2 + 12\xi^2\hbar^4 a_1^2 x_2^2 +\right.$
$4\gamma^2\xi^3\hbar^5 x_1 x_2^2 - 12\gamma\xi^3\hbar^5 a_1 x_1 x_2^2 + 3\gamma^2\xi^4\hbar^6 x_1^2 x_2^2 + 12\eta\hbar^3 a_1^2 T_1 y_2 +$
$24\eta\xi\hbar^4 a_1^2 T_1 x_2 y_2 - 12\gamma\eta\xi^2\hbar^5 a_1 T_1 x_1 x_2 y_2 + 6\gamma^2\eta^2\hbar^4 T_1 y_1 y_2 - 12\gamma\eta^2\hbar^4 a_1 T_1 y_1 y_2 -$
$12\gamma\eta^2\xi\hbar^5 a_1 T_1 x_2 y_1 y_2 + 6\gamma^2\eta^2\xi^2\hbar^6 T_1 x_1 x_2 y_1 y_2 + 4\gamma^2\eta^3\hbar^5 T_1 y_1^2 y_2 + 12\eta^2\hbar^4 a_1^2 T_1^2 y_2^2 +$
$\left.4\gamma^2\eta^3\hbar^5 T_1^2 y_1 y_2^2 - 12\gamma\eta^3\hbar^5 a_1 T_1^2 y_1 y_2^2 + 3\gamma^2\eta^4\hbar^6 T_1^2 y_1^2 y_2^2\right)\epsilon^2 + O[\epsilon]^3],\ 0\Big\}\Big\}$

Zip and Bind

E

In[∘]:=
```
𝔼 /: 𝔼[L1_, Q1_, P1_] ≡ 𝔼[L2_, Q2_, P2_] :=
  Simplify[L1 == L2] ∧ Simplify[Q1 == Q2] ∧ Simplify[Normal[P1 - P2] == 0];
𝔼 /: 𝔼[L1_, Q1_, P1_] 𝔼[L2_, Q2_, P2_] := 𝔼[L1 + L2, Q1 + Q2, P1 * P2];
```

Zip

In[∘]:=
```
{t*, y*, a*, x*, z*} = {τ, η, α, ξ, ζ};
{τ*, η*, α*, ξ*, ζ*} = {t, y, a, x, z}; (u_i_)* := (u*)i;
```

Zip

In[∘]:=
```
Zip{}[P_] := P; Zip{ζ_,ζs___}[P_] := (Expand[P // Zip{ζs}] /. f_. ζ^d_. ⧴ ∂{ζ*,d} f) /. ζ* → 0
```

QZip implements the “Q-level zips” on $\mathbb{E}(L, Q, P) = Pe^{L+Q}$. Such zips regard the L variables as scalars.

Zip

In[∘]:=
```
𝔼 /: QZipζs_List@𝔼[L_, Q_, P_] := Module[{ζ, z, zs, c, ys, ηs, qt, zrule, Q1, Q2},
   zs = Table[ζ*, {ζ, ζs}];
   c = Q /. Alternatives @@ (ζs ⋃ zs) → 0;
   ys = Table[∂ζ (Q /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = Table[∂z (Q /. Alternatives @@ ζs → 0), {z, zs}];
   qt = Inverse@Table[Kδz,ζ* - ∂z,ζ Q, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → qt.(zs + ys)];
   Q2 = (Q1 = c + ηs.zs /. zrule) /. Alternatives @@ zs → 0;
   Simplify /@ 𝔼[L, Q2, Det[qt] e^-Q2 Zipζs[e^Q1 (P /. zrule)]]];
```

In[∘]:=
```
Timing@{E0 = 𝔼[0, Sum[a10 i+j xi ξj, {i, 3}, {j, 3}],
    1 + ϵ Sum[fi[x1, x2, x3] ξi, {i, 3}] + ϵ Sum[f10 i+j[x1, x2, x3] ξi ξj, {i, 3}, {j, 3}]],
  lhs = QZip{ξ1,ξ2}@E0,
  HL[lhs == QZip{ξ1}@QZip{ξ2}@E0]}
```

Out[∘]=
```
{38.6875, {𝔼[0, ⋯1⋯, 1 + ϵ (ξ1 ⋯1⋯ + ⋯1⋯ + ⋯1⋯) +
    ϵ (ξ1² f11[x1, x2, x3] + ⋯7⋯ + ξ3² f33[x1, x2, x3])], ⋯1⋯, True}}
```

large output | show less | show more | show all | set size limit...

In[∘]:=
```
Timing@{
  Eh = 𝔼[0, h Sum[a10 i+j xi ξj, {i, 3}, {j, 3}],
    1 + ϵ Sum[fi[x1, x2, x3] ξi, {i, 3}] + ϵ Sum[f10 i+j[x1, x2, x3] ξi ξj, {i, 3}, {j, 3}]],
  lhs = Normal[Eh /. 𝔼[L_, Q_, P_] :> Series[P e^(L+Q), {h, 0, 2}]] // Zip{ξ1},
  HL@Simplify[lhs == Normal[QZip{ξ1}[Eh] /. 𝔼[L_, Q_, P_] :> Series[P e^(L+Q), {h, 0, 2}]]]}
```

Out[∘]=
```
{18.4375, {𝔼[0, h ⋯1⋯, 1 + ϵ (ξ1 ⋯1⋯ + ⋯1⋯ + ⋯1⋯) +
    ϵ (ξ1² f11[x1, x2, x3] + ⋯7⋯ + ξ3² f33[x1, x2, x3])], ⋯1⋯, True}}
```

large output | show less | show more | show all | set size limit...

LZip implements the "L-level zips" on $\mathbb{E}(L, Q, P) = P e^{L+Q}$. Such zips regard all of $P e^{Q}$ as a single"P". Here the z's are t and α and the ζ's are τ and a.

Zip

In[∘]:=
```
𝔼 /: LZipζs_List@𝔼[L_, Q_, P_] := Module[{ζ, z, zs, c, ys, ηs, lt, zrule, L1, L2, Q1, Q2},
   zs = Table[ζ*, {ζ, ζs}];
   c = L /. Alternatives @@ (ζs ⋃ zs) → 0;
   ys = Table[∂ζ (L /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = Table[∂z (L /. Alternatives @@ ζs → 0), {z, zs}];
   lt = Inverse@Table[Kδz,ζ* - ∂z,ζ L, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → lt.(zs + ys)];
   L2 = (L1 = c + ηs.zs /. zrule) /. Alternatives @@ zs → 0;
   Q2 = (Q1 = Q /. T2t /. zrule) /. Alternatives @@ zs → 0;
   Simplify /@ 𝔼[L2, Q2, Det[lt] e^(-L2-Q2) Zipζs[e^(L1+Q1) (P /. T2t /. zrule)]] //. t2T];
```

Bind

In[∘]:=
```
Bind{}[L_, R_] := L R;
Bind{is__}[L_𝔼, R_𝔼] := Module[{n},
   Times[
     L /. Table[(v : T | t | a | x | y)i → vn@i, {i, {is}}],
     R /. Table[(v : τ | α | ξ | η)i → vn@i, {i, {is}}]
    ] // LZipFlatten@Table[{τn@i,an@i},{i,{is}}] // QZipFlatten@Table[{ξn@i,yn@i},{i,{is}}]];
BL_List := BindL; Bis___ := Bind{is};
Bind[ℰ_𝔼] := ℰ;
Bind[Ls__, ζs_List, R_] := Bindζs[Bind[Ls], R];
```

```
In[•]:= Bind{2}[𝔼[0, ξ (x1 + x2), 1], 𝔼[0, ξ2 (x2 + x3), 1]]
Out[•]= 𝔼[0, ξ (x1 + x2 + x3), 1]

In[•]:= Bind{2}[𝔼[0, (ξ2 + ξ3) x2, 1], 𝔼[0, (ξ1 + ξ2) x, 1]]
Out[•]= 𝔼[0, x (ξ1 + ξ2 + ξ3), 1]

In[•]:= Bind{1,2}[𝔼[0, (ξ2 + ξ3) x2 + ξ1 x1, 1], 𝔼[0, (ξ1 + ξ2) x, 1]]
Out[•]= 𝔼[0, x (ξ1 + ξ2 + ξ3), 1]
```

An xay → axy → ayx –> yax ≡ xay → xya → yxa → yax test:

```
In[•]:= Bind[𝔼[α1 a1 + τ1 t1, e^(γ α1) ξ1 x1 + η1 y1, 1], {1}, 𝔼[τ1 t1 + α1 a1, ξ1 x1 + η1 y1 + ξ1 η1 t1, 1]]
Out[•]= 𝔼[a1 α1 + t1 τ1, y1 η1 + e^(γ α1) (x1 + t1 η1) ξ1, 1]

In[•]:= Column@{Cord[ŒCU[{x1, a1}1, ξ1 x1 + α1 a1, 1 + 0_0]],
          Cord[ŒCU[{x1, y1}1, ξ1 x1 + η1 y1, 1 + 0_0]],
          Cord[ŒCU[{a1, y1}1, α1 a1 + η1 y1, 1 + 0_0]]}
        ŒCU[{a1, x1}1, e^(-γ α1) (e^(γ α1) a1 α1 + x1 ξ1), 1 + O[ϵ]^1]
Out[•]= ŒCU[{y1, a1, x1}1, y1 η1 + x1 ξ1 - t1 η1 ξ1, 1 + O[ϵ]^1]
        ŒCU[{y1, a1}1, e^(-γ α1) (e^(γ α1) a1 α1 + y1 η1), 1 + O[ϵ]^1]

In[•]:= {rxa = 𝔼[τ1 t1 + α1 a1, e^(-γ α1) ξ1 x1 + η1 y1, 1];
         rxy = 𝔼[τ1 t1 + α1 a1, ξ1 x1 + η1 y1 - ξ1 η1 t1, 1];
         ray = 𝔼[τ1 t1 + α1 a1, e^(-γ α1) η1 y1 + ξ1 x1, 1];
         lhs = Expand /@ Bind[rxa, {1}, rxy, {1}, ray],
         HL[lhs == Expand /@ Bind[ray, {1}, rxy, {1}, rxa] ]}
Out[•]= {𝔼[a1 α1 + t1 τ1, e^(-γ α1) y1 η1 + e^(-γ α1) x1 ξ1 - e^(-γ α1) t1 η1 ξ1, 1], True}

In[•]:= Simplify /@ m_{i,j→k}@ŒCU[{yi, ai, xi}i, {yj, aj, xj}j, ηi yi + αi ai + ξi xi + ηj yj + αj aj + ξj xj, 1 + 0_1]
Out[•]= ŒCU[{yk, ak, xk}k, ak (αi + αj) + yk (ηi + e^(-γ αi) ηj) + e^(-γ αj) xk ξi - tk ηj ξi + xk ξj,
         1 + 1/2 ηj ξi (4 ak - 2 e^(-γ αi) γ yk ηj + γ (-2 e^(-γ αj) xk + tk ηj) ξi) ϵ + O[ϵ]^2]
```

Tensorial Representations

t1

```
In[•]:= tη = t𝟙 = 𝔼[0, 0, 1 + 0_$k];
```

tm

```
In[•]:= m[U_, kk_] := m[U, kk] = Module[{OE},
          OE = Simplify /@
            m_{1,2→1}@Œ_U[{y1, a1, x1}1, {y2, a2, x2}2, η1 y1 + α1 a1 + ξ1 x1 + η2 y2 + α2 a2 + ξ2 x2, 1 + 0_kk];
          𝔼[t1 (τ1 + τ2) + (OE⟦2⟧ /. (ξ | η)_{1|2} → 0), OE⟦2⟧ /. a1 → 0, OE⟦3⟧] ];
        tm_{i_,j_→k_} :=
          m[$U, $k] /. {(v : τ | η | α | ξ)1 → v_i, (v : τ | η | α | ξ)2 → v_j, (v : t | T | y | a | x)1 → v_k};
```

In[∘]:= $tm_{1,2\to 3}$

Out[∘]= $\mathbb{E}\Big[a_3\,(\alpha_1+\alpha_2)+t_3\,(\tau_1+\tau_2),\ y_3\,\left(\eta_1+e^{-\gamma\alpha_1}\eta_2\right)-\frac{(-1+T_3)\,\eta_2\,\xi_1}{\hbar}+x_3\,\left(e^{-\gamma\alpha_2}\xi_1+\xi_2\right),$
$1+\frac{1}{4\hbar}e^{-\gamma(\alpha_1+\alpha_2)}\eta_2\,\xi_1\,\left(8\,e^{\gamma(\alpha_1+\alpha_2)}\hbar\,a_3\,T_3+e^{\gamma\alpha_2}\gamma\,(-1+3\,T_3)\,\eta_2\,\left(-2\,\hbar\,y_3+e^{\gamma\alpha_1}(-1+T_3)\,\xi_1\right)+\right.$
$\left.2\,\gamma\,\hbar\,x_3\,\left(2\,\hbar\,y_3-e^{\gamma\alpha_1}(-1+3\,T_3)\,\xi_1\right)\right)\epsilon+O[\epsilon]^2\Big]$

Associativity of tm.

```
In[∘]:= Table[Block[{$U = U, $k = kk},
          {lhs = Bind[tm1,2→2, {2}, tm2,3→1];
           {$U, $k} -> HL[lhs ≡ Bind[tm2,3→2, {2}, tm1,2→1]]}
        ], {U, {CU, QU}}, {kk, 0, 1}]
```

Out[∘]= {{{{CU, 0} → True}, {{CU, 1} → True}}, {{{QU, 0} → True}, {{QU, 1} → True}}}

```
In[∘]:= Block[{$U = CU, $k = 2}, Timing@{lhs = Bind[tm1,2→2, {2}, tm2,3→1];
          HL[lhs ≡ Bind[tm2,3→2, {2}, tm1,2→1]]}]
```

Out[∘]= {65.75, {True}}

tS

```
In[∘]:= S[U_, kk_] := S[U, kk] = Module[{OE},
          OE = m3,2,1→1[ExpU1,$k[η, S1[U[y1]] /. U → Times]
             ExpU2,$k[α, S2[U[a2]] /. U → Times] ExpU3,$k[ξ, S3[U[x3]] /. U → Times]];
          𝔼[-t1 τ1 + (OE⟦2⟧ /. ξ | η → 0), OE⟦2⟧ /. a1 → 0, OE⟦3⟧] /. {η → η1, α → α1, ξ → ξ1}];
        tSi_ := S[$U, $k] /. {(v : τ | η | α | ξ)1 → vi, (v : t | T | y | a | x)1 → vi};
```

In[∘]:= tS_1

Out[∘]= $\mathbb{E}\Big[-a_1\,\alpha_1-t_1\,\tau_1,\ \frac{1}{\hbar\,T_1}\left(-e^{\gamma\alpha_1}\hbar\,y_1\,\eta_1-e^{\gamma\alpha_1}\hbar\,T_1\,x_1\,\xi_1+e^{\gamma\alpha_1}\eta_1\,\xi_1-e^{\gamma\alpha_1}T_1\,\eta_1\,\xi_1\right),$
$1+\frac{1}{4\,\hbar\,T_1^2}\left(4\,e^{\gamma\alpha_1}\gamma\,\hbar^2\,T_1\,y_1\,\eta_1-4\,e^{\gamma\alpha_1}\hbar^2\,a_1\,T_1\,y_1\,\eta_1-2\,e^{2\gamma\alpha_1}\gamma\,\hbar^2\,y_1^2\,\eta_1^2-4\,e^{\gamma\alpha_1}\hbar^2\,a_1\,T_1^2\,x_1\,\xi_1-\right.$
$4\,e^{\gamma\alpha_1}\gamma\,\hbar\,T_1\,\eta_1\,\xi_1+8\,e^{\gamma\alpha_1}\hbar\,a_1\,T_1\,\eta_1\,\xi_1+4\,e^{\gamma\alpha_1}\gamma\,\hbar\,T_1^2\,\eta_1\,\xi_1-4\,e^{2\gamma\alpha_1}\gamma\,\hbar^2\,T_1\,x_1\,y_1\,\eta_1\,\xi_1+$
$6\,e^{2\gamma\alpha_1}\gamma\,\hbar\,y_1\,\eta_1^2\,\xi_1-2\,e^{2\gamma\alpha_1}\gamma\,\hbar\,T_1\,y_1\,\eta_1^2\,\xi_1-2\,e^{2\gamma\alpha_1}\gamma\,\hbar^2\,T_1^2\,x_1^2\,\xi_1^2+6\,e^{2\gamma\alpha_1}\gamma\,\hbar\,T_1\,x_1\,\eta_1\,\xi_1^2-$
$\left.2\,e^{2\gamma\alpha_1}\gamma\,\hbar\,T_1^2\,x_1\,\eta_1\,\xi_1^2-3\,e^{2\gamma\alpha_1}\gamma\,\eta_1^2\,\xi_1^2+4\,e^{2\gamma\alpha_1}\gamma\,T_1\,\eta_1^2\,\xi_1^2-e^{2\gamma\alpha_1}\gamma\,T_1^2\,\eta_1^2\,\xi_1^2\right)\epsilon+O[\epsilon]^2\Big]$

tS is an anti-homomorphism for tm.

```
In[∘]:= HL[(tS1 tS2) ~B1,2~ tm1,2→1 ≡ tm2,1→1 ~B1~ tS1]
```

Out[∘]= True

tDelta

In[∘]:=
```
Δ[U_, kk_] := Δ[U, kk] = Module[{OE},
    OE = Block[{$k = kk, $p = kk + 1}, m_{1,3,5→1}@m_{2,4,6→2}@Times[
          Prepend[{y_2}_2]@Exp_{U_1,$k}[η, Δ_{1→1,2}[U[y_1]] /. U → Times],
          Prepend[{a_4}_4]@Exp_{U_3,$k}[α, Δ_{3→3,4}[U[a_3]] /. U → Times],
          Prepend[{x_6}_6]@Exp_{U_5,$k}[ξ, Δ_{5→5,6}[U[x_5]] /. U → Times]
        ] /. {η → η_1, α → α_1, ξ → ξ_1}];
    𝔼[τ_1 (t_1 + t_2) + α_1 (a_1 + a_2), OE⟦3⟧ /. α_1 → 0, OE⟦4⟧]];
tΔ_{i_→j_,k_} :=
  Δ[$U, $k] /. {(v : τ | η | α | ξ)_1 → v_i, (v : t | T | y | a | x)_1 → v_j, (v : t | T | y | a | x)_2 → v_k};
```

In[∘]:=
```
tΔ_{1→1,2}
```

Out[∘]= $\mathbb{E}\Big[(a_1 + a_2)\,\alpha_1 + (t_1 + t_2)\,\tau_1,\ y_1\,\eta_1 + T_1\,y_2\,\eta_1 + x_1\,\xi_1 + x_2\,\xi_1,$
$1 + \frac{1}{2}\left(-2\,\hbar\,a_1\,T_1\,y_2\,\eta_1 + \gamma\,\hbar\,T_1\,y_1\,y_2\,\eta_1^2 - 2\,\hbar\,a_1\,x_2\,\xi_1 + \gamma\,\hbar\,x_1\,x_2\,\xi_1^2\right)\,\epsilon + O[\epsilon]^2\Big]$

Testing co-associativity.

In[∘]:=
```
HL[tΔ_{1→1,2} ~ B_2 ~ tΔ_{2→2,3} ≡ tΔ_{1→1,3} ~ B_1 ~ tΔ_{1→1,2}]
```

Out[∘]= True

Testing S is an anti-co-homomorphism

In[∘]:=
```
HL[tS_1 ~ B_1 ~ tΔ_{1→1,2} ≡ tΔ_{1→2,1} ~ B_{1,2} ~ (tS_1 tS_2)]
```

Out[∘]= True

Testing convolution inverse:

In[∘]:=
```
{HL[tΔ_{1→1,2} ~ B_1 ~ tS_1 ~ B_{1,2} ~ tm_{1,2→1} ≡ tη ~ B_{} ~ t𝟙],
 HL[tΔ_{1→1,2} ~ B_2 ~ tS_2 ~ B_{1,2} ~ tm_{1,2→1} ≡ tη ~ B_{} ~ t𝟙]}
```

Out[∘]= {True, True}

tR

In[∘]:=
```
R[QU, kk_] := R[QU, kk] = Module[{OE},
    OE = Simplify /@ Œ_{QU,kk}@R_{1,2};
    𝔼[-ħ a_2 t_1/γ, ħ x_2 y_1, Last@OE]];
tR_{i_,j_} := R[$U, $k] /. {(v : t | T | y | a | x)_1 → v_i, (v : t | T | y | a | x)_2 → v_j};
tR̄_{i_,j_} := tR̄_{i,j} = tR_{i,j} ~ B_j ~ tS_j;
```

In[∘]:=
```
{tR_{1,2}, tR̄_{1,2}}
```

Out[∘]= $\Big\{\mathbb{E}\Big[-\frac{\hbar\,a_2\,t_1}{\gamma},\ \hbar\,x_2\,y_1,\ 1 + \left(\frac{\hbar\,a_1\,a_2}{\gamma} - \frac{1}{4}\,\gamma\,\hbar^3\,x_2^2\,y_1^2\right)\epsilon + O[\epsilon]^2\Big],\ \mathbb{E}\Big[\frac{\hbar\,a_2\,t_1}{\gamma},\ -\frac{\hbar\,x_2\,y_1}{T_1},$
$1 - \frac{1}{4\,(\gamma\,T_1^2)}\left(\hbar\,\left(4\,a_1\,T_1\,(a_2\,T_1 + \gamma\,\hbar\,x_2\,y_1) + \gamma\,\hbar\,x_2\,y_1\,(4\,a_2\,T_1 + 3\,\gamma\,\hbar\,x_2\,y_1)\right)\right)\epsilon + O[\epsilon]^2\Big]\Big\}$

Testing R2

In[∘]:= $HL\left[\left(\overline{tR}_{1,2}\,tR_{3,4}\right)\sim B_{1,2,3,4}\sim\left(tm_{1,3\to 1}\,tm_{2,4\to 2}\right) == t\mathbb{1}\right]$

Out[∘]= True

Testing R3

In[∘]:= $HL\left[\left(tR_{2,3}\,tR_{1,4}\,tR_{5,6}\right)\sim B_{Range@6}\sim\left(tm_{1,5\to 1}\,tm_{2,6\to 2}\,tm_{3,4\to 3}\right) \equiv \left(tR_{1,2}\,tR_{5,3}\,tR_{6,4}\right)\sim B_{Range@6}\sim\left(tm_{1,5\to 1}\,tm_{2,6\to 2}\,tm_{3,4\to 3}\right)\right]$

Out[∘]= True

tC

In[∘]:=
$tC_{i\_} := \mathbb{E}\left[-t_i/2,\,0,\,e^{\epsilon\,a_i\,\hbar} + 0_{\$k}\right];$
$\overline{tC}_{i\_} := \mathbb{E}\left[t_i/2,\,0,\,e^{-\epsilon\,a_i\,\hbar} + 0_{\$k}\right];$

In[∘]:= $Block\left[\{\$k = 3\},\,\{tC_1,\,\overline{tC}_2\}\right]$

Out[∘]= $\left\{\mathbb{E}\left[-\frac{t_1}{2},\,0,\,1 + a_1\,\epsilon + \frac{1}{2}\,a_1^2\,\epsilon^2 + \frac{1}{6}\,a_1^3\,\epsilon^3 + O[\epsilon]^4\right],\,\mathbb{E}\left[\frac{t_2}{2},\,0,\,1 - a_2\,\epsilon + \frac{1}{2}\,a_2^2\,\epsilon^2 - \frac{1}{6}\,a_2^3\,\epsilon^3 + O[\epsilon]^4\right]\right\}$

tC, $\overline{tC}$ are inverses

In[∘]:= $Block\left[\{\$k = 2\},\,HL\left[\left(tC_1\,\overline{tC}_2\right)\sim B_{1,2}\sim tm_{1,2\to 1} \equiv t\mathbb{1}\right]\right]$

Out[∘]= True

Cyclic R2 as in the PP1 paper:

In[∘]:= $Block\left[\{\$k = 2\},\,HL\left[\left(tR_{1,2}\,\overline{tR}_{3,4}\,tC_5\,\overline{tC}_6\right)\sim B_{\{1,2,3,4,5,6\}}\sim\left(tm_{1,3\to 1}\,tm_{4,5\to 4}\,tm_{2,6\to 2}\right)\sim B_{2,4}\sim tm_{4,2\to 2} \equiv t\mathbb{1}\right]\right]$ // Timing

Out[∘]= {17.4688, True}

Swirl relation as in the PP1 paper:

In[∘]:= $Block\left[\{\$k = 2\},\,HL\left[tR_{1,2} \equiv \left(tC_1\,tC_2\,tR_{3,4}\,\overline{tC}_5\,\overline{tC}_6\right)\sim B_{1,2,3,4}\sim\left(tm_{1,3\to 1}\,tm_{2,4\to 2}\right)\sim B_{1,2,5,6}\sim\left(tm_{1,5\to 1}\,tm_{2,6\to 2}\right)\right]\right]$ // Timing

Out[∘]= {210.547, True}

Trefoil

tKink

In[∘]:=
$Kink[QU,\,kk\_] := Block\left[\{\$k = kk\},\,\left(tR_{1,3}\,tC_2\right)\sim B_{1,2}\sim tm_{1,2\to 1}\sim B_{1,3}\sim tm_{1,3\to 1}\right];$
$tKink_{i\_} := Kink[\$U,\,\$k]$ /. $\left\{(v : t\,|\,T\,|\,y\,|\,a\,|\,x)_1 \to v_i\right\}$

In[∘]:= $tKink_1$

Out[∘]= $\mathbb{E}\left[-\frac{\left(\gamma + 2\,\hbar\,a_1\right)\,t_1}{2\,\gamma},\,\hbar\,x_1\,y_1,\,1 + \left(\hbar\,a_1 + \frac{\hbar\,a_1^2}{\gamma} - \frac{1}{4}\,\gamma\,\hbar^3\,x_1^2\,y_1^2\right)\epsilon + O[\epsilon]^2\right]$

In[∘]:=
$Timing\Big[$
$Z = tR_{1,5}\,tR_{6,2}\,tR_{3,7}\,\overline{tC}_4\,tKink_8\,tKink_9\,tKink_{10};$
$Do\left[Echo@Z;\,Z = Z\sim B_{1,k}\sim tm_{1,k\to 1},\,\{k,\,2,\,10\}\right];$
$Z\Big]$

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$$\mathbb{E}\Big[-\frac{\hbar\,a_5\,t_1}{\gamma}-\frac{\hbar\,a_7\,t_3}{\gamma}+\frac{t_4}{2}-\frac{\hbar\,a_2\,t_6}{\gamma}-\frac{(\gamma+2\,\hbar\,a_8)\,t_8}{2\,\gamma}-\frac{(\gamma+2\,\hbar\,a_9)\,t_9}{2\,\gamma}-\frac{(\gamma+2\,\hbar\,a_{10})\,t_{10}}{2\,\gamma},$$
$$\hbar\,x_5\,y_1+\hbar\,x_7\,y_3+\hbar\,x_2\,y_6+\hbar\,x_8\,y_8+\hbar\,x_9\,y_9+\hbar\,x_{10}\,y_{10},$$
$$1+\Big(-\hbar\,a_4+\frac{\hbar\,a_1\,a_5}{\gamma}+\frac{\hbar\,a_2\,a_6}{\gamma}+\frac{\hbar\,a_3\,a_7}{\gamma}+\hbar\,a_8+\frac{\hbar\,a_8^2}{\gamma}+\hbar\,a_9+\frac{\hbar\,a_9^2}{\gamma}+\hbar\,a_{10}+\frac{\hbar\,a_{10}^2}{\gamma}-\frac{1}{4}\gamma\,\hbar^3\,x_5^2\,y_1^2-$$
$$\frac{1}{4}\gamma\,\hbar^3\,x_7^2\,y_3^2-\frac{1}{4}\gamma\,\hbar^3\,x_2^2\,y_6^2-\frac{1}{4}\gamma\,\hbar^3\,x_8^2\,y_8^2-\frac{1}{4}\gamma\,\hbar^3\,x_9^2\,y_9^2-\frac{1}{4}\gamma\,\hbar^3\,x_{10}^2\,y_{10}^2\Big)\,\epsilon+O[\epsilon]^2\Big]$$

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$$\mathbb{E}\Big[-\frac{1}{2\,\gamma}(2\,\hbar\,a_5\,t_1+2\,\hbar\,a_7\,t_3-\gamma\,t_4+2\,\hbar\,a_1\,t_6+\gamma\,t_8+2\,\hbar\,a_8\,t_8+\gamma\,t_9+2\,\hbar\,a_9\,t_9+\gamma\,t_{10}+2\,\hbar\,a_{10}\,t_{10}),$$
$$\hbar\,(x_5\,y_1+x_7\,y_3+x_1\,y_6+x_8\,y_8+x_9\,y_9+x_{10}\,y_{10}),$$
$$1-\frac{1}{4\,\gamma}\Big(\hbar\,\big(4\,\gamma\,a_4-4\,a_1\,(a_5+a_6)-4\,a_3\,a_7-4\,\gamma\,a_8-4\,a_8^2-4\,\gamma\,a_9-4\,a_9^2-4\,\gamma\,a_{10}-4\,a_{10}^2+$$
$$\gamma^2\,\hbar^2\,x_5^2\,y_1^2+\gamma^2\,\hbar^2\,x_7^2\,y_3^2+\gamma^2\,\hbar^2\,x_1^2\,y_6^2+\gamma^2\,\hbar^2\,x_8^2\,y_8^2+\gamma^2\,\hbar^2\,x_9^2\,y_9^2+\gamma^2\,\hbar^2\,x_{10}^2\,y_{10}^2\big)\Big)\,\epsilon+O[\epsilon]^2\Big]$$

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$$\mathbb{E}\Big[-\frac{1}{2\,\gamma}(2\,\hbar\,a_5\,t_1+2\,\hbar\,a_7\,t_1-\gamma\,t_4+2\,\hbar\,a_1\,t_6+\gamma\,t_8+2\,\hbar\,a_8\,t_8+\gamma\,t_9+2\,\hbar\,a_9\,t_9+\gamma\,t_{10}+2\,\hbar\,a_{10}\,t_{10}),$$
$$\hbar\,(x_5\,y_1+T_6\,x_7\,y_1+x_1\,y_6+x_7\,y_6-T_1\,x_7\,y_6+x_8\,y_8+x_9\,y_9+x_{10}\,y_{10}),$$
$$1-\frac{1}{4\,\gamma}\Big(\hbar\,\big(4\,\gamma\,a_4-4\,\gamma\,a_8-4\,a_8^2-4\,\gamma\,a_9-4\,a_9^2-4\,\gamma\,a_{10}-4\,a_{10}^2+4\,\gamma\,\hbar\,a_5\,T_6\,x_7\,y_1+$$
$$4\,\gamma\,\hbar\,a_6\,T_6\,x_7\,y_1+\gamma^2\,\hbar^2\,x_5^2\,y_1^2+\gamma^2\,\hbar^2\,T_6^2\,x_7^2\,y_1^2+4\,\gamma\,\hbar\,a_7\,x_1\,y_6-4\,\gamma^2\,\hbar^2\,T_6\,x_1\,x_7\,y_1\,y_6+$$
$$4\,\gamma^2\,\hbar^2\,T_1\,T_6\,x_7^2\,y_1\,y_6+\gamma^2\,\hbar^2\,x_1^2\,y_6^2+4\,\gamma^2\,\hbar^2\,T_1\,x_1\,x_7\,y_6^2+\gamma^2\,\hbar^2\,x_7^2\,y_6^2-\gamma^2\,\hbar^2\,T_1^2\,x_7^2\,y_6^2-$$
$$4\,a_1\,(a_5+a_6+a_7+2\,\gamma\,\hbar\,T_1\,x_7\,y_6)+\gamma^2\,\hbar^2\,x_8^2\,y_8^2+\gamma^2\,\hbar^2\,x_9^2\,y_9^2+\gamma^2\,\hbar^2\,x_{10}^2\,y_{10}^2\big)\Big)\,\epsilon+O[\epsilon]^2\Big]$$

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$$\mathbb{E}\Big[-\frac{1}{2\,\gamma}((-\gamma+2\,\hbar\,a_5+2\,\hbar\,a_7)\,t_1+2\,\hbar\,a_1\,t_6+\gamma\,t_8+2\,\hbar\,a_8\,t_8+\gamma\,t_9+2\,\hbar\,a_9\,t_9+\gamma\,t_{10}+2\,\hbar\,a_{10}\,t_{10}),$$
$$\hbar\,((x_5+T_6\,x_7)\,y_1+x_1\,y_6-(-1+T_1)\,x_7\,y_6+x_8\,y_8+x_9\,y_9+x_{10}\,y_{10}),$$
$$1-\frac{1}{4\,\gamma}\Big(\hbar\,\big(-4\,\gamma\,a_8-4\,a_8^2-4\,\gamma\,a_9-4\,a_9^2-4\,\gamma\,a_{10}-4\,a_{10}^2+4\,\gamma\,\hbar\,a_5\,T_6\,x_7\,y_1+4\,\gamma\,\hbar\,a_6\,T_6\,x_7\,y_1+$$
$$\gamma^2\,\hbar^2\,x_5^2\,y_1^2+\gamma^2\,\hbar^2\,T_6^2\,x_7^2\,y_1^2-4\,\gamma^2\,\hbar\,x_1\,y_6+4\,\gamma\,\hbar\,a_7\,x_1\,y_6-4\,\gamma^2\,\hbar^2\,T_6\,x_1\,x_7\,y_1\,y_6+$$
$$4\,\gamma^2\,\hbar^2\,T_1\,T_6\,x_7^2\,y_1\,y_6+\gamma^2\,\hbar^2\,x_1^2\,y_6^2+4\,\gamma^2\,\hbar^2\,T_1\,x_1\,x_7\,y_6^2+\gamma^2\,\hbar^2\,x_7^2\,y_6^2-\gamma^2\,\hbar^2\,T_1^2\,x_7^2\,y_6^2-$$
$$4\,a_1\,(-\gamma+a_5+a_6+a_7+2\,\gamma\,\hbar\,T_1\,x_7\,y_6)+\gamma^2\,\hbar^2\,x_8^2\,y_8^2+\gamma^2\,\hbar^2\,x_9^2\,y_9^2+\gamma^2\,\hbar^2\,x_{10}^2\,y_{10}^2\big)\Big)\,\epsilon+O[\epsilon]^2\Big]$$

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$$\mathbb{E}\Big[-\frac{1}{2\,\gamma}((-\gamma+2\,\hbar\,a_1+2\,\hbar\,a_7)\,t_1+2\,\hbar\,a_1\,t_6+\gamma\,t_8+2\,\hbar\,a_8\,t_8+\gamma\,t_9+2\,\hbar\,a_9\,t_9+\gamma\,t_{10}+2\,\hbar\,a_{10}\,t_{10}),$$
$$\hbar\,((x_1+T_6\,x_7)\,y_1+T_1\,x_1\,y_6-(-1+T_1)\,x_7\,y_6+x_8\,y_8+x_9\,y_9+x_{10}\,y_{10}),$$
$$1-\frac{1}{4\,\gamma}\Big(\hbar\,\big(-4\,a_1^2-4\,\gamma\,a_8-4\,a_8^2-4\,\gamma\,a_9-4\,a_9^2-4\,\gamma\,a_{10}-4\,a_{10}^2+4\,\gamma\,\hbar\,a_6\,T_6\,x_7\,y_1+\gamma^2\,\hbar^2\,x_1^2\,y_1^2+\gamma^2\,\hbar^2\,T_6^2\,x_7^2\,y_1^2-$$
$$4\,\gamma^2\,\hbar\,T_1\,x_1\,y_6+4\,\gamma\,\hbar\,a_7\,T_1\,x_1\,y_6-8\,\gamma^2\,\hbar^2\,T_1\,T_6\,x_1\,x_7\,y_1\,y_6+4\,\gamma^2\,\hbar^2\,T_1\,T_6\,x_7^2\,y_1\,y_6+\gamma^2\,\hbar^2\,T_1^2\,x_1^2\,y_6^2+4\,\gamma^2\,\hbar^2$$
$$T_1^2\,x_1\,x_7\,y_6^2+\gamma^2\,\hbar^2\,x_7^2\,y_6^2-\gamma^2\,\hbar^2\,T_1^2\,x_7^2\,y_6^2+4\,a_1\,(\gamma-a_6-a_7+\gamma\,\hbar\,T_6\,x_7\,y_1+\gamma\,\hbar\,T_1\,x_1\,y_6-2\,\gamma\,\hbar\,T_1\,x_7\,y_6)+$$
$$\gamma^2\,\hbar^2\,x_8^2\,y_8^2+\gamma^2\,\hbar^2\,x_9^2\,y_9^2+\gamma^2\,\hbar^2\,x_{10}^2\,y_{10}^2\big)\Big)\,\epsilon+O[\epsilon]^2\Big]$$

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$$\mathbb{E}\Big[-\frac{1}{2\gamma}\left(\left(-\gamma+4\hbar a_1+2\hbar a_7\right)t_1+\left(\gamma+2\hbar a_8\right)t_8+\gamma t_9+2\hbar a_9 t_9+\gamma t_{10}+2\hbar a_{10}t_{10}\right),$$
$$\hbar\left(\left(1+T_1\right)x_1 y_1+x_7 y_1+x_8 y_8+x_9 y_9+x_{10}y_{10}\right),$$
$$\frac{1}{1-T_1+T_1^2}+\frac{1}{4\gamma\left(1-T_1+T_1^2\right)^3}\hbar\Big(4\gamma a_9+4a_9^2+4\gamma a_{10}+4a_{10}^2+4\gamma^2T_1-4\gamma a_7T_1-8\gamma a_9T_1-8a_9^2T_1-8\gamma a_{10}T_1-$$
$$8a_{10}^2T_1-8\gamma^2T_1^2+8\gamma a_7T_1^2+12\gamma a_9T_1^2+12a_9^2T_1^2+12\gamma a_{10}T_1^2+12a_{10}^2T_1^2+4\gamma^2T_1^3-8\gamma a_7T_1^3-$$
$$8\gamma a_9T_1^3-8a_9^2T_1^3-8\gamma a_{10}T_1^3-8a_{10}^2T_1^3+4\gamma a_7T_1^4+4\gamma a_9T_1^4+4a_9^2T_1^4+4\gamma a_{10}T_1^4+4a_{10}^2T_1^4+$$
$$8a_1^2\left(1-T_1+T_1^2\right)^2+4\gamma a_8\left(1-T_1+T_1^2\right)^2+4a_8^2\left(1-T_1+T_1^2\right)^2+8\gamma^2\hbar T_1x_1y_1-4\gamma\hbar a_7T_1x_1y_1-$$
$$16\gamma^2\hbar T_1^2x_1y_1+4\gamma\hbar a_7T_1^2x_1y_1+16\gamma^2\hbar T_1^3x_1y_1-8\gamma\hbar a_7T_1^3x_1y_1-8\gamma^2\hbar T_1^4x_1y_1+4\gamma\hbar a_7T_1^4x_1y_1-$$
$$4\gamma\hbar a_7T_1^5x_1y_1+8\gamma^2\hbar T_1x_7y_1-4\gamma\hbar a_7T_1x_7y_1-24\gamma^2\hbar T_1^2x_7y_1+8\gamma\hbar a_7T_1^2x_7y_1+$$
$$24\gamma^2\hbar T_1^3x_7y_1-8\gamma\hbar a_7T_1^3x_7y_1-16\gamma^2\hbar T_1^4x_7y_1+4\gamma\hbar a_7T_1^4x_7y_1-\gamma^2\hbar^2x_1^2y_1^2+2\gamma^2\hbar^2T_1x_1^2y_1^2-$$
$$4\gamma^2\hbar^2T_1^2x_1^2y_1^2+4\gamma^2\hbar^2T_1^4x_1^2y_1^2-6\gamma^2\hbar^2T_1^5x_1^2y_1^2+3\gamma^2\hbar^2T_1^6x_1^2y_1^2+4\gamma^2\hbar^2T_1x_1x_7y_1^2-$$
$$4\gamma^2\hbar^2T_1^2x_1x_7y_1^2+8\gamma^2\hbar^2T_1^3x_1x_7y_1^2-4\gamma^2\hbar^2T_1^4x_1x_7y_1^2+4\gamma^2\hbar^2T_1^5x_1x_7y_1^2-\gamma^2\hbar^2x_7^2y_1^2+$$
$$2\gamma^2\hbar^2T_1x_7^2y_1^2-7\gamma^2\hbar^2T_1^2x_7^2y_1^2+6\gamma^2\hbar^2T_1^3x_7^2y_1^2-5\gamma^2\hbar^2T_1^4x_7^2y_1^2-4a_1\left(1-T_1+T_1^2\right)$$
$$\left(a_7\left(-1+T_1-T_1^2\right)+\gamma\left(1+\hbar x_1y_1+2\hbar T_1^3x_1y_1+\hbar T_1\left(x_1-x_7\right)y_1+T_1^2\left(-2-2\hbar x_1y_1+\hbar x_7y_1\right)\right)\right)-$$
$$\gamma^2\hbar^2x_8^2y_8^2+2\gamma^2\hbar^2T_1x_8^2y_8^2-3\gamma^2\hbar^2T_1^2x_8^2y_8^2+2\gamma^2\hbar^2T_1^3x_8^2y_8^2-\gamma^2\hbar^2T_1^4x_8^2y_8^2-\gamma^2\hbar^2x_9^2y_9^2+$$
$$2\gamma^2\hbar^2T_1x_9^2y_9^2-3\gamma^2\hbar^2T_1^2x_9^2y_9^2+2\gamma^2\hbar^2T_1^3x_9^2y_9^2-\gamma^2\hbar^2T_1^4x_9^2y_9^2-\gamma^2\hbar^2x_{10}^2y_{10}^2+$$
$$2\gamma^2\hbar^2T_1x_{10}^2y_{10}^2-3\gamma^2\hbar^2T_1^2x_{10}^2y_{10}^2+2\gamma^2\hbar^2T_1^3x_{10}^2y_{10}^2-\gamma^2\hbar^2T_1^4x_{10}^2y_{10}^2\Big)\epsilon+O[\epsilon]^2\Big]$$

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$$\mathbb{E}\Big[-\frac{1}{2\gamma}\left(-\left(\gamma-6\hbar a_1\right)t_1+\left(\gamma+2\hbar a_8\right)t_8+\gamma t_9+2\hbar a_9t_9+\gamma t_{10}+2\hbar a_{10}t_{10}\right),$$
$$\hbar\left(\left(1+T_1+T_1^2\right)x_1y_1+x_8y_8+x_9y_9+x_{10}y_{10}\right),$$
$$\frac{1}{1-T_1+T_1^2}+\frac{1}{4\gamma\left(1-T_1+T_1^2\right)^3}\hbar\Big(4\gamma a_9+4a_9^2+4\gamma a_{10}+4a_{10}^2+4\gamma^2T_1-8\gamma a_9T_1-8a_9^2T_1-8\gamma a_{10}T_1-8a_{10}^2T_1-$$
$$8\gamma^2T_1^2+12\gamma a_9T_1^2+12a_9^2T_1^2+12\gamma a_{10}T_1^2+12a_{10}^2T_1^2+4\gamma^2T_1^3-8\gamma a_9T_1^3-8a_9^2T_1^3-8\gamma a_{10}T_1^3-8a_{10}^2T_1^3+$$
$$4\gamma a_9T_1^4+4a_9^2T_1^4+4\gamma a_{10}T_1^4+4a_{10}^2T_1^4+12a_1^2\left(1-T_1+T_1^2\right)^2+4\gamma a_8\left(1-T_1+T_1^2\right)^2+4a_8^2\left(1-T_1+T_1^2\right)^2+$$
$$8\gamma^2\hbar T_1x_1y_1-8\gamma^2\hbar T_1^2x_1y_1+8\gamma^2\hbar T_1^4x_1y_1-8\gamma^2\hbar T_1^5x_1y_1-\gamma^2\hbar^2x_1^2y_1^2+2\gamma^2\hbar^2T_1x_1^2y_1^2+$$
$$4\gamma^2\hbar^2T_1^3x_1^2y_1^2-\gamma^2\hbar^2T_1^4x_1^2y_1^2+4\gamma^2\hbar^2T_1^5x_1^2y_1^2+8\gamma^2\hbar^2T_1^6x_1^2y_1^2-6\gamma^2\hbar^2T_1^7x_1^2y_1^2+7\gamma^2\hbar^2T_1^8x_1^2y_1^2-$$
$$4\gamma a_1\left(1-T_1+T_1^2\right)\left(1-2\hbar T_1^3x_1y_1+4\hbar T_1^4x_1y_1+T_1^2\left(-3+2\hbar x_1y_1\right)+T_1\left(1+2\hbar x_1y_1\right)\right)-$$
$$\gamma^2\hbar^2x_8^2y_8^2+2\gamma^2\hbar^2T_1x_8^2y_8^2-3\gamma^2\hbar^2T_1^2x_8^2y_8^2+2\gamma^2\hbar^2T_1^3x_8^2y_8^2-\gamma^2\hbar^2T_1^4x_8^2y_8^2-\gamma^2\hbar^2x_9^2y_9^2+$$
$$2\gamma^2\hbar^2T_1x_9^2y_9^2-3\gamma^2\hbar^2T_1^2x_9^2y_9^2+2\gamma^2\hbar^2T_1^3x_9^2y_9^2-\gamma^2\hbar^2T_1^4x_9^2y_9^2-\gamma^2\hbar^2x_{10}^2y_{10}^2+$$
$$2\gamma^2\hbar^2T_1x_{10}^2y_{10}^2-3\gamma^2\hbar^2T_1^2x_{10}^2y_{10}^2+2\gamma^2\hbar^2T_1^3x_{10}^2y_{10}^2-\gamma^2\hbar^2T_1^4x_{10}^2y_{10}^2\Big)\epsilon+O[\epsilon]^2\Big]$$

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$$\mathbb{E}\Big[-\frac{8\hbar a_1t_1+\left(\gamma+2\hbar a_9\right)t_9+\left(\gamma+2\hbar a_{10}\right)t_{10}}{2\gamma},$$
$$\hbar\left(\left(1+T_1+T_1^2+T_1^3\right)x_1y_1+x_9y_9+x_{10}y_{10}\right),\ \frac{1}{1-T_1+T_1^2}+\frac{1}{4\gamma\left(1-T_1+T_1^2\right)^3}$$
$$\hbar\Big(4\gamma a_{10}+4a_{10}^2+4\gamma^2T_1-8\gamma a_{10}T_1-8a_{10}^2T_1-8\gamma^2T_1^2+12\gamma a_{10}T_1^2+12a_{10}^2T_1^2+4\gamma^2T_1^3-8\gamma a_{10}T_1^3-$$
$$8a_{10}^2T_1^3+4\gamma a_{10}T_1^4+4a_{10}^2T_1^4+16a_1^2\left(1-T_1+T_1^2\right)^2+4\gamma a_9\left(1-T_1+T_1^2\right)^2+4a_9^2\left(1-T_1+T_1^2\right)^2+$$
$$8\gamma^2\hbar T_1x_1y_1-8\gamma^2\hbar T_1^2x_1y_1+16\gamma^2\hbar T_1^3x_1y_1-16\gamma^2\hbar T_1^4x_1y_1+16\gamma^2\hbar T_1^5x_1y_1-$$
$$8\gamma^2\hbar T_1^6x_1y_1-\gamma^2\hbar^2x_1^2y_1^2+2\gamma^2\hbar^2T_1x_1^2y_1^2+4\gamma^2\hbar^2T_1^3x_1^2y_1^2+7\gamma^2\hbar^2T_1^4x_1^2y_1^2+4\gamma^2\hbar^2T_1^5x_1^2y_1^2+$$
$$11\gamma^2\hbar^2T_1^6x_1^2y_1^2+4\gamma^2\hbar^2T_1^7x_1^2y_1^2+16\gamma^2\hbar^2T_1^8x_1^2y_1^2-6\gamma^2\hbar^2T_1^9x_1^2y_1^2+11\gamma^2\hbar^2T_1^{10}x_1^2y_1^2-$$
$$8\gamma a_1T_1\left(1-T_1+T_1^2\right)\left(1+\hbar x_1y_1+2\hbar T_1^2x_1y_1-\hbar T_1^3x_1y_1+3\hbar T_1^4x_1y_1+T_1\left(-2+\hbar x_1y_1\right)\right)-$$
$$\gamma^2\hbar^2x_9^2y_9^2+2\gamma^2\hbar^2T_1x_9^2y_9^2-3\gamma^2\hbar^2T_1^2x_9^2y_9^2+2\gamma^2\hbar^2T_1^3x_9^2y_9^2-\gamma^2\hbar^2T_1^4x_9^2y_9^2-\gamma^2\hbar^2x_{10}^2y_{10}^2+$$
$$2\gamma^2\hbar^2T_1x_{10}^2y_{10}^2-3\gamma^2\hbar^2T_1^2x_{10}^2y_{10}^2+2\gamma^2\hbar^2T_1^3x_{10}^2y_{10}^2-\gamma^2\hbar^2T_1^4x_{10}^2y_{10}^2\Big)\epsilon+O[\epsilon]^2\Big]$$

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$$\mathbb{E}\Big[-\frac{(\gamma+10\hbar a_1)\,t_1+(\gamma+2\hbar a_{10})\,t_{10}}{2\gamma},\ \hbar\left(\left(1+T_1+T_1^2+T_1^3+T_1^4\right)x_1\,y_1+x_{10}\,y_{10}\right),\ \frac{1}{1-T_1+T_1^2}+$$
$$\frac{1}{4\gamma\left(1-T_1+T_1^2\right)^3}\,\hbar\,\Big(20\,a_1^2\left(1-T_1+T_1^2\right)^2+4\gamma\,a_{10}\left(1-T_1+T_1^2\right)^2+4\,a_{10}^2\left(1-T_1+T_1^2\right)^2-4\gamma\,a_1\left(1-T_1+T_1^2\right)$$
$$\left(-1+4\hbar T_1^3 x_1 y_1+6\hbar T_1^4 x_1 y_1-2\hbar T_1^5 x_1 y_1+8\hbar T_1^6 x_1 y_1+T_1^2\left(-5+2\hbar x_1 y_1\right)+T_1\left(3+2\hbar x_1 y_1\right)\right)+$$
$$\gamma^2\Big(15\hbar^2 T_1^8 x_1^2 y_1^2+12\hbar^2 T_1^9 x_1^2 y_1^2+24\hbar^2 T_1^{10} x_1^2 y_1^2-6\hbar^2 T_1^{11} x_1^2 y_1^2+15\hbar^2 T_1^{12} x_1^2 y_1^2+$$
$$4\hbar T_1^5 x_1 y_1\left(-2+3\hbar x_1 y_1\right)+4\hbar T_1^7 x_1 y_1\left(-2+5\hbar x_1 y_1\right)+\hbar T_1^6 x_1 y_1\left(16+11\hbar x_1 y_1\right)+$$
$$\hbar^2 T_1^4\left(7x_1^2 y_1^2-x_{10}^2 y_{10}^2\right)-\hbar^2\left(x_1^2 y_1^2+x_{10}^2 y_{10}^2\right)+2T_1\left(2+4\hbar x_1 y_1+\hbar^2 x_1^2 y_1^2+\hbar^2 x_{10}^2 y_{10}^2\right)+$$
$$2T_1^3\left(2+8\hbar x_1 y_1+2\hbar^2 x_1^2 y_1^2+\hbar^2 x_{10}^2 y_{10}^2\right)-T_1^2\left(8+8\hbar x_1 y_1+3\hbar^2 x_{10}^2 y_{10}^2\right)\Big)\Big)\,\epsilon+O[\epsilon]^2\Big]$$

Out[∘]=

$$\Big\{76.7031,\ \mathbb{E}\Big[-\frac{(\gamma+6\hbar a_1)\,t_1}{\gamma},\ \hbar\left(1+T_1+T_1^2+T_1^3+T_1^4+T_1^5\right)x_1\,y_1,$$
$$\frac{1}{1-T_1+T_1^2}+\frac{1}{4\gamma\left(1-T_1+T_1^2\right)^3}\,\hbar\,\Big(24\,a_1^2\left(1-T_1+T_1^2\right)^2-8\gamma\,a_1\left(1-T_1+T_1^2\right)\Big(-1+2\hbar T_1^3 x_1 y_1+$$
$$3\hbar T_1^4 x_1 y_1+4\hbar T_1^5 x_1 y_1-\hbar T_1^6 x_1 y_1+5\hbar T_1^7 x_1 y_1+T_1^2\left(-3+\hbar x_1 y_1\right)+T_1\left(2+\hbar x_1 y_1\right)\Big)+$$
$$\gamma^2\Big(-\hbar^2 x_1^2 y_1^2+7\hbar^2 T_1^4 x_1^2 y_1^2+28\hbar^2 T_1^9 x_1^2 y_1^2+27\hbar^2 T_1^{10} x_1^2 y_1^2+20\hbar^2 T_1^{11} x_1^2 y_1^2+$$
$$32\hbar^2 T_1^{12} x_1^2 y_1^2-6\hbar^2 T_1^{13} x_1^2 y_1^2+19\hbar^2 T_1^{14} x_1^2 y_1^2-8T_1^2\left(1+\hbar x_1 y_1\right)+4\hbar T_1^5 x_1 y_1\left(2+3\hbar x_1 y_1\right)+$$
$$4\hbar T_1^7 x_1 y_1\left(4+5\hbar x_1 y_1\right)+\hbar T_1^6 x_1 y_1\left(-8+19\hbar x_1 y_1\right)+\hbar T_1^8 x_1 y_1\left(-8+31\hbar x_1 y_1\right)+$$
$$4T_1^3\left(1+4\hbar x_1 y_1+\hbar^2 x_1^2 y_1^2\right)+2T_1\left(2+4\hbar x_1 y_1+\hbar^2 x_1^2 y_1^2\right)\Big)\Big)\,\epsilon+O[\epsilon]^2\Big]\Big\}$$

Alternative Algorithms

AltLogos

In[∘]:=

```
λalt,k_[CU] := If[k == 0, 1, Module[{eq, d, b, c, so},
    eq = ρ@e^(ξ xCU).ρ@e^(η yCU) == ρ@e^(d yCU).ρ@e^(c (t 1CU - 2 ϵ aCU)).ρ@e^(b xCU);
    {so} = Solve[Thread[Flatten /@ eq], {d, b, c}] /. C@1 → 0;
    Series[e^(-η y-ξ x+η ξ t+c t+d y-2 ϵ c a+b x) /. so, {ϵ, 0, k}]]];
```

In[∘]:=

```
{λalt,2[CU], HL@Simplify@Normal[λalt,2[CU] == Last[ΛCU,2[{ξ, η}, {x, y}]]]}
```

Out[∘]=

$$\Big\{1+\left(2\,a\,\eta\,\xi-y\,\gamma\,\eta^2\,\xi-x\,\gamma\,\eta\,\xi^2+\frac{1}{2}\,t\,\gamma\,\eta^2\,\xi^2\right)\epsilon+$$
$$\frac{1}{2}\left(\left(2\,a\,\eta\,\xi-y\,\gamma\,\eta^2\,\xi-x\,\gamma\,\eta\,\xi^2+\frac{1}{2}\,t\,\gamma\,\eta^2\,\xi^2\right)^2+2\left(-a\,\gamma\,\eta^2\,\xi^2+y\,\gamma^2\,\eta^3\,\xi^2+x\,\gamma^2\,\eta^2\,\xi^3-\frac{1}{3}\,t\,\gamma^2\,\eta^3\,\xi^3\right)\right)$$
$$\epsilon^2+O[\epsilon]^3,\ \text{True}\Big\}$$
