Pensieve header: A unified verification notebook for the sl_2-portfolio project, Uxi version. Continues pensieve://Projects/PPSA/nb/Verification.pdf.

Continues pensieve://2017-06/ and pensieve://2017-08/.

Prolog

```
wdir = SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\SL2Portfolio"];
NotebookOpen[wdir <> "\\MakeSnips.nb"];
```

In[∘]:=
```
HL[ℰ_] := Style[ℰ, Background → Yellow];
```

DocileQ

DocileQ

In[∘]:=
```
DQ[ℰ_] := (Exponent[Normal@ℰ /.
    {a → a / ϵ, a_i_ → a_i / ϵ, (u : x | y) :→ ϵ^(-1/2) u, (u : x | y)_i_ :→ ϵ^(-1/2) u_i}, ϵ, Min] ≥ 0);
```

In[∘]:=
```
DQ /@ {ϵ^2 x y a_2, ϵ^2 x^2 y^3}
```

Out[∘]=
```
{True, False}
```

Initialization / Utilities

It is verification-risky to work with low $E!

TD

In[∘]:=
```
$p = 2; $k = 1; $E := {$k, $p};
$trim := {ℏ^p_· /; p > $p → 0, ϵ^k_· /; k > $k → 0};
SetAttributes[{SS, SST}, HoldAll];
TRule = {T_i_ → e^(ℏ t_i), T → e^(ℏ t)}; q_ℏ = e^(γ ϵ ℏ);
SS[ℰ_, op_] := Collect[
   Normal@Series[If[$p > 0, ℰ, ℰ /. TRule], {ℏ, 0, $p}],
   ℏ, op];
SS[ℰ_] := SS[ℰ, Together];
SST[ℰ_, op___] := SS[ℰ /. TRule, op];
Simp[ℰ_, op_] := Collect[ℰ, _CU | _QU, op];
Simp[ℰ_] := Simp[ℰ, SS[#, Expand] &];
SimpT[ℰ_] := Collect[ℰ, _CU | _QU, SST[#, Expand] &];
Kδ /: Kδ_i_,j_ := If[i === j, 1, 0];
```

Differential polynomials (DP):

Utils

In[∘]:=
```
DP_(α_→D_x_, β_→D_y_)[P_][λ_] :=
  Total[CoefficientRules[Normal@P, {α, β}] /. ({m_, n_} → c_) :→ c ∂_({x,m},{y,n}) λ]
```

```
HL[DP_(x→D_ξ, y→D_η)[x^2 y^3][e^(δ ξ η)] == 6 e^(δ η ξ) δ^3 ξ + 6 e^(δ η ξ) δ^4 η ξ^2 + e^(δ η ξ) δ^5 η^2 ξ^3]
```

```
True
```

CF

In[∘]:=
```
CF[ℰ_] := ExpandDenominator@
  ExpandNumerator@Together[Expand[ℰ] //. ⅇ^x_ ⅇ^y_ ⧴ ⅇ^(x+y) /. ⅇ^x_ ⧴ ⅇ^CF[x]];
```

SeriesData

In[∘]:=
```
Unprotect[SeriesData];
SeriesData /: CF[sd_SeriesData] := MapAt[CF, sd, 3];
SeriesData /: Expand[sd_SeriesData] := MapAt[Expand, sd, 3];
SeriesData /: Simplify[sd_SeriesData] := MapAt[Simplify, sd, 3];
SeriesData /: Together[sd_SeriesData] := MapAt[Together, sd, 3];
SeriesData /: Collect[sd_SeriesData, specs__] := MapAt[Collect[#, specs] &, sd, 3];
Protect[SeriesData];
```

Self-Pair (SP):

SP

In[∘]:=
```
SP{}[P_] := P; SP{ξ_→x_,ps___}[P_] := Expand[P // SP{ps}] /. f_. ξ^d_. ⧴ ∂{x,d} f
```

```
SP{ξ→x}[(ξ^2 + ξ + 3) (x^5 ⅇ^x + 7 x) + 99 a]
```

$7 + 99\,a + 21\,x + 20\,e^x x^3 + 15\,e^x x^4 + 5\,e^x x^5$

```
SP{ξ→x,η→y}[(ξ^2 + ξ + 3 + 2 ξ η) (x^5 ⅇ^x + 7 x) + 99 a + ⅇ^(δ x y) ξ η]
```

$7 + 99\,a + 21\,x + 20\,e^x x^3 + 15\,e^x x^4 + 5\,e^x x^5 + e^{x y \delta}\,\delta + e^{x y \delta}\,x\,y\,\delta^2$

DeclareAlgebra

QLImplementation

In[∘]:=
```
Unprotect[NonCommutativeMultiply]; Attributes[NonCommutativeMultiply] = {};
(NCM = NonCommutativeMultiply)[x_] := x;
NCM[x_, y_, z__] := (x ** y) ** z;
0 ** _ = _ ** 0 = 0;
(x_Plus) ** y_ := (# ** y) & /@ x; x_ ** (y_Plus) := (x ** #) & /@ y;
B[x_, x_] = 0; B[x_, y_] := x ** y - y ** x;
B[x_, y_, e_] := B[x, y, e] = B[x, y];
```

QLImplementation

In[∘]:=

```
DeclareAlgebra[U_Symbol, opts__Rule] := Module[{gp, sr, g, cp, M, CE, pow, k = 0,
    gs = Generators /. {opts},
    cs = Centrals /. {opts} /. Centrals → {} },
  (#_U = U@#) & /@ gs;
  gp = Alternatives @@ gs; gp = gp | gp_; (* gens *)
  sr = Flatten@Table[{g → ++k, g_i_ → {i, k}}, {g, gs}]; (* sorting → *)
  cp = Alternatives @@ cs; (* cents *)
  SetAttributes[M, HoldRest]; M[0, _] = 0; M[a_, x_] := a x;
  CE[ℰ_] := Collect[ℰ, _U, Expand] /. $trim;
  U_i_[ℰ_] := ℰ /. {t : cp ⧴ t_i, u_U ⧴ (#_i &) /@ u};
  U_i_[NCM[]] = pow[ℰ_, 0] = U@{} = 1_U = U[];
  B[U@(x_)_i_, U@(y_)_i_] := U_i@B[U@x, U@y];
  B[U@(x_)_i_, U@(y_)_j_] /; i =!= j := 0;
  B[U@y_, U@x_] := CE[-B[U@x, U@y]];
  x_ ** (c_. 1_U) := CE[c x]; (c_. 1_U) ** x_ := CE[c x];
  (a_. U[xx___, x_]) ** (b_. U[y_, yy___]) := If[OrderedQ[{x, y} /. sr],
    CE@M[a b /. $trim, U[xx, x, y, yy]],
    U@xx ** CE@M[a b /. $trim, U@y ** U@x + B[U@x, U@y, $E]] ** U@yy ];
  U@{c_. * (l : gp)^n_, r___} /; FreeQ[c, gp] := CE[c U@Table[l, {n}] ** U@{r}];
  U@{c_. * l : gp, r___} := CE[c U[l] ** U@{r}];
  U@{c_, r___} /; FreeQ[c, gp] := CE[c U@{r}];
  U@{l_Plus, r___} := CE[U@{#, r} & /@ l];
  U@{l_, r___} := U@{Expand[l], r};
  U[ℰ_NonCommutativeMultiply] := U /@ ℰ;
  𝕆_U[specs___, poly_] := Module[{sp, null, vs, us},
    sp = Replace[{specs}, l_List ⧴ l_null, {1}];
    vs = Join @@ (First /@ sp);
    us = Join @@ (sp /. l_s_ ⧴ (l /. x_i_ ⧴ x_s));
    CE[Total[
       CoefficientRules[poly, vs] /. (p_ → c_) ⧴ c U@(us^p)
      ]] /. x_null ⧴ x];
  pow[ℰ_, n_] := pow[ℰ, n - 1] ** ℰ;
  𝕊_U[ℰ_, ss___Rule] := CE@Total[
     CoefficientRules[ℰ, First /@ {ss}] /.
      (p_ → c_) ⧴ c NCM @@ MapThread[pow, {Last /@ {ss}, p}]];
  σ_rs___[c_. * u_U] := (c /. (t : cp)_j_ ⧴ t_{j /. {rs}}) U[List @@ (u /. v_j_ ⧴ v_{j /. {rs}})];
  m_{j_→k_}[c_. * u_U] := CE[((c /. (t : cp)_j → t_k) DeleteCases[u, _j|k]) **
     U @@ Cases[u, w_j ⧴ w_k] ** U @@ Cases[u, _k]];
  U /: c_. * u_U * v_U := CE[c u ** v];
  S_i_[c_. * u_U] := CE[((c /. S_i[U, Centrals]) DeleteCases[u, _i]) **
     U_i[NCM @@ Reverse@Cases[u, x_i ⧴ S@U@x]]];
  Δ_{i_→j_,k_}[c_. * u_U] := CE[((c /. Δ_{i→j,k}[U, Centrals]) DeleteCases[u, _i]) **
     (NCM @@ Cases[u, x_i ⧴ σ_{1→j,2→k}@Δ@U@x] /. NCM[] → U[])]; ]
```

DeclareMorphism

QLImplementation

In[∘]:=
```
DeclareMorphism[m_, U_ → V_, ongs_List, oncs_List: {}] := (
  Replace[ongs,
   {(g_ → img_) ⧴ (m[U[g]] = img), (g_ ⧴ img_) ⧴ (m[U[g]] := img /. $trim)}, {1}];
  m[1_U] = 1_V;
  m[U[g_i_]] := V_i[m[U@g]];
  m[U[vs__]] := NCM @@ (m /@ U /@ {vs});
  m[ℰ_] := Simp[ℰ /. oncs /. u_U ⧴ m[u]] /. $trim; )
```

Meta-Operations

QLImplementation

In[∘]:=
```
σ_rs___[ℰ_Plus] := σ_rs /@ ℰ;
m_j_→j_ = Identity; m_j_→k_[0] = 0;
m_j_→k_[ℰ_Plus] := Simp[m_j→k /@ ℰ];
m_is___,i_,j_→k_[ℰ_] := m_j→k@m_is,i→j@ℰ;
S_i_[ℰ_Plus] := Simp[S_i /@ ℰ];
Δ_is___[ℰ_Plus] := Simp[Δ_is /@ ℰ];
```

Implementing CU = $\mathcal{U}(sl_2^{\gamma\epsilon})$

Verify σ and Δ! Also Generalize Δ to $\Delta_{i,j_1,j_2,\ldots}$.

CU

In[∘]:=
```
DeclareAlgebra[CU, Generators → {y, a, x}, Centrals → {t}];
B[a_CU, y_CU] = -γ y_CU; B[x_CU, a_CU] = -γ x_CU;
B[x_CU, y_CU] = 2 ϵ a_CU - t 1_CU;
(S@y_CU = -y_CU; S@a_CU = -a_CU; S@x_CU = -x_CU;)
S_i_[CU, Centrals] = {t_i → -t_i};
Δ@y_CU = CU@y_1 + CU@y_2; Δ@a_CU = CU@a_1 + CU@a_2; Δ@x_CU = CU@x_1 + CU@x_2;
Δ_i_→j_,k_[CU, Centrals] = {t_i → t_j + t_k};
```

Verifying associativity on triples of generators:

```
With[{bas = CU /@ {y, a, x}},
 Table[z1 ** (z2 ** z3) - (z1 ** z2) ** z3 // Simp // HL,
  {z1, bas}, {z2, bas}, {z3, bas}] ]
```

```
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying associativity on a “random” triple:

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
   (rhs = (z1 ** z2) ** z3 // Simp) // Short,
   z1 ** (z2 ** z3) - rhs // Simp // HL
  }] // Timing
```

```
{1.32813,
 {(28 t^2 γ^4 + 116 t γ^5 ϵ) CU[y, y, y, x, x] + <<21>> + CU[y, y, y, y, y, a, a, a, a, x, x, x, x], 0}}
```

Implementing QU = $\mathcal{U}_q(\mathrm{sl}_2^{\gamma\epsilon})$

Aside

```
Series[(1 - T ⅇ^(-2 ϵ a ℏ)) / ℏ, {a, 0, 3}]
```

Aside

$$\frac{1-T}{\hbar} + 2 T \epsilon a - 2 \left(T \epsilon^2 \hbar\right) a^2 + \frac{4}{3} T \epsilon^3 \hbar^2 a^3 + O[a]^4$$

In[∘]:=
```
HL /@ DQ /@ Series[{(1 - T ⅇ^(-2 ϵ a ℏ)) / ℏ, ⅇ^(ℏ ϵ a)}, {ϵ, 0, 5}]
```

Out[∘]=
```
{True, True}
```

QU

In[∘]:=
```
DeclareAlgebra[QU, Generators → {y, a, x}, Centrals → {t, T}];
B[a_QU, y_QU] = -γ y_QU; B[x_QU, a_QU] = -γ QU@x;
B[x_QU, y_QU] := SS[q_ℏ - 1] QU@{y, x} + 𝕆_QU[{a}, SS[(1 - T ⅇ^(-2 ϵ a ℏ)) / ℏ]];
(S@y_QU := 𝕆_QU[{a, y}, SS[-T^-1 ⅇ^(ℏ ϵ a) y]]; S@a_QU = -a_QU; S@x_QU := 𝕆_QU[{a, x}, SS[-ⅇ^(ℏ ϵ a) x]];)
S_i_[QU, Centrals] = {t_i → -t_i, T_i → T_i^-1};
Δ@y_QU := 𝕆_QU[{y_1, a_1}_1, {y_2}_2, SS[y_1 + T_1 ⅇ^(-ℏ ϵ a_1) y_2]];
Δ@a_QU = QU@a_1 + QU@a_2; Δ@x_QU := 𝕆_QU[{a_1, x_1}_1, {x_2}_2, SS[x_1 + ⅇ^(-ℏ ϵ a_1) x_2]];
Δ_i_→j_,k_[QU, Centrals] = {t_i → t_j + t_k, T_i → T_j T_k};
```

```
With[{bas = QU /@ {y, a, x}}, Table[{z1, z2} → Simp[z1 ** z2 - z2 ** z1], {z1, bas}, {z2, bas}] ]
```

$$\Big\{\Big\{\{QU[y], QU[y]\} \to 0,\ \{QU[y], QU[a]\} \to \gamma\, QU[y],\ \{QU[y], QU[x]\} \to \frac{(-1+T)\, QU[]}{\hbar} - 2 T \epsilon\, QU[a] - \gamma \epsilon \hbar\, QU[y, x]\Big\},$$
$$\{\{QU[a], QU[y]\} \to -\gamma\, QU[y],\ \{QU[a], QU[a]\} \to 0,\ \{QU[a], QU[x]\} \to \gamma\, QU[x]\},$$
$$\Big\{\{QU[x], QU[y]\} \to \frac{(1-T)\, QU[]}{\hbar} + 2 T \epsilon\, QU[a] + \gamma \epsilon \hbar\, QU[y, x],\ \{QU[x], QU[a]\} \to -\gamma\, QU[x],\ \{QU[x], QU[x]\} \to 0\Big\}\Big\}$$

Verifying associativity on triples of generators:

```
With[{bas = QU /@ {y, a, x}},
 Table[HL[z1 ** (z2 ** z3) - (z1 ** z2) ** z3] // Simp,
  {z1, bas}, {z2, bas}, {z3, bas} ] ]
```

```
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying associativity on a “random” triple (~34 secs @ $p=5, $k=2):

```
With[{z1 = QU[y, y, a, a, x, x], z2 = QU[y, a, x], z3 = QU[y, y, a, x]}, {
    (rhs = (z1 ** z2) ** z3 // Simp) // Short,
   HL[z1 ** (z2 ** z3) - rhs // Simp]
  }] // Timing
```

$\{3.78125, \{\left(\frac{28\,\gamma^4 - 56\,T\,\gamma^4 + 28\,T^2\,\gamma^4}{\hbar^2} + \frac{82\,\gamma^5\,\epsilon - 280\,T \ll 1 \gg \epsilon + 198\,T^2\,\gamma^5\,\epsilon}{\hbar}\right)$ QU[y, y, y, x, x] + $\ll 18 \gg + (1 + 8\,\gamma\,\epsilon\,\hbar)$ QU[y, ≪11≫, x], 0}}

Verifying that $\text{Lim}_{\hbar\to 0}$ QU = CU using a "random" product (~23 secs @ $p=5, $k=2):

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
    Short[lhs = z1 ** (z2 ** z3)],
    Short[rhs = (QU @@ z1) ** ((QU @@ z2) ** (QU @@ z3))],
    Expand[Limit[rhs /. TRule ⋃ {QU → CU}, ℏ → 0] - lhs] // HL
  }] // Timing
```

$\{10.125, \{28\,t^2\,\gamma^4$ CU[y, y, y, x, x] + $116\,t\,\gamma^5\,\epsilon$ CU[y, y, y, x, x] + ≪44≫ + CU[y, y, y, y, y, a, a, a, a, x, x, x, x],

$2\left(\frac{\gamma^4}{\hbar^2} - \frac{2\,T\,\gamma^4}{\hbar^2} + \frac{T^2\,\gamma^4}{\hbar^2} + \frac{\gamma^5\,\epsilon}{\hbar} - \frac{2\,T\,\gamma^5\,\epsilon}{\hbar} + \frac{T^2\,\gamma^5\,\epsilon}{\hbar}\right)$ QU[y, y, y, x, x] +

$\ll 209 \gg + (1 + 8\,\gamma\,\epsilon\,\hbar)$ QU[y, y, y, ≪7≫, x, x, x], 0}}

Verifying *σ*, *m*, *S*, and Δ.

Verifying $\sigma_{i\to j,k\to l}$:

In[∘]:= $\text{CU@}x_1 + \text{CU@}x_2$ // $\sigma_{1\to 3,2\to 4}$

Out[∘]= CU[x_3] + CU[x_4]

Verifying relabeling using *m*:

In[∘]:= $t_1\,t_3\,\text{CU}[y_1, a_1, x_2] + t_1\,t_1\,\text{CU}[y_1, a_2, x_2]$ // $m_{1\to 3}$

Out[∘]= $\text{CU}[a_2, x_2, y_3]\,t_3^2 + \text{CU}[x_2, y_3, a_3]\,t_3^2$

Verifying the meta-associativity of *m*:

```
In[∘]:= Module[{z, u},
         Table[u = CU[z⟦1⟧₁, z⟦2⟧₂, z⟦3⟧₃]; HL[m₁,₃→₃@m₂,₃→₃@u == m₂,₃→₃@m₁,₂→₂@u],
          {z, Tuples[{y, a, x}, 3]}, {U, {CU, QU}}]]
```

Out[∘]= {{True, True}, {True, True}}

Verifying the involutivity of *S* on CU on products of triples:

```
In[•]:= With[{bas = CU /@ {y₁, a₁, x₁}},
          Table[HL@Simp[z1 ** z2 ** z3 - S₁@S₁[z1 ** z2 ** z3]],
           {z1, bas}, {z2, bas}, {z3, bas} ] ]

Out[•]= {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
          {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying that S is an anti-homomorphism on CU/QU:

```
In[•]:= With[{bas = U /@ {y₁, a₁, x₁}},
          Table[HL@Simp[S₁[z1 ** z2] - S₁[z2] ** S₁[z1]],
           {z1, bas}, {z2, bas} , {U, {CU, QU}}] ]

Out[•]= {{{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}}
```

Verifying the co-associativity of Δ:

```
In[•]:= Block[{bas = U /@ {y₁, a₁, x₁}},
          Table[(z1 ** z2 ** z3 // Δ1→1,2 // Δ2→2,3) - (z1 ** z2 ** z3 // Δ1→1,3 // Δ1→1,2) // Simp // HL,
           {z1, bas}, {z2, bas}, {z3, bas} , {U, {CU, QU}}] ]

Out[•]= {{{{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}},
          {{{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}},
          {{{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}}}
```

Verifying S-Δ compatibility:

```
In[•]:= Block[{bas = U /@ {y₁, a₁, x₁}},
          Table[z1 ** z2 ** z3 // Δ1→1,2 // Si // m1,2→1 // Simp // HL,
           {U, {CU, QU}}, {i, 2}, {z1, bas}, {z2, bas}, {z3, bas}] ]

Out[•]= {{{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
            {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}, {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
            {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}},
          {{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
            {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}, {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
            {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}}}
```

Verifying S-Δ compatibility for opposite m, only for CU:

```
In[•]:= Block[{bas = CU /@ {y₁, a₁, x₁}},
          Table[z1 ** z2 ** z3 // Δ1→1,2 // Si // m2,1→1 // Simp // HL,
           {i, 2}, {z1, bas}, {z2, bas}, {z3, bas} ]]

Out[•]= {{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
           {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}, {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
           {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}}
```

Verifying m-Δ compatibility:

In[∘]:=
```
Block[{bas1 = U /@ {y1, a1, x1}, bas2 = U /@ {y2, a2, x2}},
 Table[(z1 ** z2 ** z3 ** z4 // m1,2→1 // Δ1→1,2) -
    (z1 ** z2 ** z3 ** z4 // Δ1→3,4 // Δ2→5,6 // m3,5→1 // m4,6→2) // Simp // HL,
  {U, {CU, QU}}, {z1, bas1}, {z2, bas1}, {z3, bas2}, {z4, bas2} ] ]
```

Out[∘]=
```
{{{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
   {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}},
  {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
   {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}, {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
   {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}},
 {{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
   {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}, {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
   {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}},
  {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
   {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}}}
```

Implementing θ

theta

In[∘]:=
```
DeclareMorphism[Cθ, CU → CU, {y → -x_CU, a → -a_CU, x → -y_CU}, {t → -t, T → T^-1}];
DeclareMorphism[Qθ, QU → QU, {y :→ 𝕆_QU[{a, x}, SS[-T^(-1/2) ℯ^(ℏ ϵ a) x]],
  a → -a_QU, x :→ 𝕆_QU[{a, y}, SS[-T^(-1/2) ℯ^(ℏ ϵ a) y]]}, {t → -t, T → T^-1}]
```

Verifying involutivity on CU:

```
With[{bas = CU /@ {y, a, x}},
 Table[z → Cθ[z] → HL[Cθ[Cθ[z]]], {z, bas}] ]
```

{CU[y] → -CU[x] → CU[y], CU[a] → -CU[a] → CU[a], CU[x] → -CU[y] → CU[x]}

Verifying that θ is a multiplicative homomorphism on CU:

```
With[{bas = CU /@ {y, a, x}},
 Table[Cθ[z1 ** z2] - Cθ[z1] ** Cθ[z2] // HL, {z1, bas}, {z2, bas}] ]
```

{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

Verifying involutivity on QU:

```
With[{bas = QU /@ {y, a, x}},
 Table[z → Qθ[z] → HL[Simp[Qθ[Qθ[z]], PowerExpand]], {z, bas}] ]
```

$\left\{QU[y] \to -\frac{QU[x]}{\sqrt{T}} - \frac{\epsilon\hbar\, QU[a, x]}{\sqrt{T}} \to QU[y],\ QU[a] \to -QU[a] \to QU[a],\right.$

$\left. QU[x] \to \left(-\frac{1}{\sqrt{T}} + \frac{\gamma\epsilon\hbar}{\sqrt{T}}\right) QU[y] - \frac{\epsilon\hbar\, QU[y, a]}{\sqrt{T}} \to QU[x]\right\}$

Verifying that θ is a multiplicative homomorphism on QU:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL[Simp[Qθ[z1 ** z2] - Qθ[z1] ** Qθ[z2]]], {z1, bas}, {z2, bas}] ]
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

The Asymmetric Dequantizator

Following pensieve://People/VanDerVeen/Dequant1.pdf.

ADeq

In[∘]:=

$$A\mathbb{D}\$f = \gamma \left(\left(\mathrm{Cosh}\left[\hbar\left(a\,\epsilon + \frac{\gamma\,\epsilon}{2} - \frac{t}{2}\right)\right] - \mathrm{Cosh}\left[\hbar\sqrt{\left(\frac{t-\gamma\,\epsilon}{2}\right)^2 + \epsilon\,\omega}\,\right]\right) \Big/ \left(\hbar\, e^{\hbar\,((a+\gamma)\,\epsilon - t/2)}\,\mathrm{Sinh}\left[\frac{\gamma\,\epsilon\,\hbar}{2}\right]\left(a^2\,\epsilon + a\,\gamma\,\epsilon - a\,t - \omega\right)\right)\right);$$

Docility of A𝔻$f:

In[∘]:=

```
HL@DQ@Block[{$p = 4}, Collect[SS@A𝔻$f /. ω → a₁, ϵ]]
```

Out[∘]= True

Scaling behaviour of A𝔻$f:

```
HL@Simplify[A𝔻$f == ((A𝔻$f /. γ → 1) /. {ϵ → γ ϵ, a → γ^-1 a, ω → γ^-1 ω})]
True
```

```
HL@FullSimplify[
  A𝔻$f == ((A𝔻$f /. γ → 1) /. {ℏ → γ^2 ℏ, ϵ → ϵ / γ, a → a / γ, t → γ^-2 t, ω → γ^-3 ω})]
True
```

ADeq

In[∘]:=

```
A𝔻$ω = γ CU[y, x] + ϵ CU[a, a] - (t - γ ϵ) CU[a];
```

ADeq

In[∘]:=

```
DeclareMorphism[A𝔻, QU → CU,
  {a → a_CU, x → CU@x, y :→ 𝕊_CU[SS[A𝔻$f], a → a_CU, ω → A𝔻$ω] ** y_CU}]
```

Verifying that the asymmetric dequantizator is a homomorphism:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL[SimpT[A𝔻[z1 ** z2] - A𝔻[z1] ** A𝔻[z2]]], {z1, bas}, {z2, bas}] ]
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

The Symmetric Dequantizator

Following pensieve://People/VanDerVeen/Dequant1.pdf.

SDeq

In[∘]:=
```
SD$g = Sqrt[((2 γ (Cosh[ħ/2 Sqrt[t^2 + γ^2 ϵ^2 + 4 ϵ ϖ]] - Cosh[(t - ϵ γ - 2 ϵ a)/(2/ħ)]))/
    (Sinh[γ ϵ ħ/2] (t (2 a + γ) - 2 a (a + γ) ϵ + 2 ϖ) ħ))];
```

SDeq

In[∘]:=
```
SD$f = Simplify[e^(ħ (t/2-ϵ a)) (SD$g /. {a → -a, t → -t})];
```

Verify agreement with the formulas in pensieve://People/VanDerVeen/Dequant1.pdf:

In[∘]:=
```
{SD$P = (Cosh[ħ ((ϵ-t)/2 + ϵ a)] - Cosh[ħ Sqrt[(t^2+ϵ^2)/4 + ϵ ϖ]])/
    (ħ Sinh[(-ϵ ħ)/2] (ϖ - ϵ a^2 + (t - ϵ) a + t/2)),
  Simplify[SD$P == (SD$P /. {a → -a - 1, t → -t})] // HL,
  PowerExpand@Simplify[(SD$P /. {ħ → γ^2 ħ, ϵ → ϵ/γ, a → a/γ, t → γ^-2 t, ϖ → γ^-3 ϖ}) ==
     SD$g (SD$g /. {a → -a - γ, t → -t})] // HL,
  SD$Q = Simplify[SD$P /. {a → c - 1/2}],
  Simplify[SD$Q == (SD$Q /. {c → -c, t → -t})] // HL,
  FullSimplify[SD$g == FullSimplify[
     Sqrt[SD$Q] /. c → a + 1/2 /. {ħ → γ^2 ħ, ϵ → ϵ/γ, a → a/γ, t → γ^-2 t, ϖ → γ^-3 ϖ}]] // HL,
  HL@DQ@Block[{$p = 4}, Collect[SS@SD$g /. ω → a1, ϵ]],
  HL@DQ@Block[{$p = 4}, Collect[SS@SD$f /. ω → a1, ϵ]]
}
```

Out[∘]=
$$\left\{-\left(\left(\left(\mathrm{Cosh}\left[\left(a\epsilon+\frac{1}{2}(-t+\epsilon)\right)\hbar\right]-\mathrm{Cosh}\left[\sqrt{\frac{1}{4}(t^2+\epsilon^2)+\epsilon\varpi}\;\hbar\right]\right)\mathrm{Csch}\left[\frac{\epsilon\hbar}{2}\right]\right)\Big/\left(\left(\frac{t}{2}+a(t-\epsilon)-a^2\epsilon+\varpi\right)\hbar\right)\right),\ \text{True},\ \text{True},\right.$$
$$\left.-\left(\left(4\left(\mathrm{Cosh}\left[\frac{1}{2}(t-2c\epsilon)\hbar\right]-\mathrm{Cosh}\left[\frac{1}{2}\sqrt{t^2+\epsilon^2+4\epsilon\varpi}\;\hbar\right]\right)\mathrm{Csch}\left[\frac{\epsilon\hbar}{2}\right]\right)\Big/\left((4ct+\epsilon-4c^2\epsilon+4\varpi)\hbar\right)\right),\ \text{True},\ \text{True},\ \text{True},\ \text{True}\right\}$$

SDeq

In[∘]:=
```
SD$ϖ = γ CU[y, x] + ϵ CU[a, a] - (t - γ ϵ) CU[a] - t γ 1_CU/2;
```

SDeq

In[∘]:=
```
DeclareMorphism[S𝔻, QU → CU, {a → a_CU,
   x :→ 𝕊_CU[SS[S𝔻$f], a → a_CU, ϖ → S𝔻$ϖ] ** x_CU,
   y :→ 𝕊_CU[SS[S𝔻$g], a → a_CU, ϖ → S𝔻$ϖ] ** y_CU }]
```

Verifying the θ-symmetry:

```
Table[HL@SimpT[Cθ[S𝔻[z]] == S𝔻[Qθ[z]]], {z, QU /@ {y, a, x}}]
```

```
{True, True, True}
```

Verifying that the symmetric dequantizator is a homomorphism:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL@SimpT[S𝔻[z1 ** z2] - S𝔻[z1] ** S𝔻[z2]], {z1, bas}, {z2, bas}] ]
```

```
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

The representation ρ

rho

In[∘]:=

ρ@y_{CU} = ρ@y_{QU} = $\begin{pmatrix} 0 & 0 \\ \epsilon & 0 \end{pmatrix}$; ρ@a_{CU} = ρ@a_{QU} = $\begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix}$;

ρ@x_{CU} = $\begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix}$; ρ@x_{QU} = $\begin{pmatrix} 0 & \left(1 - e^{-\gamma \epsilon \hbar}\right) / (\epsilon \hbar) \\ 0 & 0 \end{pmatrix}$;

$\rho\left[e^{\mathcal{E}\_}\right]$:= MatrixExp[ρ[$\mathcal{E}$]];

ρ[$\mathcal{E}$_] := ($\mathcal{E}$ /. TRule /. t → γ ϵ /. (U : CU | QU) [u___] :→ Fold[Dot, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, ρ /@ U /@ {u}])

Verifying that ρ represents CU and QU:

Table[HL[SS[ρ[z1 ** z2] == ρ[z1].ρ[z2]] /. $\epsilon^{k\_\cdot}$ /; k > $k → 0],
 {U, {CU, QU}}, {z1, U /@ {y, a, x}}, {z2, U /@ {y, a, x}}]

```
{{{True, True, True}, {True, True, True}, {True, True, True}},
 {{True, True, True}, {True, True, True}, {True, True, True}}}
```

Commuting $e^{\alpha a}$ with $e^{\xi x}$:

Table[HL[ρ[$e^{\xi\, U@x}$].ρ[$e^{\alpha\, U@a}$] == ρ[$e^{\alpha\, U@a}$].ρ[$e^{e^{-\gamma\alpha}\xi\, U@x}$]], {U, {CU, QU}}]

```
{True, True}
```

Œ and the logoi Λ

Logoi from Pensieve://Talks/Toulouse-1705/DogmaDemo.nb and from Pensieve://Talks/Sydney-1708/ExtraDetails@@.nb.

MultiplyingOEs

In[∘]:=
```
Œ_U_[s1__, Q1_, P1_] Œ_U_[s2__, Q2_, P2_] ^:= Œ_U[s1, s2, Q1 + Q2, P1 P2];
```

CdsO

In[∘]:=

```
CU@𝔼CU[specs___, Q_, P_] := 𝕆CU[specs, SS[𝕖^Q P]];
QU@𝔼QU[specs___, Q_, P_] := 𝕆QU[specs, SS[𝕖^Q P]];
```

Logos

In[∘]:=

```
c_Integer_{k_Integer} := c + O[ϵ]^(k+1);
Λ_{U_,k_}[{α_, β_}, {x_, x_}] := 𝔼_U[{x}, (α + β) x, 1_k];
Λ_{U_,k_}[{ξ_, α_}, {x, a}] := 𝔼_U[{a, x}, α a + 𝕖^(-γ α) ξ x, 1_k];
Λ_{U_,k_}[{α_, η_}, {a, y}] := 𝔼_U[{y, a}, α a + 𝕖^(-γ α) η y, 1_k];
```

```
Table[
 {Λ_{U,1}[{α, β}, {u, u}],
  lhs = U@𝔼_U[{u1, u2}, ℏ (α u1 + β u2), 1], HL[lhs == U@Λ_{U,1}[ℏ {α, β}, {u, u}]]},
 {U, {CU, QU}}, {u, {y, a, x}}]
```

{{{$\mathbb{E}_{CU}[\{y\}, y(\alpha+\beta), 1+O[\epsilon]^2]$,
$CU[] + (\alpha\hbar+\beta\hbar)\,CU[y] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right)CU[y, y]$, True},
{$\mathbb{E}_{CU}[\{a\}, a(\alpha+\beta), 1+O[\epsilon]^2]$, $CU[] + (\alpha\hbar+\beta\hbar)\,CU[a] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right)CU[a, a]$,
True}, {$\mathbb{E}_{CU}[\{x\}, x(\alpha+\beta), 1+O[\epsilon]^2]$,
$CU[] + (\alpha\hbar+\beta\hbar)\,CU[x] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right)CU[x, x]$, True}},
{{$\mathbb{E}_{QU}[\{y\}, y(\alpha+\beta), 1+O[\epsilon]^2]$, $QU[] + (\alpha\hbar+\beta\hbar)\,QU[y] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right)QU[y, y]$,
True}, {$\mathbb{E}_{QU}[\{a\}, a(\alpha+\beta), 1+O[\epsilon]^2]$, $QU[] + (\alpha\hbar+\beta\hbar)\,QU[a] +$
$\left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right)QU[a, a]$, True}, {$\mathbb{E}_{QU}[\{x\}, x(\alpha+\beta), 1+O[\epsilon]^2]$,
$QU[] + (\alpha\hbar+\beta\hbar)\,QU[x] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right)QU[x, x]$, True}}}

```
{Λ_{#,1}[{ξ, α}, {x, a}], lhs = #@𝔼_#[{x, a}, ℏ (ξ x + α a), 1],
   HL[lhs == #@Λ_{#,1}[ℏ {ξ, α}, {x, a}]]} & /@ {CU, QU}
```

{{$\mathbb{E}_{CU}[\{a, x\}, a\alpha + e^{-\alpha\gamma} x\xi, 1+O[\epsilon]^2]$,
$CU[] + \alpha\hbar\,CU[a] + (\xi\hbar - \alpha\gamma\xi\hbar^2)\,CU[x] + \frac{1}{2}\alpha^2\hbar^2\,CU[a, a] + \alpha\xi\hbar^2\,CU[a, x] + \frac{1}{2}\xi^2\hbar^2\,CU[x, x]$,
True}, {$\mathbb{E}_{QU}[\{a, x\}, a\alpha + e^{-\alpha\gamma} x\xi, 1+O[\epsilon]^2]$, $QU[] + \alpha\hbar\,QU[a] +$
$(\xi\hbar - \alpha\gamma\xi\hbar^2)\,QU[x] + \frac{1}{2}\alpha^2\hbar^2\,QU[a, a] + \alpha\xi\hbar^2\,QU[a, x] + \frac{1}{2}\xi^2\hbar^2\,QU[x, x]$, True}}

```
{Λ#,2[{α, η}, {a, y}], lhs = #@Œ#[{a, y}, ℏ (η y + α a), 1],
   HL[lhs == #@Λ#,2[ℏ {α, η}, {a, y}]]} & /@ {CU, QU}
```

```
{{Œ_CU[{y, a}, a α + e^(-α γ) y η, 1 + O[ϵ]^3],
  CU[] + α ℏ CU[a] + (η ℏ - α γ η ℏ^2) CU[y] + 1/2 α^2 ℏ^2 CU[a, a] + α η ℏ^2 CU[y, a] + 1/2 η^2 ℏ^2 CU[y, y],
  True}, {Œ_QU[{y, a}, a α + e^(-α γ) y η, 1 + O[ϵ]^3], QU[] + α ℏ QU[a] +
   (η ℏ - α γ η ℏ^2) QU[y] + 1/2 α^2 ℏ^2 QU[a, a] + α η ℏ^2 QU[y, a] + 1/2 η^2 ℏ^2 QU[y, y], True}}
```

Goal. In either U, compute $F = e^{-\eta y} e^{\xi x} e^{\eta y} e^{-\xi x}$. First compute $G = e^{\xi x} y e^{-\xi x}$, a finite sum. Now F satisfies the ODE $\partial_\eta F = \partial_\eta \left(e^{-\eta y} e^{\eta G}\right) = -yF + FG$ with initial conditions $F(\eta = 0) = 1$. So we set it up and solve:

```
If[$k > 0, With[{U = CU},
  Module[{G, F, fs, bs, e, b, es, sol},
   G = Echo@Simp[Table[ξ^k / k!, {k, 0, $k + 1}].NestList[Simp[B[x_U , #]] &, y_U, $k + 1]];
   fs = Echo@Flatten@Table[f_{l,i,j,k}[η], {l, 0, $k}, {i, 0, 1}, {j, 0, 1}, {k, 0, 1}];
   F = Echo[fs.(bs = fs /. f_{l_,i_,j_,k_}[η] :> ϵ^l U@{y^i, a^j, x^k})];
   es = Flatten[
     Table[Coefficient[e, b] == 0, {e, {F - 1_U /. η → 0, F ** G - y_U ** F - ∂_η F}}, {b, bs}]];
   sol = Echo@First[F /. DSolve[es, fs, η]];
   Echo[sol /. {e- → 1, U → Times}];
   Collect[sol /. {e- → 1, U → Times}, ϵ, Simplify]
  ]]]
```

```
» -t ξ CU[] + 2 ϵ ξ CU[a] - γ ϵ ξ^2 CU[x] + CU[y]

» {f_{0,0,0,0}[η], f_{1,0,0,0}[η], f_{1,0,0,1}[η], f_{1,0,1,0}[η],
   f_{1,0,1,1}[η], f_{1,1,0,0}[η], f_{1,1,0,1}[η], f_{1,1,1,0}[η], f_{1,1,1,1}[η]}

» CU[] f_{0,0,0,0}[η] + ϵ CU[] f_{1,0,0,0}[η] + ϵ CU[x] f_{1,0,0,1}[η] + ϵ CU[a] f_{1,0,1,0}[η] + ϵ CU[a, x] f_{1,0,1,1}[η] +
   ϵ CU[y] f_{1,1,0,0}[η] + ϵ CU[y, x] f_{1,1,0,1}[η] + ϵ CU[y, a] f_{1,1,1,0}[η] + ϵ CU[y, a, x] f_{1,1,1,1}[η]

» e^(-t η ξ) CU[] + 1/2 e^(-t η ξ) t γ ϵ η^2 ξ^2 CU[] + 2 e^(-t η ξ) ϵ η ξ CU[a] - e^(-t η ξ) γ ϵ η ξ^2 CU[x] - e^(-t η ξ) γ ϵ η^2 ξ CU[y]

» 1 + 2 a ϵ η ξ - y γ ϵ η^2 ξ - x γ ϵ η ξ^2 + 1/2 t γ ϵ η^2 ξ^2
```

$$1 + \frac{1}{2} \epsilon \eta \xi \left(4 a + \gamma \left(-2 y \eta - 2 x \xi + t \eta \xi\right)\right)$$

Logos

```
In[∘]:= Λ_{U_,kk_}[{ξ1_, η1_}, {x, y}] := Λ_{U,kk}[{ξ1, η1}, {x, y}] =
    Block[{$k = kk, $p = kk}, Module[{ξ, η, G, F, fs, f, bs, e, b, es},
      G = Simp[Table[ξ^k / k!, {k, 0, $k + 1}].NestList[Simp[B[x_U, #]] &, y_U, $k + 1]];
      fs = Flatten@Table[f_{l,i,j,k}[η], {l, 0, $k}, {i, 0, 1}, {j, 0, 1}, {k, 0, 1}];
      F = fs.(bs = fs /. f_{l_,i_,j_,k_}[η] ↦ ϵ^l U@{y^i, a^j, x^k});
      es = Flatten[
        Table[Coefficient[e, b] == 0, {e, {F - 1_U /. η → 0, F ** G - y_U ** F - ∂_η F}}, {b, bs}]];
      F = F /. DSolve[es, fs, η][[1]];
      Œ_U[{y, a, x},
        ξ x + η y + (U /. {CU → -t η ξ, QU → η ξ (1 - T) / ℏ}),
        F + 0_{$k} /. {e- → 1, U → Times}
      ] /. {ξ → ξ1, η → η1}]];
```

```
In[∘]:= Timing@Λ_{QU,2}[{ξ, η}, {x, y}]
```

Out[∘]=

$$\Big\{1.64063,\ Œ_{QU}\Big[\{y,a,x\},\ y\,\eta + x\,\xi + \frac{(1-T)\,\eta\,\xi}{\hbar},\ 1 + \frac{1}{4\hbar}\,\eta\,\xi\,\big(\gamma\,\eta\,\xi - 4\,T\,\gamma\,\eta\,\xi + 3\,T^2\,\gamma\,\eta\,\xi + 8\,a\,T\,\hbar + 2\,y\,\gamma\,\eta\,\hbar - 6\,T\,y\,\gamma\,\eta\,\hbar + 2\,x\,\gamma\,\xi\,\hbar - 6\,T\,x\,\gamma\,\xi\,\hbar + 4\,x\,y\,\gamma\,\hbar^2\big)\,\epsilon +$$
$$\Big(-a\,T\,y\,\gamma\,\eta^2\,\xi\,(-\eta\,\xi + 3\,T\,\eta\,\xi - 3\,\hbar) - a\,T\,x\,\gamma\,\eta\,\xi^2\,(-\eta\,\xi + 3\,T\,\eta\,\xi - 3\,\hbar) + 2\,a^2\,T\,\eta\,\xi\,(T\,\eta\,\xi - \hbar) +$$
$$2\,a\,T\,x\,y\,\gamma\,\eta^2\,\xi^2\,\hbar - \frac{1}{2}\,x\,y^2\,\gamma^2\,\eta^2\,\xi\,(-\eta\,\xi + 3\,T\,\eta\,\xi - \hbar)\,\hbar - \frac{1}{2}\,x^2\,y\,\gamma^2\,\eta\,\xi^2\,(-\eta\,\xi + 3\,T\,\eta\,\xi - \hbar)\,\hbar +$$
$$\frac{1}{2}\,x^2\,y^2\,\gamma^2\,\eta^2\,\xi^2\,\hbar^2 + \frac{1}{24}\,y^2\,\gamma^2\,\eta^3\,\xi\,(3\,\eta\,\xi - 18\,T\,\eta\,\xi + 27\,T^2\,\eta\,\xi + 4\,\hbar - 28\,T\,\hbar) + \frac{1}{24}\,x^2\,\gamma^2\,\eta\,\xi^3$$
$$(3\,\eta\,\xi - 18\,T\,\eta\,\xi + 27\,T^2\,\eta\,\xi + 4\,\hbar - 28\,T\,\hbar) + \frac{1}{2\hbar}\,a\,T\,\gamma\,\eta^2\,\xi^2\,(\eta\,\xi - 4\,T\,\eta\,\xi + 3\,T^2\,\eta\,\xi + 4\,\hbar - 6\,T\,\hbar) +$$
$$\frac{1}{4}\,x\,y\,\gamma^2\,\eta\,\xi\,(2\,\eta^2\,\xi^2 - 10\,T\,\eta^2\,\xi^2 + 12\,T^2\,\eta^2\,\xi^2 + 5\,\eta\,\xi\,\hbar - 21\,T\,\eta\,\xi\,\hbar + 2\,\hbar^2) - \frac{1}{24\hbar}$$
$$y\,\gamma^2\,\eta^2\,\xi\,(-3\,\eta^2\,\xi^2 + 21\,T\,\eta^2\,\xi^2 - 45\,T^2\,\eta^2\,\xi^2 + 27\,T^3\,\eta^2\,\xi^2 - 10\,\eta\,\xi\,\hbar + 68\,T\,\eta\,\xi\,\hbar - 82\,T^2\,\eta\,\xi\,\hbar -$$
$$6\,\hbar^2 + 30\,T\,\hbar^2) - \frac{1}{24\hbar}\,x\,\gamma^2\,\eta\,\xi^2\,(-3\,\eta^2\,\xi^2 + 21\,T\,\eta^2\,\xi^2 - 45\,T^2\,\eta^2\,\xi^2 + 27\,T^3\,\eta^2\,\xi^2 - 10\,\eta\,\xi\,\hbar +$$
$$68\,T\,\eta\,\xi\,\hbar - 82\,T^2\,\eta\,\xi\,\hbar - 6\,\hbar^2 + 30\,T\,\hbar^2) + \frac{1}{288\,\hbar^2}\,(-1+T)\,\gamma^2\,\eta^2\,\xi^2\,(-9\,\eta^2\,\xi^2 + 63\,T\,\eta^2\,\xi^2 -$$
$$135\,T^2\,\eta^2\,\xi^2 + 81\,T^3\,\eta^2\,\xi^2 - 40\,\eta\,\xi\,\hbar + 272\,T\,\eta\,\xi\,\hbar - 328\,T^2\,\eta\,\xi\,\hbar - 36\,\hbar^2 + 180\,T\,\hbar^2)\Big)\,\epsilon^2 + O[\epsilon]^3\Big]\Big\}$$

```
{Λ_{CU,1}[{ξ, η}, {x, y}], lhs = CU@Œ_{CU}[{x, y}, ℏ (ξ x + η y), 1],
 HL[lhs == CU@Λ_{CU,1}[ℏ {ξ, η}, {x, y}]]}
```

$$\Big\{Œ_{CU}\Big[\{y,a,x\},\ y\,\eta + x\,\xi - t\,\eta\,\xi,\ 1 + \frac{1}{2}\,\eta\,\xi\,(4\,a - 2\,y\,\gamma\,\eta - 2\,x\,\gamma\,\xi + t\,\gamma\,\eta\,\xi)\,\epsilon + O[\epsilon]^2\Big],$$
$$(1 - t\,\eta\,\xi\,\hbar^2)\,CU[\,] + 2\,\epsilon\,\eta\,\xi\,\hbar^2\,CU[a] + \xi\,\hbar\,CU[x] + \eta\,\hbar\,CU[y] +$$
$$\frac{1}{2}\,\xi^2\,\hbar^2\,CU[x,x] + \eta\,\xi\,\hbar^2\,CU[y,x] + \frac{1}{2}\,\eta^2\,\hbar^2\,CU[y,y],\ \text{True}\Big\}$$

In[∘]:=
```
{Λ_{QU,1}[{ξ, η}, {x, y}], lhs = QU@Œ_{QU}[{x, y}, ℏ (ξ x + η y), 1],
 HL@SimpT[lhs == QU@Λ_{QU,1}[ℏ {ξ, η}, {x, y}]]}
```

Out[∘]=

$\{Œ_{QU}[\{y, a, x\}, y\eta + x\xi + \frac{(1-T)\eta\xi}{\hbar}, 1 + \frac{1}{4\hbar}\eta\xi(\gamma\eta\xi - 4T\gamma\eta\xi + 3T^2\gamma\eta\xi + 8aT\hbar + 2y\gamma\eta\hbar - 6Ty\gamma\eta\hbar + 2x\gamma\xi\hbar - 6Tx\gamma\xi\hbar + 4xy\gamma\hbar^2)\epsilon + O[\epsilon]^2], (1 + \eta\xi\hbar - T\eta\xi\hbar) QU[] + 2T\epsilon\eta\xi\hbar^2 QU[a] + \xi\hbar QU[x] + \eta\hbar QU[y] + \frac{1}{2}\xi^2\hbar^2 QU[x, x] + \eta\xi\hbar^2 QU[y, x] + \frac{1}{2}\eta^2\hbar^2 QU[y, y], True\}$

```
{tt = Last[Λ_{CU,2}[{ξ, η}, {x, y}]], Log[tt],
  Exponent[Normal@Log[tt] /. {ξ → ℏ ξ, η → ℏ η, x → ℏ x, y → ℏ y}, ℏ]} // Expand
```

$\{1 + (2a\eta\xi - y\gamma\eta^2\xi - x\gamma\eta\xi^2 + \frac{1}{2}t\gamma\eta^2\xi^2)\epsilon + (2a^2\eta^2\xi^2 - a\gamma\eta^2\xi^2 - 2ay\gamma\eta^3\xi^2 + y\gamma^2\eta^3\xi^2 + \frac{1}{2}y^2\gamma^2\eta^4\xi^2 - 2ax\gamma\eta^2\xi^3 + x\gamma^2\eta^2\xi^3 + at\gamma\eta^3\xi^3 - \frac{1}{3}t\gamma^2\eta^3\xi^3 + xy\gamma^2\eta^3\xi^3 - \frac{1}{2}ty\gamma^2\eta^4\xi^3 + \frac{1}{2}x^2\gamma^2\eta^2\xi^4 - \frac{1}{2}tx\gamma^2\eta^3\xi^4 + \frac{1}{8}t^2\gamma^2\eta^4\xi^4)\epsilon^2 + O[\epsilon]^3, (2a\eta\xi - y\gamma\eta^2\xi - x\gamma\eta\xi^2 + \frac{1}{2}t\gamma\eta^2\xi^2)\epsilon + (-a\gamma\eta^2\xi^2 + y\gamma^2\eta^3\xi^2 + x\gamma^2\eta^2\xi^3 - \frac{1}{3}t\gamma^2\eta^3\xi^3)\epsilon^2 + O[\epsilon]^3, 6\}$

```
{tt = Last[Λ_{QU,2}[{ξ, η}, {x, y}]], Log[tt],
  Exponent[Normal@Log[tt] /. {ξ → d ξ, η → d η, x → d x, y → d y}, d]} // Expand
```

$$\Big\{1+\Big(2\,a\,T\,\eta\,\xi+\frac{1}{2}\,y\,\gamma\,\eta^2\,\xi-\frac{3}{2}\,T\,y\,\gamma\,\eta^2\,\xi+$$

$$\frac{1}{2}\,x\,\gamma\,\eta\,\xi^2-\frac{3}{2}\,T\,x\,\gamma\,\eta\,\xi^2+\frac{\gamma\,\eta^2\,\xi^2}{4\,\hbar}-\frac{T\,\gamma\,\eta^2\,\xi^2}{\hbar}+\frac{3\,T^2\,\gamma\,\eta^2\,\xi^2}{4\,\hbar}+x\,y\,\gamma\,\eta\,\xi\,\hbar\Big)\,\epsilon+$$

$$\Big(2\,a^2\,T^2\,\eta^2\,\xi^2+2\,a\,T\,\gamma\,\eta^2\,\xi^2-3\,a\,T^2\,\gamma\,\eta^2\,\xi^2+\frac{1}{8}\,\gamma^2\,\eta^2\,\xi^2-\frac{3}{4}\,T\,\gamma^2\,\eta^2\,\xi^2+\frac{5}{8}\,T^2\,\gamma^2\,\eta^2\,\xi^2+$$

$$a\,T\,y\,\gamma\,\eta^3\,\xi^2-3\,a\,T^2\,y\,\gamma\,\eta^3\,\xi^2+\frac{5}{12}\,y\,\gamma^2\,\eta^3\,\xi^2-\frac{17}{6}\,T\,y\,\gamma^2\,\eta^3\,\xi^2+\frac{41}{12}\,T^2\,y\,\gamma^2\,\eta^3\,\xi^2+\frac{1}{8}\,y^2\,\gamma^2\,\eta^4\,\xi^2-$$

$$\frac{3}{4}\,T\,y^2\,\gamma^2\,\eta^4\,\xi^2+\frac{9}{8}\,T^2\,y^2\,\gamma^2\,\eta^4\,\xi^2+a\,T\,x\,\gamma\,\eta^2\,\xi^3-3\,a\,T^2\,x\,\gamma\,\eta^2\,\xi^3+\frac{5}{12}\,x\,\gamma^2\,\eta^2\,\xi^3-\frac{17}{6}\,T\,x\,\gamma^2\,\eta^2\,\xi^3+$$

$$\frac{41}{12}\,T^2\,x\,\gamma^2\,\eta^2\,\xi^3+\frac{1}{2}\,x\,y\,\gamma^2\,\eta^3\,\xi^3-\frac{5}{2}\,T\,x\,y\,\gamma^2\,\eta^3\,\xi^3+3\,T^2\,x\,y\,\gamma^2\,\eta^3\,\xi^3+\frac{1}{8}\,x^2\,\gamma^2\,\eta^2\,\xi^4-$$

$$\frac{3}{4}\,T\,x^2\,\gamma^2\,\eta^2\,\xi^4+\frac{9}{8}\,T^2\,x^2\,\gamma^2\,\eta^2\,\xi^4+\frac{\gamma^2\,\eta^4\,\xi^4}{32\,\hbar^2}-\frac{T\,\gamma^2\,\eta^4\,\xi^4}{4\,\hbar^2}+\frac{11\,T^2\,\gamma^2\,\eta^4\,\xi^4}{16\,\hbar^2}-\frac{3\,T^3\,\gamma^2\,\eta^4\,\xi^4}{4\,\hbar^2}+$$

$$\frac{9\,T^4\,\gamma^2\,\eta^4\,\xi^4}{32\,\hbar^2}+\frac{a\,T\,\gamma\,\eta^3\,\xi^3}{2\,\hbar}-\frac{2\,a\,T^2\,\gamma\,\eta^3\,\xi^3}{\hbar}+\frac{3\,a\,T^3\,\gamma\,\eta^3\,\xi^3}{2\,\hbar}+\frac{5\,\gamma^2\,\eta^3\,\xi^3}{36\,\hbar}-\frac{13\,T\,\gamma^2\,\eta^3\,\xi^3}{12\,\hbar}+$$

$$\frac{25\,T^2\,\gamma^2\,\eta^3\,\xi^3}{12\,\hbar}-\frac{41\,T^3\,\gamma^2\,\eta^3\,\xi^3}{36\,\hbar}+\frac{y\,\gamma^2\,\eta^4\,\xi^3}{8\,\hbar}-\frac{7\,T\,y\,\gamma^2\,\eta^4\,\xi^3}{8\,\hbar}+\frac{15\,T^2\,y\,\gamma^2\,\eta^4\,\xi^3}{8\,\hbar}-\frac{9\,T^3\,y\,\gamma^2\,\eta^4\,\xi^3}{8\,\hbar}+$$

$$\frac{x\,\gamma^2\,\eta^3\,\xi^4}{8\,\hbar}-\frac{7\,T\,x\,\gamma^2\,\eta^3\,\xi^4}{8\,\hbar}+\frac{15\,T^2\,x\,\gamma^2\,\eta^3\,\xi^4}{8\,\hbar}-\frac{9\,T^3\,x\,\gamma^2\,\eta^3\,\xi^4}{8\,\hbar}-2\,a^2\,T\,\eta\,\xi\,\hbar+3\,a\,T\,y\,\gamma\,\eta^2\,\xi\,\hbar+$$

$$\frac{1}{4}\,y\,\gamma^2\,\eta^2\,\xi\,\hbar-\frac{5}{4}\,T\,y\,\gamma^2\,\eta^2\,\xi\,\hbar+\frac{1}{6}\,y^2\,\gamma^2\,\eta^3\,\xi\,\hbar-\frac{7}{6}\,T\,y^2\,\gamma^2\,\eta^3\,\xi\,\hbar+3\,a\,T\,x\,\gamma\,\eta\,\xi^2\,\hbar+\frac{1}{4}\,x\,\gamma^2\,\eta\,\xi^2\,\hbar-$$

$$\frac{5}{4}\,T\,x\,\gamma^2\,\eta\,\xi^2\,\hbar+2\,a\,T\,x\,y\,\gamma\,\eta^2\,\xi^2\,\hbar+\frac{5}{4}\,x\,y\,\gamma^2\,\eta^2\,\xi^2\,\hbar-\frac{21}{4}\,T\,x\,y\,\gamma^2\,\eta^2\,\xi^2\,\hbar+\frac{1}{2}\,x\,y^2\,\gamma^2\,\eta^3\,\xi^2\,\hbar-$$

$$\frac{3}{2}\,T\,x\,y^2\,\gamma^2\,\eta^3\,\xi^2\,\hbar+\frac{1}{6}\,x^2\,\gamma^2\,\eta\,\xi^3\,\hbar-\frac{7}{6}\,T\,x^2\,\gamma^2\,\eta\,\xi^3\,\hbar+\frac{1}{2}\,x^2\,y\,\gamma^2\,\eta^2\,\xi^3\,\hbar-\frac{3}{2}\,T\,x^2\,y\,\gamma^2\,\eta^2\,\xi^3\,\hbar+$$

$$\frac{1}{2}\,x\,y\,\gamma^2\,\eta\,\xi\,\hbar^2+\frac{1}{2}\,x\,y^2\,\gamma^2\,\eta^2\,\xi\,\hbar^2+\frac{1}{2}\,x^2\,y\,\gamma^2\,\eta\,\xi^2\,\hbar^2+\frac{1}{2}\,x^2\,y^2\,\gamma^2\,\eta^2\,\xi^2\,\hbar^2\Big)\,\epsilon^2+O[\epsilon]^3,$$

$$\Big(2\,a\,T\,\eta\,\xi+\frac{1}{2}\,y\,\gamma\,\eta^2\,\xi-\frac{3}{2}\,T\,y\,\gamma\,\eta^2\,\xi+\frac{1}{2}\,x\,\gamma\,\eta\,\xi^2-\frac{3}{2}\,T\,x\,\gamma\,\eta\,\xi^2+\frac{\gamma\,\eta^2\,\xi^2}{4\,\hbar}-$$

$$\frac{T\,\gamma\,\eta^2\,\xi^2}{\hbar}+\frac{3\,T^2\,\gamma\,\eta^2\,\xi^2}{4\,\hbar}+x\,y\,\gamma\,\eta\,\xi\,\hbar\Big)\,\epsilon+$$

$$\Big(2\,a\,T\,\gamma\,\eta^2\,\xi^2-3\,a\,T^2\,\gamma\,\eta^2\,\xi^2+\frac{1}{8}\,\gamma^2\,\eta^2\,\xi^2-\frac{3}{4}\,T\,\gamma^2\,\eta^2\,\xi^2+\frac{5}{8}\,T^2\,\gamma^2\,\eta^2\,\xi^2+\frac{5}{12}\,y\,\gamma^2\,\eta^3\,\xi^2-$$

$$\frac{17}{6}\,T\,y\,\gamma^2\,\eta^3\,\xi^2+\frac{41}{12}\,T^2\,y\,\gamma^2\,\eta^3\,\xi^2+\frac{5}{12}\,x\,\gamma^2\,\eta^2\,\xi^3-\frac{17}{6}\,T\,x\,\gamma^2\,\eta^2\,\xi^3+\frac{41}{12}\,T^2\,x\,\gamma^2\,\eta^2\,\xi^3+$$

$$\frac{5\,\gamma^2\,\eta^3\,\xi^3}{36\,\hbar}-\frac{13\,T\,\gamma^2\,\eta^3\,\xi^3}{12\,\hbar}+\frac{25\,T^2\,\gamma^2\,\eta^3\,\xi^3}{12\,\hbar}-\frac{41\,T^3\,\gamma^2\,\eta^3\,\xi^3}{36\,\hbar}-2\,a^2\,T\,\eta\,\xi\,\hbar+3\,a\,T\,y\,\gamma\,\eta^2\,\xi\,\hbar+$$

$$\frac{1}{4}\,y\,\gamma^2\,\eta^2\,\xi\,\hbar-\frac{5}{4}\,T\,y\,\gamma^2\,\eta^2\,\xi\,\hbar+\frac{1}{6}\,y^2\,\gamma^2\,\eta^3\,\xi\,\hbar-\frac{7}{6}\,T\,y^2\,\gamma^2\,\eta^3\,\xi\,\hbar+3\,a\,T\,x\,\gamma\,\eta\,\xi^2\,\hbar+$$

$$\frac{1}{4}\,x\,\gamma^2\,\eta\,\xi^2\,\hbar-\frac{5}{4}\,T\,x\,\gamma^2\,\eta\,\xi^2\,\hbar+\frac{5}{4}\,x\,y\,\gamma^2\,\eta^2\,\xi^2\,\hbar-\frac{21}{4}\,T\,x\,y\,\gamma^2\,\eta^2\,\xi^2\,\hbar+\frac{1}{6}\,x^2\,\gamma^2\,\eta\,\xi^3\,\hbar-$$

$$\frac{7}{6}\,T\,x^2\,\gamma^2\,\eta\,\xi^3\,\hbar+\frac{1}{2}\,x\,y\,\gamma^2\,\eta\,\xi\,\hbar^2+\frac{1}{2}\,x\,y^2\,\gamma^2\,\eta^2\,\xi\,\hbar^2+\frac{1}{2}\,x^2\,y\,\gamma^2\,\eta\,\xi^2\,\hbar^2\Big)\,\epsilon^2+O[\epsilon]^3,\ 6\Big\}$$

Logos

```
In[]:= Simp[Œ_U_[specs___, Q_, P_]] := Œ_U[specs, CF[Q], CF[P]];
```

Logos

```
In[]:= Λ_U_,k_[{υ1_, ω1_, δ_}, {u_, w_}] := Simp@Module[{υ, ω, yax, q, p, Q, d},
         {yax, q, p} = List @@ Λ_U,k[{υ, ω}, {u, w}];
         Œ_U[yax, Q = (υ u + ω w + δ u w + d υ ω) / (1 - d δ),
           Expand[(1 - d δ)^-1 e^-Q DP_{υ→D_u, ω→D_w}[p][e^Q]] + 0_k] /. {d → ∂_{υ,ω} q} /. {υ → υ1, ω → ω1}];
```

```
Block[{$p = 4, $k = 1},
 {Λ_CU,$k[ℏ {ξ, η, δ}, {x, y}],
  Short[lhs = CU@Œ_CU[{x, y}, ℏ (ξ x + η y + δ x y), 1_$k], 5],
  HL@Simp[lhs - CU@Λ_CU,$k[ℏ {ξ, η, δ}, {x, y}]]}
]
```

$$\Big\{Œ_{CU}\Big[\{y, a, x\}, \frac{x y \delta \hbar + y \eta \hbar + x \xi \hbar - t \eta \xi \hbar^2}{1 + t \delta \hbar},$$

$$\frac{1}{1 + t\delta\hbar} + \Big(\big(4 a \delta \hbar + 12 a t \delta^2 \hbar^2 + 4 a x y \delta^2 \hbar^2 + 2 t \gamma \delta^2 \hbar^2 - 8 x y \gamma \delta^2 \hbar^2 + 4 a y \delta \eta \hbar^2 -$$
$$4 y \gamma \delta \eta \hbar^2 + 4 a x \delta \xi \hbar^2 - 4 x \gamma \delta \xi \hbar^2 + 4 a \eta \xi \hbar^2 + 12 a t^2 \delta^3 \hbar^3 + 8 a t x y \delta^3 \hbar^3 +$$
$$4 t^2 \gamma \delta^3 \hbar^3 - 12 t x y \gamma \delta^3 \hbar^3 - 4 x^2 y^2 \gamma \delta^3 \hbar^3 + 8 a t y \delta^2 \eta \hbar^3 - 4 t y \gamma \delta^2 \eta \hbar^3 -$$
$$6 x y^2 \gamma \delta^2 \eta \hbar^3 - 2 y^2 \gamma \delta \eta^2 \hbar^3 + 8 a t x \delta^2 \xi \hbar^3 - 4 t x \gamma \delta^2 \xi \hbar^3 - 6 x^2 y \gamma \delta^2 \xi \hbar^3 +$$
$$8 a t \delta \eta \xi \hbar^3 + 4 t \gamma \delta \eta \xi \hbar^3 - 8 x y \gamma \delta \eta \xi \hbar^3 - 2 y \gamma \eta^2 \xi \hbar^3 - 2 x^2 \gamma \delta \xi^2 \hbar^3 - 2 x \gamma \eta \xi^2 \hbar^3 +$$
$$4 a t^3 \delta^4 \hbar^4 + 4 a t^2 x y \delta^4 \hbar^4 + 2 t^3 \gamma \delta^4 \hbar^4 - 4 t^2 x y \gamma \delta^4 \hbar^4 - 3 t x^2 y^2 \gamma \delta^4 \hbar^4 +$$
$$4 a t^2 y \delta^3 \eta \hbar^4 - 4 t x y^2 \gamma \delta^3 \eta \hbar^4 - t y^2 \gamma \delta^2 \eta^2 \hbar^4 + 4 a t^2 x \delta^3 \xi \hbar^4 - 4 t x^2 y \gamma \delta^3 \xi \hbar^4 +$$
$$4 a t^2 \delta^2 \eta \xi \hbar^4 + 4 t^2 \gamma \delta^2 \eta \xi \hbar^4 - 4 t x y \gamma \delta^2 \eta \xi \hbar^4 - t x^2 \gamma \delta^2 \xi^2 \hbar^4 + t \gamma \eta^2 \xi^2 \hbar^4\big) \epsilon\Big) \Big/$$
$$\big(2 + 10 t \delta \hbar + 20 t^2 \delta^2 \hbar^2 + 20 t^3 \delta^3 \hbar^3 + 10 t^4 \delta^4 \hbar^4 + 2 t^5 \delta^5 \hbar^5\big) + O[\epsilon]^2\Big],$$

$$\Big(1 - t \delta \hbar + t^2 \delta^2 \hbar^2 + t \gamma \delta^2 \epsilon \hbar^2 - t \eta \xi \hbar^2 - t^3 \delta^3 \hbar^3 - 3 t^2 \gamma \delta^3 \epsilon \hbar^3 + 2 t^2 \delta \eta \xi \hbar^3 +$$
$$2 t \gamma \delta \epsilon \eta \xi \hbar^3 + t^4 \delta^4 \hbar^4 + 6 t^3 \gamma \delta^4 \epsilon \hbar^4 - 3 t^3 \delta^2 \eta \xi \hbar^4 -$$
$$9 t^2 \gamma \delta^2 \epsilon \eta \xi \hbar^4 + \frac{1}{2} t^2 \eta^2 \xi^2 \hbar^4 + \frac{1}{2} t \gamma \epsilon \eta^2 \xi^2 \hbar^4\Big) CU[\,] +$$
$$\big(2 \delta \epsilon \hbar - 4 t \delta^2 \epsilon \hbar^2 + 2 \epsilon \eta \xi \hbar^2 + 6 t^2 \delta^3 \epsilon \hbar^3 - 8 t \delta \epsilon \eta \xi \hbar^3 - 8 t^3 \delta^4 \epsilon \hbar^4 +$$
$$18 t^2 \delta^2 \epsilon \eta \xi \hbar^4 - 2 t \epsilon \eta^2 \xi^2 \hbar^4\big) CU[a] +$$
$$\ll 37 \gg + \frac{1}{6} \delta^3 \eta \hbar^4 CU[y, y, y, y, x, x, x] +$$
$$\frac{1}{24} \delta^4 \hbar^4$$
$$CU[y, y, y, y, x, x, x, x], 0\Big\}$$

```
{Λ_QU,2[{ξ, η, δ}, {x, y}], lhs = QU@Œ_QU[{x, y}, ℏ (ξ x + η y + δ x y), 1],
 HL@SimpT[lhs == QU@Λ_QU,1[ℏ {ξ, η, δ}, {x, y}]]}
```

$\{\mathbb{E}_{QU}[\{y, a, x\}, \frac{\cdots 1 \cdots}{\cdots 1 \cdots},$
$\frac{\hbar}{-\delta + T\,\delta + \hbar} + ((-8\,a\,T\,\delta^4\,\hbar^2 + 24\,a\,T^2\,\delta^4\,\hbar^2 - 24\,a\,T^3\,\delta^4\,\hbar^2 + 8\,a\,T^4\,\delta^4\,\hbar^2 + \cdots 149 \cdots +$
$4\,x^2\,y^2\,\gamma\,\delta^2\,\hbar^6 + 4\,x\,y^2\,\gamma\,\delta\,\eta\,\hbar^6 + 4\,x^2\,y\,\gamma\,\delta\,\xi\,\hbar^6 + 4\,x\,y\,\gamma\,\eta\,\xi\,\hbar^6)\,\epsilon) /$
$(-4\,\delta^5 + 20\,T\,\delta^5 - 40\,T^2\,\delta^5 + 40\,T^3\,\delta^5 - 20\,T^4\,\delta^5 + 4\,T^5\,\delta^5 + \cdots 12 \cdots + 40\,T^3\,\delta^3\,\hbar^2 +$
$40\,\delta^2\,\hbar^3 - 80\,T\,\delta^2\,\hbar^3 + 40\,T^2\,\delta^2\,\hbar^3 - 20\,\delta\,\hbar^4 + 20\,T\,\delta\,\hbar^4 + 4\,\hbar^5) +$
$\frac{(\cdots 1 \cdots)\,\cdots 1 \cdots}{\cdots 1 \cdots} + O[\epsilon]^3],\ \cdots 1 \cdots,\ \text{True}\}$

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```
{tt = ComposeSeries[(1 + t δ) Last[Λ_CU,2[{ξ, η, δ}, {x, y}]], (1 + t δ)^4 ϵ + O[ϵ]^18];
  Together@Log[tt],
  Exponent[Normal@Together@Log[tt] /. {ξ → d ξ, η → d η, x → d x, y → d y}, d],
  Exponent[Normal@Together@Log[tt] /. {x → d x, y → d y}, d]
 } // Expand
```

$\{(2\,a\,\delta + 6\,a\,t\,\delta^2 + 2\,a\,x\,y\,\delta^2 + t\,\gamma\,\delta^2 - 4\,x\,y\,\gamma\,\delta^2 + 6\,a\,t^2\,\delta^3 + 4\,a\,t\,x\,y\,\delta^3 + 2\,t^2\,\gamma\,\delta^3 - 6\,t\,x\,y\,\gamma\,\delta^3 -$
$2\,x^2\,y^2\,\gamma\,\delta^3 + 2\,a\,t^3\,\delta^4 + 2\,a\,t^2\,x\,y\,\delta^4 + t^3\,\gamma\,\delta^4 - 2\,t^2\,x\,y\,\gamma\,\delta^4 - \frac{3}{2}\,t\,x^2\,y^2\,\gamma\,\delta^4 + 2\,a\,y\,\delta\,\eta -$
$2\,y\,\gamma\,\delta\,\eta + 4\,a\,t\,y\,\delta^2\,\eta - 2\,t\,y\,\gamma\,\delta^2\,\eta - 3\,x\,y^2\,\gamma\,\delta^2\,\eta + 2\,a\,t^2\,y\,\delta^3\,\eta - 2\,t\,x\,y^2\,\gamma\,\delta^3\,\eta -$
$y^2\,\gamma\,\delta\,\eta^2 - \frac{1}{2}\,t\,y^2\,\gamma\,\delta^2\,\eta^2 + 2\,a\,x\,\delta\,\xi - 2\,x\,\gamma\,\delta\,\xi + 4\,a\,t\,x\,\delta^2\,\xi - 2\,t\,x\,\gamma\,\delta^2\,\xi - 3\,x^2\,y\,\gamma\,\delta^2\,\xi +$
$2\,a\,t^2\,x\,\delta^3\,\xi - 2\,t\,x^2\,y\,\gamma\,\delta^3\,\xi + 2\,a\,\eta\,\xi + 4\,a\,t\,\delta\,\eta\,\xi + 2\,t\,\gamma\,\delta\,\eta\,\xi - 4\,x\,y\,\gamma\,\delta\,\eta\,\xi + 2\,a\,t^2\,\delta^2\,\eta\,\xi +$
$2\,t^2\,\gamma\,\delta^2\,\eta\,\xi - 2\,t\,x\,y\,\gamma\,\delta^2\,\eta\,\xi - y\,\gamma\,\eta^2\,\xi - x^2\,\gamma\,\delta\,\xi^2 - \frac{1}{2}\,t\,x^2\,\gamma\,\delta^2\,\xi^2 - x\,\gamma\,\eta\,\xi^2 + \frac{1}{2}\,t\,\gamma\,\eta^2\,\xi^2)\,\epsilon +$
$(2\,a^2\,\delta^2 - 2\,a\,\gamma\,\delta^2 + 12\,a^2\,t\,\delta^3 + 4\,a^2\,x\,y\,\delta^3 - 8\,a\,t\,\gamma\,\delta^3 - 20\,a\,x\,y\,\gamma\,\delta^3 - 2\,t\,\gamma^2\,\delta^3 + 18\,x\,y\,\gamma^2\,\delta^3 +$
$30\,a^2\,t^2\,\delta^4 + 20\,a^2\,t\,x\,y\,\delta^4 - 10\,a\,t^2\,\gamma\,\delta^4 - 88\,a\,t\,x\,y\,\gamma\,\delta^4 - 13\,a\,x^2\,y^2\,\gamma\,\delta^4 - \frac{15}{2}\,t^2\,\gamma^2\,\delta^4 +$
$64\,t\,x\,y\,\gamma^2\,\delta^4 + 34\,x^2\,y^2\,\gamma^2\,\delta^4 + 40\,a^2\,t^3\,\delta^5 + 40\,a^2\,t^2\,x\,y\,\delta^5 - 152\,a\,t^2\,x\,y\,\gamma\,\delta^5 - 48\,a\,t\,x^2\,y^2\,\gamma\,\delta^5 -$
$10\,t^3\,\gamma^2\,\delta^5 + 86\,t^2\,x\,y\,\gamma^2\,\delta^5 + 107\,t\,x^2\,y^2\,\gamma^2\,\delta^5 + 11\,x^3\,y^3\,\gamma^2\,\delta^5 + 30\,a^2\,t^4\,\delta^6 + 40\,a^2\,t^3\,x\,y\,\delta^6 +$
$10\,a\,t^4\,\gamma\,\delta^6 - 128\,a\,t^3\,x\,y\,\gamma\,\delta^6 - 66\,a\,t^2\,x^2\,y^2\,\gamma\,\delta^6 - 5\,t^4\,\gamma^2\,\delta^6 + 54\,t^3\,x\,y\,\gamma^2\,\delta^6 + \frac{247}{2}\,t^2\,x^2\,y^2\,\gamma^2\,\delta^6 +$
$\frac{80}{3}\,t\,x^3\,y^3\,\gamma^2\,\delta^6 + 12\,a^2\,t^5\,\delta^7 + 20\,a^2\,t^4\,x\,y\,\delta^7 + 8\,a\,t^5\,\gamma\,\delta^7 - 52\,a\,t^4\,x\,y\,\gamma\,\delta^7 - 40\,a\,t^3\,x^2\,y^2\,\gamma\,\delta^7 +$
$16\,t^4\,x\,y\,\gamma^2\,\delta^7 + 62\,t^3\,x^2\,y^2\,\gamma^2\,\delta^7 + \frac{64}{3}\,t^2\,x^3\,y^3\,\gamma^2\,\delta^7 + 2\,a^2\,t^6\,\delta^8 + 4\,a^2\,t^5\,x\,y\,\delta^8 + 2\,a\,t^6\,\gamma\,\delta^8 -$
$8\,a\,t^5\,x\,y\,\gamma\,\delta^8 - 9\,a\,t^4\,x^2\,y^2\,\gamma\,\delta^8 + \frac{1}{2}\,t^6\,\gamma^2\,\delta^8 + 2\,t^5\,x\,y\,\gamma^2\,\delta^8 + \frac{23}{2}\,t^4\,x^2\,y^2\,\gamma^2\,\delta^8 + \frac{17}{3}\,t^3\,x^3\,y^3\,\gamma^2\,\delta^8 +$
$4\,a^2\,y\,\delta^2\,\eta - 12\,a\,y\,\gamma\,\delta^2\,\eta + 6\,y\,\gamma^2\,\delta^2\,\eta + 20\,a^2\,t\,y\,\delta^3\,\eta - 48\,a\,t\,y\,\gamma\,\delta^3\,\eta - 20\,a\,x\,y^2\,\gamma\,\delta^3\,\eta +$
$14\,t\,y\,\gamma^2\,\delta^3\,\eta + 40\,x\,y^2\,\gamma^2\,\delta^3\,\eta + 40\,a^2\,t^2\,y\,\delta^4\,\eta - 72\,a\,t^2\,y\,\gamma\,\delta^4\,\eta - 72\,a\,t\,x\,y^2\,\gamma\,\delta^4\,\eta + 6\,t^2\,y\,\gamma^2\,\delta^4\,\eta +$
$115\,t\,x\,y^2\,\gamma^2\,\delta^4\,\eta + 23\,x^2\,y^3\,\gamma^2\,\delta^4\,\eta + 40\,a^2\,t^3\,y\,\delta^5\,\eta - 48\,a\,t^3\,y\,\gamma\,\delta^5\,\eta - 96\,a\,t^2\,x\,y^2\,\gamma\,\delta^5\,\eta -$
$6\,t^3\,y\,\gamma^2\,\delta^5\,\eta + 118\,t^2\,x\,y^2\,\gamma^2\,\delta^5\,\eta + 53\,t\,x^2\,y^3\,\gamma^2\,\delta^5\,\eta + 20\,a^2\,t^4\,y\,\delta^6\,\eta - 12\,a\,t^4\,y\,\gamma\,\delta^6\,\eta -$
$56\,a\,t^3\,x\,y^2\,\gamma\,\delta^6\,\eta - 4\,t^4\,y\,\gamma^2\,\delta^6\,\eta + 51\,t^3\,x\,y^2\,\gamma^2\,\delta^6\,\eta + 40\,t^2\,x^2\,y^3\,\gamma^2\,\delta^6\,\eta + 4\,a^2\,t^5\,y\,\delta^7\,\eta -$

$$
\begin{aligned}
&12\,a\,t^4\,x\,y^2\,\gamma\,\delta^7\,\eta + 8\,t^4\,x\,y^2\,\gamma^2\,\delta^7\,\eta + 10\,t^3\,x^2\,y^3\,\gamma^2\,\delta^7\,\eta - 7\,a\,y^2\,\gamma\,\delta^2\,\eta^2 + 10\,y^2\,\gamma^2\,\delta^2\,\eta^2 - \\
&24\,a\,t\,y^2\,\gamma\,\delta^3\,\eta^2 + 24\,t\,y^2\,\gamma^2\,\delta^3\,\eta^2 + 15\,x\,y^3\,\gamma^2\,\delta^3\,\eta^2 - 30\,a\,t^2\,y^2\,\gamma\,\delta^4\,\eta^2 + \frac{37}{2}\,t^2\,y^2\,\gamma^2\,\delta^4\,\eta^2 + \\
&32\,t\,x\,y^3\,\gamma^2\,\delta^4\,\eta^2 - 16\,a\,t^3\,y^2\,\gamma\,\delta^5\,\eta^2 + 5\,t^3\,y^2\,\gamma^2\,\delta^5\,\eta^2 + 22\,t^2\,x\,y^3\,\gamma^2\,\delta^5\,\eta^2 - 3\,a\,t^4\,y^2\,\gamma\,\delta^6\,\eta^2 + \\
&\frac{1}{2}\,t^4\,y^2\,\gamma^2\,\delta^6\,\eta^2 + 5\,t^3\,x\,y^3\,\gamma^2\,\delta^6\,\eta^2 + 3\,y^3\,\gamma^2\,\delta^2\,\eta^3 + \frac{17}{3}\,t\,y^3\,\gamma^2\,\delta^3\,\eta^3 + \frac{10}{3}\,t^2\,y^3\,\gamma^2\,\delta^4\,\eta^3 + \\
&\frac{2}{3}\,t^3\,y^3\,\gamma^2\,\delta^5\,\eta^3 + 4\,a^2\,x\,\delta^2\,\xi - 12\,a\,x\,\gamma\,\delta^2\,\xi + 6\,x\,\gamma^2\,\delta^2\,\xi + 20\,a^2\,t\,x\,\delta^3\,\xi - 48\,a\,t\,x\,\gamma\,\delta^3\,\xi - \\
&20\,a\,x^2\,y\,\gamma\,\delta^3\,\xi + 14\,t\,x\,\gamma^2\,\delta^3\,\xi + 40\,x^2\,y\,\gamma^2\,\delta^3\,\xi + 40\,a^2\,t^2\,x\,\delta^4\,\xi - 72\,a\,t^2\,x\,\gamma\,\delta^4\,\xi - 72\,a\,t\,x^2\,y\,\gamma\,\delta^4\,\xi + \\
&6\,t^2\,x\,\gamma^2\,\delta^4\,\xi + 115\,t\,x^2\,y\,\gamma^2\,\delta^4\,\xi + 23\,x^3\,y^2\,\gamma^2\,\delta^4\,\xi + 40\,a^2\,t^3\,x\,\delta^5\,\xi - 48\,a\,t^3\,x\,\gamma\,\delta^5\,\xi - \\
&96\,a\,t^2\,x^2\,y\,\gamma\,\delta^5\,\xi - 6\,t^3\,x\,\gamma^2\,\delta^5\,\xi + 118\,t^2\,x^2\,y\,\gamma^2\,\delta^5\,\xi + 53\,t\,x^3\,y^2\,\gamma^2\,\delta^5\,\xi + 20\,a^2\,t^4\,x\,\delta^6\,\xi - \\
&12\,a\,t^4\,x\,\gamma\,\delta^6\,\xi - 56\,a\,t^3\,x^2\,y\,\gamma\,\delta^6\,\xi - 4\,t^4\,x\,\gamma^2\,\delta^6\,\xi + 51\,t^3\,x^2\,y\,\gamma^2\,\delta^6\,\xi + 40\,t^2\,x^3\,y^2\,\gamma^2\,\delta^6\,\xi + \\
&4\,a^2\,t^5\,x\,\delta^7\,\xi - 12\,a\,t^4\,x^2\,y\,\gamma\,\delta^7\,\xi + 8\,t^4\,x^2\,y\,\gamma^2\,\delta^7\,\xi + 10\,t^3\,x^3\,y^2\,\gamma^2\,\delta^7\,\xi + 4\,a^2\,\delta\,\eta\,\xi - 4\,a\,\gamma\,\delta\,\eta\,\xi + \\
&20\,a^2\,t\,\delta^2\,\eta\,\xi - 8\,a\,t\,\gamma\,\delta^2\,\eta\,\xi - 28\,a\,x\,y\,\gamma\,\delta^2\,\eta\,\xi - 6\,t\,\gamma^2\,\delta^2\,\eta\,\xi + 38\,x\,y\,\gamma^2\,\delta^2\,\eta\,\xi + 40\,a^2\,t^2\,\delta^3\,\eta\,\xi + \\
&8\,a\,t^2\,\gamma\,\delta^3\,\eta\,\xi - 96\,a\,t\,x\,y\,\gamma\,\delta^3\,\eta\,\xi - 14\,t^2\,\gamma^2\,\delta^3\,\eta\,\xi + 88\,t\,x\,y\,\gamma^2\,\delta^3\,\eta\,\xi + 44\,x^2\,y^2\,\gamma^2\,\delta^3\,\eta\,\xi + \\
&40\,a^2\,t^3\,\delta^4\,\eta\,\xi + 32\,a\,t^3\,\gamma\,\delta^4\,\eta\,\xi - 120\,a\,t^2\,x\,y\,\gamma\,\delta^4\,\eta\,\xi - 6\,t^3\,\gamma^2\,\delta^4\,\eta\,\xi + 62\,t^2\,x\,y\,\gamma^2\,\delta^4\,\eta\,\xi + \\
&93\,t\,x^2\,y^2\,\gamma^2\,\delta^4\,\eta\,\xi + 20\,a^2\,t^4\,\delta^5\,\eta\,\xi + 28\,a\,t^4\,\gamma\,\delta^5\,\eta\,\xi - 64\,a\,t^3\,x\,y\,\gamma\,\delta^5\,\eta\,\xi + 6\,t^4\,\gamma^2\,\delta^5\,\eta\,\xi + \\
&12\,t^3\,x\,y\,\gamma^2\,\delta^5\,\eta\,\xi + 63\,t^2\,x^2\,y^2\,\gamma^2\,\delta^5\,\eta\,\xi + 4\,a^2\,t^5\,\delta^6\,\eta\,\xi + 8\,a\,t^5\,\gamma\,\delta^6\,\eta\,\xi - 12\,a\,t^4\,x\,y\,\gamma\,\delta^6\,\eta\,\xi + \\
&4\,t^5\,\gamma^2\,\delta^6\,\eta\,\xi + 14\,t^3\,x^2\,y^2\,\gamma^2\,\delta^6\,\eta\,\xi - 8\,a\,y\,\gamma\,\delta\,\eta^2\,\xi + 6\,y\,\gamma^2\,\delta\,\eta^2\,\xi - 24\,a\,t\,y\,\gamma\,\delta^2\,\eta^2\,\xi + \\
&5\,t\,y\,\gamma^2\,\delta^2\,\eta^2\,\xi + 25\,x\,y^2\,\gamma^2\,\delta^2\,\eta^2\,\xi - 24\,a\,t^2\,y\,\gamma\,\delta^3\,\eta^2\,\xi - 8\,t^2\,y\,\gamma^2\,\delta^3\,\eta^2\,\xi + 45\,t\,x\,y^2\,\gamma^2\,\delta^3\,\eta^2\,\xi - \\
&8\,a\,t^3\,y\,\gamma\,\delta^4\,\eta^2\,\xi - 7\,t^3\,y\,\gamma^2\,\delta^4\,\eta^2\,\xi + 24\,t^2\,x\,y^2\,\gamma^2\,\delta^4\,\eta^2\,\xi + 4\,t^3\,x\,y^2\,\gamma^2\,\delta^5\,\eta^2\,\xi + 4\,y^2\,\gamma^2\,\delta\,\eta^3\,\xi + \\
&5\,t\,y^2\,\gamma^2\,\delta^2\,\eta^3\,\xi + t^2\,y^2\,\gamma^2\,\delta^3\,\eta^3\,\xi - 7\,a\,x^2\,\gamma\,\delta^2\,\xi^2 + 10\,x^2\,\gamma^2\,\delta^2\,\xi^2 - 24\,a\,t\,x^2\,\gamma\,\delta^3\,\xi^2 + \\
&24\,t\,x^2\,\gamma^2\,\delta^3\,\xi^2 + 15\,x^3\,y\,\gamma^2\,\delta^3\,\xi^2 - 30\,a\,t^2\,x^2\,\gamma\,\delta^4\,\xi^2 + \frac{37}{2}\,t^2\,x^2\,\gamma^2\,\delta^4\,\xi^2 + 32\,t\,x^3\,y\,\gamma^2\,\delta^4\,\xi^2 - \\
&16\,a\,t^3\,x^2\,\gamma\,\delta^5\,\xi^2 + 5\,t^3\,x^2\,\gamma^2\,\delta^5\,\xi^2 + 22\,t^2\,x^3\,y\,\gamma^2\,\delta^5\,\xi^2 - 3\,a\,t^4\,x^2\,\gamma\,\delta^6\,\xi^2 + \frac{1}{2}\,t^4\,x^2\,\gamma^2\,\delta^6\,\xi^2 + \\
&5\,t^3\,x^3\,y\,\gamma^2\,\delta^6\,\xi^2 - 8\,a\,x\,\gamma\,\delta\,\eta\,\xi^2 + 6\,x\,\gamma^2\,\delta\,\eta\,\xi^2 - 24\,a\,t\,x\,\gamma\,\delta^2\,\eta\,\xi^2 + 5\,t\,x\,\gamma^2\,\delta^2\,\eta\,\xi^2 + \\
&25\,x^2\,y\,\gamma^2\,\delta^2\,\eta\,\xi^2 - 24\,a\,t^2\,x\,\gamma\,\delta^3\,\eta\,\xi^2 - 8\,t^2\,x\,\gamma^2\,\delta^3\,\eta\,\xi^2 + 45\,t\,x^2\,y\,\gamma^2\,\delta^3\,\eta\,\xi^2 - 8\,a\,t^3\,x\,\gamma\,\delta^4\,\eta\,\xi^2 - \\
&7\,t^3\,x\,\gamma^2\,\delta^4\,\eta\,\xi^2 + 24\,t^2\,x^2\,y\,\gamma^2\,\delta^4\,\eta\,\xi^2 + 4\,t^3\,x^2\,y\,\gamma^2\,\delta^5\,\eta\,\xi^2 - a\,\gamma\,\eta^2\,\xi^2 - 3\,t\,\gamma^2\,\delta\,\eta^2\,\xi^2 + \\
&11\,x\,y\,\gamma^2\,\delta\,\eta^2\,\xi^2 + 6\,a\,t^2\,\gamma\,\delta^2\,\eta^2\,\xi^2 - \frac{5}{2}\,t^2\,\gamma^2\,\delta^2\,\eta^2\,\xi^2 + 12\,t\,x\,y\,\gamma^2\,\delta^2\,\eta^2\,\xi^2 + 8\,a\,t^3\,\gamma\,\delta^3\,\eta^2\,\xi^2 + \\
&4\,t^3\,\gamma^2\,\delta^3\,\eta^2\,\xi^2 + 3\,a\,t^4\,\gamma\,\delta^4\,\eta^2\,\xi^2 + \frac{7}{2}\,t^4\,\gamma^2\,\delta^4\,\eta^2\,\xi^2 - t^3\,x\,y\,\gamma^2\,\delta^4\,\eta^2\,\xi^2 + y\,\gamma^2\,\eta^3\,\xi^2 - \\
&t\,y\,\gamma^2\,\delta\,\eta^3\,\xi^2 - 2\,t^2\,y\,\gamma^2\,\delta^2\,\eta^3\,\xi^2 + 3\,x^3\,\gamma^2\,\delta^2\,\xi^3 + \frac{17}{3}\,t\,x^3\,\gamma^2\,\delta^3\,\xi^3 + \frac{10}{3}\,t^2\,x^3\,\gamma^2\,\delta^4\,\xi^3 + \\
&\frac{2}{3}\,t^3\,x^3\,\gamma^2\,\delta^5\,\xi^3 + 4\,x^2\,\gamma^2\,\delta\,\eta\,\xi^3 + 5\,t\,x^2\,\gamma^2\,\delta^2\,\eta\,\xi^3 + t^2\,x^2\,\gamma^2\,\delta^3\,\eta\,\xi^3 + x\,\gamma^2\,\eta^2\,\xi^3 - t\,x\,\gamma^2\,\delta\,\eta^2\,\xi^3 - \\
&2\,t^2\,x\,\gamma^2\,\delta^2\,\eta^2\,\xi^3 - \frac{1}{3}\,t\,\gamma^2\,\eta^3\,\xi^3 + \frac{1}{3}\,t^2\,\gamma^2\,\delta\,\eta^3\,\xi^3 + \frac{2}{3}\,t^3\,\gamma^2\,\delta^2\,\eta^3\,\xi^3\Big)\,\epsilon^2 + O[\epsilon]^3,\ 6,\ 6\Big\}
\end{aligned}
$$

```
{tt = Last[Λ_{QU,2}[{ξ, η, δ}, {x, y}]];
  Log[tt],
  Exponent[Normal@Together@Log[tt] /. {ξ → d ξ, η → d η, x → d x, y → d y}, d]} // Expand
```

$$\left\{\text{Log}\left[\frac{\hbar}{-\delta+T\,\delta+\hbar}\right]+\left(\frac{2\,a\,T\,\delta^5\,\hbar}{(-\delta+T\,\delta+\hbar)^5}-\frac{8\,a\,T^2\,\delta^5\,\hbar}{(-\delta+T\,\delta+\hbar)^5}+\frac{12\,a\,T^3\,\delta^5\,\hbar}{(-\delta+T\,\delta+\hbar)^5}-\frac{8\,a\,T^4\,\delta^5\,\hbar}{(-\delta+T\,\delta+\hbar)^5}+\cdots 267\cdots+\frac{x^2\,y^2\,\gamma\,\delta^2\,\hbar^6}{(-\delta+T\,\delta+\hbar)^5}+\frac{x\,y^2\,\gamma\,\delta\,\eta\,\hbar^6}{(-\delta+T\,\delta+\hbar)^5}+\frac{x^2\,y\,\gamma\,\delta\,\xi\,\hbar^6}{(-\delta+T\,\delta+\hbar)^5}+\frac{x\,y\,\gamma\,\eta\,\xi\,\hbar^6}{(-\delta+T\,\delta+\hbar)^5}\right)\epsilon+\left(-\frac{32\,a^2\,T^2\,\delta^{10}\,\hbar^2}{(\cdots 1\cdots)^2}+\cdots 8307\cdots+\cdots 1\cdots+\frac{144\,x^2\,y^2\,\gamma^2\,\eta^2\,\xi^2\,\hbar^{11}}{\cdots 1\cdots}\right)\epsilon^2+O[\epsilon]^3,\ 6\right\}$$

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Reorderings with Rord

Rord

In[∘]:=

```
Rord_{u_i_,w_j_→k_}[𝔼_U_[L___, {l___, u_i_, w_j_, r___}_s_, R___, Q_, P_]] :=
  Simp@Module[{υ, ω, δ, Λ1, yax, q, p, kk = P⟦5⟧, δ1 = ∂_{u_i,w_j} Q},
    {yax, q, p} = Echo[List @@ If[δ1 === 0, Λ_{U,kk}[{υ, ω}, {u, w}],
          Λ_{U,kk}[{υ, ω, δ}, {u, w}]] /. {y → y_k, a → a_k, x → x_k, t → t_s, T → T_s}];
    𝔼_U[L, {l, Sequence @@ yax, r}_s, R, q + (Q /. u_i | w_j → 0), e^{-q} DP_{u_i→D_υ, w_j→D_ω}[P][p e^q]] /.
     {υ → ∂_{u_i} Q /. w_j → 0, ω → ∂_{w_j} Q /. u_i → 0, δ → δ1}];
```

Rord

In[∘]:=

```
Rord_{u_i_,w_j_→k_}[𝔼_U_[L___, {l___, u_i_, w_j_, r___}_s_, R___, Q_, P_]] :=
  Simp@Module[{υ, ω, δ, Λ1, yax, q, p, n, kk = P⟦5⟧, δ1 = ∂_{u_i,w_j} Q},
    {yax, q, p} = List @@ If[δ1 === 0, Λ_{U,kk}[{υ, ω}, {u, w}], Λ_{U,kk}[{υ, ω, δ}, {u, w}]] /.
      {y → y_n, a → a_n, x → x_n, t → t_s, T → T_s};
    (*Echo@{{{u_i,υ},{w_j,ω}},P,p e^q};*)
    𝔼_U[L, {l, Sequence @@ yax, r}_s, R, q + (Q /. u_i | w_j → 0), e^{-q} SP_{u_i→υ, w_j→ω}[P p e^q]] /.
     {n → k, υ → ∂_{u_i} Q /. w_j → 0, ω → ∂_{w_j} Q /. u_i → 0, δ → δ1}];
```

```
With[{c𝜙 = 𝔼_CU[{y1, x1}_1, {x2, a2, y2}_2, ℏ t1 a2 + ℏ t1^-1 (e^t1 - 1) y1 x2, 1_2 + ϵ x1 y2]},
 {Short[rhs = c𝜙 // Rord_{x2,a2→3}, 3], HL[CU[c𝜙] == CU[rhs]]}]
```

$$\left\{\mathbb{E}_{CU}\left[\{y_1,\,x_1\}_1,\ \{a_3,\,x_3,\,y_2\}_2,\ \frac{e^{-\gamma\,\hbar\,t_1}\left(e^{\gamma\,\hbar\,t_1}\,\hbar\,a_3\,t_1^2-\hbar\,x_3\,y_1+e^{t_1}\,\hbar\,x_3\,y_1\right)}{t_1},\ 1+x_1\,y_2\,\epsilon+O[\epsilon]^3\right],\ \text{True}\right\}$$

```
With[{c𝜙 = 𝔼_CU[{y1, a1, a2}_1, {x2, x1, y2}_2,
     ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
     1_2 + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 {Short[rhs = c𝜙 // Rord_{a1,a2→3} // Rord_{x2,x1→4}, 3], HL[CU[c𝜙] == CU[rhs]]}]
```

$$\{\mathbb{E}_{CU}[\{y_1,\,a_3\}_1,\ \{x_4,\,y_2\}_2,\ \hbar\,a_3\,l_{11}\,t_1+\hbar\,a_3\,l_{12}\,t_1+\hbar\,a_3\,l_{21}\,t_2+\hbar\,a_3\,l_{22}\,t_2+\hbar\,x_4\,y_1\,\gamma_{11}+\hbar\,x_4\,y_2\,\gamma_{12}+\hbar\,x_4\,y_1\,\gamma_{21}+\hbar\,x_4\,y_2\,\gamma_{22},\ 1+(a_3\,l_1+a_3\,l_2+p_{11}\,x_4\,y_1+p_{21}\,x_4\,y_1+p_{12}\,x_4\,y_2+p_{22}\,x_4\,y_2)\,\epsilon+O[\epsilon]^3],\ \text{True}\}$$

```
ℏ a3 l11 t1 + ℏ a3 l12 t1 + ℏ a3 l21 t2 + ℏ a3 l22 t2 +
  ℏ x4 y1 γ11 + ℏ x4 y2 γ12 + ℏ x4 y1 γ21 + ℏ x4 y2 γ22 // Simplify
```

$$\hbar\,(a_3\,(l_{11}\,t_1+l_{12}\,t_1+(l_{21}+l_{22})\,t_2)+x_4\,(y_1\,(\gamma_{11}+\gamma_{21})+y_2\,(\gamma_{12}+\gamma_{22})))$$

```
With[{c⊕ = 𝔈CU[{y1, a1, x1}1, {x2, a2, y2}2,
     ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
     12 + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 {Short[rhs = c⊕ // Rord_{x2,a2→3}, 3], HL[CU[c⊕] ≡ CU[rhs]]}]
```

```
{𝔈CU[{y1, a1, x1}1, <<1>>2, <<1>> <<1>>,
  1 + e^(-γ ℏ l12 t1 - γ ℏ l22 t2) (e^(γ ℏ l12 t1 + γ ℏ l22 t2) a1 l1 + e^(γ ℏ l12 t1 + γ ℏ l22 t2) a3 l2 + e^(γ ℏ l12 t1 + γ ℏ l22 t2) p11 x1 y1 + p21 x3 y1 +
      e^(<<1>> + <<1>>) p12 x1 y2 + p22 x3 y2 - γ ℏ l2 x3 y1 γ21 - γ ℏ l2 x3 y2 γ22) ϵ + O[ϵ]^3], True}
```

```
With[{q⊕ = 𝔈QU[{y1, a1, x1}1, {x2, a2, y2}2,
     ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
     12 + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 {Short[rhs = q⊕ // Rord_{x2,a2→3}, 3], HL[QU[q⊕] ≡ QU[rhs]]}]
```

```
{𝔈QU[{y1, a1, x1}1, <<1>>2, <<1>> <<1>>,
  1 + e^(-γ ℏ l12 t1 - γ ℏ l22 t2) (e^(γ ℏ l12 t1 + γ ℏ l22 t2) a1 l1 + e^(γ ℏ l12 t1 + γ ℏ l22 t2) a3 l2 + e^(γ ℏ l12 t1 + γ ℏ l22 t2) p11 x1 y1 + p21 x3 y1 +
      e^(<<1>> + <<1>>) p12 x1 y2 + p22 x3 y2 - γ ℏ l2 x3 y1 γ21 - γ ℏ l2 x3 y2 γ22) ϵ + O[ϵ]^3], True}
```

```
With[{q⊕ = 𝔈QU[{y1, a1, x1}1, {x2, a2, y2}2,
     ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
     12 + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 {Short[rhs = q⊕ // Rord_{a2,y2→3}, 3], HL[QU[q⊕] ≡ QU[rhs]]}]
```

```
{<<1>>, True}
```

```
Timing@With[{q⊕ = 𝔈QU[{x1, y1}1, {x2, a2, y2}2,
      ℏ (l12 t1 a2 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
      02 + ϵ (l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
  {Short[rhs = q⊕ // Rord_{x1,y1→3}, 5]}]
```

```
{116.156, {𝔈QU[{y3, a3, x3}1, <<1>>2, <<1>> / (1 - <<1>> + <<1>>),
   ((ℏ a2 l2 + p11 - p11 T1 + ℏ p22 x2 y2 + ℏ p12 x3 y2 + <<46>> + 2 ℏ p12 T1 x2 y2 γ11 γ21 -
         ℏ p12 T1^2 x2 y2 γ11 γ21 + ℏ p11 x2 y2 γ12 γ21 - 2 ℏ p11 T1 x2 y2 γ12 γ21 + ℏ p11 T1^2 x2 y2 γ12 γ21) ϵ) /
     (ℏ - 3 ℏ γ11 + 3 ℏ T1 γ11 + 3 ℏ γ11^2 - 6 ℏ T1 γ11^2 + 3 ℏ T1^2 γ11^2 - ℏ γ11^3 + 3 ℏ T1 γ11^3 - 3 ℏ T1^2 γ11^3 + ℏ T1^3 γ11^3) +
    ((8 a3 p11 T1 + <<1>> + <<2726>> + 3 γ <<6>> γ21^3) <<1>>) /
     (4 - 28 γ11 + <<48>> + 4 T1^7 γ11^7) + O[ϵ]^3]}}
```

```
Timing@With[{qΦ = 𝔈QU[{x1, y1}1, {x2, a2, y2}2,
      ℏ (l12 t1 a2 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
      12 + ϵ (l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
  {Short[rhs = qΦ // Rord_{x1,y1→3}, 5], HL@SimpT[QU[qΦ] == QU[rhs]]}]
```

```
{388.922,
 {𝔈QU[{y3, a3, x3}1, {<<1>>}2, <<1>>/<<1>>, 1/(1 - γ11 + T1 γ11) + ((4 ℏ a2 l2 + 4 p11 - 4 p11 T1 + 4 ℏ p22 x2 y2 +
        <<339>> + γ ℏ^4 x2^2 y2^2 γ12^2 γ21^2 - 4 γ ℏ^4 T1 x2^2 y2^2 γ12^2 γ21^2 + 3 γ ℏ^4 T1^2 x2^2 y2^2 γ12^2 γ21^2) ϵ) /
    (4 ℏ - 20 ℏ γ11 + 20 ℏ T1 γ11 + 40 ℏ γ11^2 - 80 ℏ T1 γ11^2 + 40 ℏ T1^2 γ11^2 - 40 ℏ γ11^3 + <<13>> +
      20 ℏ T1 γ11^5 - 40 ℏ T1^2 γ11^5 + 40 ℏ T1^3 γ11^5 - 20 ℏ T1^4 γ11^5 + 4 ℏ T1^5 γ11^5) +
    (576 a3 p11 T1 + <<8073>> + <<1>>) <<1>> / (<<79>> + 288 T1^9 γ11^9) + O[ϵ]^3], True}}
```

```
Timing@With[{qΦ = 𝔈QU[{x1, y1}1, {x2, a2, y2}2,
      ℏ (l12 t1 a2 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
      12 + ϵ (l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
  {Short[rhs = qΦ // Rord_{x1,y1→1}, 5], HL@SimpT[QU[qΦ] == QU[rhs]]}]
```

```
{336.781,
 {𝔈QU[{y1, a1, x1}1, {<<1>>}2, <<1>>/<<1>>, 1/(1 - γ11 + T1 γ11) + ((4 ℏ a2 l2 + 4 p11 - 4 p11 T1 + 4 ℏ p11 x1 y1 +
        4 ℏ p21 x2 y1 + <<338>> + γ ℏ^4 x2^2 y2^2 γ12^2 γ21^2 - 4 γ ℏ^4 T1 x2^2 y2^2 γ12^2 γ21^2 + 3 γ ℏ^4 T1^2 x2^2 y2^2 γ12^2 γ21^2) ϵ) /
    (4 ℏ - 20 ℏ γ11 + 20 ℏ T1 γ11 + 40 ℏ γ11^2 - 80 ℏ T1 γ11^2 + 40 ℏ T1^2 γ11^2 - 40 ℏ γ11^3 + <<10>> +
      20 ℏ T1^4 γ11^4 - 4 ℏ γ11^5 + 20 ℏ T1 γ11^5 - 40 ℏ T1^2 γ11^5 + 40 ℏ T1^3 γ11^5 - 20 ℏ T1^4 γ11^5 + 4 ℏ T1^5 γ11^5) +
    (576 a1 p11 T1 + <<8073>> + <<1>>) <<1>> / (<<79>> + 288 T1^9 γ11^9) + O[ϵ]^3], True}}
```

Canonical ordering with Cord

Cord

In[∘]:=
```
Cord[𝔈U_[L___, {l___, u_i_, w_j_, r___}s_, R___, Q_, P_]] /;
    OrderedQ[{w, u} /. {y → 1, a → 2, x → 3}] :=
  ((*Echo@{ui,wj};*) Cord[Rord_{ui,wj→Unique[]}[𝔈U[L, {l, ui, wj, r}s, R, Q, P]]]);
Cord[𝔈U_[specs__, Q_, P_]] := 𝔈U[Sequence @@ Sort@{specs}, Q, P] /.
  Flatten[{specs} /. {yax__}s_ :> ({yax} /. u_i_ :> (ui → us))]
```

```
Cord@𝔈CU[{x1, y1}1, 0, 01 + x1 y1]
```

```
𝔈CU[{y1, a1, x1}1, 0, (-t1 + x1 y1) + 2 a1 ϵ + O[ϵ]^2]
```

```
Block[{$p = 4, $k = 0, c⊙ = ŒCU[{y1, a1, x1, x2, a2, y2}1,
    ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    1θ + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 Timing@{Short[Cord[c⊙], 8], HL@Simp[CU[c⊙] - CU[Cord[c⊙]]]}]
```

{4.53125,
 {ŒCU[{y1, a1, x1}1, (e^(γℏ l11 t1+2γℏ l12 t1+γℏ l21 t2+2γℏ l22 t2) ℏ a1 l11 t1 + e^(γℏ l11 t1+2γℏ l12 t1+γℏ l21 t2+2γℏ l22 t2) ℏ a1 l12 t1 + <<12>> + e^(γℏ l11 t1+γℏ l12 t1+γℏ l21 t2+γℏ l22 t2) ℏ^2 a1 l22 t1 t2 γ22 + ℏ x1 y1 γ22) /
 (e^(γℏ l11 t1+2γℏ l12 t1+γℏ l21 t2+2γℏ l22 t2) + e^(γℏ l11 t1+γℏ l12 t1+γℏ l21 t2+γℏ l22 t2) ℏ t1 γ12 + e^(γℏ l11 t1+γℏ l12 t1+γℏ l21 t2+γℏ l22 t2) ℏ t1 γ22), e^(γℏ l12 t1+γℏ l22 t2) / (e^(γℏ l12 t1+γℏ l22 t2) + ℏ t1 γ12 + ℏ t1 γ22) + O[ϵ]^1], 0}}

```
Block[{$p = 4, $k = 1, c⊙ = ŒCU[{y1, a1, x1, x2, a2, y2}1,
    ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    11 + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 Timing@{Short[Cord[c⊙], 8], HL@Simp[CU[c⊙] - CU[Cord[c⊙]]]}]
```

{81.2656, {ŒCU[{y1, a1, x1}1, (e^(γℏ l11 t1+2γℏ l12 t1+γℏ l21 t2+2γℏ l22 t2) ℏ a1 l11 t1 + <<14>> + ℏ x1 y1 γ22) /
 (e^(γℏ l11 t1+2γℏ l12 t1+γℏ l21 t2+2γℏ l22 t2) + e^(γℏ l11 t1+<<2>>+γℏ <<1>> t2) ℏ t1 γ12 + e^(γℏ l11 t1+γℏ l12 t1+γℏ l21 t2+γℏ l22 t2) ℏ t1 γ22),
 e^(γℏ l12 t1+γℏ l22 t2) / (e^(γℏ l12 t1+γℏ l22 t2) + ℏ t1 γ12 + ℏ t1 γ22) + ((2 e^(2γℏ l11 t1+6γℏ l12 t1+2γℏ l21 t2+6γℏ l22 t2) a1 l1 + <<419>>) ϵ) /
 (2 e^(2γℏ l11 t1+6γℏ l12 t1+2γℏ l21 t2+6γℏ l22 t2) + 10 e^(2γℏ l11 t1+5γℏ l12 t1+2γℏ l21 t2+5γℏ l22 t2) ℏ t1 γ12 + <<18>> + 2 e^(2γℏ l11 t1+γℏ l12 t1+2γℏ l21 t2+γℏ l22 t2) ℏ^5 t1^5 γ22^5) + O[ϵ]^2], 0}}

```
With[{q⊙ = ŒQU[{y1, a1, x1, x2, a2, y2}1,
    ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    1θ + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 Cord[
  q⊙]]
```

ŒQU[{y1, a1, x1}1,
 (e^(γℏ l11 t1+2γℏ l12 t1+γℏ l21 t2+2γℏ l22 t2) ℏ a1 l11 t1 + e^(γℏ l11 t1+2γℏ l12 t1+γℏ l21 t2+2γℏ l22 t2) ℏ a1 l12 t1 +
 e^(γℏ l11 t1+2γℏ l12 t1+γℏ l21 t2+2γℏ l22 t2) ℏ a1 l21 t2 + e^(γℏ l11 t1+2γℏ l12 t1+γℏ l21 t2+2γℏ l22 t2) ℏ a1 l22 t2 +
 e^(γℏ l11 t1+γℏ l12 t1+γℏ l21 t2+γℏ l22 t2) ℏ x1 y1 γ11 − e^(γℏ l11 t1+γℏ l12 t1+γℏ l21 t2+γℏ l22 t2) ℏ a1 l11 t1 γ12 −
 e^(γℏ l11 t1+γℏ l12 t1+γℏ l21 t2+γℏ l22 t2) ℏ a1 l12 t1 γ12 − e^(γℏ l11 t1+γℏ l12 t1+γℏ l21 t2+γℏ l22 t2) ℏ a1 l21 t2 γ12 −
 e^(γℏ l11 t1+γℏ l12 t1+γℏ l21 t2+γℏ l22 t2) ℏ a1 l22 t2 γ12 + e^(γℏ l11 t1+γℏ l12 t1+γℏ l21 t2+γℏ l22 t2) ℏ a1 l11 t1 T1 γ12 +
 e^(γℏ l11 t1+γℏ l12 t1+γℏ l21 t2+γℏ l22 t2) ℏ a1 l12 t1 T1 γ12 + e^(γℏ l11 t1+γℏ l12 t1+γℏ l21 t2+γℏ l22 t2) ℏ a1 l21 t2 T1 γ12 +
 e^(γℏ l11 t1+γℏ l12 t1+γℏ l21 t2+γℏ l22 t2) ℏ a1 l22 t2 T1 γ12 + ℏ x1 y1 γ12 +
 e^(γℏ l11 t1+γℏ l12 t1+γℏ l21 t2+γℏ l22 t2) ℏ x1 y1 γ21 − e^(γℏ l11 t1+γℏ l12 t1+γℏ l21 t2+γℏ l22 t2) ℏ a1 l11 t1 γ22 −
 e^(γℏ l11 t1+γℏ l12 t1+γℏ l21 t2+γℏ l22 t2) ℏ a1 l12 t1 γ22 − e^(γℏ l11 t1+γℏ l12 t1+γℏ l21 t2+γℏ l22 t2) ℏ a1 l21 t2 γ22 −
 e^(γℏ l11 t1+γℏ l12 t1+γℏ l21 t2+γℏ l22 t2) ℏ a1 l22 t2 γ22 + e^(γℏ l11 t1+γℏ l12 t1+γℏ l21 t2+γℏ l22 t2) ℏ a1 l11 t1 T1 γ22 +
 e^(γℏ l11 t1+γℏ l12 t1+γℏ l21 t2+γℏ l22 t2) ℏ a1 l12 t1 T1 γ22 + e^(γℏ l11 t1+γℏ l12 t1+γℏ l21 t2+γℏ l22 t2) ℏ a1 l21 t2 T1 γ22 +
 e^(γℏ l11 t1+γℏ l12 t1+γℏ l21 t2+γℏ l22 t2) ℏ a1 l22 t2 T1 γ22 + ℏ x1 y1 γ22) /
 (e^(γℏ l11 t1+2γℏ l12 t1+γℏ l21 t2+2γℏ l22 t2) − e^(γℏ l11 t1+γℏ l12 t1+γℏ l21 t2+γℏ l22 t2) γ12 + e^(γℏ l11 t1+γℏ l12 t1+γℏ l21 t2+γℏ l22 t2) T1 γ12 − e^(γℏ l11 t1+γℏ l12 t1+γℏ l21 t2+γℏ l22 t2) γ22 + e^(γℏ l11 t1+γℏ l12 t1+γℏ l21 t2+γℏ l22 t2) T1 γ22),
 e^(γℏ l12 t1+γℏ l22 t2) / (e^(γℏ l12 t1+γℏ l22 t2) − γ12 + T1 γ12 − γ22 + T1 γ22) + O[ϵ]^1]

Stitching Œ's.

StitchingOEs

In[∘]:=
```
m_{j_→k_}[Œ_{U_}[specs__, Q_, P_]] := Cord[Œ_U[Sequence @@ Append[DeleteCases[{specs}, {__}_{j|k}],
      Flatten[{Cases[{specs}, {us__}_j ⧴ {us}], Cases[{specs}, {us__}_k ⧴ {us}]}]_k],
    Q, P] /. {t_j → t_k, T_j → T_k}]
```

```
cœ = Œ_CU[{y1, a1, x1}1, {y2, a2, x2}2,
   {y3, a3, x3}3, ℏ Sum[l_{10 i+j} t_i a_j + γ_{10 i+j} y_i x_j, {i, 3}, {j, 3}], 1_2];
{cœ // m_{3→4}, HL@Simp[CU[m_{3→4}[cœ]] - m_{3→4}[CU[cœ]]]}
```

$\{Œ_{CU}[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, \{y_4, a_4, x_4\}_4,$
$\hbar\,(a_1 l_{11} t_1 + a_2 l_{12} t_1 + a_4 l_{13} t_1 + a_1 l_{21} t_2 + a_2 l_{22} t_2 + a_4 l_{23} t_2 + a_1 l_{31} t_4 + a_2 l_{32} t_4 + a_4 l_{33} t_4 + x_1 y_1 \gamma_{11} + x_2 y_1 \gamma_{12} + x_4 y_1 \gamma_{13} + x_1 y_2 \gamma_{21} + x_2 y_2 \gamma_{22} + x_4 y_2 \gamma_{23} + x_1 y_4 \gamma_{31} + x_2 y_4 \gamma_{32} + x_4 y_4 \gamma_{33}), 1 + O[\epsilon]^3], 0\}$

Verifying that *m* commutes with evaluation, in CU:

```
cœ = Œ_CU[{y1, a1, x1}1, {y2, a2, x2}2,
   {y3, a3, x3}3, ℏ Sum[l_{10 i+j} t_i a_j + γ_{10 i+j} y_i x_j, {i, 3}, {j, 3}], 1_2];
Timing@{cœ // m_{2→3}, HL@Simp[CU[m_{2→3}[cœ]] - m_{2→3}[CU[cœ]]]}
```

$\{513.453, \{Œ_{CU}[\{y_1, a_1, x_1\}_1, \{y_3, a_3, x_3\}_3, \frac{\cdots 1 \cdots}{e^{\cdots 1 \cdots}\ \cdots 1 \cdots\ \cdots 1 \cdots},$
$\frac{1}{1+\hbar t_3 \gamma_{32}} + \left(\left(4\, e^{2\gamma\hbar l_{12} t_1 + \cdots 4 \cdots + 2\gamma\hbar l_{33} t_3} \hbar^2 a_3 x_1 y_1 \gamma_{12} \gamma_{31} - 2 \cdots 7 \cdots \gamma_{31} + \cdots 154 \cdots\right) \epsilon\right) /$
$\left(2\, e^{2\gamma\hbar l_{12} t_1 + 2\gamma\hbar l_{13} t_1 + \cdots 3 \cdots + 2\gamma\hbar l_{33} t_3} + 10\, e^{\cdots 1 \cdots} \hbar t_3 \gamma_{32} + \cdots 2 \cdots + \cdots 1 \cdots + 2\, e^{2\gamma\hbar l_{12} t_1 + \cdots 4 \cdots + \cdots 1 \cdots} \hbar^5 t_3^5 \gamma_{32}^5\right) + \frac{(\cdots 1 \cdots)\, \epsilon^2}{\cdots 1 \cdots} + O[\epsilon]^3], 0\}\}$

large output | show less | show more | show all | set size limit...

Verifying that *m* commutes with evaluation, in QU:

```
qœ = Œ_QU[{y1, a1, x1}1, {y2, a2, x2}2,
   {y3, a3, x3}3, ℏ Sum[l_{10 i+j} t_i a_j + γ_{10 i+j} y_i x_j, {i, 3}, {j, 3}], 1_2];
Timing@{qœ // m_{2→3}, HL@SimpT[QU[m_{2→3}[qœ]] - m_{2→3}[QU[qœ]]]}
```

$\{7831.47, \{Œ_{QU}[\{y_1, a_1, x_1\}_1, \{y_3, a_3, x_3\}_3, \frac{\cdots 1 \cdots}{\cdots 1 \cdots}, \frac{1}{1 - \gamma_{32} + T_3 \gamma_{32}} +$
$\left(\left(8\, e^{2\gamma\hbar l_{12} t_1 + \cdots 4 \cdots + 2\gamma\hbar l_{33} t_3} \hbar^2 a_3 T_3 x_1 y_1 \gamma_{12} \gamma_{31} + 4 \cdots 8 \cdots \gamma_{31} + \cdots 371 \cdots\right) \epsilon\right) /$
$\left(4\, e^{2\gamma\hbar l_{12} t_1 + 2\gamma\hbar l_{13} t_1 + \cdots 3 \cdots + 2\gamma\hbar l_{33} t_3} - 20\, e^{\cdots 1 \cdots} \gamma_{32} + \cdots 26 \cdots + 4\, e^{2\gamma\hbar l_{12} t_1 + \cdots 4 \cdots + \cdots 1 \cdots} T_3^5 \gamma_{32}^5\right) + \frac{(\cdots 1 \cdots)\, \epsilon^2}{\cdots 79 \cdots + \cdots 1 \cdots} + O[\epsilon]^3], 0\}\}$

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In[∘]:=
```
ℭ_U_[sp1__, Q1_, P1_] ≡ ℭ_U_[sp2__, Q2_, P2_] :=
  Sort[{sp1}] == Sort[{sp2}] ∧ Simplify[Q1 == Q2] ∧ Simplify[Normal[P1 - P2] == 0]
```

Verifying meta-associativity in CU:

```
c∞ = ℭ_CU[{y_1, a_1, x_1}_1, {y_2, a_2, x_2}_2,
   {y_3, a_3, x_3}_3, ℏ Sum[λ_{10 i+j} t_i a_j + γ_{10 i+j} y_i x_j, {i, 3}, {j, 3}], 1_0];
Timing@HL[(lhs = c∞ // m_{1,2→1} // m_{1,3→1}) ≡ (rhs = c∞ // m_{2,3→2} // m_{1,2→1})]
```

{41.9219, True}

```
c∞ = ℭ_CU[{y_1, a_1, x_1}_1, {y_2, a_2, x_2}_2,
   {y_3, a_3, x_3}_3, ℏ Sum[λ_{10 i+j} t_i a_j + γ_{10 i+j} y_i x_j, {i, 3}, {j, 3}], 1_1];
Timing@HL[(lhs = c∞ // m_{1,2→1} // m_{1,3→1}) ≡ (rhs = c∞ // m_{2,3→2} // m_{1,2→1})]
```

{30 119.8, True}

mexamples

```
c∞ = ℭ_CU[{y_1, a_1, x_1}_1, {y_2, a_2, x_2}_2, ℏ Sum[l_{10 i+j} t_i a_j + γ_{10 i+j} y_i x_j, {i, 2}, {j, 2}], 1_1];
Short[Simplify /@ (cexample = c∞ // m_{1→2}), 12]
Short[Simplify /@ (qexample = (q∞ = c∞ /. CU → QU) // m_{1→2}), 12]
```

mexamples

$$\mathfrak{C}_{CU}\Big[\{y_2, a_2, x_2\}_2,\ \hbar\, a_2\,(l_{11}+l_{12}+l_{21}+l_{22})\, t_2 + \frac{1}{1+\hbar t_2 \gamma_{21}} e^{-\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar\, x_2 y_2 \Big(\gamma_{21} + e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\gamma_{12}\,(1+\hbar t_2\gamma_{21}) + e^{\gamma\hbar(l_{12}+l_{22})t_2}\gamma_{22} + \gamma_{11}\big(e^{\gamma\hbar(l_{11}+l_{21})t_2} - e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar t_2\gamma_{22}\big)\Big),$$
$$\frac{1}{1+\hbar t_2\gamma_{21}} + \frac{1}{2\,(1+\hbar t_2\gamma_{21})^5} e^{-2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar\Big(4 a_2\,(1+\hbar t_2\gamma_{21})^2 \big(e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar\,(e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2} t_2 + x_2 y_2)\gamma_{21}^2 + e^{2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar\, x_2 y_2\gamma_{11}\gamma_{22} + \gamma_{21}\big(e^{2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2} + \hbar x_2 y_2\,(e^{\gamma\hbar(2 l_{11}+l_{12}+2 l_{21}+l_{22})t_2}\gamma_{11} + e^{\gamma\hbar(l_{11}+2 l_{12}+l_{21}+2 l_{22})t_2}\gamma_{22})\big)\big) - \gamma\hbar\Big(-2\, e^{2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2} t_2\gamma_{21}^2\,(1+\hbar t_2\gamma_{21})^2 + 4 \ll 5 \gg (\ll 1 \gg) + \hbar \ll 4 \gg \big(3\hbar t_2\gamma_{21}^2 + 2\, e^{\gamma\hbar(l_{12}+l_{\ll 2 \gg})t_2}\gamma_{22} + \gamma_{21}\,(4 + e^{\gamma \ll 3 \gg}\hbar t_2\gamma_{22}) + e^{\gamma\hbar(l_{11}+l_{21})t_2}\gamma_{11}\,(2+\hbar t_2\,(\gamma_{21} - e^{\gamma\hbar(l_{12}+l_{22})t_2}\gamma_{22}))\big)\Big)\Big)\,\epsilon + O[\epsilon]^2\Big]$$

mexamples

$$\mathfrak{C}_{QU}\Big[\{y_2, a_2, x_2\}_2,\ \hbar\, a_2\,(l_{11}+l_{12}+l_{21}+l_{22})\, t_2 + \frac{1}{1+(-1+T_2)\gamma_{21}} e^{-\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar\, x_2 y_2\Big(\gamma_{21} + e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\gamma_{12}\,(1+(-1+T_2)\gamma_{21}) + e^{\gamma\hbar(l_{12}+l_{22})t_2}\gamma_{22} + \gamma_{11}\big(e^{\gamma\hbar(l_{11}+l_{21})t_2} - e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}(-1+T_2)\gamma_{22}\big)\Big),$$
$$\frac{1}{1+(-1+T_2)\gamma_{21}} + \frac{1}{4\,(1+(-1+T_2)\gamma_{21})^5} e^{-2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar\Big(8 a_2 T_2\,(1+(-1+T_2)\gamma_{21})^2 \big(e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}(-e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2} + e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2} T_2 + \hbar x_2 y_2)\gamma_{21}^2 + e^{2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar\, x_2 y_2\gamma_{11}\gamma_{22} + \gamma_{21}\big(e^{2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2} + \hbar x_2 y_2\,(e^{\gamma\hbar(2 l_{11}+l_{12}+2 l_{21}+l_{22})t_2}\gamma_{11} + e^{\gamma\hbar(l_{11}+2 l_{12}+l_{21}+2 l_{22})t_2}\gamma_{22})\big)\big) + \gamma\Big(2\, e^{2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}(1 - 4 T_2 + 3 T_2^2)\gamma_{21}^2\,(1+(-1+T_2)\gamma_{21})^2 + 4\, e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar\, x_2 y_2\gamma_{21}\,(1+(-1+T_2)\gamma_{21})\,(\ll 1 \gg) - \ll 1 \gg\Big)\Big)\,\epsilon + O[\epsilon]^2\Big]$$

R in QU.

The Faddeev-Quesne formula:

Faddeev

In[ ]:=
```
e_{q_,k_}[x_] := e^(Sum[(1 - q)^j x^j / (j (1 - q^j)), {j, 1, k+1}]); e_{q_}[x_] := e_{q,$k}[x]
```

```
Table[Series[e_{q_ħ,k}[x], {ϵ, 0, 4}], {k, 0, 5}] // Column
```

$$e^x$$

$$e^x - \frac{1}{4}\left(e^x x^2 \gamma \hbar\right) \epsilon + \frac{1}{32} e^x x^4 \gamma^2 \hbar^2 \epsilon^2 - \frac{1}{384}\left(e^x x^2 \left(-8 + x^4\right) \gamma^3 \hbar^3\right) \epsilon^3 + \frac{e^x x^4 \left(-32+x^4\right) \gamma^4 \hbar^4 \epsilon^4}{6144} + O[\epsilon]^5$$

$$e^x - \frac{1}{4}\left(e^x x^2 \gamma \hbar\right) \epsilon + \frac{1}{288} e^x x^3 \left(32 + 9 x\right) \gamma^2 \hbar^2 \epsilon^2 - \frac{\left(e^x x^2 \left(-24+32 x^3+3 x^4\right) \gamma^3 \hbar^3\right) \epsilon^3}{1152} + \frac{e^x x^3 \left(-4608-864 x+1024 x^3+576 x^4+27 x^5\right) \gamma^4 \hbar^4 \epsilon^4}{165888} + O[\epsilon]^5$$

$$e^x - \frac{1}{4}\left(e^x x^2 \gamma \hbar\right) \epsilon + \frac{1}{288} e^x x^3 \left(32 + 9 x\right) \gamma^2 \hbar^2 \epsilon^2 - \frac{\left(e^x x^2 \left(-24+72 x^2+32 x^3+3 x^4\right) \gamma^3 \hbar^3\right) \epsilon^3}{1152} + \frac{e^x x^3 \left(-4608-864 x+3616 x^3+576 x^4+27 x^5\right) \gamma^4 \hbar^4 \epsilon^4}{165888} + O[\epsilon]^5$$

$$e^x - \frac{1}{4}\left(e^x x^2 \gamma \hbar\right) \epsilon + \frac{1}{288} e^x x^3 \left(32 + 9 x\right) \gamma^2 \hbar^2 \epsilon^2 - \frac{\left(e^x x^2 \left(-24+72 x^2+32 x^3+3 x^4\right) \gamma^3 \hbar^3\right) \epsilon^3}{1152} + \frac{1}{4147200} e^x x^3 \left(-115200 - 21600 x + 165888 x^2 + 90400 x^3 + 14400 x^4 + 675 x^5\right) \gamma^4 \hbar^4 \epsilon^4 + O[\epsilon]^5$$

$$e^x - \frac{1}{4}\left(e^x x^2 \gamma \hbar\right) \epsilon + \frac{1}{288} e^x x^3 \left(32 + 9 x\right) \gamma^2 \hbar^2 \epsilon^2 - \frac{\left(e^x x^2 \left(-24+72 x^2+32 x^3+3 x^4\right) \gamma^3 \hbar^3\right) \epsilon^3}{1152} + \frac{1}{4147200} e^x x^3 \left(-115200 - 21600 x + 165888 x^2 + 90400 x^3 + 14400 x^4 + 675 x^5\right) \gamma^4 \hbar^4 \epsilon^4 + O[\epsilon]^5$$

```
Table[Together@SeriesCoefficient[e_{q,5}[x], {x, 0, n}], {n, 0, 5}]
```

$$\left\{1, 1, \frac{1}{1+q}, \frac{1}{(1+q)\left(1+q+q^2\right)}, \frac{1}{(1+q)^2\left(1+q^2\right)\left(1+q+q^2\right)}, 1\Big/\left((1+q)^2\left(1+q^2\right)\left(1+q+q^2\right)\left(1+q+q^2+q^3+q^4\right)\right)\right\}$$

```
Table[HL@FunctionExpand[QFactorial[n, q] SeriesCoefficient[e_{q,5}[x], {x, 0, n}]], {n, 0, 5}]
```

{1, 1, 1, 1, 1, 1}

R

In[ ]:=
```
QU[R_{i_,j_}] := O_QU[{y_1, a_1}_i, {a_2, x_2}_j, SS[e^(ħ b_1 a_2) e_{q_ħ}[ħ y_1 x_2] /. b_1 → γ^-1 (ϵ a_1 - t_i)]];
QU[R^-1_{i_,j_}] := S_j@QU[R_{i,j}];
```

```
QU[R_{3,4}] // Short
```

$$QU[\,] + \frac{\epsilon \hbar\, QU[a_3, a_4]}{\gamma} + \hbar\, QU[y_3, x_4] + \frac{\epsilon \ll 1 \gg \ll 1 \gg}{\gamma} + \frac{1}{2} \ll 1 \gg \ll 1 \gg - \frac{\ll 1 \gg}{\gamma} - \frac{\epsilon \hbar^2 \ll 1 \gg t_3}{\gamma^2} - \frac{\hbar^2 QU[y_3, a_4, x_4]\, t_3}{\gamma} + \frac{\hbar^2 QU[a_4, a_4]\, t_3^2}{2 \gamma^2}$$

Verifying R2 (~2 secs @ $p=4, $k=2):

```
QU[R_{1,2} ** R^-1_{1,2}] // Simp // HL // Timing
```

{0.078125, QU[]}

Verifying R3 (~156 secs @ $p=4, $k=2):

```
{Short[lhs = QU[R1,2 ** R1,3 ** R2,3]], HL@SimpT[lhs - QU[R2,3 ** R1,3 ** R1,2]]} // Timing
```

$\{0.203125, \{\text{QU}[\,] + \frac{\epsilon\hbar\,\text{QU}[a_1, a_2]}{\gamma} + \frac{\epsilon\hbar\,\text{QU}[a_1, a_3]}{\gamma} + \ll 73 \gg + 2\epsilon\hbar^2\,\text{QU}[y_1, a_2, x_3]\,T_2 + \text{QU}[y_1, x_3]\,(\hbar - \hbar T_2), 0\}\}$

R in $Œ_{QU}$.

RinOE

In[∘]:=

$Œ_{QU,k\_}[R_{i\_,j\_}] := Œ_{QU}\left[\{y_i, a_i, x_i\}_i, \{y_j, a_j, x_j\}_j, -\hbar\gamma^{-1} t_i a_j + \hbar y_i x_j, \text{Series}\left[e^{\hbar\gamma^{-1} t_i a_j - \hbar y_i x_j} \left(e^{\hbar b_i a_j}\, e_{q_\hbar,k}[\hbar y_i x_j] \;/.\; b_i \to \gamma^{-1}(\epsilon a_i - t_i)\right), \{\epsilon, 0, k\}\right]\right]$

$\{Œ_{QU,1}[R_{1,2}],\ Œ_{QU,2}[R_{1,2}]\}$

$\{Œ_{QU}[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, -\frac{\hbar a_2 t_1}{\gamma} + \hbar x_2 y_1, 1 + \left(\frac{\hbar a_1 a_2}{\gamma} - \frac{1}{4}\gamma\hbar^3 x_2^2 y_1^2\right)\epsilon + O[\epsilon]^2],$
$Œ_{QU}[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, -\frac{\hbar a_2 t_1}{\gamma} + \hbar x_2 y_1, 1 + \left(\frac{\hbar a_1 a_2}{\gamma} - \frac{1}{4}\gamma\hbar^3 x_2^2 y_1^2\right)\epsilon + \frac{1}{288\gamma^2}\left(144\hbar^2 a_1^2 a_2^2 - 72\gamma^2\hbar^4 a_1 a_2 x_2^2 y_1^2 + 32\gamma^4\hbar^5 x_2^3 y_1^3 + 9\gamma^4\hbar^6 x_2^4 y_1^4\right)\epsilon^2 + O[\epsilon]^3]\}$

The morphism $Œ_{U,k}$.

MorphismOE

In[∘]:=

```
ŒU_,k_[a_ * b_] := ŒU,k[a] ŒU,k[b];
ŒU_,k_[mis__[a_]] := mis[ŒU,k[a]];
```

$Œ_{QU,1}[R_{1,2}\, R_{3,4}\, R_{5,6}]$

$Œ_{QU}[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, \{y_3, a_3, x_3\}_3, \{y_4, a_4, x_4\}_4, \{y_5, a_5, x_5\}_5, \{y_6, a_6, x_6\}_6, -\frac{\hbar a_2 t_1}{\gamma} - \frac{\hbar a_4 t_3}{\gamma} - \frac{\hbar a_6 t_5}{\gamma} + \hbar x_2 y_1 + \hbar x_4 y_3 + \hbar x_6 y_5,$
$1 + \left(\frac{\hbar a_1 a_2}{\gamma} + \frac{\hbar a_3 a_4}{\gamma} + \frac{\hbar a_5 a_6}{\gamma} - \frac{1}{4}\gamma\hbar^3 x_2^2 y_1^2 - \frac{1}{4}\gamma\hbar^3 x_4^2 y_3^2 - \frac{1}{4}\gamma\hbar^3 x_6^2 y_5^2\right)\epsilon + O[\epsilon]^2]$

$Œ_{QU,1}[R_{1,2}\, R_{3,4}\, R_{5,6}\; // \; m_{1,3\to 1}\; // \; m_{2,5\to 2}\; // \; m_{4,6\to 4}]$

$Œ_{QU}[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, \{y_4, a_4, x_4\}_4, \frac{1}{\gamma}\left(-\hbar a_2 t_1 - \hbar a_4 t_1 - \hbar a_4 t_2 + \gamma\hbar x_2 y_1 + \gamma\hbar x_4 y_1 + e^{\hbar t_2}\gamma\hbar x_4 y_1 - \gamma\hbar T_2 x_4 y_1 + e^{\hbar t_1}\gamma\hbar x_4 y_2\right),$
$1 + \frac{1}{4\gamma}\left(4\hbar a_1 a_2 + 4\hbar a_1 a_4 + 4\hbar a_2 a_4 - 4\gamma\hbar^2 a_4 x_2 y_1 - 8 e^{\hbar t_2}\gamma\hbar^2 a_2 x_4 y_1 + 8\gamma\hbar^2 a_2 T_2 x_4 y_1 - \gamma^2\hbar^3 x_2^2 y_1^2 + 4 e^{\hbar t_2}\gamma^2\hbar^3 x_2 x_4 y_1^2 - 4\gamma^2\hbar^3 T_2 x_2 x_4 y_1^2 - \gamma^2\hbar^3 x_4^2 y_1^2 - e^{2\hbar t_2}\gamma^2\hbar^3 x_4^2 y_1^2 + \gamma^2\hbar^3 T_2^2 x_4^2 y_1^2 - 4 e^{\hbar t_1}\gamma\hbar^2 a_1 x_4 y_2 + 4 e^{\hbar t_1}\gamma^2\hbar^3 x_2 x_4 y_1 y_2 + 4 e^{\hbar t_1 + \hbar t_2}\gamma^2\hbar^3 x_4^2 y_1 y_2 - 4 e^{\hbar t_1}\gamma^2\hbar^3 T_2 x_4^2 y_1 y_2 - e^{2\hbar t_1}\gamma^2\hbar^3 x_4^2 y_2^2\right)\epsilon + O[\epsilon]^2]$

$Œ_{QU,1}[R_{1,2}\, R_{3,4}\, R_{5,6}$ // $m_{3,1\to 1}$ // $m_{5,2\to 2}$ // $m_{6,4\to 4}]$

$$Œ_{QU}\Big[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, \{y_4, a_4, x_4\}_4, \frac{1}{\gamma}\left(-\hbar\, a_2\, t_1 - \hbar\, a_4\, t_1 - \hbar\, a_4\, t_2 + \gamma\,\hbar\, x_2\, y_1 + \gamma\,\hbar\, x_4\, y_1 + e^{\hbar\, t_1}\,\gamma\,\hbar\, x_4\, y_2\right),$$
$$1 + \frac{1}{4\gamma}\Big(4\,\hbar\, a_1\, a_2 + 4\,\hbar\, a_1\, a_4 + 4\,\hbar\, a_2\, a_4 - 4\,\gamma\,\hbar^2\, a_4\, x_2\, y_1 - \gamma^2\,\hbar^3\, x_2^2\, y_1^2 - \gamma^2\,\hbar^3\, x_4^2\, y_1^2 -$$
$$4\, e^{\hbar\, t_1}\,\gamma\,\hbar^2\, a_1\, x_4\, y_2 + 4\, e^{\hbar\, t_1}\,\gamma^2\,\hbar^3\, x_2\, x_4\, y_1\, y_2 - e^{2\,\hbar\, t_1}\,\gamma^2\,\hbar^3\, x_4^2\, y_2^2\Big)\,\epsilon + O[\epsilon]^2\Big]$$

$Œ_{QU,1}[R_{1,2}\, R_{3,4}\, R_{5,6}$ // $m_{1,3\to 1}$ // $m_{2,5\to 2}$ // $m_{4,6\to 4}] \equiv Œ_{QU,1}[R_{1,2}\, R_{3,4}\, R_{5,6}$ // $m_{3,1\to 1}$ // $m_{5,2\to 2}$ // $m_{6,4\to 4}]$

$$\hbar\left(e^{\hbar\, t_2} - T_2\right) x_4\, y_1 == 0\ \&\&\ \epsilon\,\hbar\left(e^{\hbar\, t_2} - T_2\right) x_4\, y_1\left(8\, a_2 + \gamma\,\hbar\left(-4\, x_2\, y_1 + x_4\left(\left(e^{\hbar\, t_2} + T_2\right) y_1 - 4\, e^{\hbar\, t_1}\, y_2\right)\right)\right) == 0$$

Exponentials as needed.

In[∘]:=
```
Block[{$p = 2, $k = 2}, TableForm[StringSplit[
  "y | a | x | Cθ@y_CU | Cθ@a_CU | Cθ@x_CU | Qθ@y_QU | Qθ@a_QU | Qθ@x_QU | AD@y_QU | AD@a_QU | AD@x_QU | SD@y_QU | SD@a_QU | SD
   @x_QU | S@y_CU | S@a_CU | S@x_CU | S@y_QU | S@a_QU | S@x_QU | Δ@y_CU | Δ@a_CU | Δ@x_CU | Δ@y_QU | Δ@a_QU | Δ@x_QU",
  "|"] /. s_String ↦
  {s, Normal@Simplify@Series[ToExpression[s] /. CU | QU → Times, {ϵ, 0, $k}]}]]
```

Out[∘]//TableForm=

y	y
a	a
x	x
$C\theta@y_{CU}$	$-x$
$C\theta@a_{CU}$	$-a$
$C\theta@x_{CU}$	$-y$
$Q\theta@y_{QU}$	$-\frac{x}{\sqrt{T}}-\frac{a\,x\,\epsilon\,\hbar}{\sqrt{T}}-\frac{a^2\,x\,\epsilon^2\,\hbar^2}{2\sqrt{T}}$
$Q\theta@a_{QU}$	$-a$
$Q\theta@x_{QU}$	$-\frac{y}{\sqrt{T}}+\frac{y\,(-a+\gamma)\,\epsilon\,\hbar}{\sqrt{T}}-\frac{y\,(a-\gamma)^2\,\epsilon^2\,\hbar^2}{2\sqrt{T}}$
$A\mathbb{D}@y_{QU}$	$\frac{2}{3}a^2\,y\,\epsilon^2\,\hbar^2+\frac{1}{6}y\left(6+3\,t\,\hbar+t^2\,\hbar^2\right)+\frac{1}{12}y\,\epsilon\,\hbar\left(x\,y\,\gamma\,\hbar-4\,a\left(3+2\,t\,\hbar\right)\right)$
$A\mathbb{D}@a_{QU}$	a
$A\mathbb{D}@x_{QU}$	x
$S\mathbb{D}@y_{QU}$	$y+\frac{1}{48}t^2\,y\,\hbar^2+\frac{1}{24}y\left(-2\,a\,t+x\,y\,\gamma\right)\epsilon\,\hbar^2+\frac{1}{12}a^2\,y\,\epsilon^2\,\hbar^2$
$S\mathbb{D}@a_{QU}$	a
$S\mathbb{D}@x_{QU}$	$\frac{7}{12}a^2\,x\,\epsilon^2\,\hbar^2+x\left(1+\frac{t\,\hbar}{2}+\frac{7\,t^2\,\hbar^2}{48}\right)+\frac{1}{24}x\,\epsilon\,\hbar\left(x\,y\,\gamma\,\hbar-2\,a\left(12+7\,t\,\hbar\right)\right)$
$S@y_{CU}$	$-y$
$S@a_{CU}$	$-a$
$S@x_{CU}$	$-x$
$S@y_{QU}$	$-\frac{y}{T}+\frac{y\,(-a+\gamma)\,\epsilon\,\hbar}{T}-\frac{y\,(a-\gamma)^2\,\epsilon^2\,\hbar^2}{2\,T}$
$S@a_{QU}$	$-a$
$S@x_{QU}$	$-x-a\,x\,\epsilon\,\hbar-\frac{1}{2}a^2\,x\,\epsilon^2\,\hbar^2$
$\Delta@y_{CU}$	y_1+y_2
$\Delta@a_{CU}$	a_1+a_2
$\Delta@x_{CU}$	x_1+x_2
$\Delta@y_{QU}$	$y_1+T_1\,y_2-\epsilon\,\hbar\,a_1\,T_1\,y_2+\frac{1}{2}\epsilon^2\,\hbar^2\,a_1^2\,T_1\,y_2$
$\Delta@a_{QU}$	a_1+a_2
$\Delta@x_{QU}$	$x_1+x_2-\epsilon\,\hbar\,a_1\,x_2+\frac{1}{2}\epsilon^2\,\hbar^2\,a_1^2\,x_2$

Exp

Task. Define $\mathrm{Exp}_{U_i,k}[\xi, P]$ which computes $e^{\xi\,\mathbb{O}(P)}$ to ϵ^k in the algebra U_i, where ξ is a scalar, X is x_i or y_i, and P is an ϵ-dependent near-docile element, giving the answer in Œ-form. Should satisfy $U@\mathrm{Exp}_{U_i,k}[\xi, P] == \mathbb{S}_U\left[e^{\xi x},\ x \to \mathbb{O}(P)\right]$.

Methodology. If $P_0 := P_{\epsilon=0}$ and $e^{\xi\,\mathbb{O}(P)} = \mathbb{O}\left(e^{\xi P_0} F(\xi)\right)$, then $F(\xi = 0) = 1$ and we have:
$\mathbb{O}\left(e^{\xi P_0}(P_0 F(\xi) + \partial_\xi F\right) = \mathbb{O}\left(\partial_\xi e^{\xi P_0} F(\xi)\right) = \partial_\xi \mathbb{O}\left(e^{\xi P_0} F(\xi)\right) = \partial_\xi e^{\xi\,\mathbb{O}(P)} = e^{\xi\,\mathbb{O}(P)}\,\mathbb{O}(P) = \mathbb{O}\left(e^{\xi P_0} F(\xi)\right)\mathbb{O}(P)$.
This is an ODE for F. Setting inductively $F_k = F_{k-1} + \epsilon^k \varphi$ we find that $F_0 = 1$ and solve for φ.

Exp

In[∘]:=
```
(* Bug: The first line is valid only if 𝕆(ℯ^P0)==ℯ^𝕆(P0). *)
(* Bug: ξ must be a symbol. *)
Exp_{U_i_,0}[ξ_, P_] := 𝔼_U[{y_i, a_i, x_i}_i, Normal@P /. ϵ → 0, 1 + 0_0];
Exp_{U_i_,k_}[ξ_, P_] := Module[{yax = {y_i, a_i, x_i}, P0, φ, φs, F, j, rhs, at0, atξ},
  P0 = Normal@P /. ϵ → 0;
  φs =
   Flatten@Table[φ_{j1,j2,j3}[ξ], {j2, 0, k}, {j1, 0, 2 k + 1 - j2}, {j3, 0, 2 k + 1 - j2 - j1}];
  F = Normal@Last@Exp_{U_i,k-1}[ξ, P] + ϵ^k φs.(φs /. φ_js__[ξ] ⧴ Times @@ yax^{js});
  rhs = Normal@Last@m_{i,j→i}[𝔼_U[yax_i, ξ P0, F + 0_k] m_{i→j}@𝔼_U[{y_i, a_i, x_i}_i, 0, P + 0_k]];
  at0 = (# == 0) & /@ Flatten@CoefficientList[F - 1 /. ξ → 0, yax];
  atξ = (# == 0) & /@ Flatten@CoefficientList[(∂_ξ F) + P0 F - rhs, yax];
  𝔼_U[yax_i, ξ P0, F + 0_k] /. DSolve[And @@ (at0 ⋃ atξ), φs, ξ][[1]] ]
```

In[∘]:=
```
Timing@Block[{$p = 4, $k = 2}, {
   s = S_1[QU[y_1]] /. QU → Times,
   exps = Exp_{QU_1,$k}[η, s], (* Warning: wrong unless $p≥$k+1! *)
   HL@Simp[S_1@𝕆_QU[{y_1}_1, SS[ℯ^{ℏ η y_1}]] - QU@(exps /. η → ℏ η)]
  }]
```

Out[∘]= $\left\{35.8281, \left\{a_1\left(-\frac{\epsilon\hbar}{T_1}+\frac{\gamma\epsilon^2\hbar^2}{T_1}\right)y_1+\left(-\frac{1}{T_1}+\frac{\gamma\epsilon\hbar}{T_1}-\frac{\gamma^2\epsilon^2\hbar^2}{2T_1}\right)y_1-\frac{\epsilon^2\hbar^2a_1^2y_1}{2T_1},\right.\right.$
$\mathbb{E}_{QU}\left[\{y_1,a_1,x_1\}_1,-\frac{\eta y_1}{T_1},1+\frac{\left(2\gamma\eta\hbar T_1y_1-2\eta\hbar a_1T_1y_1-\gamma\eta^2\hbar y_1^2\right)\epsilon}{2T_1^2}+\right.$
$\left(-\frac{\gamma^2\eta\hbar^2y_1}{2T_1}+\frac{\gamma\eta\hbar^2a_1y_1}{T_1}-\frac{\eta\hbar^2a_1^2y_1}{2T_1}+\frac{7\gamma^2\eta^2\hbar^2y_1^2}{4T_1^2}-\frac{2\gamma\eta^2\hbar^2a_1y_1^2}{T_1^2}+\right.$
$\left.\left.\left.\frac{\eta^2\hbar^2a_1^2y_1^2}{2T_1^2}-\frac{\gamma^2\eta^3\hbar^2y_1^3}{T_1^3}+\frac{\gamma\eta^3\hbar^2a_1y_1^3}{2T_1^3}+\frac{\gamma^2\eta^4\hbar^2y_1^4}{8T_1^4}\right)\epsilon^2+O[\epsilon]^3\right],0\right\}\right\}$

In[∘]:=
```
Timing@Block[{$p = 4, $k = 2}, {
   s = S_1[QU[a_1]] /. QU → Times,
   exps = Exp_{QU_1,$k}[α, s], (* Warning: wrong unless $p≥$k+1! *)
   HL@Simp[S_1@𝕆_QU[{a_1}_1, SS[ℯ^{ℏ α a_1}]] - QU@(exps /. α → ℏ α)]
  }]
```

Out[∘]= $\left\{33.5938, \left\{-a_1, \mathbb{E}_{QU}\left[\{y_1,a_1,x_1\}_1,-\alpha a_1,1+O[\epsilon]^3\right],0\right\}\right\}$

```
In[]:= Timing@Block[{$p = 4, $k = 2}, {
    s = S₁[QU[x₁]] /. QU → Times,
    exps = Exp_{QU₁,$k}[ξ, s], (* Warning: wrong unless $p≥$k+1! *)
    HL@Simp[S₁@O_QU[{x₁}₁, SS[e^{ħ ξ x₁}]] - QU@(exps /. ξ → ħ ξ)]
  }]
```

Out[]= $\{34.0625, \{-x_1 - \epsilon\hbar a_1 x_1 - \frac{1}{2}\epsilon^2\hbar^2 a_1^2 x_1,$
$\mathbb{E}_{QU}[\{y_1, a_1, x_1\}_1, -\xi x_1, 1 + \left(-\xi\hbar a_1 x_1 - \frac{1}{2}\gamma\xi^2\hbar x_1^2\right)\epsilon + \left(-\frac{1}{2}\xi\hbar^2 a_1^2 x_1 + \frac{1}{4}\gamma^2\xi^2\hbar^2 x_1^2 - \gamma\xi^2\hbar^2 a_1 x_1^2 + \frac{1}{2}\xi^2\hbar^2 a_1^2 x_1^2 - \frac{1}{2}\gamma^2\xi^3\hbar^2 x_1^3 + \frac{1}{2}\gamma\xi^3\hbar^2 a_1 x_1^3 + \frac{1}{8}\gamma^2\xi^4\hbar^2 x_1^4\right)\epsilon^2 + O[\epsilon]^3], 0\}\}$

$S(e^{\eta y}\, e^{\alpha a}\, e^{\xi x})$

```
In[]:= Timing@Block[{$p = 3, $k = 1}, {
    sexp = m_{3,2,1→1}[Exp_{QU₁,$k}[η, S₁[QU[y₁]] /. QU → Times] Exp_{QU₂,$k}[α, S₂[QU[a₂]] /. QU → Times]
        Exp_{QU₃,$k}[ξ, S₃[QU[x₃]] /. QU → Times]] /. {η → ħ η, α → ħ α, ξ → ħ ξ},
    HL@SimpT[QU@sexp - S₁@O_QU[{y₁, a₁, x₁}₁, SS[e^{ħ (η y₁+α a₁+ξ x₁)}]]]
  }]
```

Out[]= $\{9.34375,$
$\{\mathbb{E}_{QU}[\{y_1, a_1, x_1\}_1, \frac{1}{\hbar T_1}\left(e^{\alpha\gamma\hbar}\eta\xi\hbar^2 - e^{\alpha\gamma\hbar}\eta\xi\hbar^2 T_1 - \alpha\hbar^2 a_1 T_1 - e^{\alpha\gamma\hbar}\xi\hbar^2 T_1 x_1 - e^{\alpha\gamma\hbar}\eta\hbar^2 y_1\right),$
$1 + \frac{1}{4\hbar T_1^2}\left(-3e^{2\alpha\gamma\hbar}\gamma\eta^2\xi^2\hbar^4 - 4e^{\alpha\gamma\hbar}\gamma\eta\xi\hbar^3 T_1 + 4e^{2\alpha\gamma\hbar}\gamma\eta^2\xi^2\hbar^4 T_1 + 8e^{\alpha\gamma\hbar}\eta\xi\hbar^3 a_1 T_1 + 4e^{\alpha\gamma\hbar}\gamma\eta\xi\hbar^3 T_1^2 - e^{2\alpha\gamma\hbar}\gamma\eta^2\xi^2\hbar^4 T_1^2 + 6e^{2\alpha\gamma\hbar}\gamma\eta\xi^2\hbar^4 T_1 x_1 - 2e^{2\alpha\gamma\hbar}\gamma\eta\xi^2\hbar^4 T_1^2 x_1 - 4e^{\alpha\gamma\hbar}\xi\hbar^3 a_1 T_1^2 x_1 - 2e^{2\alpha\gamma\hbar}\gamma\xi^2\hbar^4 T_1^2 x_1^2 + 6e^{2\alpha\gamma\hbar}\gamma\eta^2\xi\hbar^4 y_1 + 4e^{\alpha\gamma\hbar}\gamma\eta\hbar^3 T_1 y_1 - 2e^{2\alpha\gamma\hbar}\gamma\eta^2\xi\hbar^4 T_1 y_1 - 4e^{\alpha\gamma\hbar}\eta\hbar^3 a_1 T_1 y_1 - 4e^{2\alpha\gamma\hbar}\gamma\eta\xi\hbar^4 T_1 x_1 y_1 - 2e^{2\alpha\gamma\hbar}\gamma\eta^2\hbar^4 y_1^2\right)\epsilon + O[\epsilon]^2], 0\}\}$

LinearLambda

```
In[]:= Timing@Block[{$p = 3, $k = 1}, {
    (sexp = m_{3,2,1→1}[Exp_{QU₁,$k}[η, S₁[QU[y₁]] /. QU → Times] Exp_{QU₂,$k}[α, S₂[QU[a₂]] /. QU → Times]
        Exp_{QU₃,$k}[ξ, S₃[QU[x₃]] /. QU → Times]]) /. u_₁ :→ u,
    HL@SimpT[QU@(sexp /. {η → ħ η, α → ħ α, ξ → ħ ξ}) -
      S₁@O_QU[{y₁, a₁, x₁}₁, SS[e^{ħ (η y₁+α a₁+ξ x₁)}]]]
  }]
```

LinearLambda

Out[]= $\{15.2969, \{\mathbb{E}_{QU}[\{y_1, a_1, x_1\}, \frac{1}{T\hbar}\left(e^{\alpha\gamma}\eta\xi - e^{\alpha\gamma}T\eta\xi - aT\alpha\hbar - e^{\alpha\gamma}y\eta\hbar - e^{\alpha\gamma}Tx\xi\hbar\right),$
$1 + \frac{1}{4T^2\hbar}\left(-3e^{2\alpha\gamma}\gamma\eta^2\xi^2 + 4e^{2\alpha\gamma}T\gamma\eta^2\xi^2 - e^{2\alpha\gamma}T^2\gamma\eta^2\xi^2 + 8ae^{\alpha\gamma}T\eta\xi\hbar - 4e^{\alpha\gamma}T\gamma\eta\xi\hbar + 4e^{\alpha\gamma}T^2\gamma\eta\xi\hbar + 6e^{2\alpha\gamma}y\gamma\eta^2\xi\hbar - 2e^{2\alpha\gamma}Ty\gamma\eta^2\xi\hbar + 6e^{2\alpha\gamma}Tx\gamma\eta\xi^2\hbar - 2e^{2\alpha\gamma}T^2x\gamma\eta\xi^2\hbar - 4ae^{\alpha\gamma}Ty\eta\hbar^2 + 4e^{\alpha\gamma}Ty\gamma\eta\hbar^2 - 2e^{2\alpha\gamma}y^2\gamma\eta^2\hbar^2 - 4ae^{\alpha\gamma}T^2x\xi\hbar^2 - 4e^{2\alpha\gamma}Txy\gamma\eta\xi\hbar^2 - 2e^{2\alpha\gamma}T^2x^2\gamma\xi^2\hbar^2\right)\epsilon + O[\epsilon]^2], 0\}\}$

$\Delta_{1\to 1,2}\left(e^{\eta y_1}\, e^{\alpha a_1}\, e^{\xi x_1}\right)$

In[∘]:=
```
Timing@Block[{$p = 3, $k = 2}, {
   s = Δ1→1,2[QU[y1]] /. QU → Times,
   exps = Prepend[{y2}2]@ExpQU1,$k[η, s], (* Warning: wrong unless $p≥$k+1! *)
   HL@Simp[Δ1→1,2@𝕆QU[{y1}1, SS[e^(ħ η y1)]] - QU@(exps /. η → ħ η)]
  }]
```

Out[∘]= $\{35.9531, \{y_1 + T_1 y_2 - \epsilon\hbar a_1 T_1 y_2 + \frac{1}{2}\epsilon^2\hbar^2 a_1^2 T_1 y_2,$
$\mathbb{E}_{QU}[\{y_2\}_2, \{y_1, a_1, x_1\}_1, \eta(y_1 + T_1 y_2), 1 + \left(-\eta\hbar a_1 T_1 y_2 + \frac{1}{2}\gamma\eta^2\hbar T_1 y_1 y_2\right)\epsilon +$
$\left(\frac{1}{2}\hbar^2 a_1^2 T_1 y_2\left(\eta + \eta^2 T_1 y_2\right) + \frac{1}{2} a_1 y_1\left(-\gamma\eta^2\hbar^2 T_1 y_2 - \gamma\eta^3\hbar^2 T_1^2 y_2^2\right) + \frac{1}{12} y_1\right.$
$\left.\left(3\gamma^2\eta^2\hbar^2 T_1 y_2 + 2\gamma^2\eta^3\hbar^2 T_1^2 y_2^2\right) + \frac{1}{24} y_1^2\left(4\gamma^2\eta^3\hbar^2 T_1 y_2 + 3\gamma^2\eta^4\hbar^2 T_1^2 y_2^2\right)\right)\epsilon^2 + O[\epsilon]^3], 0\}\}$

In[∘]:=
```
Timing@Block[{$p = 3, $k = 2}, {
   s = Δ1→1,2[QU[a1]] /. QU → Times,
   exps = Prepend[{a2}2]@ExpQU1,$k[α, s], (* Warning: wrong unless $p≥$k+1! *)
   HL@Simp[Δ1→1,2@𝕆QU[{a1}1, SS[e^(ħ α a1)]] - QU@(exps /. α → ħ α)]
  }]
```

Out[∘]= $\{30.5313, \{a_1 + a_2, \mathbb{E}_{QU}[\{a_2\}_2, \{y_1, a_1, x_1\}_1, \alpha(a_1 + a_2), 1 + O[\epsilon]^3], 0\}\}$

In[∘]:=
```
Timing@Block[{$p = 3, $k = 2}, {
   s = Δ1→1,2[QU[x1]] /. QU → Times,
   exps = Prepend[{x2}2]@ExpQU1,$k[ξ, s], (* Warning: wrong unless $p≥$k+1! *)
   HL@Simp[Δ1→1,2@𝕆QU[{x1}1, SS[e^(ħ ξ x1)]] - QU@(exps /. ξ → ħ ξ)]
  }]
```

Out[∘]= $\{34.2656, \{x_1 + x_2 - \epsilon\hbar a_1 x_2 + \frac{1}{2}\epsilon^2\hbar^2 a_1^2 x_2, \mathbb{E}_{QU}[\{x_2\}_2, \{y_1, a_1, x_1\}_1, \xi(x_1 + x_2),$
$1 + \left(-\xi\hbar a_1 x_2 + \frac{1}{2}\gamma\xi^2\hbar x_1 x_2\right)\epsilon + \left(\frac{1}{2}\hbar^2 a_1^2 x_2\left(\xi + \xi^2 x_2\right) - \frac{1}{2}\gamma\hbar^2 a_1 x_1 x_2\left(\xi^2 + \xi^3 x_2\right) +\right.$
$\left.\frac{1}{12}\gamma^2\hbar^2 x_1 x_2\left(3\xi^2 + 2\xi^3 x_2\right) + \frac{1}{24}\gamma^2\hbar^2 x_1^2 x_2\left(4\xi^3 + 3\xi^4 x_2\right)\right)\epsilon^2 + O[\epsilon]^3], 0\}\}$

LinearLambda

In[∘]:=
```
Timing@Block[{$p = 4, $k = 2}, {
    sexp = m1,3,5→1@m2,4,6→2@Times[
        Prepend[{y2}2]@ExpQU1,$k[η, Δ1→1,2[QU[y1]] /. QU → Times],
        Prepend[{a4}4]@ExpQU3,$k[α, Δ3→3,4[QU[a3]] /. QU → Times],
        Prepend[{x6}6]@ExpQU5,$k[ξ, Δ5→5,6[QU[x5]] /. QU → Times]
      ] /. {η → ħ η, α → ħ α, ξ → ħ ξ},
    HL@SimpT[QU@sexp - Δ1→1,2@OQU[{y1, a1, x1}1, SS[e^(ħ (η y1+α a1+ξ x1))]]]
  }]
```

LinearLambda

Out[∘]= $\{162., \{\mathbb{E}_{QU}[\{y_2, a_2, x_2\}_2, \{y_1, a_1, x_1\}_1, \alpha\hbar a_1 + \alpha\hbar a_2 + \xi\hbar x_1 + \xi\hbar x_2 + \eta\hbar y_1 + \eta\hbar T_1 y_2,$
$1 + \frac{1}{2}\left(-2\xi\hbar^2 a_1 x_2 + \gamma\xi^2\hbar^3 x_1 x_2 - 2\eta\hbar^2 a_1 T_1 y_2 + \gamma\eta^2\hbar^3 T_1 y_1 y_2\right)\epsilon +$
$\frac{1}{24}\left(12\xi\hbar^3 a_1^2 x_2 + 6\gamma^2\xi^2\hbar^4 x_1 x_2 - 12\gamma\xi^2\hbar^4 a_1 x_1 x_2 + 4\gamma^2\xi^3\hbar^5 x_1^2 x_2 + 12\xi^2\hbar^4 a_1^2 x_2^2 +\right.$
$4\gamma^2\xi^3\hbar^5 x_1 x_2^2 - 12\gamma\xi^3\hbar^5 a_1 x_1 x_2^2 + 3\gamma^2\xi^4\hbar^6 x_1^2 x_2^2 + 12\eta\hbar^3 a_1^2 T_1 y_2 +$
$24\eta\xi\hbar^4 a_1^2 T_1 x_2 y_2 - 12\gamma\eta\xi^2\hbar^5 a_1 T_1 x_1 x_2 y_2 + 6\gamma^2\eta^2\hbar^4 T_1 y_1 y_2 - 12\gamma\eta^2\hbar^4 a_1 T_1 y_1 y_2 -$
$12\gamma\eta^2\xi\hbar^5 a_1 T_1 x_2 y_1 y_2 + 6\gamma^2\eta^2\xi^2\hbar^6 T_1 x_1 x_2 y_1 y_2 + 4\gamma^2\eta^3\hbar^5 T_1 y_1^2 y_2 + 12\eta^2\hbar^4 a_1^2 T_1^2 y_2^2 +$
$\left.4\gamma^2\eta^3\hbar^5 T_1^2 y_1 y_2^2 - 12\gamma\eta^3\hbar^5 a_1 T_1^2 y_1 y_2^2 + 3\gamma^2\eta^4\hbar^6 T_1^2 y_1^2 y_2^2\right)\epsilon^2 + O[\epsilon]^3], 0\}\}$

Zip and Bind

E

In[∘]:=
```
𝔼 /: 𝔼[L1_, Q1_, P1_] 𝔼[L2_, Q2_, P2_] := 𝔼[L1 + L2, Q1 + Q2, P1 * P2];
```

Zip

In[∘]:=
```
{t*, y*, a*, x*, z*} = {τ, η, α, ξ, ζ};
{τ*, η*, α*, ξ*, ζ*} = {t, y, a, x, z};  (u_i_)* := (u*)i;
```

Zip

In[∘]:=
```
Zip{}[P_] := P;  Zip{ζ_,ζs___}[P_]  :=  (Expand[P // Zip{ζs}] /. f_. ζ^d_. ⧴ ∂{ζ*,d} f)  /. ζ* → 0
```

QZip implements the “Q-level zips” on $\mathbb{E}(L, Q, P) = Pe^{L+Q}$. Such zips regard the L variables as scalars.

Zip

In[∘]:=
```
𝔼 /: QZipζs_List@𝔼[L_, Q_, P_]  := Module[{ζ, z, zs, c, ys, ηs, qt, zrule, Q1, Q2},
   zs = Table[ζ*, {ζ, ζs}];
   c = Q /. Alternatives @@ (ζs ⋃ zs) → 0;
   ys = Table[∂ζ (Q /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = Table[∂z (Q /. Alternatives @@ ζs → 0), {z, zs}];
   qt = Inverse@Table[Kδz,ζ* - ∂z,ζ Q, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → qt.(zs + ys)];
   Q1 = c + ηs.zs /. zrule;
   Q2 = Q1 /. Alternatives @@ zs → 0;
   Simplify /@ 𝔼[L, Q2, Det[qt] e^-Q2 Zipζs[e^Q1 (P /. zrule)]] ];
```

In[∘]:=
```
Timing@{E0 = 𝔼[0, Sum[a10 i+j xi ξj, {i, 3}, {j, 3}],
    1 + ϵ Sum[fi[x1, x2, x3] ξi, {i, 3}] + ϵ Sum[f10 i+j[x1, x2, x3] ξi ξj, {i, 3}, {j, 3}]],
  lhs = QZip{ξ1,ξ2}@E0,
  HL[lhs ⩵ QZip{ξ1}@QZip{ξ2}@E0]}
```

Out[∘]=
```
{38.6875, {𝔼[0, … 1 …, 1 + ϵ (ξ1 … 1 … + … 1 … + … 1 …) +
    ϵ (ξ1² f11[x1, x2, x3] + … 7 … + ξ3² f33[x1, x2, x3])], … 1 …, True}}
large output   show less   show more   show all   set size limit...
```

In[∘]:=
```
Timing@{
  Eh = 𝔼[0, h Sum[a10 i+j xi ξj, {i, 3}, {j, 3}],
    1 + ϵ Sum[fi[x1, x2, x3] ξi, {i, 3}] + ϵ Sum[f10 i+j[x1, x2, x3] ξi ξj, {i, 3}, {j, 3}]],
  lhs = Normal[Eh /. 𝔼[L_, Q_, P_] ⧴ Series[P e^(L+Q), {h, 0, 2}]] // Zip{ξ1},
  HL@Simplify[lhs == Normal[QZip{ξ1}[Eh] /. 𝔼[L_, Q_, P_] ⧴ Series[P e^(L+Q), {h, 0, 2}]]]}
```

Out[∘]=
```
{18.4375, {𝔼[0, h … 1 …, 1 + ϵ (ξ1 … 1 … + … 1 … + … 1 …) +
    ϵ (ξ1² f11[x1, x2, x3] + … 7 … + ξ3² f33[x1, x2, x3])], … 1 …, True}}
large output   show less   show more   show all   set size limit...
```

LZip implements the "L-level zips" on $\mathbb{E}(L, Q, P) = Pe^{L+Q}$. Such zips regard all of Pe^{Q} as a single"P". Here the z's are t and α and the ζ's are τ and a.

Zip

```
𝔼 /: LZipζs_List@𝔼[L_, Q_, P_] := Module[{ζ, z, zs, c, ys, ηs, lt, zrule, L1, L2},
   zs = Table[ζ*, {ζ, ζs}];
   c = L /. Alternatives @@ (ζs ⋃ zs) → 0;
   ys = Table[∂ζ (L /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = Table[∂z (L /. Alternatives @@ ζs → 0), {z, zs}];
   lt = Inverse@Table[Kδz,ζ* - ∂z,ζ L, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → lt.(zs + ys)];
   L1 = c + ηs.zs /. zrule;
   L2 = L1 /. Alternatives @@ zs → 0;
   (* Warning: The "/. Alternatives@@zs→0" in the line below may be fishy *)
   Simplify /@
    𝔼[L2, Q /. zrule /. Alternatives @@ zs → 0, Det[lt] e^-L2 Zipζs[e^L1 (P /. zrule)]]];
```

Bind

In[∘]:=
```
Bind{is___Integer}[L_𝔼, R_𝔼] := Module[{n},
   Times[
     L /. Table[(v : t | a | x | y)i → vn@i, {i, {is}}],
     R /. Table[(v : τ | α | ξ | η)i → vn@i, {i, {is}}]
    ] // LZipFlatten@Table[{τn@i,an@i},{i,{is}}] // QZipFlatten@Table[{ξn@i,yn@i},{i,{is}}]
   ];
Bind[ℰ_𝔼] := ℰ;
Bind[Ls__, ζs_List, R_] := Bindζs[Bind[Ls], R];
```

In[∘]:= $\text{Bind}_{\{2\}}[\mathbb{E}[0, \xi\,(x_1 + x_2), 1], \mathbb{E}[0, \xi_2\,(x_2 + x_3), 1]]$

Out[∘]= $\mathbb{E}[0, \xi\,(x_1 + x_2 + x_3), 1]$

In[∘]:= $\text{Bind}_{\{2\}}[\mathbb{E}[0, (\xi_2 + \xi_3)\,x_2, 1], \mathbb{E}[0, (\xi_1 + \xi_2)\,x, 1]]$

Out[∘]= $\mathbb{E}[0, x\,(\xi_1 + \xi_2 + \xi_3), 1]$

In[∘]:= $\text{Bind}_{\{1,2\}}[\mathbb{E}[0, (\xi_2 + \xi_3)\,x_2 + \xi_1\,x_1, 1], \mathbb{E}[0, (\xi_1 + \xi_2)\,x, 1]]$

Out[∘]= $\mathbb{E}[0, x\,(\xi_1 + \xi_2 + \xi_3), 1]$

An xay → axy → ayx –> yax ≡ xay → xya → yxa → yax test:

In[∘]:= $\text{Bind}\left[\mathbb{E}\left[\alpha_1\,a_1 + \tau_1\,t_1, e^{\gamma\,\alpha_1}\,\xi_1\,x_1 + \eta_1\,y_1, 1\right], \{1\}, \mathbb{E}[\tau_1\,t_1 + \alpha_1\,a_1, \xi_1\,x_1 + \eta_1\,y_1 + \xi_1\,\eta_1\,t_1, 1]\right]$

Out[∘]= $\mathbb{E}\left[a_1\,\alpha_1 + t_1\,\tau_1, y_1\,\eta_1 + e^{\gamma\,\alpha_1}\left(x_1 + t_1\,\eta_1\right)\xi_1, 1\right]$

In[∘]:= $\text{Column@}\{\text{Cord}[\mathbb{C}_{CU}[\{x_1, a_1\}_1, \xi_1\,x_1 + \alpha_1\,a_1, 1 + 0_0]],$
$\quad \text{Cord}[\mathbb{C}_{CU}[\{x_1, y_1\}_1, \xi_1\,x_1 + \eta_1\,y_1, 1 + 0_0]],$
$\quad \text{Cord}[\mathbb{C}_{CU}[\{a_1, y_1\}_1, \alpha_1\,a_1 + \eta_1\,y_1, 1 + 0_0]]\}$

Out[∘]=
$\mathbb{C}_{CU}\left[\{a_1, x_1\}_1, e^{-\gamma\,\alpha_1}\left(e^{\gamma\,\alpha_1}\,a_1\,\alpha_1 + x_1\,\xi_1\right), 1 + O[\epsilon]^1\right]$
$\mathbb{C}_{CU}\left[\{y_1, a_1, x_1\}_1, y_1\,\eta_1 + x_1\,\xi_1 - t_1\,\eta_1\,\xi_1, 1 + O[\epsilon]^1\right]$
$\mathbb{C}_{CU}\left[\{y_1, a_1\}_1, e^{-\gamma\,\alpha_1}\left(e^{\gamma\,\alpha_1}\,a_1\,\alpha_1 + y_1\,\eta_1\right), 1 + O[\epsilon]^1\right]$

In[∘]:= $\text{rxa} = \mathbb{E}\left[\tau_1\,t_1 + \alpha_1\,a_1, e^{-\gamma\,\alpha_1}\,\xi_1\,x_1 + \eta_1\,y_1, 1\right];$
$\text{rxy} = \mathbb{E}[\tau_1\,t_1 + \alpha_1\,a_1, \xi_1\,x_1 + \eta_1\,y_1 - \xi_1\,\eta_1\,t_1, 1];$
$\text{ray} = \mathbb{E}\left[\tau_1\,t_1 + \alpha_1\,a_1, e^{-\gamma\,\alpha_1}\,\eta_1\,y_1 + \xi_1\,x_1, 1\right];$
$\text{Bind}[\text{rxa}, \{1\}, \text{rxy}]$

Out[∘]= $\mathbb{E}\left[a_1\,\alpha_1 + t_1\,\tau_1, y_1\,\eta_1 + e^{-\gamma\,\alpha_1}\left(x_1 - t_1\,\eta_1\right)\xi_1, 1\right]$

In[∘]:= Expand /@ Bind[rxa, {1}, rxy, {1}, ray]

Out[∘]= $\mathbb{E}\left[a_1\,\alpha_1 + t_1\,\tau_1, e^{-\gamma\,\alpha_1}\,y_1\,\eta_1 + e^{-\gamma\,\alpha_1}\,x_1\,\xi_1 - e^{-\gamma\,\alpha_1}\,t_1\,\eta_1\,\xi_1, 1\right]$

In[∘]:= Expand /@ Bind[ray, {1}, rxy, {1}, rxa]

Out[∘]= $\mathbb{E}\left[a_1\,\alpha_1 + t_1\,\tau_1, e^{-\gamma\,\alpha_1}\,y_1\,\eta_1 + e^{-\gamma\,\alpha_1}\,x_1\,\xi_1 - e^{-\gamma\,\alpha_1}\,t_1\,\eta_1\,\xi_1, 1\right]$

Alternative Algorithms

AltLogos

In[∘]:=
$\lambda_{\text{alt},k\_}[\text{CU}] := \text{If}\Big[k == 0, 1, \text{Module}\Big[\{\text{eq}, d, b, c, \text{so}\},$
$\quad \text{eq} = \rho@e^{\xi\,x_{CU}}.\rho@e^{\eta\,y_{CU}} == \rho@e^{d\,y_{CU}}.\rho@e^{c\,(t\,1_{CU} - 2\,\epsilon\,a_{CU})}.\rho@e^{b\,x_{CU}};$
$\quad \{\text{so}\} = \text{Solve}[\text{Thread}[\text{Flatten}\,/@\,\text{eq}], \{d, b, c\}]\ /.\ C@1 \to 0;$
$\quad \text{Series}\left[e^{-\eta\,y - \xi\,x + \eta\,\xi\,t + c\,t + d\,y - 2\,\epsilon\,c\,a + b\,x}\ /.\ \text{so}, \{\epsilon, 0, k\}\right]\Big]\Big];$

```
{λalt,2[CU], HL@Simplify@Normal[λalt,2[CU] == Last[ΛCU,2[{ξ, η}, {x, y}]]]}
```

$$\left\{1+\left(2\,a\,\eta\,\xi-y\,\gamma\,\eta^2\,\xi-x\,\gamma\,\eta\,\xi^2+\frac{1}{2}\,t\,\gamma\,\eta^2\,\xi^2\right)\epsilon+\frac{1}{2}\left(\left(2\,a\,\eta\,\xi-y\,\gamma\,\eta^2\,\xi-x\,\gamma\,\eta\,\xi^2+\frac{1}{2}\,t\,\gamma\,\eta^2\,\xi^2\right)^2+2\left(-a\,\gamma\,\eta^2\,\xi^2+y\,\gamma^2\,\eta^3\,\xi^2+x\,\gamma^2\,\eta^2\,\xi^3-\frac{1}{3}\,t\,\gamma^2\,\eta^3\,\xi^3\right)\right)\epsilon^2+O[\epsilon]^3,\ \text{True}\right\}$$