Pensieve header: A unified verification notebook for the sl_2-portfolio project; continues pensieve://Projects/PPSA/nb/Verification.pdf.

Continues pensieve://2017-06/ and pensieve://2017-08/.

Prolog

```
Go;
```

```
wdir = SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\SL2Portfolio"];
NotebookOpen[wdir <> "\\MakeSnips.nb"];
```

In[∘]:=
```
HL[ℰ_] := Style[ℰ, Background → Yellow];
```

Initialization / Utilities

It is verification-risky to work with low $E!

TD

In[∘]:=
```
$p = 2; $k = 1; $E := {$k, $p};
$trim := {ℏ^p_. /; p > $p → 0, ϵ^k_. /; k > $k → 0};
SetAttributes[{SS, SST}, HoldAll];
TRule = {T_i_ → ⅇ^(ℏ t_i), T → ⅇ^(ℏ t)}; q_ℏ = ⅇ^(γ ϵ ℏ);
SS[ℰ_, op_] := Collect[
   Normal@Series[If[$p > 0, ℰ, ℰ /. TRule], {ℏ, 0, $p}],
   ℏ, op];
SS[ℰ_] := SS[ℰ, Together];
SST[ℰ_, op___] := SS[ℰ /. TRule, op];
Simp[ℰ_, op_] := Collect[ℰ, _CU | _QU, op];
Simp[ℰ_] := Simp[ℰ, SS[#, Expand] &];
SimpT[ℰ_] := Collect[ℰ, _CU | _QU, SST[#, Expand] &];
```

Differential polynomials (DP):

Utils

In[∘]:=
```
DP_{α_→D_x_, β_→D_y_}[P_][λ_] :=
 Total[CoefficientRules[Normal@P, {α, β}] /. ({m_, n_} → c_) :→ c ∂_{{x,m},{y,n}} λ]
```

```
HL[DP_{x→D_ξ, y→D_η}[x^2 y^3][ⅇ^(δ ξ η)] == 6 ⅇ^(δ η ξ) δ^3 ξ + 6 ⅇ^(δ η ξ) δ^4 η ξ^2 + ⅇ^(δ η ξ) δ^5 η^2 ξ^3]
```

True

CF

In[∘]:=
```
CF[ℰ_] := ExpandDenominator@
   ExpandNumerator@Together[Expand[ℰ] //. ⅇ^x_ ⅇ^y_ :→ ⅇ^(x+y) /. ⅇ^x_ :→ ⅇ^CF[x]];
```

SeriesData

In[ ]:=
```
Unprotect[SeriesData];
SeriesData /: CF[sd_SeriesData] := MapAt[CF, sd, 3];
SeriesData /: Expand[sd_SeriesData] := MapAt[Expand, sd, 3];
SeriesData /: Simplify[sd_SeriesData] := MapAt[Simplify, sd, 3];
SeriesData /: Together[sd_SeriesData] := MapAt[Together, sd, 3];
SeriesData /: Collect[sd_SeriesData, specs__] := MapAt[Collect[#, specs] &, sd, 3];
Protect[SeriesData];
```

Self-Pair (SP):

SP

In[ ]:=
```
SP{}[P_] := P; SP{ξ_→x_,ps___}[P_] := Expand[P // SP{ps}] /. f_. ξ^d_. ⇒ ∂{x,d} f
```

$SP_{\{\xi\to x\}}\left[\left(\xi^2+\xi+3\right)\left(x^5 e^x+7x\right)+99a\right]$

$7+99a+21x+20e^x x^3+15e^x x^4+5e^x x^5$

$SP_{\{\xi\to x,\eta\to y\}}\left[\left(\xi^2+\xi+3+2\xi\eta\right)\left(x^5 e^x+7x\right)+99a+e^{\delta x y}\xi\eta\right]$

$7+99a+21x+20e^x x^3+15e^x x^4+5e^x x^5+e^{x y \delta}\delta+e^{x y \delta}x y \delta^2$

DeclareAlgebra

QLImplementation

In[ ]:=
```
Unprotect[NonCommutativeMultiply]; Attributes[NonCommutativeMultiply] = {};
(NCM = NonCommutativeMultiply)[x_] := x;
NCM[x_, y_, z__] := (x ** y) ** z;
0 ** _ = _ ** 0 = 0;
(x_Plus) ** y_ := (# ** y) & /@ x; x_ ** (y_Plus) := (x ** #) & /@ y;
B[x_, x_] = 0; B[x_, y_] := x ** y - y ** x;
B[x_, y_, e_] := B[x, y, e] = B[x, y];
```

QLImplementation

In[•]:=

```
DeclareAlgebra[U_Symbol, opts__Rule] := Module[{gp, sr, g, cp, M, CE, pow, k = 0,
    gs = Generators /. {opts},
    cs = Centrals /. {opts} /. Centrals → {} },
   (#_U = U@#) & /@ gs;
   gp = Alternatives @@ gs; gp = gp | gp_; (* gens *)
   sr = Flatten@Table[{g → ++k, g_i_ → {i, k}}, {g, gs}]; (* sorting → *)
   cp = Alternatives @@ cs; (* cents *)
   SetAttributes[M, HoldRest]; M[0, _] = 0; M[a_, x_] := a x;
   CE[ℰ_] := Collect[ℰ, _U, Expand] /. $trim;
   U_i_ [ℰ_] := ℰ /. {t : cp ⧴ t_i, u_U ⧴ (#_i &) /@ u};
   U_i_ [NCM[]] = pow[ℰ_, 0] = U@{} = 1_U = U[];
   B[U@(x_)_i_, U@(y_)_i_] := U_i@B[U@x, U@y];
   B[U@(x_)_i_, U@(y_)_j_] /; i =!= j := 0;
   B[U@y_, U@x_] := CE[-B[U@x, U@y]];
   x_ ** (c_. 1_U) := CE[c x]; (c_. 1_U) ** x_ := CE[c x];
   (a_. U[xx___, x_]) ** (b_. U[y_, yy___]) := If[OrderedQ[{x, y} /. sr],
     CE@M[a b /. $trim, U[xx, x, y, yy]],
     U@xx ** CE@M[a b /. $trim, U@y ** U@x + B[U@x, U@y, $E]] ** U@yy ];
   U@{c_. * (l : gp)^n_, r___} /; FreeQ[c, gp] := CE[c U@Table[l, {n}] ** U@{r}];
   U@{c_. * l : gp, r___} := CE[c U[l] ** U@{r}];
   U@{c_, r___} /; FreeQ[c, gp] := CE[c U@{r}];
   U@{l_Plus, r___} := CE[U@{#, r} & /@ l];
   U@{l_, r___} := U@{Expand[l], r};
   U[ℰ_NonCommutativeMultiply] := U /@ ℰ;
   𝕆_U[specs___, poly_] := Module[{sp, null, vs, us},
     sp = Replace[{specs}, l_List ⧴ l_null, {1}];
     vs = Join @@ (First /@ sp);
     us = Join @@ (sp /. l_s_ ⧴ (l /. x_i_ ⧴ x_s));
     CE[Total[
        CoefficientRules[poly, vs] /. (p_ → c_) ⧴ c U@(us^p)
       ]] /. x_null ⧴ x];
   pow[ℰ_, n_] := pow[ℰ, n - 1] ** ℰ;
   𝕊_U[ℰ_, ss___Rule] := CE@Total[
      CoefficientRules[ℰ, First /@ {ss}] /.
       (p_ → c_) ⧴ c NCM @@ MapThread[pow, {Last /@ {ss}, p}]];
   m_j_→k_ [c_. * u_U] := CE[((c /. (t : cp)_j → t_k) DeleteCases[u, _j|k]) **
      U @@ Cases[u, w_j ⧴ w_k] ** U @@ Cases[u, _k]];
   U /: c_. * u_U * v_U := CE[c u ** v];
   S_i_ [c_. * u_U] := CE[((c /. S_i[U, Centrals]) DeleteCases[u, _i]) **
      U_i[NCM @@ Reverse@Cases[u, x_i ⧴ S@U@x]] ]
```

DeclareMorphism

QLImplementation

```
In[∘]:= DeclareMorphism[m_, U_ → V_, ongs_List, oncs_List: {}] := (
          Replace[ongs,
            {(g_ → img_) ⧴ (m[U[g]] = img), (g_ ⧴ img_) ⧴ (m[U[g]] := img /. $trim)}, {1}];
          m[1_U] = 1_V;
          m[U[g_i_]] := V_i[m[U@g]];
          m[U[vs__]] := NCM @@ (m /@ U /@ {vs});
          m[ℰ_] := Simp[ℰ /. oncs /. u_U ⧴ m[u]] /. $trim; )
```

Meta-Operations

QLImplementation

```
In[∘]:= m_{j_→j_} = Identity;
        m_{j_→k_}[ℰ_Plus] := Simp[m_{j→k} /@ ℰ];
        m_{is___,i_,j_→k_}[ℰ_] := m_{j→k}@m_{is,i→j}@ℰ;
        S_{i_}[ℰ_Plus] := Simp[S_i /@ ℰ];
```

Implementing CU = $\mathcal{U}(sl_2^{\gamma\epsilon})$

CU

```
In[∘]:= DeclareAlgebra[CU, Generators → {y, a, x}, Centrals → {t}];
        B[a_CU, y_CU] = -γ y_CU; B[x_CU, a_CU] = -γ x_CU;
        B[x_CU, y_CU] = 2 ϵ a_CU - t 1_CU;
        (S@CU@y = -y_CU; S@a_CU = -a_CU; S@x_CU = -x_CU;)
        S_{i_}[CU, Centrals] = {t_i → -t_i};
```

Verifying associativity on triples of generators:

```
With[{bas = CU /@ {y, a, x}},
 Table[z1 ** (z2 ** z3) - (z1 ** z2) ** z3 // Simp // HL,
  {z1, bas}, {z2, bas}, {z3, bas}] ]
```

```
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying associativity on a “random” triple:

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
   (rhs = (z1 ** z2) ** z3 // Simp) // Short,
   z1 ** (z2 ** z3) - rhs // Simp // HL
  }] // Timing
```

```
{1.32813,
 {(28 t^2 γ^4 + 116 t γ^5 ϵ) CU[y, y, y, x, x] + <<21>> + CU[y, y, y, y, y, a, a, a, a, x, x, x, x], 0}}
```

Verifying that *S* is an anti-homomorphism on CU:

```
With[{bas = CU /@ {y₁, a₁, x₁}},
 Table[HL@Simp[S₁[z1 ** z2] - S₁[z2] ** S₁[z1]],
  {z1, bas}, {z2, bas}]]
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

Verifying the involutivity of S on products of triples:

```
With[{bas = CU /@ {y₁, a₁, x₁}},
 Table[HL@Simp[z1 ** z2 ** z3 - S₁@S₁[z1 ** z2 ** z3]],
  {z1, bas}, {z2, bas}, {z3, bas}]]
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying relabeling:

```
t₁ t₃ CU[y₁, a₁, x₂] + t₁ t₁ CU[y₁, a₂, x₂] // m₁→₃
```

$\text{CU}[a_2, x_2, y_3]\, t_3^2 + \text{CU}[x_2, y_3, a_3]\, t_3^2$

Verifying meta-associativity:

```
Module[{z, u},
 Table[u = CU[z[[1]]₁, z[[2]]₂, z[[3]]₃]; z → HL[m₁,₃→₃@m₂,₃→₃@u == m₂,₃→₃@m₁,₂→₂@u],
  {z, Tuples[{y, a, x}, 3]}]]
{{y, y, y} → True, {y, y, a} → True, {y, y, x} → True, {y, a, y} → True,
 {y, a, a} → True, {y, a, x} → True, {y, x, y} → True, {y, x, a} → True,
 {y, x, x} → True, {a, y, y} → True, {a, y, a} → True, {a, y, x} → True, {a, a, y} → True,
 {a, a, a} → True, {a, a, x} → True, {a, x, y} → True, {a, x, a} → True, {a, x, x} → True,
 {x, y, y} → True, {x, y, a} → True, {x, y, x} → True, {x, a, y} → True, {x, a, a} → True,
 {x, a, x} → True, {x, x, y} → True, {x, x, a} → True, {x, x, x} → True}
```

Implementing QU = $\mathcal{U}_q(sl_2^{\gamma\epsilon})$

Aside

```
Series[(1 - T ⅇ^(-2 ϵ a ℏ)) / ℏ, {a, 0, 3}]
```

Aside

$$\frac{1-T}{\hbar} + 2\,T\,\epsilon\,a - 2\left(T\,\epsilon^2\,\hbar\right) a^2 + \frac{4}{3}\,T\,\epsilon^3\,\hbar^2\,a^3 + O[a]^4$$

QU

In[∘]:=

```
DeclareAlgebra[QU, Generators → {y, a, x}, Centrals → {t, T}];
B[a_QU, y_QU] = -γ y_QU; B[x_QU, a_QU] = -γ QU@x;
B[x_QU, y_QU] := SS[q_ℏ - 1] QU@{y, x} + 𝕆_QU[{a}, SS[(1 - T ⅇ^(-2 ϵ a ℏ)) / ℏ]];
(S@y_QU := 𝕆_QU[{a, y}, SS[-T^-1 ⅇ^(ℏ ϵ a) y]]; S@a_QU = -a_QU; S@x_QU := 𝕆_QU[{a, x}, SS[-ⅇ^(ℏ ϵ a) x]];)
S_i_[QU, Centrals] = {t_i → -t_i, T_i → T_i^-1};
```

```
With[{bas = QU /@ {y, a, x}}, Table[{z1, z2} → Simp[z1 ** z2 - z2 ** z1], {z1, bas}, {z2, bas}] ]
```

$\{\{\{QU[y], QU[y]\} \to 0, \{QU[y], QU[a]\} \to \gamma\, QU[y],$
$\{QU[y], QU[x]\} \to \frac{(-1+T)\, QU[]}{\hbar} - 2\, T \in QU[a] - \gamma \in \hbar\, QU[y, x]\},$
$\{\{QU[a], QU[y]\} \to -\gamma\, QU[y], \{QU[a], QU[a]\} \to 0, \{QU[a], QU[x]\} \to \gamma\, QU[x]\},$
$\{\{QU[x], QU[y]\} \to \frac{(1-T)\, QU[]}{\hbar} + 2\, T \in QU[a] + \gamma \in \hbar\, QU[y, x],$
$\{QU[x], QU[a]\} \to -\gamma\, QU[x], \{QU[x], QU[x]\} \to 0\}\}$

Verifying associativity on triples of generators:

```
With[{bas = QU /@ {y, a, x}},
 Table[HL[z1 ** (z2 ** z3) - (z1 ** z2) ** z3] // Simp,
  {z1, bas}, {z2, bas}, {z3, bas} ] ]
```

{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}

Verifying associativity on a “random” triple (~34 secs @ $p=5, $k=2):

```
With[{z1 = QU[y, y, a, a, x, x], z2 = QU[y, a, x], z3 = QU[y, y, a, x]}, {
   (rhs = (z1 ** z2) ** z3 // Simp) // Short,
   HL[z1 ** (z2 ** z3) - rhs // Simp ]
  }] // Timing
```

$\{3.78125, \{\left(\frac{28\,\gamma^4 - 56\,T\,\gamma^4 + 28\,T^2\,\gamma^4}{\hbar^2} + \frac{82\,\gamma^5 \in - 280\,T \ll 1 \gg \in + 198\,T^2\,\gamma^5 \in}{\hbar}\right) QU[y, y, y, x, x] +$
$\ll 18 \gg + (1 + 8\,\gamma \in \hbar)\, QU[y, \ll 11 \gg, x], 0\}\}$

Verifying that *S* is an anti-homomorphism on QU:

```
With[{bas = QU /@ {y₁, a₁, x₁}},
 Table[{z1, z2} → HL@Simp[S₁[z1 ** z2] - S₁[z2] ** S₁[z1]],
  {z1, bas}, {z2, bas} ] ]
```

$\{\{\{QU[y_1], QU[y_1]\} \to 0, \{QU[y_1], QU[a_1]\} \to 0, \{QU[y_1], QU[x_1]\} \to 0\},$
$\{\{QU[a_1], QU[y_1]\} \to 0, \{QU[a_1], QU[a_1]\} \to 0, \{QU[a_1], QU[x_1]\} \to 0\},$
$\{\{QU[x_1], QU[y_1]\} \to 0, \{QU[x_1], QU[a_1]\} \to 0, \{QU[x_1], QU[x_1]\} \to 0\}\}$

Verifying that $\mathrm{Lim}_{\hbar\to 0}$ QU = CU using a “random” product (~23 secs @ $p=5, $k=2):

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
   Short[lhs = z1 ** (z2 ** z3)],
   Short[rhs = (QU @@ z1) ** ((QU @@ z2) ** (QU @@ z3))],
   Expand[Limit[rhs /. TRule ⋃ {QU → CU}, ℏ → 0] - lhs] // HL
  }] // Timing
```

$\{10.125, \{28\,t^2\,\gamma^4\, CU[y, y, y, x, x] +$
$116\, t\,\gamma^5 \in CU[y, y, y, x, x] + \ll 44 \gg + CU[y, y, y, y, y, a, a, a, a, x, x, x, x],$
$2\left(\frac{\gamma^4}{\hbar^2} - \frac{2\,T\,\gamma^4}{\hbar^2} + \frac{T^2\,\gamma^4}{\hbar^2} + \frac{\gamma^5 \in}{\hbar} - \frac{2\,T\,\gamma^5 \in}{\hbar} + \frac{T^2\,\gamma^5 \in}{\hbar}\right) QU[y, y, y, x, x] +$
$\ll 209 \gg + (1 + 8\,\gamma \in \hbar)\, QU[y, y, y, \ll 7 \gg, x, x, x], 0\}\}$

Implementing θ

theta

In[ ]:=
```
DeclareMorphism[Cθ, CU → CU, {y → -x_CU, a → -a_CU, x → -y_CU}, {t → -t, T → T^-1}];
DeclareMorphism[Qθ, QU → QU, {y :→ 𝕆_QU[{a, x}, SS[-T^(-1/2) ℯ^(ℏ ϵ a) x]],
   a → -a_QU, x :→ 𝕆_QU[{a, y}, SS[-T^(-1/2) ℯ^(ℏ ϵ a) y]]}, {t → -t, T → T^-1}]
```

Verifying involutivity on CU:

```
With[{bas = CU /@ {y, a, x}},
 Table[z → Cθ[z] → HL[Cθ[Cθ[z]]], {z, bas}] ]
```

{CU[y] → -CU[x] → CU[y], CU[a] → -CU[a] → CU[a], CU[x] → -CU[y] → CU[x]}

Verifying that θ is a multiplicative homomorphism on CU:

```
With[{bas = CU /@ {y, a, x}},
 Table[Cθ[z1 ** z2] - Cθ[z1] ** Cθ[z2] // HL, {z1, bas}, {z2, bas}] ]
```

{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

Verifying involutivity on QU:

```
With[{bas = QU /@ {y, a, x}},
 Table[z → Qθ[z] → HL[Simp[Qθ[Qθ[z]], PowerExpand]], {z, bas}] ]
```

$\{QU[y] \to -\frac{QU[x]}{\sqrt{T}} - \frac{\epsilon\hbar\, QU[a,x]}{\sqrt{T}} \to QU[y],\ QU[a] \to -QU[a] \to QU[a],$

$QU[x] \to \left(-\frac{1}{\sqrt{T}} + \frac{\gamma\epsilon\hbar}{\sqrt{T}}\right) QU[y] - \frac{\epsilon\hbar\, QU[y,a]}{\sqrt{T}} \to QU[x]\}$

Verifying that θ is a multiplicative homomorphism on QU:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL[Simp[Qθ[z1 ** z2] - Qθ[z1] ** Qθ[z2]]], {z1, bas}, {z2, bas}] ]
```

{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}

The Asymmetric Dequantizator

Following pensieve://People/VanDerVeen/Dequant1.pdf.

ADeq

In[ ]:=

$$\text{A}\mathbb{D}\$f = \gamma\left(\left(\text{Cosh}\left[\hbar\left(a\,\epsilon + \frac{\gamma\,\epsilon}{2} - \frac{t}{2}\right)\right] - \text{Cosh}\left[\hbar\sqrt{\left(\frac{t-\gamma\,\epsilon}{2}\right)^2 + \epsilon\,\omega}\right]\right)\Big/\left(\hbar\, e^{\hbar\,((a+\gamma)\,\epsilon - t/2)}\,\text{Sinh}\left[\frac{\gamma\,\epsilon\,\hbar}{2}\right]\left(a^2\,\epsilon + a\,\gamma\,\epsilon - a\,t - \omega\right)\right)\right);$$

Scaling behaviour of A𝔻$f:

```
HL@Simplify[A𝔻$f == ((A𝔻$f /. γ → 1) /. {ϵ → γ ϵ, a → γ^-1 a, ω → γ^-1 ω})]
```

True

```
HL@FullSimplify[
  A𝔻$f == ((A𝔻$f /. γ → 1) /. {ℏ → γ^2 ℏ, ϵ → ϵ / γ, a → a / γ, t → γ^-2 t, ω → γ^-3 ω})]
```

True

ADeq

```
In[∘]:= A𝔻$ω = γ CU[y, x] + ϵ CU[a, a] - (t - γ ϵ) CU[a];
```

ADeq

```
In[∘]:= DeclareMorphism[A𝔻, QU → CU,
  {a → a_CU, x → CU@x, y :→ 𝕊_CU[SS[A𝔻$f], a → a_CU, ω → A𝔻$ω] ** y_CU}]
```

Verifying that the asymmetric dequantizator is a homomorphism:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL[SimpT[A𝔻[z1 ** z2] - A𝔻[z1] ** A𝔻[z2]]], {z1, bas}, {z2, bas}] ]
```

```
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

The Symmetric Dequantizator

Following pensieve://People/VanDerVeen/Dequant1.pdf.

SDeq

In[∘]:= $$S\mathbb{D}\$g = \sqrt{\left(\left(2\gamma\left(\mathrm{Cosh}\left[\frac{\hbar}{2}\sqrt{t^2+\gamma^2\epsilon^2+4\epsilon\varpi}\right]-\mathrm{Cosh}\left[\frac{t-\epsilon\gamma-2\epsilon a}{2/\hbar}\right]\right)\right)\Big/\left(\mathrm{Sinh}\left[\frac{\gamma\epsilon\hbar}{2}\right]\left(t\left(2a+\gamma\right)-2a\left(a+\gamma\right)\epsilon+2\varpi\right)\hbar\right)\right)};$$

Verify agreement with the formulas in pensieve://People/VanDerVeen/Dequant1.pdf:

```
{SD$P = (Cosh[ℏ ((ϵ-t)/2 + ϵ a)] - Cosh[ℏ Sqrt[(t^2+ϵ^2)/4 + ϵ ϖ]]) / (ℏ Sinh[-ϵ ℏ/2] (ϖ - ϵ a^2 + (t - ϵ) a + t/2)),
 Simplify[SD$P == (SD$P /. {a → -a - 1, t → -t})] // HL,
 PowerExpand@Simplify[(SD$P /. {ℏ → γ^2 ℏ, ϵ → ϵ/γ, a → a/γ, t → γ^-2 t, ϖ → γ^-3 ϖ}) ==
    SD$g (SD$g /. {a → -a - γ, t → -t})] // HL,
 SD$Q = Simplify[SD$P /. {a → c - 1/2}],
 Simplify[SD$Q == (SD$Q /. {c → -c, t → -t})] // HL,
 FullSimplify[SD$g == FullSimplify[
    Sqrt[SD$Q] /. c → a + 1/2 /. {ℏ → γ^2 ℏ, ϵ → ϵ/γ, a → a/γ, t → γ^-2 t, ϖ → γ^-3 ϖ}]] // HL
}
```

$\{-\left(\left(\left(\mathrm{Cosh}\left[\left(a\,\epsilon+\frac{1}{2}(-t+\epsilon)\right)\hbar\right]-\mathrm{Cosh}\left[\sqrt{\frac{1}{4}(t^2+\epsilon^2)+\epsilon\,\varpi}\;\hbar\right]\right)\mathrm{Csch}\left[\frac{\epsilon\,\hbar}{2}\right]\right)\Big/\left(\left(\frac{t}{2}+a\,(t-\epsilon)-a^2\,\epsilon+\varpi\right)\hbar\right)\right)$, True, True,

$-\left(\left(4\left(\mathrm{Cosh}\left[\frac{1}{2}(t-2\,c\,\epsilon)\,\hbar\right]-\mathrm{Cosh}\left[\frac{1}{2}\sqrt{t^2+\epsilon^2+4\,\epsilon\,\varpi}\;\hbar\right]\right)\mathrm{Csch}\left[\frac{\epsilon\,\hbar}{2}\right]\right)\Big/\left(\left(4\,c\,t+\epsilon-4\,c^2\,\epsilon+4\,\varpi\right)\hbar\right)\right)$, True, True$\}$

SDeq

In[∘]:=
```
SD$f = Simplify[e^(ℏ (t/2-ϵ a)) (SD$g /. {a → -a, t → -t})];
```

SDeq

In[∘]:=
```
SD$ϖ = γ CU[y, x] + ϵ CU[a, a] - (t - γ ϵ) CU[a] - t γ 1_CU / 2;
```

SDeq

In[∘]:=
```
DeclareMorphism[SD, QU → CU, {a → a_CU,
  x :→ S_CU[SS[SD$f], a → a_CU, ϖ → SD$ϖ] ** x_CU,
  y :→ S_CU[SS[SD$g], a → a_CU, ϖ → SD$ϖ] ** y_CU }]
```

Verifying the θ-symmetry:

```
Table[HL@SimpT[Cθ[SD[z]] == SD[Qθ[z]]], {z, QU /@ {y, a, x}}]
```

{True, True, True}

Verifying that the symmetric dequantizator is a homomorphism:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL@SimpT[SD[z1 ** z2] - SD[z1] ** SD[z2]], {z1, bas}, {z2, bas}] ]
```

```
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

The representation ρ

rho

In[ ]:=
```
ρ@y_CU = ρ@y_QU = (0 0; ϵ 0); ρ@a_CU = ρ@a_QU = (ɣ 0; 0 0);
ρ@x_CU = (0 ɣ; 0 0); ρ@x_QU = (0 (1 - e^(-ɣ ϵ ℏ)) / (ϵ ℏ); 0 0);
ρ[e^ℰ_] := MatrixExp[ρ[ℰ]];
ρ[ℰ_] := (ℰ /. TRule /. t → ɣ ϵ /. (U : CU | QU) [u___] ⧴ Fold[Dot, (1 0; 0 1), ρ /@ U /@ {u}])
```

Verifying that ρ represents CU and QU:

```
Table[HL[SS[ρ[z1 ** z2] == ρ[z1].ρ[z2]] /. ϵ^k_· /; k > $k → 0],
 {U, {CU, QU}}, {z1, U /@ {y, a, x}}, {z2, U /@ {y, a, x}} ]
{{{True, True, True}, {True, True, True}, {True, True, True}},
 {{True, True, True}, {True, True, True}, {True, True, True}}}
```

Commuting $e^{\alpha a}$ with $e^{\xi x}$:

```
Table[HL[ρ[e^(ξ U@x)].ρ[e^(α U@a)] == ρ[e^(α U@a)].ρ[e^(e^(-ɣ α) ξ U@x)]], {U, {CU, QU}}]
{True, True}
```

Œ and the logoi Λ

Logoi from Pensieve://Talks/Toulouse-1705/DogmaDemo.nb and from Pensieve://Talks/Sydney-1708/ExtraDetails@@.nb.

MultiplyingOEs

In[ ]:=
```
Œ_U_[s1__, Q1_, P1_] Œ_U_[s2__, Q2_, P2_] ^:= Œ_U[s1, s2, Q1 + Q2, P1 P2];
```

CdsO

In[ ]:=
```
CU@Œ_CU[specs___, Q_, P_] := 𝕆_CU[specs, SS[e^Q P]];
QU@Œ_QU[specs___, Q_, P_] := 𝕆_QU[specs, SS[e^Q P]];
```

Logos

In[ ]:=
```
c_Integer_(k_Integer) := c + O[ϵ]^(k+1);
Λ_(U_,k_)[{α_, β_}, {x_, x_}] := Œ_U[{x}, (α + β) x, 1_k];
Λ_(U_,k_)[{ξ_, α_}, {x, a}] := Œ_U[{a, x}, α a + e^(-ɣ α) ξ x, 1_k];
Λ_(U_,k_)[{α_, η_}, {a, y}] := Œ_U[{y, a}, α a + e^(-ɣ α) η y, 1_k];
```

```
Table[
 {ΛU,1[{α, β}, {u, u}],
  lhs = U@ŒU[{u1, u2}, ℏ (α u1 + β u2), 1], HL[lhs == U@ΛU,1[ℏ {α, β}, {u, u}]]},
 {U, {CU, QU}}, {u, {y, a, x}}]
```

$\{\{\{\text{Œ}_{CU}[\{y\}, y(\alpha+\beta), 1+O[\epsilon]^2],$
$CU[] + (\alpha\hbar+\beta\hbar)\,CU[y] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right)CU[y,y], \text{True}\},$
$\{\text{Œ}_{CU}[\{a\}, a(\alpha+\beta), 1+O[\epsilon]^2], CU[] + (\alpha\hbar+\beta\hbar)\,CU[a] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right)CU[a,a],$
$\text{True}\}, \{\text{Œ}_{CU}[\{x\}, x(\alpha+\beta), 1+O[\epsilon]^2],$
$CU[] + (\alpha\hbar+\beta\hbar)\,CU[x] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right)CU[x,x], \text{True}\}\},$
$\{\{\text{Œ}_{QU}[\{y\}, y(\alpha+\beta), 1+O[\epsilon]^2], QU[] + (\alpha\hbar+\beta\hbar)\,QU[y] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right)QU[y,y],$
$\text{True}\}, \{\text{Œ}_{QU}[\{a\}, a(\alpha+\beta), 1+O[\epsilon]^2], QU[] + (\alpha\hbar+\beta\hbar)\,QU[a] +$
$\left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right)QU[a,a], \text{True}\}, \{\text{Œ}_{QU}[\{x\}, x(\alpha+\beta), 1+O[\epsilon]^2],$
$QU[] + (\alpha\hbar+\beta\hbar)\,QU[x] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right)QU[x,x], \text{True}\}\}\}$

```
{Λ#,1[{ξ, α}, {x, a}], lhs = #@Œ#[{x, a}, ℏ (ξ x + α a), 1],
   HL[lhs == #@Λ#,1[ℏ {ξ, α}, {x, a}]]} & /@ {CU, QU}
```

$\{\{\text{Œ}_{CU}[\{a, x\}, a\alpha + e^{-\alpha\gamma}x\xi, 1+O[\epsilon]^2],$
$CU[] + \alpha\hbar\,CU[a] + (\xi\hbar-\alpha\gamma\xi\hbar^2)\,CU[x] + \frac{1}{2}\alpha^2\hbar^2\,CU[a,a] + \alpha\xi\hbar^2\,CU[a,x] + \frac{1}{2}\xi^2\hbar^2\,CU[x,x],$
$\text{True}\}, \{\text{Œ}_{QU}[\{a, x\}, a\alpha + e^{-\alpha\gamma}x\xi, 1+O[\epsilon]^2], QU[] + \alpha\hbar\,QU[a] +$
$(\xi\hbar-\alpha\gamma\xi\hbar^2)\,QU[x] + \frac{1}{2}\alpha^2\hbar^2\,QU[a,a] + \alpha\xi\hbar^2\,QU[a,x] + \frac{1}{2}\xi^2\hbar^2\,QU[x,x], \text{True}\}\}$

```
{Λ#,2[{α, η}, {a, y}], lhs = #@Œ#[{a, y}, ℏ (η y + α a), 1],
   HL[lhs == #@Λ#,2[ℏ {α, η}, {a, y}]]} & /@ {CU, QU}
```

$\{\{\text{Œ}_{CU}[\{y, a\}, a\alpha + e^{-\alpha\gamma}y\eta, 1+O[\epsilon]^3],$
$CU[] + \alpha\hbar\,CU[a] + (\eta\hbar-\alpha\gamma\eta\hbar^2)\,CU[y] + \frac{1}{2}\alpha^2\hbar^2\,CU[a,a] + \alpha\eta\hbar^2\,CU[y,a] + \frac{1}{2}\eta^2\hbar^2\,CU[y,y],$
$\text{True}\}, \{\text{Œ}_{QU}[\{y, a\}, a\alpha + e^{-\alpha\gamma}y\eta, 1+O[\epsilon]^3], QU[] + \alpha\hbar\,QU[a] +$
$(\eta\hbar-\alpha\gamma\eta\hbar^2)\,QU[y] + \frac{1}{2}\alpha^2\hbar^2\,QU[a,a] + \alpha\eta\hbar^2\,QU[y,a] + \frac{1}{2}\eta^2\hbar^2\,QU[y,y], \text{True}\}\}$

Goal. In either U, compute $F = e^{-\eta y} e^{\xi x} e^{\eta y} e^{-\xi x}$. First compute $G = e^{\xi x}\, y\, e^{-\xi x}$, a finite sum. Now F satisfies the ODE $\partial_\eta F = \partial_\eta\left(e^{-\eta y} e^{\eta G}\right) = -yF + FG$ with initial conditions $F(\eta = 0) = 1$. So we set it up and solve:

```
If[$k > 0, With[{U = CU},
  Module[{G, F, fs, bs, e, b, es, sol},
   G = Echo@Simp[Table[ξ^k / k!, {k, 0, $k + 1}].NestList[Simp[B[x_U , #]] &, y_U, $k + 1]];
   fs = Echo@Flatten@Table[f_{l,i,j,k}[η], {l, 0, $k}, {i, 0, 1}, {j, 0, 1}, {k, 0, 1}];
   F = Echo[fs.(bs = fs /. f_{l_,i_,j_,k_}[η] :> ϵ^l U@{y^i, a^j, x^k})];
   es = Flatten[
     Table[Coefficient[e, b] == 0, {e, {F - 1_U /. η → 0, F ** G - y_U ** F - ∂_η F}}, {b, bs}]];
   sol = Echo@First[F /. DSolve[es, fs, η]];
   Echo[sol /. {ⅇ- → 1, U → Times}];
   Collect[sol /. {ⅇ- → 1, U → Times}, ϵ, Simplify]
  ]]]
```

$-t\,\xi\,CU[\,] + 2\,\epsilon\,\xi\,CU[a] - \gamma\,\epsilon\,\xi^2\,CU[x] + CU[y]$

$\{f_{0,0,0,0}[\eta], f_{1,0,0,0}[\eta], f_{1,0,0,1}[\eta], f_{1,0,1,0}[\eta], f_{1,0,1,1}[\eta], f_{1,1,0,0}[\eta], f_{1,1,0,1}[\eta], f_{1,1,1,0}[\eta], f_{1,1,1,1}[\eta]\}$

$CU[\,]\,f_{0,0,0,0}[\eta] + \epsilon\,CU[\,]\,f_{1,0,0,0}[\eta] + \epsilon\,CU[x]\,f_{1,0,0,1}[\eta] + \epsilon\,CU[a]\,f_{1,0,1,0}[\eta] + \epsilon\,CU[a, x]\,f_{1,0,1,1}[\eta] + \epsilon\,CU[y]\,f_{1,1,0,0}[\eta] + \epsilon\,CU[y, x]\,f_{1,1,0,1}[\eta] + \epsilon\,CU[y, a]\,f_{1,1,1,0}[\eta] + \epsilon\,CU[y, a, x]\,f_{1,1,1,1}[\eta]$

» $e^{-t\,\eta\,\xi}\,CU[\,] + \frac{1}{2}\,e^{-t\,\eta\,\xi}\,t\,\gamma\,\epsilon\,\eta^2\,\xi^2\,CU[\,] + 2\,e^{-t\,\eta\,\xi}\,\epsilon\,\eta\,\xi\,CU[a] - e^{-t\,\eta\,\xi}\,\gamma\,\epsilon\,\eta\,\xi^2\,CU[x] - e^{-t\,\eta\,\xi}\,\gamma\,\epsilon\,\eta^2\,\xi\,CU[y]$

» $1 + 2\,a\,\epsilon\,\eta\,\xi - y\,\gamma\,\epsilon\,\eta^2\,\xi - x\,\gamma\,\epsilon\,\eta\,\xi^2 + \frac{1}{2}\,t\,\gamma\,\epsilon\,\eta^2\,\xi^2$

$1 + \frac{1}{2}\,\epsilon\,\eta\,\xi\,(4\,a + \gamma\,(-2\,y\,\eta - 2\,x\,\xi + t\,\eta\,\xi))$

Logos

In[∘]:=
```
Λ_{U_,kk_}[{ξ1_, η1_}, {x, y}] :=
  Λ_U[{ξ1, η1}, {x, y}] = Block[{$k = kk, $p = kk}, Module[{ξ, η, G, F, fs, f, bs, e, b, es},
     G = Simp[Table[ξ^k / k!, {k, 0, $k + 1}].NestList[Simp[B[x_U , #]] &, y_U, $k + 1]];
     fs = Flatten@Table[f_{l,i,j,k}[η], {l, 0, $k}, {i, 0, 1}, {j, 0, 1}, {k, 0, 1}];
     F = fs.(bs = fs /. f_{l_,i_,j_,k_}[η] :> ϵ^l U@{y^i, a^j, x^k});
     es = Flatten[
       Table[Coefficient[e, b] == 0, {e, {F - 1_U /. η → 0, F ** G - y_U ** F - ∂_η F}}, {b, bs}]];
     F = F /. DSolve[es, fs, η][[1]];
     Œ_U[{y, a, x},
      ξ x + η y + (U /. {CU → -t η ξ, QU → η ξ (1 - T) / ℏ}),
      F + O_$k /. {ⅇ- → 1, U → Times}
     ] /. {ξ → ξ1, η → η1}]];
```

```
{Λ_{CU,1}[{ξ, η}, {x, y}], lhs = CU@Œ_CU[{x, y}, ℏ (ξ x + η y), 1],
 HL[lhs == CU@Λ_{CU,1}[ℏ {ξ, η}, {x, y}]]}
```

$\{Œ_{CU}[\{y, a, x\}, y\,\eta + x\,\xi - t\,\eta\,\xi, 1 + \frac{1}{2}\,\eta\,\xi\,(4\,a - 2\,y\,\gamma\,\eta - 2\,x\,\gamma\,\xi + t\,\gamma\,\eta\,\xi)\,\epsilon + O[\epsilon]^2],$
$(1 - t\,\eta\,\xi\,\hbar^2)\,CU[\,] + 2\,\epsilon\,\eta\,\xi\,\hbar^2\,CU[a] + \xi\,\hbar\,CU[x] + \eta\,\hbar\,CU[y] +$
$\frac{1}{2}\,\xi^2\,\hbar^2\,CU[x, x] + \eta\,\xi\,\hbar^2\,CU[y, x] + \frac{1}{2}\,\eta^2\,\hbar^2\,CU[y, y], \text{True}\}$

```
{Λ_{QU,1}[{ξ, η}, {x, y}], lhs = QU@Œ_{QU}[{x, y}, ℏ (ξ x + η y), 1],
 HL@SimpT[lhs == QU@Λ_{QU,1}[ℏ {ξ, η}, {x, y}]]}
```

$$\left\{\text{Œ}_{QU}\left[\{y, a, x\}, y\,\eta + x\,\xi + \frac{(1-T)\,\eta\,\xi}{\hbar}, 1 + \frac{1}{4\hbar}\eta\,\xi\left(\gamma\,\eta\,\xi - 4\,T\,\gamma\,\eta\,\xi + 3\,T^2\,\gamma\,\eta\,\xi + 8\,a\,T\,\hbar + 2\,y\,\gamma\,\eta\,\hbar - 6\,T\,y\,\gamma\,\eta\,\hbar + 2\,x\,\gamma\,\xi\,\hbar - 6\,T\,x\,\gamma\,\xi\,\hbar + 4\,x\,y\,\gamma\,\hbar^2\right)\epsilon + O[\epsilon]^2\right], (1 + \eta\,\xi\,\hbar - T\,\eta\,\xi\,\hbar)\,QU[\,] + 2\,T\,\epsilon\,\eta\,\xi\,\hbar^2\,QU[a] + \xi\,\hbar\,QU[x] + \eta\,\hbar\,QU[y] + \frac{1}{2}\xi^2\,\hbar^2\,QU[x, x] + \eta\,\xi\,\hbar^2\,QU[y, x] + \frac{1}{2}\eta^2\,\hbar^2\,QU[y, y], \text{True}\right\}$$

```
{tt = Last[Λ_{CU,2}[{ξ, η}, {x, y}]], Log[tt],
  Exponent[Normal@Log[tt] /. {ξ → ℏ ξ, η → ℏ η, x → ℏ x, y → ℏ y}, ℏ]} // Expand
```

$$\left\{1 + \left(2\,a\,\eta\,\xi - y\,\gamma\,\eta^2\,\xi - x\,\gamma\,\eta\,\xi^2 + \frac{1}{2}t\,\gamma\,\eta^2\,\xi^2\right)\epsilon + \left(2\,a^2\,\eta^2\,\xi^2 - a\,\gamma\,\eta^2\,\xi^2 - 2\,a\,y\,\gamma\,\eta^3\,\xi^2 + y\,\gamma^2\,\eta^3\,\xi^2 + \frac{1}{2}y^2\,\gamma^2\,\eta^4\,\xi^2 - 2\,a\,x\,\gamma\,\eta^2\,\xi^3 + x\,\gamma^2\,\eta^2\,\xi^3 + a\,t\,\gamma\,\eta^3\,\xi^3 - \frac{1}{3}t\,\gamma^2\,\eta^3\,\xi^3 + x\,y\,\gamma^2\,\eta^3\,\xi^3 - \frac{1}{2}t\,y\,\gamma^2\,\eta^4\,\xi^3 + \frac{1}{2}x^2\,\gamma^2\,\eta^2\,\xi^4 - \frac{1}{2}t\,x\,\gamma^2\,\eta^3\,\xi^4 + \frac{1}{8}t^2\,\gamma^2\,\eta^4\,\xi^4\right)\epsilon^2 + O[\epsilon]^3,\right.$$
$$\left.\left(2\,a\,\eta\,\xi - y\,\gamma\,\eta^2\,\xi - x\,\gamma\,\eta\,\xi^2 + \frac{1}{2}t\,\gamma\,\eta^2\,\xi^2\right)\epsilon + \left(-a\,\gamma\,\eta^2\,\xi^2 + y\,\gamma^2\,\eta^3\,\xi^2 + x\,\gamma^2\,\eta^2\,\xi^3 - \frac{1}{3}t\,\gamma^2\,\eta^3\,\xi^3\right)\epsilon^2 + O[\epsilon]^3, 6\right\}$$

```
{tt = Last[Λ_{QU,2}[{ξ, η}, {x, y}]], Log[tt],
  Exponent[Normal@Log[tt] /. {ξ → d ξ, η → d η, x → d x, y → d y}, d]} // Expand
```

$$\Big\{1+\Big(2\,a\,T\,\eta\,\xi+\frac{1}{2}\,y\,\gamma\,\eta^2\,\xi-\frac{3}{2}\,T\,y\,\gamma\,\eta^2\,\xi+$$
$$\frac{1}{2}\,x\,\gamma\,\eta\,\xi^2-\frac{3}{2}\,T\,x\,\gamma\,\eta\,\xi^2+\frac{\gamma\,\eta^2\,\xi^2}{4\,\hbar}-\frac{T\,\gamma\,\eta^2\,\xi^2}{\hbar}+\frac{3\,T^2\,\gamma\,\eta^2\,\xi^2}{4\,\hbar}+x\,y\,\gamma\,\eta\,\xi\,\hbar\Big)\,\epsilon+$$
$$\Big(2\,a^2\,T^2\,\eta^2\,\xi^2+2\,a\,T\,\gamma\,\eta^2\,\xi^2-3\,a\,T^2\,\gamma\,\eta^2\,\xi^2+\frac{1}{8}\,\gamma^2\,\eta^2\,\xi^2-\frac{3}{4}\,T\,\gamma^2\,\eta^2\,\xi^2+\frac{5}{8}\,T^2\,\gamma^2\,\eta^2\,\xi^2+$$
$$a\,T\,y\,\gamma\,\eta^3\,\xi^2-3\,a\,T^2\,y\,\gamma\,\eta^3\,\xi^2+\frac{5}{12}\,y\,\gamma^2\,\eta^3\,\xi^2-\frac{17}{6}\,T\,y\,\gamma^2\,\eta^3\,\xi^2+\frac{41}{12}\,T^2\,y\,\gamma^2\,\eta^3\,\xi^2+\frac{1}{8}\,y^2\,\gamma^2\,\eta^4\,\xi^2-$$
$$\frac{3}{4}\,T\,y^2\,\gamma^2\,\eta^4\,\xi^2+\frac{9}{8}\,T^2\,y^2\,\gamma^2\,\eta^4\,\xi^2+a\,T\,x\,\gamma\,\eta^2\,\xi^3-3\,a\,T^2\,x\,\gamma\,\eta^2\,\xi^3+\frac{5}{12}\,x\,\gamma^2\,\eta^2\,\xi^3-\frac{17}{6}\,T\,x\,\gamma^2\,\eta^2\,\xi^3+$$
$$\frac{41}{12}\,T^2\,x\,\gamma^2\,\eta^2\,\xi^3+\frac{1}{2}\,x\,y\,\gamma^2\,\eta^3\,\xi^3-\frac{5}{2}\,T\,x\,y\,\gamma^2\,\eta^3\,\xi^3+3\,T^2\,x\,y\,\gamma^2\,\eta^3\,\xi^3+\frac{1}{8}\,x^2\,\gamma^2\,\eta^2\,\xi^4-$$
$$\frac{3}{4}\,T\,x^2\,\gamma^2\,\eta^2\,\xi^4+\frac{9}{8}\,T^2\,x^2\,\gamma^2\,\eta^2\,\xi^4+\frac{\gamma^2\,\eta^4\,\xi^4}{32\,\hbar^2}-\frac{T\,\gamma^2\,\eta^4\,\xi^4}{4\,\hbar^2}+\frac{11\,T^2\,\gamma^2\,\eta^4\,\xi^4}{16\,\hbar^2}-\frac{3\,T^3\,\gamma^2\,\eta^4\,\xi^4}{4\,\hbar^2}+$$
$$\frac{9\,T^4\,\gamma^2\,\eta^4\,\xi^4}{32\,\hbar^2}+\frac{a\,T\,\gamma\,\eta^3\,\xi^3}{2\,\hbar}-\frac{2\,a\,T^2\,\gamma\,\eta^3\,\xi^3}{\hbar}+\frac{3\,a\,T^3\,\gamma\,\eta^3\,\xi^3}{2\,\hbar}+\frac{5\,\gamma^2\,\eta^3\,\xi^3}{36\,\hbar}-\frac{13\,T\,\gamma^2\,\eta^3\,\xi^3}{12\,\hbar}+$$
$$\frac{25\,T^2\,\gamma^2\,\eta^3\,\xi^3}{12\,\hbar}-\frac{41\,T^3\,\gamma^2\,\eta^3\,\xi^3}{36\,\hbar}+\frac{y\,\gamma^2\,\eta^4\,\xi^3}{8\,\hbar}-\frac{7\,T\,y\,\gamma^2\,\eta^4\,\xi^3}{8\,\hbar}+\frac{15\,T^2\,y\,\gamma^2\,\eta^4\,\xi^3}{8\,\hbar}-\frac{9\,T^3\,y\,\gamma^2\,\eta^4\,\xi^3}{8\,\hbar}+$$
$$\frac{x\,\gamma^2\,\eta^3\,\xi^4}{8\,\hbar}-\frac{7\,T\,x\,\gamma^2\,\eta^3\,\xi^4}{8\,\hbar}+\frac{15\,T^2\,x\,\gamma^2\,\eta^3\,\xi^4}{8\,\hbar}-\frac{9\,T^3\,x\,\gamma^2\,\eta^3\,\xi^4}{8\,\hbar}-2\,a^2\,T\,\eta\,\xi\,\hbar+3\,a\,T\,y\,\gamma\,\eta^2\,\xi\,\hbar+$$
$$\frac{1}{4}\,y\,\gamma^2\,\eta^2\,\xi\,\hbar-\frac{5}{4}\,T\,y\,\gamma^2\,\eta^2\,\xi\,\hbar+\frac{1}{6}\,y^2\,\gamma^2\,\eta^3\,\xi\,\hbar-\frac{7}{6}\,T\,y^2\,\gamma^2\,\eta^3\,\xi\,\hbar+3\,a\,T\,x\,\gamma\,\eta\,\xi^2\,\hbar+\frac{1}{4}\,x\,\gamma^2\,\eta\,\xi^2\,\hbar-$$
$$\frac{5}{4}\,T\,x\,\gamma^2\,\eta\,\xi^2\,\hbar+2\,a\,T\,x\,y\,\gamma\,\eta^2\,\xi^2\,\hbar+\frac{5}{4}\,x\,y\,\gamma^2\,\eta^2\,\xi^2\,\hbar-\frac{21}{4}\,T\,x\,y\,\gamma^2\,\eta^2\,\xi^2\,\hbar+\frac{1}{2}\,x\,y^2\,\gamma^2\,\eta^3\,\xi^2\,\hbar-$$
$$\frac{3}{2}\,T\,x\,y^2\,\gamma^2\,\eta^3\,\xi^2\,\hbar+\frac{1}{6}\,x^2\,\gamma^2\,\eta\,\xi^3\,\hbar-\frac{7}{6}\,T\,x^2\,\gamma^2\,\eta\,\xi^3\,\hbar+\frac{1}{2}\,x^2\,y\,\gamma^2\,\eta^2\,\xi^3\,\hbar-\frac{3}{2}\,T\,x^2\,y\,\gamma^2\,\eta^2\,\xi^3\,\hbar+$$
$$\frac{1}{2}\,x\,y\,\gamma^2\,\eta\,\xi\,\hbar^2+\frac{1}{2}\,x\,y^2\,\gamma^2\,\eta^2\,\xi\,\hbar^2+\frac{1}{2}\,x^2\,y\,\gamma^2\,\eta\,\xi^2\,\hbar^2+\frac{1}{2}\,x^2\,y^2\,\gamma^2\,\eta^2\,\xi^2\,\hbar^2\Big)\,\epsilon^2+O[\epsilon]^3,$$

$$\Big(2\,a\,T\,\eta\,\xi+\frac{1}{2}\,y\,\gamma\,\eta^2\,\xi-\frac{3}{2}\,T\,y\,\gamma\,\eta^2\,\xi+\frac{1}{2}\,x\,\gamma\,\eta\,\xi^2-\frac{3}{2}\,T\,x\,\gamma\,\eta\,\xi^2+\frac{\gamma\,\eta^2\,\xi^2}{4\,\hbar}-$$
$$\frac{T\,\gamma\,\eta^2\,\xi^2}{\hbar}+\frac{3\,T^2\,\gamma\,\eta^2\,\xi^2}{4\,\hbar}+x\,y\,\gamma\,\eta\,\xi\,\hbar\Big)\,\epsilon+$$
$$\Big(2\,a\,T\,\gamma\,\eta^2\,\xi^2-3\,a\,T^2\,\gamma\,\eta^2\,\xi^2+\frac{1}{8}\,\gamma^2\,\eta^2\,\xi^2-\frac{3}{4}\,T\,\gamma^2\,\eta^2\,\xi^2+\frac{5}{8}\,T^2\,\gamma^2\,\eta^2\,\xi^2+\frac{5}{12}\,y\,\gamma^2\,\eta^3\,\xi^2-$$
$$\frac{17}{6}\,T\,y\,\gamma^2\,\eta^3\,\xi^2+\frac{41}{12}\,T^2\,y\,\gamma^2\,\eta^3\,\xi^2+\frac{5}{12}\,x\,\gamma^2\,\eta^2\,\xi^3-\frac{17}{6}\,T\,x\,\gamma^2\,\eta^2\,\xi^3+\frac{41}{12}\,T^2\,x\,\gamma^2\,\eta^2\,\xi^3+$$
$$\frac{5\,\gamma^2\,\eta^3\,\xi^3}{36\,\hbar}-\frac{13\,T\,\gamma^2\,\eta^3\,\xi^3}{12\,\hbar}+\frac{25\,T^2\,\gamma^2\,\eta^3\,\xi^3}{12\,\hbar}-\frac{41\,T^3\,\gamma^2\,\eta^3\,\xi^3}{36\,\hbar}-2\,a^2\,T\,\eta\,\xi\,\hbar+3\,a\,T\,y\,\gamma\,\eta^2\,\xi\,\hbar+$$
$$\frac{1}{4}\,y\,\gamma^2\,\eta^2\,\xi\,\hbar-\frac{5}{4}\,T\,y\,\gamma^2\,\eta^2\,\xi\,\hbar+\frac{1}{6}\,y^2\,\gamma^2\,\eta^3\,\xi\,\hbar-\frac{7}{6}\,T\,y^2\,\gamma^2\,\eta^3\,\xi\,\hbar+3\,a\,T\,x\,\gamma\,\eta\,\xi^2\,\hbar+$$
$$\frac{1}{4}\,x\,\gamma^2\,\eta\,\xi^2\,\hbar-\frac{5}{4}\,T\,x\,\gamma^2\,\eta\,\xi^2\,\hbar+\frac{5}{4}\,x\,y\,\gamma^2\,\eta^2\,\xi^2\,\hbar-\frac{21}{4}\,T\,x\,y\,\gamma^2\,\eta^2\,\xi^2\,\hbar+\frac{1}{6}\,x^2\,\gamma^2\,\eta\,\xi^3\,\hbar-$$
$$\frac{7}{6}\,T\,x^2\,\gamma^2\,\eta\,\xi^3\,\hbar+\frac{1}{2}\,x\,y\,\gamma^2\,\eta\,\xi\,\hbar^2+\frac{1}{2}\,x\,y^2\,\gamma^2\,\eta^2\,\xi\,\hbar^2+\frac{1}{2}\,x^2\,y\,\gamma^2\,\eta\,\xi^2\,\hbar^2\Big)\,\epsilon^2+O[\epsilon]^3,\ 6\Big\}$$

Logos

In[∘]:=
```
(*SimpPQ[PQ_] := Expand@Collect[PQ, {x_,y_,a_},CC]  /. e^δ_ :→ e^Expand@Together[δ] /.
   CC→ExpandDenominator@*ExpandNumerator@*Together;*)
SimpPQ[PQ_] := Simplify[PQ //. e^δ_ :→ e^Simplify[δ]];
Simp[Œ_U_[specs___, Q_, P_]] := Œ_U[specs, CF[Q], CF[P]];
```

Logos

In[∘]:=
```
Λ_U_,k_[{υ1_, ω1_, δ_}, {u_, w_}] := Simp@Module[{υ, ω, yax, q, p, Q, d},
    {yax, q, p} = List @@ Λ_U,k[{υ, ω}, {u, w}];
    Œ_U[yax, Q = (υ u + ω w + δ u w + d υ ω) / (1 - d δ),
      Expand[(1 - d δ)^-1 e^-Q DP_{υ→D_u, ω→D_w}[p][e^Q]] + θ_k] /. {d → ∂_{υ,ω} q} /. {υ → υ1, ω → ω1}];
```

```
Block[{$p = 4, $k = 1},
 {Λ_CU,$k[ħ {ξ, η, δ}, {x, y}],
  Short[lhs = CU@Œ_CU[{x, y}, ħ (ξ x + η y + δ x y), 1_$k], 5],
  HL@Simp[lhs - CU@Λ_CU,$k[ħ {ξ, η, δ}, {x, y}]]}
]
```

$$\Big\{Œ_{CU}\Big[\{y,a,x\},\ \frac{x\,y\,\delta\,\hbar+y\,\eta\,\hbar+x\,\xi\,\hbar-t\,\eta\,\xi\,\hbar^2}{1+t\,\delta\,\hbar},$$

$$\frac{1}{1+t\,\delta\,\hbar}+\Big(\big(4\,a\,\delta\,\hbar+12\,a\,t\,\delta^2\,\hbar^2+4\,a\,x\,y\,\delta^2\,\hbar^2+2\,t\,\gamma\,\delta^2\,\hbar^2-8\,x\,y\,\gamma\,\delta^2\,\hbar^2+4\,a\,y\,\delta\,\eta\,\hbar^2-$$
$$4\,y\,\gamma\,\delta\,\eta\,\hbar^2+4\,a\,x\,\delta\,\xi\,\hbar^2-4\,x\,\gamma\,\delta\,\xi\,\hbar^2+4\,a\,\eta\,\xi\,\hbar^2+12\,a\,t^2\,\delta^3\,\hbar^3+8\,a\,t\,x\,y\,\delta^3\,\hbar^3+$$
$$4\,t^2\,\gamma\,\delta^3\,\hbar^3-12\,t\,x\,y\,\gamma\,\delta^3\,\hbar^3-4\,x^2\,y^2\,\gamma\,\delta^3\,\hbar^3+8\,a\,t\,y\,\delta^2\,\eta\,\hbar^3-4\,t\,y\,\gamma\,\delta^2\,\eta\,\hbar^3-$$
$$6\,x\,y^2\,\gamma\,\delta^2\,\eta\,\hbar^3-2\,y^2\,\gamma\,\delta\,\eta^2\,\hbar^3+8\,a\,t\,x\,\delta^2\,\xi\,\hbar^3-4\,t\,x\,\gamma\,\delta^2\,\xi\,\hbar^3-6\,x^2\,y\,\gamma\,\delta^2\,\xi\,\hbar^3+$$
$$8\,a\,t\,\delta\,\eta\,\xi\,\hbar^3+4\,t\,\gamma\,\delta\,\eta\,\xi\,\hbar^3-8\,x\,y\,\gamma\,\delta\,\eta\,\xi\,\hbar^3-2\,y\,\gamma\,\eta^2\,\xi\,\hbar^3-2\,x^2\,\gamma\,\delta\,\xi^2\,\hbar^3-2\,x\,\gamma\,\eta\,\xi^2\,\hbar^3+$$
$$4\,a\,t^3\,\delta^4\,\hbar^4+4\,a\,t^2\,x\,y\,\delta^4\,\hbar^4+2\,t^3\,\gamma\,\delta^4\,\hbar^4-4\,t^2\,x\,y\,\gamma\,\delta^4\,\hbar^4-3\,t\,x^2\,y^2\,\gamma\,\delta^4\,\hbar^4+$$
$$4\,a\,t^2\,y\,\delta^3\,\eta\,\hbar^4-4\,t\,x\,y^2\,\gamma\,\delta^3\,\eta\,\hbar^4-t\,y^2\,\gamma\,\delta^2\,\eta^2\,\hbar^4+4\,a\,t^2\,x\,\delta^3\,\xi\,\hbar^4-4\,t\,x^2\,y\,\gamma\,\delta^3\,\xi\,\hbar^4+$$
$$4\,a\,t^2\,\delta^2\,\eta\,\xi\,\hbar^4+4\,t^2\,\gamma\,\delta^2\,\eta\,\xi\,\hbar^4-4\,t\,x\,y\,\gamma\,\delta^2\,\eta\,\xi\,\hbar^4-t\,x^2\,\gamma\,\delta^2\,\xi^2\,\hbar^4+t\,\gamma\,\eta^2\,\xi^2\,\hbar^4\big)\,\epsilon\Big)\Big/$$
$$\big(2+10\,t\,\delta\,\hbar+20\,t^2\,\delta^2\,\hbar^2+20\,t^3\,\delta^3\,\hbar^3+10\,t^4\,\delta^4\,\hbar^4+2\,t^5\,\delta^5\,\hbar^5\big)+O[\epsilon]^2\Big],$$

$$\Big(1-t\,\delta\,\hbar+t^2\,\delta^2\,\hbar^2+t\,\gamma\,\delta^2\,\epsilon\,\hbar^2-t\,\eta\,\xi\,\hbar^2-t^3\,\delta^3\,\hbar^3-3\,t^2\,\gamma\,\delta^3\,\epsilon\,\hbar^3+2\,t^2\,\delta\,\eta\,\xi\,\hbar^3+$$
$$2\,t\,\gamma\,\delta\,\epsilon\,\eta\,\xi\,\hbar^3+t^4\,\delta^4\,\hbar^4+6\,t^3\,\gamma\,\delta^4\,\epsilon\,\hbar^4-3\,t^3\,\delta^2\,\eta\,\xi\,\hbar^4-$$
$$9\,t^2\,\gamma\,\delta^2\,\epsilon\,\eta\,\xi\,\hbar^4+\frac{1}{2}\,t^2\,\eta^2\,\xi^2\,\hbar^4+\frac{1}{2}\,t\,\gamma\,\epsilon\,\eta^2\,\xi^2\,\hbar^4\Big)\,CU[\,]+$$
$$\big(2\,\delta\,\epsilon\,\hbar-4\,t\,\delta^2\,\epsilon\,\hbar^2+2\,\epsilon\,\eta\,\xi\,\hbar^2+6\,t^2\,\delta^3\,\epsilon\,\hbar^3-8\,t\,\delta\,\epsilon\,\eta\,\xi\,\hbar^3-8\,t^3\,\delta^4\,\epsilon\,\hbar^4+$$
$$18\,t^2\,\delta^2\,\epsilon\,\eta\,\xi\,\hbar^4-2\,t\,\epsilon\,\eta^2\,\xi^2\,\hbar^4\big)\,CU[a]+$$
$$\ll 37 \gg+\frac{1}{6}\,\delta^3\,\eta\,\hbar^4\,CU[y,y,y,y,x,x,x]+$$
$$\frac{1}{24}\,\delta^4\,\hbar^4\,CU[y,y,y,y,x,x,x,x],\ 0\Big\}$$

```
{Λ_QU,2[{ξ, η, δ}, {x, y}], lhs = QU@Œ_QU[{x, y}, ℏ (ξ x + η y + δ x y), 1],
 HL@SimpT[lhs == QU@Λ_QU,1[ℏ {ξ, η, δ}, {x, y}]]}
```

$\{\mathbb{E}_{QU}[\{y, a, x\}, \frac{\cdots 1 \cdots}{\cdots 1 \cdots},$
$\frac{\hbar}{-\delta + T\,\delta + \hbar} + ((-8\,a\,T\,\delta^4\,\hbar^2 + 24\,a\,T^2\,\delta^4\,\hbar^2 - 24\,a\,T^3\,\delta^4\,\hbar^2 + 8\,a\,T^4\,\delta^4\,\hbar^2 + \cdots 149 \cdots +$
$4\,x^2\,y^2\,\gamma\,\delta^2\,\hbar^6 + 4\,x\,y^2\,\gamma\,\delta\,\eta\,\hbar^6 + 4\,x^2\,y\,\gamma\,\delta\,\xi\,\hbar^6 + 4\,x\,y\,\gamma\,\eta\,\xi\,\hbar^6)\,\epsilon)\,/$
$(-4\,\delta^5 + 20\,T\,\delta^5 - 40\,T^2\,\delta^5 + 40\,T^3\,\delta^5 - 20\,T^4\,\delta^5 + 4\,T^5\,\delta^5 + \cdots 12 \cdots + 40\,T^3\,\delta^3\,\hbar^2 +$
$40\,\delta^2\,\hbar^3 - 80\,T\,\delta^2\,\hbar^3 + 40\,T^2\,\delta^2\,\hbar^3 - 20\,\delta\,\hbar^4 + 20\,T\,\delta\,\hbar^4 + 4\,\hbar^5) +$
$\frac{(\cdots 1 \cdots)\,\cdots 1 \cdots}{\cdots 1 \cdots} + O[\epsilon]^3],\ \cdots 1 \cdots,\ \text{True}\}$

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```
{tt = ComposeSeries[(1 + t δ) Last[Λ_CU,2[{ξ, η, δ}, {x, y}]], (1 + t δ)^4 ϵ + O[ϵ]^18];
  Together@Log[tt],
  Exponent[Normal@Together@Log[tt] /. {ξ → d ξ, η → d η, x → d x, y → d y}, d],
  Exponent[Normal@Together@Log[tt] /. {x → d x, y → d y}, d]
 } // Expand
```

$\{(2\,a\,\delta + 6\,a\,t\,\delta^2 + 2\,a\,x\,y\,\delta^2 + t\,\gamma\,\delta^2 - 4\,x\,y\,\gamma\,\delta^2 + 6\,a\,t^2\,\delta^3 + 4\,a\,t\,x\,y\,\delta^3 + 2\,t^2\,\gamma\,\delta^3 - 6\,t\,x\,y\,\gamma\,\delta^3 -$
$2\,x^2\,y^2\,\gamma\,\delta^3 + 2\,a\,t^3\,\delta^4 + 2\,a\,t^2\,x\,y\,\delta^4 + t^3\,\gamma\,\delta^4 - 2\,t^2\,x\,y\,\gamma\,\delta^4 - \frac{3}{2}\,t\,x^2\,y^2\,\gamma\,\delta^4 + 2\,a\,y\,\delta\,\eta -$
$2\,y\,\gamma\,\delta\,\eta + 4\,a\,t\,y\,\delta^2\,\eta - 2\,t\,y\,\gamma\,\delta^2\,\eta - 3\,x\,y^2\,\gamma\,\delta^2\,\eta + 2\,a\,t^2\,y\,\delta^3\,\eta - 2\,t\,x\,y^2\,\gamma\,\delta^3\,\eta -$
$y^2\,\gamma\,\delta\,\eta^2 - \frac{1}{2}\,t\,y^2\,\gamma\,\delta^2\,\eta^2 + 2\,a\,x\,\delta\,\xi - 2\,x\,\gamma\,\delta\,\xi + 4\,a\,t\,x\,\delta^2\,\xi - 2\,t\,x\,\gamma\,\delta^2\,\xi - 3\,x^2\,y\,\gamma\,\delta^2\,\xi +$
$2\,a\,t^2\,x\,\delta^3\,\xi - 2\,t\,x^2\,y\,\gamma\,\delta^3\,\xi + 2\,a\,\eta\,\xi + 4\,a\,t\,\delta\,\eta\,\xi + 2\,t\,\gamma\,\delta\,\eta\,\xi - 4\,x\,y\,\gamma\,\delta\,\eta\,\xi + 2\,a\,t^2\,\delta^2\,\eta\,\xi +$
$2\,t^2\,\gamma\,\delta^2\,\eta\,\xi - 2\,t\,x\,y\,\gamma\,\delta^2\,\eta\,\xi - y\,\gamma\,\eta^2\,\xi - x^2\,\gamma\,\delta\,\xi^2 - \frac{1}{2}\,t\,x^2\,\gamma\,\delta^2\,\xi^2 - x\,\gamma\,\eta\,\xi^2 + \frac{1}{2}\,t\,\gamma\,\eta^2\,\xi^2)\,\epsilon +$
$(2\,a^2\,\delta^2 - 2\,a\,\gamma\,\delta^2 + 12\,a^2\,t\,\delta^3 + 4\,a^2\,x\,y\,\delta^3 - 8\,a\,t\,\gamma\,\delta^3 - 20\,a\,x\,y\,\gamma\,\delta^3 - 2\,t\,\gamma^2\,\delta^3 + 18\,x\,y\,\gamma^2\,\delta^3 +$
$30\,a^2\,t^2\,\delta^4 + 20\,a^2\,t\,x\,y\,\delta^4 - 10\,a\,t^2\,\gamma\,\delta^4 - 88\,a\,t\,x\,y\,\gamma\,\delta^4 - 13\,a\,x^2\,y^2\,\gamma\,\delta^4 - \frac{15}{2}\,t^2\,\gamma^2\,\delta^4 +$
$64\,t\,x\,y\,\gamma^2\,\delta^4 + 34\,x^2\,y^2\,\gamma^2\,\delta^4 + 40\,a^2\,t^3\,\delta^5 + 40\,a^2\,t^2\,x\,y\,\delta^5 - 152\,a\,t^2\,x\,y\,\gamma\,\delta^5 - 48\,a\,t\,x^2\,y^2\,\gamma\,\delta^5 -$
$10\,t^3\,\gamma^2\,\delta^5 + 86\,t^2\,x\,y\,\gamma^2\,\delta^5 + 107\,t\,x^2\,y^2\,\gamma^2\,\delta^5 + 11\,x^3\,y^3\,\gamma^2\,\delta^5 + 30\,a^2\,t^4\,\delta^6 + 40\,a^2\,t^3\,x\,y\,\delta^6 +$
$10\,a\,t^4\,\gamma\,\delta^6 - 128\,a\,t^3\,x\,y\,\gamma\,\delta^6 - 66\,a\,t^2\,x^2\,y^2\,\gamma\,\delta^6 - 5\,t^4\,\gamma^2\,\delta^6 + 54\,t^3\,x\,y\,\gamma^2\,\delta^6 + \frac{247}{2}\,t^2\,x^2\,y^2\,\gamma^2\,\delta^6 +$
$\frac{80}{3}\,t\,x^3\,y^3\,\gamma^2\,\delta^6 + 12\,a^2\,t^5\,\delta^7 + 20\,a^2\,t^4\,x\,y\,\delta^7 + 8\,a\,t^5\,\gamma\,\delta^7 - 52\,a\,t^4\,x\,y\,\gamma\,\delta^7 - 40\,a\,t^3\,x^2\,y^2\,\gamma\,\delta^7 +$
$16\,t^4\,x\,y\,\gamma^2\,\delta^7 + 62\,t^3\,x^2\,y^2\,\gamma^2\,\delta^7 + \frac{64}{3}\,t^2\,x^3\,y^3\,\gamma^2\,\delta^7 + 2\,a^2\,t^6\,\delta^8 + 4\,a^2\,t^5\,x\,y\,\delta^8 + 2\,a\,t^6\,\gamma\,\delta^8 -$
$8\,a\,t^5\,x\,y\,\gamma\,\delta^8 - 9\,a\,t^4\,x^2\,y^2\,\gamma\,\delta^8 + \frac{1}{2}\,t^6\,\gamma^2\,\delta^8 + 2\,t^5\,x\,y\,\gamma^2\,\delta^8 + \frac{23}{2}\,t^4\,x^2\,y^2\,\gamma^2\,\delta^8 + \frac{17}{3}\,t^3\,x^3\,y^3\,\gamma^2\,\delta^8 +$
$4\,a^2\,y\,\delta^2\,\eta - 12\,a\,y\,\gamma\,\delta^2\,\eta + 6\,y\,\gamma^2\,\delta^2\,\eta + 20\,a^2\,t\,y\,\delta^3\,\eta - 48\,a\,t\,y\,\gamma\,\delta^3\,\eta - 20\,a\,x\,y^2\,\gamma\,\delta^3\,\eta +$
$14\,t\,y\,\gamma^2\,\delta^3\,\eta + 40\,x\,y^2\,\gamma^2\,\delta^3\,\eta + 40\,a^2\,t^2\,y\,\delta^4\,\eta - 72\,a\,t^2\,y\,\gamma\,\delta^4\,\eta - 72\,a\,t\,x\,y^2\,\gamma\,\delta^4\,\eta + 6\,t^2\,y\,\gamma^2\,\delta^4\,\eta +$
$115\,t\,x\,y^2\,\gamma^2\,\delta^4\,\eta + 23\,x^2\,y^3\,\gamma^2\,\delta^4\,\eta + 40\,a^2\,t^3\,y\,\delta^5\,\eta - 48\,a\,t^3\,y\,\gamma\,\delta^5\,\eta - 96\,a\,t^2\,x\,y^2\,\gamma\,\delta^5\,\eta -$
$6\,t^3\,y\,\gamma^2\,\delta^5\,\eta + 118\,t^2\,x\,y^2\,\gamma^2\,\delta^5\,\eta + 53\,t\,x^2\,y^3\,\gamma^2\,\delta^5\,\eta + 20\,a^2\,t^4\,y\,\delta^6\,\eta - 12\,a\,t^4\,y\,\gamma\,\delta^6\,\eta -$
$56\,a\,t^3\,x\,y^2\,\gamma\,\delta^6\,\eta - 4\,t^4\,y\,\gamma^2\,\delta^6\,\eta + 51\,t^3\,x\,y^2\,\gamma^2\,\delta^6\,\eta + 40\,t^2\,x^2\,y^3\,\gamma^2\,\delta^6\,\eta + 4\,a^2\,t^5\,y\,\delta^7\,\eta -$

$$12\,a\,t^4\,x\,y^2\,\gamma\,\delta^7\,\eta + 8\,t^4\,x\,y^2\,\gamma^2\,\delta^7\,\eta + 10\,t^3\,x^2\,y^3\,\gamma^2\,\delta^7\,\eta - 7\,a\,y^2\,\gamma\,\delta^2\,\eta^2 + 10\,y^2\,\gamma^2\,\delta^2\,\eta^2 -$$
$$24\,a\,t\,y^2\,\gamma\,\delta^3\,\eta^2 + 24\,t\,y^2\,\gamma^2\,\delta^3\,\eta^2 + 15\,x\,y^3\,\gamma^2\,\delta^3\,\eta^2 - 30\,a\,t^2\,y^2\,\gamma\,\delta^4\,\eta^2 + \frac{37}{2}\,t^2\,y^2\,\gamma^2\,\delta^4\,\eta^2 +$$
$$32\,t\,x\,y^3\,\gamma^2\,\delta^4\,\eta^2 - 16\,a\,t^3\,y^2\,\gamma\,\delta^5\,\eta^2 + 5\,t^3\,y^2\,\gamma^2\,\delta^5\,\eta^2 + 22\,t^2\,x\,y^3\,\gamma^2\,\delta^5\,\eta^2 - 3\,a\,t^4\,y^2\,\gamma\,\delta^6\,\eta^2 +$$
$$\frac{1}{2}\,t^4\,y^2\,\gamma^2\,\delta^6\,\eta^2 + 5\,t^3\,x\,y^3\,\gamma^2\,\delta^6\,\eta^2 + 3\,y^3\,\gamma^2\,\delta^2\,\eta^3 + \frac{17}{3}\,t\,y^3\,\gamma^2\,\delta^3\,\eta^3 + \frac{10}{3}\,t^2\,y^3\,\gamma^2\,\delta^4\,\eta^3 +$$
$$\frac{2}{3}\,t^3\,y^3\,\gamma^2\,\delta^5\,\eta^3 + 4\,a^2\,x\,\delta^2\,\xi - 12\,a\,x\,\gamma\,\delta^2\,\xi + 6\,x\,\gamma^2\,\delta^2\,\xi + 20\,a^2\,t\,x\,\delta^3\,\xi - 48\,a\,t\,x\,\gamma\,\delta^3\,\xi -$$
$$20\,a\,x^2\,y\,\gamma\,\delta^3\,\xi + 14\,t\,x\,\gamma^2\,\delta^3\,\xi + 40\,x^2\,y\,\gamma^2\,\delta^3\,\xi + 40\,a^2\,t^2\,x\,\delta^4\,\xi - 72\,a\,t^2\,x\,\gamma\,\delta^4\,\xi - 72\,a\,t\,x^2\,y\,\gamma\,\delta^4\,\xi +$$
$$6\,t^2\,x\,\gamma^2\,\delta^4\,\xi + 115\,t\,x^2\,y\,\gamma^2\,\delta^4\,\xi + 23\,x^3\,y^2\,\gamma^2\,\delta^4\,\xi + 40\,a^2\,t^3\,x\,\delta^5\,\xi - 48\,a\,t^3\,x\,\gamma\,\delta^5\,\xi -$$
$$96\,a\,t^2\,x^2\,y\,\gamma\,\delta^5\,\xi - 6\,t^3\,x\,\gamma^2\,\delta^5\,\xi + 118\,t^2\,x^2\,y\,\gamma^2\,\delta^5\,\xi + 53\,t\,x^3\,y^2\,\gamma^2\,\delta^5\,\xi + 20\,a^2\,t^4\,x\,\delta^6\,\xi -$$
$$12\,a\,t^4\,x\,\gamma\,\delta^6\,\xi - 56\,a\,t^3\,x^2\,y\,\gamma\,\delta^6\,\xi - 4\,t^4\,x\,\gamma^2\,\delta^6\,\xi + 51\,t^3\,x^2\,y\,\gamma^2\,\delta^6\,\xi + 40\,t^2\,x^3\,y^2\,\gamma^2\,\delta^6\,\xi +$$
$$4\,a^2\,t^5\,x\,\delta^7\,\xi - 12\,a\,t^4\,x^2\,y\,\gamma\,\delta^7\,\xi + 8\,t^4\,x^2\,y\,\gamma^2\,\delta^7\,\xi + 10\,t^3\,x^3\,y^2\,\gamma^2\,\delta^7\,\xi + 4\,a^2\,\delta\,\eta\,\xi - 4\,a\,\gamma\,\delta\,\eta\,\xi +$$
$$20\,a^2\,t\,\delta^2\,\eta\,\xi - 8\,a\,t\,\gamma\,\delta^2\,\eta\,\xi - 28\,a\,x\,y\,\gamma\,\delta^2\,\eta\,\xi - 6\,t\,\gamma^2\,\delta^2\,\eta\,\xi + 38\,x\,y\,\gamma^2\,\delta^2\,\eta\,\xi + 40\,a^2\,t^2\,\delta^3\,\eta\,\xi +$$
$$8\,a\,t^2\,\gamma\,\delta^3\,\eta\,\xi - 96\,a\,t\,x\,y\,\gamma\,\delta^3\,\eta\,\xi - 14\,t^2\,\gamma^2\,\delta^3\,\eta\,\xi + 88\,t\,x\,y\,\gamma^2\,\delta^3\,\eta\,\xi + 44\,x^2\,y^2\,\gamma^2\,\delta^3\,\eta\,\xi +$$
$$40\,a^2\,t^3\,\delta^4\,\eta\,\xi + 32\,a\,t^3\,\gamma\,\delta^4\,\eta\,\xi - 120\,a\,t^2\,x\,y\,\gamma\,\delta^4\,\eta\,\xi - 6\,t^3\,\gamma^2\,\delta^4\,\eta\,\xi + 62\,t^2\,x\,y\,\gamma^2\,\delta^4\,\eta\,\xi +$$
$$93\,t\,x^2\,y^2\,\gamma^2\,\delta^4\,\eta\,\xi + 20\,a^2\,t^4\,\delta^5\,\eta\,\xi + 28\,a\,t^4\,\gamma\,\delta^5\,\eta\,\xi - 64\,a\,t^3\,x\,y\,\gamma\,\delta^5\,\eta\,\xi + 6\,t^4\,\gamma^2\,\delta^5\,\eta\,\xi +$$
$$12\,t^3\,x\,y\,\gamma^2\,\delta^5\,\eta\,\xi + 63\,t^2\,x^2\,y^2\,\gamma^2\,\delta^5\,\eta\,\xi + 4\,a^2\,t^5\,\delta^6\,\eta\,\xi + 8\,a\,t^5\,\gamma\,\delta^6\,\eta\,\xi - 12\,a\,t^4\,x\,y\,\gamma\,\delta^6\,\eta\,\xi +$$
$$4\,t^5\,\gamma^2\,\delta^6\,\eta\,\xi + 14\,t^3\,x^2\,y^2\,\gamma^2\,\delta^6\,\eta\,\xi - 8\,a\,y\,\gamma\,\delta\,\eta^2\,\xi + 6\,y\,\gamma^2\,\delta\,\eta^2\,\xi - 24\,a\,t\,y\,\gamma\,\delta^2\,\eta^2\,\xi +$$
$$5\,t\,y\,\gamma^2\,\delta^2\,\eta^2\,\xi + 25\,x\,y^2\,\gamma^2\,\delta^2\,\eta^2\,\xi - 24\,a\,t^2\,y\,\gamma\,\delta^3\,\eta^2\,\xi - 8\,t^2\,y\,\gamma^2\,\delta^3\,\eta^2\,\xi + 45\,t\,x\,y^2\,\gamma^2\,\delta^3\,\eta^2\,\xi -$$
$$8\,a\,t^3\,y\,\gamma\,\delta^4\,\eta^2\,\xi - 7\,t^3\,y\,\gamma^2\,\delta^4\,\eta^2\,\xi + 24\,t^2\,x\,y^2\,\gamma^2\,\delta^4\,\eta^2\,\xi + 4\,t^3\,x\,y^2\,\gamma^2\,\delta^5\,\eta^2\,\xi + 4\,y^2\,\gamma^2\,\delta\,\eta^3\,\xi +$$
$$5\,t\,y^2\,\gamma^2\,\delta^2\,\eta^3\,\xi + t^2\,y^2\,\gamma^2\,\delta^3\,\eta^3\,\xi - 7\,a\,x^2\,\gamma\,\delta^2\,\xi^2 + 10\,x^2\,\gamma^2\,\delta^2\,\xi^2 - 24\,a\,t\,x^2\,\gamma\,\delta^3\,\xi^2 +$$
$$24\,t\,x^2\,\gamma^2\,\delta^3\,\xi^2 + 15\,x^3\,y\,\gamma^2\,\delta^3\,\xi^2 - 30\,a\,t^2\,x^2\,\gamma\,\delta^4\,\xi^2 + \frac{37}{2}\,t^2\,x^2\,\gamma^2\,\delta^4\,\xi^2 + 32\,t\,x^3\,y\,\gamma^2\,\delta^4\,\xi^2 -$$
$$16\,a\,t^3\,x^2\,\gamma\,\delta^5\,\xi^2 + 5\,t^3\,x^2\,\gamma^2\,\delta^5\,\xi^2 + 22\,t^2\,x^3\,y\,\gamma^2\,\delta^5\,\xi^2 - 3\,a\,t^4\,x^2\,\gamma\,\delta^6\,\xi^2 + \frac{1}{2}\,t^4\,x^2\,\gamma^2\,\delta^6\,\xi^2 +$$
$$5\,t^3\,x^3\,y\,\gamma^2\,\delta^6\,\xi^2 - 8\,a\,x\,\gamma\,\delta\,\eta\,\xi^2 + 6\,x\,\gamma^2\,\delta\,\eta\,\xi^2 - 24\,a\,t\,x\,\gamma\,\delta^2\,\eta\,\xi^2 + 5\,t\,x\,\gamma^2\,\delta^2\,\eta\,\xi^2 +$$
$$25\,x^2\,y\,\gamma^2\,\delta^2\,\eta\,\xi^2 - 24\,a\,t^2\,x\,\gamma\,\delta^3\,\eta\,\xi^2 - 8\,t^2\,x\,\gamma^2\,\delta^3\,\eta\,\xi^2 + 45\,t\,x^2\,y\,\gamma^2\,\delta^3\,\eta\,\xi^2 - 8\,a\,t^3\,x\,\gamma\,\delta^4\,\eta\,\xi^2 -$$
$$7\,t^3\,x\,\gamma^2\,\delta^4\,\eta\,\xi^2 + 24\,t^2\,x^2\,y\,\gamma^2\,\delta^4\,\eta\,\xi^2 + 4\,t^3\,x^2\,y\,\gamma^2\,\delta^5\,\eta\,\xi^2 - a\,\gamma\,\eta^2\,\xi^2 - 3\,t\,\gamma^2\,\delta\,\eta^2\,\xi^2 +$$
$$11\,x\,y\,\gamma^2\,\delta\,\eta^2\,\xi^2 + 6\,a\,t^2\,\gamma\,\delta^2\,\eta^2\,\xi^2 - \frac{5}{2}\,t^2\,\gamma^2\,\delta^2\,\eta^2\,\xi^2 + 12\,t\,x\,y\,\gamma^2\,\delta^2\,\eta^2\,\xi^2 + 8\,a\,t^3\,\gamma\,\delta^3\,\eta^2\,\xi^2 +$$
$$4\,t^3\,\gamma^2\,\delta^3\,\eta^2\,\xi^2 + 3\,a\,t^4\,\gamma\,\delta^4\,\eta^2\,\xi^2 + \frac{7}{2}\,t^4\,\gamma^2\,\delta^4\,\eta^2\,\xi^2 - t^3\,x\,y\,\gamma^2\,\delta^4\,\eta^2\,\xi^2 + y\,\gamma^2\,\eta^3\,\xi^2 -$$
$$t\,y\,\gamma^2\,\delta\,\eta^3\,\xi^2 - 2\,t^2\,y\,\gamma^2\,\delta^2\,\eta^3\,\xi^2 + 3\,x^3\,\gamma^2\,\delta^2\,\xi^3 + \frac{17}{3}\,t\,x^3\,\gamma^2\,\delta^3\,\xi^3 + \frac{10}{3}\,t^2\,x^3\,\gamma^2\,\delta^4\,\xi^3 +$$
$$\frac{2}{3}\,t^3\,x^3\,\gamma^2\,\delta^5\,\xi^3 + 4\,x^2\,\gamma^2\,\delta\,\eta\,\xi^3 + 5\,t\,x^2\,\gamma^2\,\delta^2\,\eta\,\xi^3 + t^2\,x^2\,\gamma^2\,\delta^3\,\eta\,\xi^3 + x\,\gamma^2\,\eta^2\,\xi^3 - t\,x\,\gamma^2\,\delta\,\eta^2\,\xi^3 -$$
$$2\,t^2\,x\,\gamma^2\,\delta^2\,\eta^2\,\xi^3 - \frac{1}{3}\,t\,\gamma^2\,\eta^3\,\xi^3 + \frac{1}{3}\,t^2\,\gamma^2\,\delta\,\eta^3\,\xi^3 + \frac{2}{3}\,t^3\,\gamma^2\,\delta^2\,\eta^3\,\xi^3\Big)\,\epsilon^2 + O[\epsilon]^3,\ 6,\ 6\Big\}$$

```
{tt = Last[ΛQU,2[{ξ, η, δ}, {x, y}]];
  Log[tt],
  Exponent[Normal@Together@Log[tt] /. {ξ → d ξ, η → d η, x → d x, y → d y}, d]} // Expand
```

$\{\text{Log}\left[\frac{\hbar}{-\delta+T\,\delta+\hbar}\right]+\left(\frac{2\,a\,T\,\delta^5\,\hbar}{(-\delta+T\,\delta+\hbar)^5}-\frac{8\,a\,T^2\,\delta^5\,\hbar}{(-\delta+T\,\delta+\hbar)^5}+\frac{12\,a\,T^3\,\delta^5\,\hbar}{(-\delta+T\,\delta+\hbar)^5}-\frac{8\,a\,T^4\,\delta^5\,\hbar}{(-\delta+T\,\delta+\hbar)^5}+\cdots 267\cdots+\frac{x^2\,y^2\,\gamma\,\delta^2\,\hbar^6}{(-\delta+T\,\delta+\hbar)^5}+\frac{x\,y^2\,\gamma\,\delta\,\eta\,\hbar^6}{(-\delta+T\,\delta+\hbar)^5}+\frac{x^2\,y\,\gamma\,\delta\,\xi\,\hbar^6}{(-\delta+T\,\delta+\hbar)^5}+\frac{x\,y\,\gamma\,\eta\,\xi\,\hbar^6}{(-\delta+T\,\delta+\hbar)^5}\right)\epsilon+\left(-\frac{32\,a^2\,T^2\,\delta^{10}\,\hbar^2}{(\cdots 1\cdots)^2}+\cdots 8307\cdots+\cdots 1\cdots+\frac{144\,x^2\,y^2\,\gamma^2\,\eta^2\,\xi^2\,\hbar^{11}}{\cdots 1\cdots}\right)\epsilon^2+O[\epsilon]^3,\ 6\}$

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Reorderings with Rord

Rord

In[∘]:=

```
Rord_{u_i_,w_j_→k_}[𝔼_U_[L___, {l___, u_i_, w_j_, r___}_s_, R___, Q_, P_]] :=
  Simp@Module[{υ, ω, δ, Λ1, yax, q, p, kk = P⟦5⟧, δ1 = ∂_{u_i,w_j} Q},
    {yax, q, p} = Echo[List @@ If[δ1 === 0, Λ_{U,kk}[{υ, ω}, {u, w}],
         Λ_{U,kk}[{υ, ω, δ}, {u, w}]] /. {y → y_k, a → a_k, x → x_k, t → t_s, T → T_s}];
    𝔼_U[L, {l, Sequence @@ yax, r}_s, R, q + (Q /. u_i | w_j → 0), e^{-q} DP_{u_i→D_υ, w_j→D_ω}[P][p e^q]] /.
     {υ → ∂_{u_i} Q /. w_j → 0, ω → ∂_{w_j} Q /. u_i → 0, δ → δ1}];
```

Rord

In[∘]:=

```
Rord_{u_i_,w_j_→k_}[𝔼_U_[L___, {l___, u_i_, w_j_, r___}_s_, R___, Q_, P_]] :=
  Simp@Module[{υ, ω, δ, Λ1, yax, q, p, n, kk = P⟦5⟧, δ1 = ∂_{u_i,w_j} Q},
    {yax, q, p} = List @@ If[δ1 === 0, Λ_{U,kk}[{υ, ω}, {u, w}], Λ_{U,kk}[{υ, ω, δ}, {u, w}]] /.
      {y → y_n, a → a_n, x → x_n, t → t_s, T → T_s};
    (*Echo@{{{u_i,υ},{w_j,ω}},P,p e^q};*)
    𝔼_U[L, {l, Sequence @@ yax, r}_s, R, q + (Q /. u_i | w_j → 0), e^{-q} SP_{{u_i→υ, w_j→ω}}[P p e^q]] /.
     {n → k, υ → ∂_{u_i} Q /. w_j → 0, ω → ∂_{w_j} Q /. u_i → 0, δ → δ1}];
```

```
With[{c𝔬 = 𝔼_CU[{y_1, x_1}_1, {x_2, a_2, y_2}_2, ℏ t_1 a_2 + ℏ t_1^{-1} (e^{t_1} - 1) y_1 x_2, 1_2 + ϵ x_1 y_2]},
 {Short[rhs = c𝔬 // Rord_{x_2,a_2→3}, 3], HL[CU[c𝔬] == CU[rhs]]}]
```

$\left\{\mathbb{E}_{CU}\left[\{y_1,\ x_1\}_1,\ \{a_3,\ x_3,\ y_2\}_2,\ \frac{e^{-\gamma\,\hbar\,t_1}\left(e^{\gamma\,\hbar\,t_1}\,\hbar\,a_3\,t_1^2-\hbar\,x_3\,y_1+e^{t_1}\,\hbar\,x_3\,y_1\right)}{t_1},\ 1+x_1\,y_2\,\epsilon+O[\epsilon]^3\right],\ \text{True}\right\}$

```
With[{c𝔬 = 𝔼_CU[{y_1, a_1, a_2}_1, {x_2, x_1, y_2}_2,
    ℏ (l_11 t_1 a_1 + l_12 t_1 a_2 + l_21 t_2 a_1 + l_22 t_2 a_2 + γ_11 x_1 y_1 + γ_12 x_1 y_2 + γ_21 x_2 y_1 + γ_22 x_2 y_2),
    1_2 + ϵ (l_1 a_1 + l_2 a_2 + p_11 x_1 y_1 + p_12 x_1 y_2 + p_21 x_2 y_1 + p_22 x_2 y_2)]},
 {Short[rhs = c𝔬 // Rord_{a_1,a_2→3} // Rord_{x_2,x_1→4}, 3], HL[CU[c𝔬] == CU[rhs]]}]
```

$\{\mathbb{E}_{CU}[\{y_1,\ a_3\}_1,\ \{x_4,\ y_2\}_2,$
$\hbar\,a_3\,l_{11}\,t_1+\hbar\,a_3\,l_{12}\,t_1+\hbar\,a_3\,l_{21}\,t_2+\hbar\,a_3\,l_{22}\,t_2+\hbar\,x_4\,y_1\,\gamma_{11}+\hbar\,x_4\,y_2\,\gamma_{12}+\hbar\,x_4\,y_1\,\gamma_{21}+\hbar\,x_4\,y_2\,\gamma_{22},$
$1+(a_3\,l_1+a_3\,l_2+p_{11}\,x_4\,y_1+p_{21}\,x_4\,y_1+p_{12}\,x_4\,y_2+p_{22}\,x_4\,y_2)\,\epsilon+O[\epsilon]^3],\ \text{True}\}$

```
ℏ a_3 l_11 t_1 + ℏ a_3 l_12 t_1 + ℏ a_3 l_21 t_2 + ℏ a_3 l_22 t_2 +
  ℏ x_4 y_1 γ_11 + ℏ x_4 y_2 γ_12 + ℏ x_4 y_1 γ_21 + ℏ x_4 y_2 γ_22 // Simplify
```

$\hbar\,(a_3\,(l_{11}\,t_1+l_{12}\,t_1+(l_{21}+l_{22})\,t_2)+x_4\,(y_1\,(\gamma_{11}+\gamma_{21})+y_2\,(\gamma_{12}+\gamma_{22})))$

```
With[{c⊕ = 𝔈CU[{y1, a1, x1}1, {x2, a2, y2}2,
     ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
     12 + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
  {Short[rhs = c⊕ // Rord_{x2,a2→3}, 3], HL[CU[c⊕] == CU[rhs]]}]
```

```
{𝔈CU[{y1, a1, x1}1, <<1>>2, <<1>> <<1>>,
  1 + e^(-γ ℏ l12 t1 - γ ℏ l22 t2) (e^(γ ℏ l12 t1 + γ ℏ l22 t2) a1 l1 + e^(γ ℏ l12 t1 + γ ℏ l22 t2) a3 l2 + e^(γ ℏ l12 t1 + γ ℏ l22 t2) p11 x1 y1 + p21 x3 y1 +
      e^(<<1>> + <<1>>) p12 x1 y2 + p22 x3 y2 - γ ℏ l2 x3 y1 γ21 - γ ℏ l2 x3 y2 γ22) ϵ + O[ϵ]^3], True}
```

```
With[{q⊕ = 𝔈QU[{y1, a1, x1}1, {x2, a2, y2}2,
     ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
     12 + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
  {Short[rhs = q⊕ // Rord_{x2,a2→3}, 3], HL[QU[q⊕] == QU[rhs]]}]
```

```
{𝔈QU[{y1, a1, x1}1, <<1>>2, <<1>> <<1>>,
  1 + e^(-γ ℏ l12 t1 - γ ℏ l22 t2) (e^(γ ℏ l12 t1 + γ ℏ l22 t2) a1 l1 + e^(γ ℏ l12 t1 + γ ℏ l22 t2) a3 l2 + e^(γ ℏ l12 t1 + γ ℏ l22 t2) p11 x1 y1 + p21 x3 y1 +
      e^(<<1>> + <<1>>) p12 x1 y2 + p22 x3 y2 - γ ℏ l2 x3 y1 γ21 - γ ℏ l2 x3 y2 γ22) ϵ + O[ϵ]^3], True}
```

```
With[{q⊕ = 𝔈QU[{y1, a1, x1}1, {x2, a2, y2}2,
     ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
     12 + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
  {Short[rhs = q⊕ // Rord_{a2,y2→3}, 3], HL[QU[q⊕] == QU[rhs]]}]
```

```
{<<1>>, True}
```

```
Timing@With[{q⊕ = 𝔈QU[{x1, y1}1, {x2, a2, y2}2,
      ℏ (l12 t1 a2 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
      02 + ϵ (l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
   {Short[rhs = q⊕ // Rord_{x1,y1→3}, 5]}]
```

```
{116.156, {𝔈QU[{y3, a3, x3}1, <<1>>2, <<1>> / (1 - <<1>> + <<1>>),
   ((ℏ a2 l2 + p11 - p11 T1 + ℏ p22 x2 y2 + ℏ p12 x3 y2 + <<46>> + 2 ℏ p12 T1 x2 y2 γ11 γ21 -
         ℏ p12 T1^2 x2 y2 γ11 γ21 + ℏ p11 x2 y2 γ12 γ21 - 2 ℏ p11 T1 x2 y2 γ12 γ21 + ℏ p11 T1^2 x2 y2 γ12 γ21) ϵ) /
     (ℏ - 3 ℏ γ11 + 3 ℏ T1 γ11 + 3 ℏ γ11^2 - 6 ℏ T1 γ11^2 + 3 ℏ T1^2 γ11^2 - ℏ γ11^3 + 3 ℏ T1 γ11^3 - 3 ℏ T1^2 γ11^3 + ℏ T1^3 γ11^3) +
    ((8 a3 p11 T1 + <<1>> + <<2726>> + 3 γ <<6>> γ21^3) <<1>>) /
    (4 - 28 γ11 + <<48>> + 4 T1^7 γ11^7) + O[ϵ]^3]}}
```

```
Timing@With[{qΦ = 𝔼QU[{x1, y1}1, {x2, a2, y2}2,
     ℏ (l12 t1 a2 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
     12 + ϵ (l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
  {Short[rhs = qΦ // Rord_{x1,y1→3}, 5], HL@SimpT[QU[qΦ] == QU[rhs]]}]
```

```
{388.922,
 {𝔼QU[{y3, a3, x3}1, {<<1>>}2, <<1>>/<<1>>, 1/(1 - γ11 + T1 γ11) + ((4 ℏ a2 l2 + 4 p11 - 4 p11 T1 + 4 ℏ p22 x2 y2 +
        <<339>> + γ ℏ^4 x2^2 y2^2 γ12^2 γ21^2 - 4 γ ℏ^4 T1 x2^2 y2^2 γ12^2 γ21^2 + 3 γ ℏ^4 T1^2 x2^2 y2^2 γ12^2 γ21^2) ϵ) /
     (4 ℏ - 20 ℏ γ11 + 20 ℏ T1 γ11 + 40 ℏ γ11^2 - 80 ℏ T1 γ11^2 + 40 ℏ T1^2 γ11^2 - 40 ℏ γ11^3 + <<13>> +
       20 ℏ T1 γ11^5 - 40 ℏ T1^2 γ11^5 + 40 ℏ T1^3 γ11^5 - 20 ℏ T1^4 γ11^5 + 4 ℏ T1^5 γ11^5) +
     (576 a3 p11 T1 + <<8073>> + <<1>>) <<1>> / (<<79>> + 288 T1^9 γ11^9) + O[ϵ]^3], True}}
```

```
Timing@With[{qΦ = 𝔼QU[{x1, y1}1, {x2, a2, y2}2,
     ℏ (l12 t1 a2 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
     12 + ϵ (l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
  {Short[rhs = qΦ // Rord_{x1,y1→1}, 5], HL@SimpT[QU[qΦ] == QU[rhs]]}]
```

```
{336.781,
 {𝔼QU[{y1, a1, x1}1, {<<1>>}2, <<1>>/<<1>>, 1/(1 - γ11 + T1 γ11) + ((4 ℏ a2 l2 + 4 p11 - 4 p11 T1 + 4 ℏ p11 x1 y1 +
        4 ℏ p21 x2 y1 + <<338>> + γ ℏ^4 x2^2 y2^2 γ12^2 γ21^2 - 4 γ ℏ^4 T1 x2^2 y2^2 γ12^2 γ21^2 + 3 γ ℏ^4 T1^2 x2^2 y2^2 γ12^2 γ21^2) ϵ) /
     (4 ℏ - 20 ℏ γ11 + 20 ℏ T1 γ11 + 40 ℏ γ11^2 - 80 ℏ T1 γ11^2 + 40 ℏ T1^2 γ11^2 - 40 ℏ γ11^3 + <<10>> +
       20 ℏ T1^4 γ11^4 - 4 ℏ γ11^5 + 20 ℏ T1 γ11^5 - 40 ℏ T1^2 γ11^5 + 40 ℏ T1^3 γ11^5 - 20 ℏ T1^4 γ11^5 + 4 ℏ T1^5 γ11^5) +
     (576 a1 p11 T1 + <<8073>> + <<1>>) <<1>> / (<<79>> + 288 T1^9 γ11^9) + O[ϵ]^3], True}}
```

Canonical ordering with Cord

Cord

In[∘]:=
```
Cord[𝔼U_[L___, {l___, u_i_, w_j_, r___}s_, R___, Q_, P_]] /;
    OrderedQ[{w, u} /. {y → 1, a → 2, x → 3}] :=
   Cord[Rord_{ui,wj→Unique[]}[𝔼U[L, {l, ui, wj, r}s, R, Q, P]]];
Cord[𝔼U_[specs__, Q_, P_]] := 𝔼U[Sequence @@ Sort@{specs}, Q, P] /.
   Flatten[{specs} /. {yax__}s_ :> ({yax} /. u_i_ :> (ui → us))]
```

```
Cord@𝔼CU[{x1, y1}1, 0, 01 + x1 y1]
```

```
𝔼CU[{y1, a1, x1}1, 0, (-t1 + x1 y1) + 2 a1 ϵ + O[ϵ]^2]
```

```
Block[{$p = 4, $k = 0, cꝏ = ℰCU[{y1, a1, x1, x2, a2, y2}1,
    ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    1θ + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
  Timing@{Short[Cord[cꝏ], 8], HL@Simp[CU[cꝏ] - CU[Cord[cꝏ]]]}]
```

```
{4.53125,
 {ℰCU[{y1, a1, x1}1, (e^(γ ℏ l11 t1+2 γ ℏ l12 t1+γ ℏ l21 t2+2 γ ℏ l22 t2) ℏ a1 l11 t1 + e^(γ ℏ l11 t1+2 γ ℏ l12 t1+γ ℏ l21 t2+2 γ ℏ l22 t2)
      ℏ a1 l12 t1 + <<12>> + e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ^2 a1 l22 t1 t2 γ22 + ℏ x1 y1 γ22) /
    (e^(γ ℏ l11 t1+2 γ ℏ l12 t1+γ ℏ l21 t2+2 γ ℏ l22 t2) + e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ t1 γ12 +
      e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ t1 γ22), e^(γ ℏ l12 t1+γ ℏ l22 t2) / (e^(γ ℏ l12 t1+γ ℏ l22 t2) + ℏ t1 γ12 + ℏ t1 γ22) + O[ϵ]^1], 0}}
```

```
Block[{$p = 4, $k = 1, cꝏ = ℰCU[{y1, a1, x1, x2, a2, y2}1,
    ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    11 + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
  Timing@{Short[Cord[cꝏ], 8], HL@Simp[CU[cꝏ] - CU[Cord[cꝏ]]]}]
```

```
{81.2656, {ℰCU[{y1, a1, x1}1, (e^(γ ℏ l11 t1+2 γ ℏ l12 t1+γ ℏ l21 t2+2 γ ℏ l22 t2) ℏ a1 l11 t1 + <<14>> + ℏ x1 y1 γ22) /
    (e^(γ ℏ l11 t1+2 γ ℏ l12 t1+γ ℏ l21 t2+2 γ ℏ l22 t2) + e^(γ ℏ l11 t1+<<2>>+γ ℏ <<1>> t2) ℏ t1 γ12 +
      e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ t1 γ22),
   e^(γ ℏ l12 t1+γ ℏ l22 t2) / (e^(γ ℏ l12 t1+γ ℏ l22 t2) + ℏ t1 γ12 + ℏ t1 γ22) + ((2 e^(2 γ ℏ l11 t1+6 γ ℏ l12 t1+2 γ ℏ l21 t2+6 γ ℏ l22 t2) a1 l1 + <<419>>) ϵ) /
    (2 e^(2 γ ℏ l11 t1+6 γ ℏ l12 t1+2 γ ℏ l21 t2+6 γ ℏ l22 t2) + 10 e^(2 γ ℏ l11 t1+5 γ ℏ l12 t1+2 γ ℏ l21 t2+5 γ ℏ l22 t2) ℏ t1 γ12 +
      <<18>> + 2 e^(2 γ ℏ l11 t1+γ ℏ l12 t1+2 γ ℏ l21 t2+γ ℏ l22 t2) ℏ^5 t1^5 γ22^5) + O[ϵ]^2], 0}}
```

```
With[{qꝏ = ℰQU[{y1, a1, x1, x2, a2, y2}1,
    ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    1θ + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
  Cord[
   qꝏ]]
```

```
ℰQU[{y1, a1, x1}1,
 (e^(γ ℏ l11 t1+2 γ ℏ l12 t1+γ ℏ l21 t2+2 γ ℏ l22 t2) ℏ a1 l11 t1 + e^(γ ℏ l11 t1+2 γ ℏ l12 t1+γ ℏ l21 t2+2 γ ℏ l22 t2) ℏ a1 l12 t1 +
    e^(γ ℏ l11 t1+2 γ ℏ l12 t1+γ ℏ l21 t2+2 γ ℏ l22 t2) ℏ a1 l21 t2 + e^(γ ℏ l11 t1+2 γ ℏ l12 t1+γ ℏ l21 t2+2 γ ℏ l22 t2) ℏ a1 l22 t2 +
    e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ x1 y1 γ11 - e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ a1 l11 t1 γ12 -
    e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ a1 l12 t1 γ12 - e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ a1 l21 t2 γ12 -
    e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ a1 l22 t2 γ12 + e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ a1 l11 t1 T1 γ12 +
    e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ a1 l12 t1 T1 γ12 + e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ a1 l21 t2 T1 γ12 +
    e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ a1 l22 t2 T1 γ12 + ℏ x1 y1 γ12 +
    e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ x1 y1 γ21 - e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ a1 l11 t1 γ22 -
    e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ a1 l12 t1 γ22 - e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ a1 l21 t2 γ22 -
    e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ a1 l22 t2 γ22 + e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ a1 l11 t1 T1 γ22 +
    e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ a1 l12 t1 T1 γ22 + e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ a1 l21 t2 T1 γ22 +
    e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ a1 l22 t2 T1 γ22 + ℏ x1 y1 γ22) /
  (e^(γ ℏ l11 t1+2 γ ℏ l12 t1+γ ℏ l21 t2+2 γ ℏ l22 t2) - e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) γ12 + e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2)
     T1 γ12 - e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) γ22 + e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) T1 γ22),
 e^(γ ℏ l12 t1+γ ℏ l22 t2) / (e^(γ ℏ l12 t1+γ ℏ l22 t2) - γ12 + T1 γ12 - γ22 + T1 γ22) + O[ϵ]^1]
```

Stitching Œ's.

StitchingOEs

In[∘]:=
```
m_{j_→k_}[Œ_U_[specs__, Q_, P_]] := Cord[Œ_U[Sequence @@ Append[DeleteCases[{specs}, {__}_{j|k}],
    Flatten[{Cases[{specs}, {us__}_j ⧴ {us}], Cases[{specs}, {us__}_k ⧴ {us}]}]_k],
  Q, P] /. {t_j → t_k, T_j → T_k}]
```

```
cω = Œ_CU[{y1, a1, x1}_1, {y2, a2, x2}_2,
   {y3, a3, x3}_3, ℏ Sum[l_{10 i+j} t_i a_j + γ_{10 i+j} y_i x_j, {i, 3}, {j, 3}], 1_2];
{cω // m_{3→4}, HL@Simp[CU[m_{3→4}[cω]] - m_{3→4}[CU[cω]]]}
```

$\{\text{Œ}_{CU}[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, \{y_4, a_4, x_4\}_4, \hbar\,(a_1 l_{11} t_1 + a_2 l_{12} t_1 + a_4 l_{13} t_1 + a_1 l_{21} t_2 + a_2 l_{22} t_2 + a_4 l_{23} t_2 + a_1 l_{31} t_4 + a_2 l_{32} t_4 + a_4 l_{33} t_4 + x_1 y_1 \gamma_{11} + x_2 y_1 \gamma_{12} + x_4 y_1 \gamma_{13} + x_1 y_2 \gamma_{21} + x_2 y_2 \gamma_{22} + x_4 y_2 \gamma_{23} + x_1 y_4 \gamma_{31} + x_2 y_4 \gamma_{32} + x_4 y_4 \gamma_{33}), 1 + O[\epsilon]^3], 0\}$

Verifying that *m* commutes with evaluation, in CU:

```
cω = Œ_CU[{y1, a1, x1}_1, {y2, a2, x2}_2,
   {y3, a3, x3}_3, ℏ Sum[l_{10 i+j} t_i a_j + γ_{10 i+j} y_i x_j, {i, 3}, {j, 3}], 1_2];
Timing@{cω // m_{2→3}, HL@Simp[CU[m_{2→3}[cω]] - m_{2→3}[CU[cω]]]}
```

```
{513.453, {Œ_CU[{y1, a1, x1}_1, {y3, a3, x3}_3, ⋯1⋯ / (e^⋯1⋯ ⋯1⋯ ⋯1⋯),
   1/(1+ℏ t3 γ32) + ((4 e^(2 γ ℏ l12 t1 + ⋯4⋯ + 2 γ ℏ l33 t3) ℏ^2 a3 x1 y1 γ12 γ31 - 2 ⋯7⋯ γ31 + ⋯154⋯) ∈) /
    (2 e^(2 γ ℏ l12 t1 + 2 γ ℏ l13 t1 + ⋯3⋯ + 2 γ ℏ l33 t3) + 10 e^⋯1⋯ ℏ t3 γ32 + ⋯2⋯ +
      ⋯1⋯ + 2 e^(2 γ ℏ l12 t1 + ⋯4⋯ + ⋯1⋯) ℏ^5 t3^5 γ32^5) + ((⋯1⋯) ∈^2)/⋯1⋯ + O[∈]^3], 0}}
```

large output | show less | show more | show all | set size limit...

Verifying that *m* commutes with evaluation, in QU:

```
qω = Œ_QU[{y1, a1, x1}_1, {y2, a2, x2}_2,
   {y3, a3, x3}_3, ℏ Sum[l_{10 i+j} t_i a_j + γ_{10 i+j} y_i x_j, {i, 3}, {j, 3}], 1_2];
Timing@{qω // m_{2→3}, HL@SimpT[QU[m_{2→3}[qω]] - m_{2→3}[QU[qω]]]}
```

```
{7831.47, {Œ_QU[{y1, a1, x1}_1, {y3, a3, x3}_3, ⋯1⋯/⋯1⋯, 1/(1-γ32+T3 γ32) +
   ((8 e^(2 γ ℏ l12 t1 + ⋯4⋯ + 2 γ ℏ l33 t3) ℏ^2 a3 T3 x1 y1 γ12 γ31 + 4 ⋯8⋯ γ31 + ⋯371⋯) ∈) /
    (4 e^(2 γ ℏ l12 t1 + 2 γ ℏ l13 t1 + ⋯3⋯ + 2 γ ℏ l33 t3) - 20 e^⋯1⋯ γ32 + ⋯26⋯ +
      4 e^(2 γ ℏ l12 t1 + ⋯4⋯ + ⋯1⋯) T3^5 γ32^5) + ((⋯1⋯) ∈^2)/(⋯79⋯ + ⋯1⋯) + O[∈]^3], 0}}
```

large output | show less | show more | show all | set size limit...

```
In[∘]:= Œ_U_[sp1__, Q1_, P1_] ≡ Œ_U_[sp2__, Q2_, P2_] :=
  Sort[{sp1}] == Sort[{sp2}] ∧ Simplify[Q1 == Q2] ∧ Simplify[Normal[P1 - P2] == 0]
```

Verifying meta-associativity in CU:

```
co = Œ_CU[{y₁, a₁, x₁}₁, {y₂, a₂, x₂}₂,
    {y₃, a₃, x₃}₃, ℏ Sum[λ_{10 i+j} t_i a_j + γ_{10 i+j} y_i x_j, {i, 3}, {j, 3}], 1₀];
Timing@HL[(lhs = co // m_{1,2→1} // m_{1,3→1}) ≡ (rhs = co // m_{2,3→2} // m_{1,2→1})]
```

{41.9219, True}

```
co = Œ_CU[{y₁, a₁, x₁}₁, {y₂, a₂, x₂}₂,
    {y₃, a₃, x₃}₃, ℏ Sum[λ_{10 i+j} t_i a_j + γ_{10 i+j} y_i x_j, {i, 3}, {j, 3}], 1₁];
Timing@HL[(lhs = co // m_{1,2→1} // m_{1,3→1}) ≡ (rhs = co // m_{2,3→2} // m_{1,2→1})]
```

{30 119.8, True}

mexamples

```
co = Œ_CU[{y₁, a₁, x₁}₁, {y₂, a₂, x₂}₂, ℏ Sum[l_{10 i+j} t_i a_j + γ_{10 i+j} y_i x_j, {i, 2}, {j, 2}], 1₁];
Short[Simplify /@ (cexample = co // m_{1→2}), 12]
Short[Simplify /@ (qexample = (qo = co /. CU → QU) // m_{1→2}), 12]
```

mexamples

$$\text{Œ}_{CU}\Big[\{y_2, a_2, x_2\}_2,\ \hbar a_2 (l_{11}+l_{12}+l_{21}+l_{22})\, t_2 + \frac{1}{1+\hbar t_2 \gamma_{21}} e^{-\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar x_2 y_2 \Big(\gamma_{21} + e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\gamma_{12}(1+\hbar t_2\gamma_{21}) + e^{\gamma\hbar(l_{12}+l_{22})t_2}\gamma_{22} + \gamma_{11}\big(e^{\gamma\hbar(l_{11}+l_{21})t_2} - e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar t_2\gamma_{22}\big)\Big),$$
$$\frac{1}{1+\hbar t_2\gamma_{21}} + \frac{1}{2(1+\hbar t_2\gamma_{21})^5} e^{-2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar \Big(4 a_2 (1+\hbar t_2\gamma_{21})^2 \Big(e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar\big(e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2} t_2 + x_2 y_2\big)\gamma_{21}^2 + e^{2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar x_2 y_2\gamma_{11}\gamma_{22} + \gamma_{21}\big(e^{2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2} + \hbar x_2 y_2 \big(e^{\gamma\hbar(2l_{11}+l_{12}+2l_{21}+l_{22})t_2}\gamma_{11} + e^{\gamma\hbar(l_{11}+2l_{12}+l_{21}+2l_{22})t_2}\gamma_{22}\big)\big)\Big) - \gamma\hbar\Big(-2e^{2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2} t_2\gamma_{21}^2(1+\hbar t_2\gamma_{21})^2 + 4 \ll 5 \gg (\ll 1 \gg) + \hbar \ll 4 \gg \big(3\hbar t_2\gamma_{21}^2 + 2e^{\gamma\hbar(l_{12}+l_{\ll 2 \gg})t_2}\gamma_{22} + \gamma_{21}(4 + e^{\gamma \ll 3 \gg}\hbar t_2\gamma_{22}) + e^{\gamma\hbar(l_{11}+l_{21})t_2}\gamma_{11}\big(2+\hbar t_2(\gamma_{21} - e^{\gamma\hbar(l_{12}+l_{22})t_2}\gamma_{22})\big)\big)\Big)\Big)\epsilon + O[\epsilon]^2\Big]$$

mexamples

$$\text{Œ}_{QU}\Big[\{y_2, a_2, x_2\}_2,\ \hbar a_2 (l_{11}+l_{12}+l_{21}+l_{22})\, t_2 + \frac{1}{1+(-1+T_2)\gamma_{21}} e^{-\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar x_2 y_2\Big(\gamma_{21} + e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\gamma_{12}\big(1+(-1+T_2)\gamma_{21}\big) + e^{\gamma\hbar(l_{12}+l_{22})t_2}\gamma_{22} + \gamma_{11}\big(e^{\gamma\hbar(l_{11}+l_{21})t_2} - e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}(-1+T_2)\gamma_{22}\big)\Big),$$
$$\frac{1}{1+(-1+T_2)\gamma_{21}} + \frac{1}{4\big(1+(-1+T_2)\gamma_{21}\big)^5} e^{-2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar \Big(8 a_2 T_2 \big(1+(-1+T_2)\gamma_{21}\big)^2 \Big(e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\big(-e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2} + e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2} T_2 + \hbar x_2 y_2\big)\gamma_{21}^2 + e^{2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar x_2 y_2\gamma_{11}\gamma_{22} + \gamma_{21}\big(e^{2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2} + \hbar x_2 y_2\big(e^{\gamma\hbar(2l_{11}+l_{12}+2l_{21}+l_{22})t_2}\gamma_{11} + e^{\gamma\hbar(l_{11}+2l_{12}+l_{21}+2l_{22})t_2}\gamma_{22}\big)\big)\Big) + \gamma\Big(2e^{2\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}(1-4T_2+3T_2^2)\gamma_{21}^2\big(1+(-1+T_2)\gamma_{21}\big)^2 + 4e^{\gamma\hbar(l_{11}+l_{12}+l_{21}+l_{22})t_2}\hbar x_2 y_2\gamma_{21}\big(1+(-1+T_2)\gamma_{21}\big)(\ll 1 \gg) - \ll 1 \gg\Big)\Big)\epsilon + O[\epsilon]^2\Big]$$

R in QU.

The Faddeev-Quesne formula:

Faddeev

In[∘]:=
$$\mathbb{e}_{q\_,k\_}[x\_] := \mathbb{e}^\wedge\left(\sum_{j=1}^{k+1} \frac{(1-q)^j x^j}{j\,(1-q^j)}\right);\ \mathbb{e}_{q\_}[x\_] := \mathbb{e}_{q,\$k}[x]$$

```
Table[Series[𝕖_{q_ℏ,k}[x], {ϵ, 0, 4}], {k, 0, 5}] // Column
```

$$\mathbb{e}^x$$

$$\mathbb{e}^x - \frac{1}{4}\left(\mathbb{e}^x x^2 \gamma \hbar\right)\epsilon + \frac{1}{32}\mathbb{e}^x x^4 \gamma^2 \hbar^2 \epsilon^2 - \frac{1}{384}\left(\mathbb{e}^x x^2\left(-8+x^4\right)\gamma^3\hbar^3\right)\epsilon^3 + \frac{\mathbb{e}^x x^4\left(-32+x^4\right)\gamma^4\hbar^4\epsilon^4}{6144} + O[\epsilon]^5$$

$$\mathbb{e}^x - \frac{1}{4}\left(\mathbb{e}^x x^2 \gamma \hbar\right)\epsilon + \frac{1}{288}\mathbb{e}^x x^3 (32+9x)\gamma^2\hbar^2\epsilon^2 - \frac{\left(\mathbb{e}^x x^2\left(-24+32x^3+3x^4\right)\gamma^3\hbar^3\right)\epsilon^3}{1152} + \frac{\mathbb{e}^x x^3\left(-4608-864x+1024x^3+576x^4+27x^5\right)\gamma^4\hbar^4\epsilon^4}{165\,888} + O[\epsilon]^5$$

$$\mathbb{e}^x - \frac{1}{4}\left(\mathbb{e}^x x^2 \gamma \hbar\right)\epsilon + \frac{1}{288}\mathbb{e}^x x^3 (32+9x)\gamma^2\hbar^2\epsilon^2 - \frac{\left(\mathbb{e}^x x^2\left(-24+72x^2+32x^3+3x^4\right)\gamma^3\hbar^3\right)\epsilon^3}{1152} + \frac{\mathbb{e}^x x^3\left(-4608-864x+3616x^3+576x^4+27x^5\right)\gamma^4\hbar^4\epsilon^4}{165\,888} + O[\epsilon]^5$$

$$\mathbb{e}^x - \frac{1}{4}\left(\mathbb{e}^x x^2 \gamma \hbar\right)\epsilon + \frac{1}{288}\mathbb{e}^x x^3 (32+9x)\gamma^2\hbar^2\epsilon^2 - \frac{\left(\mathbb{e}^x x^2\left(-24+72x^2+32x^3+3x^4\right)\gamma^3\hbar^3\right)\epsilon^3}{1152} + \frac{1}{4\,147\,200}\mathbb{e}^x x^3\left(-115\,200-21\,600x+165\,888x^2+90\,400x^3+14\,400x^4+675x^5\right)\gamma^4\hbar^4\epsilon^4 + O[\epsilon]^5$$

$$\mathbb{e}^x - \frac{1}{4}\left(\mathbb{e}^x x^2 \gamma \hbar\right)\epsilon + \frac{1}{288}\mathbb{e}^x x^3 (32+9x)\gamma^2\hbar^2\epsilon^2 - \frac{\left(\mathbb{e}^x x^2\left(-24+72x^2+32x^3+3x^4\right)\gamma^3\hbar^3\right)\epsilon^3}{1152} + \frac{1}{4\,147\,200}\mathbb{e}^x x^3\left(-115\,200-21\,600x+165\,888x^2+90\,400x^3+14\,400x^4+675x^5\right)\gamma^4\hbar^4\epsilon^4 + O[\epsilon]^5$$

```
Table[Together@SeriesCoefficient[𝕖_{q,5}[x], {x, 0, n}], {n, 0, 5}]
```

$$\left\{1,\ 1,\ \frac{1}{1+q},\ \frac{1}{(1+q)\left(1+q+q^2\right)},\ \frac{1}{(1+q)^2\left(1+q^2\right)\left(1+q+q^2\right)},\ 1\Big/\left((1+q)^2\left(1+q^2\right)\left(1+q+q^2\right)\left(1+q+q^2+q^3+q^4\right)\right)\right\}$$

```
Table[HL@FunctionExpand[QFactorial[n, q] SeriesCoefficient[𝕖_{q,5}[x], {x, 0, n}]], {n, 0, 5}]
```

{1, 1, 1, 1, 1, 1}

R

In[∘]:=
$$\mathrm{QU}[R_{i\_,j\_}] := \mathbb{O}_{QU}\left[\{y_1, a_1\}_i, \{a_2, x_2\}_j, \mathrm{SS}\left[\mathbb{e}^{\hbar b_1 a_2}\,\mathbb{e}_{q_\hbar}[\hbar\, y_1 x_2] \;/.\; b_1 \to \gamma^{-1}\left(\epsilon\, a_1 - t_i\right)\right]\right];$$
$$\mathrm{QU}\left[R^{-1}_{i\_,j\_}\right] := S_j@\mathrm{QU}[R_{i,j}];$$

```
QU[R_{3,4}] // Short
```

$$\mathrm{QU}[\,] + \frac{\epsilon\,\hbar\,\mathrm{QU}[a_3, a_4]}{\gamma} + \hbar\,\mathrm{QU}[y_3, x_4] + \frac{\epsilon \ll 1\gg \ll 1\gg}{\gamma} + \frac{1}{2}\ll 1\gg \ll 1\gg - \frac{\ll 1\gg}{\gamma} - \frac{\epsilon\,\hbar^2 \ll 1\gg t_3}{\gamma^2} - \frac{\hbar^2\,\mathrm{QU}[y_3, a_4, x_4]\,t_3}{\gamma} + \frac{\hbar^2\,\mathrm{QU}[a_4, a_4]\,t_3^2}{2\gamma^2}$$

Verifying R2 (~2 secs @ $p=4, $k=2):

```
QU[R_{1,2} ** R^{-1}_{1,2}] // Simp // HL // Timing
```

{0.078125, QU[]}

Verifying R3 (~156 secs @ $p=4, $k=2):

$$\{\text{Short}[\text{lhs} = \text{QU}[R_{1,2} ** R_{1,3} ** R_{2,3}]],\ \text{HL@SimpT}[\text{lhs} - \text{QU}[R_{2,3} ** R_{1,3} ** R_{1,2}]]\}\ //\ \text{Timing}$$

$$\left\{0.203125,\ \left\{\text{QU}[\,] + \frac{\epsilon\,\hbar\,\text{QU}[a_1, a_2]}{\gamma} + \frac{\epsilon\,\hbar\,\text{QU}[a_1, a_3]}{\gamma} + \ll 73 \gg + 2\,\epsilon\,\hbar^2\,\text{QU}[y_1, a_2, x_3]\,T_2 + \text{QU}[y_1, x_3]\,(\hbar - \hbar\,T_2),\ 0\right\}\right\}$$

R in $Œ_{QU}$.

RinOE

In[ ]:=

$$Œ_{QU,k\_}[R_{i\_,j\_}] := Œ_{QU}\Big[\{y_i, a_i, x_i\}_i,\ \{y_j, a_j, x_j\}_j,\ -\hbar\,\gamma^{-1}\,t_i\,a_j + \hbar\,y_i\,x_j,$$
$$\text{Series}\Big[e^{\hbar\gamma^{-1}t_i a_j - \hbar y_i x_j}\left(e^{\hbar\,b_i\,a_j}\,e_{q_\hbar,k}[\hbar\,y_i\,x_j]\ /.\ b_i \to \gamma^{-1}\,(\epsilon\,a_i - t_i)\right),\ \{\epsilon, 0, k\}\Big]\Big]$$

$$\{Œ_{QU,1}[R_{1,2}],\ Œ_{QU,2}[R_{1,2}]\}$$

$$\Big\{Œ_{QU}\Big[\{y_1, a_1, x_1\}_1,\ \{y_2, a_2, x_2\}_2,\ -\frac{\hbar\,a_2\,t_1}{\gamma} + \hbar\,x_2\,y_1,\ 1 + \left(\frac{\hbar\,a_1\,a_2}{\gamma} - \frac{1}{4}\gamma\,\hbar^3\,x_2^2\,y_1^2\right)\epsilon + O[\epsilon]^2\Big],$$
$$Œ_{QU}\Big[\{y_1, a_1, x_1\}_1,\ \{y_2, a_2, x_2\}_2,\ -\frac{\hbar\,a_2\,t_1}{\gamma} + \hbar\,x_2\,y_1,\ 1 + \left(\frac{\hbar\,a_1\,a_2}{\gamma} - \frac{1}{4}\gamma\,\hbar^3\,x_2^2\,y_1^2\right)\epsilon +$$
$$\frac{1}{288\,\gamma^2}\left(144\,\hbar^2\,a_1^2\,a_2^2 - 72\,\gamma^2\,\hbar^4\,a_1\,a_2\,x_2^2\,y_1^2 + 32\,\gamma^4\,\hbar^5\,x_2^3\,y_1^3 + 9\,\gamma^4\,\hbar^6\,x_2^4\,y_1^4\right)\epsilon^2 + O[\epsilon]^3\Big]\Big\}$$

The morphism $Œ_{U,k}$.

MorphismOE

In[ ]:=

$$Œ_{U\_,k\_}[a\_ * b\_] := Œ_{U,k}[a]\ Œ_{U,k}[b];$$
$$Œ_{U\_,k\_}[m_{is\_\_}[a\_]] := m_{is}[Œ_{U,k}[a]];$$

$$Œ_{QU,1}[R_{1,2}\,R_{3,4}\,R_{5,6}]$$

$$Œ_{QU}\Big[\{y_1, a_1, x_1\}_1,\ \{y_2, a_2, x_2\}_2,\ \{y_3, a_3, x_3\}_3,\ \{y_4, a_4, x_4\}_4,$$
$$\{y_5, a_5, x_5\}_5,\ \{y_6, a_6, x_6\}_6,\ -\frac{\hbar\,a_2\,t_1}{\gamma} - \frac{\hbar\,a_4\,t_3}{\gamma} - \frac{\hbar\,a_6\,t_5}{\gamma} + \hbar\,x_2\,y_1 + \hbar\,x_4\,y_3 + \hbar\,x_6\,y_5,$$
$$1 + \left(\frac{\hbar\,a_1\,a_2}{\gamma} + \frac{\hbar\,a_3\,a_4}{\gamma} + \frac{\hbar\,a_5\,a_6}{\gamma} - \frac{1}{4}\gamma\,\hbar^3\,x_2^2\,y_1^2 - \frac{1}{4}\gamma\,\hbar^3\,x_4^2\,y_3^2 - \frac{1}{4}\gamma\,\hbar^3\,x_6^2\,y_5^2\right)\epsilon + O[\epsilon]^2\Big]$$

$$Œ_{QU,1}[R_{1,2}\,R_{3,4}\,R_{5,6}\ //\ m_{1,3\to 1}\ //\ m_{2,5\to 2}\ //\ m_{4,6\to 4}]$$

$$Œ_{QU}\Big[\{y_1, a_1, x_1\}_1,\ \{y_2, a_2, x_2\}_2,\ \{y_4, a_4, x_4\}_4,\ \frac{1}{\gamma}$$
$$\left(-\hbar\,a_2\,t_1 - \hbar\,a_4\,t_1 - \hbar\,a_4\,t_2 + \gamma\,\hbar\,x_2\,y_1 + \gamma\,\hbar\,x_4\,y_1 + e^{\hbar\,t_2}\,\gamma\,\hbar\,x_4\,y_1 - \gamma\,\hbar\,T_2\,x_4\,y_1 + e^{\hbar\,t_1}\,\gamma\,\hbar\,x_4\,y_2\right),$$
$$1 + \frac{1}{4\,\gamma}\Big(4\,\hbar\,a_1\,a_2 + 4\,\hbar\,a_1\,a_4 + 4\,\hbar\,a_2\,a_4 - 4\,\gamma\,\hbar^2\,a_4\,x_2\,y_1 - 8\,e^{\hbar\,t_2}\,\gamma\,\hbar^2\,a_2\,x_4\,y_1 +$$
$$8\,\gamma\,\hbar^2\,a_2\,T_2\,x_4\,y_1 - \gamma^2\,\hbar^3\,x_2^2\,y_1^2 + 4\,e^{\hbar\,t_2}\,\gamma^2\,\hbar^3\,x_2\,x_4\,y_1^2 - 4\,\gamma^2\,\hbar^3\,T_2\,x_2\,x_4\,y_1^2 - \gamma^2\,\hbar^3\,x_4^2\,y_1^2 -$$
$$e^{2\,\hbar\,t_2}\,\gamma^2\,\hbar^3\,x_4^2\,y_1^2 + \gamma^2\,\hbar^3\,T_2^2\,x_4^2\,y_1^2 - 4\,e^{\hbar\,t_1}\,\gamma\,\hbar^2\,a_1\,x_4\,y_2 + 4\,e^{\hbar\,t_1}\,\gamma^2\,\hbar^3\,x_2\,x_4\,y_1\,y_2 +$$
$$4\,e^{\hbar\,t_1 + \hbar\,t_2}\,\gamma^2\,\hbar^3\,x_4^2\,y_1\,y_2 - 4\,e^{\hbar\,t_1}\,\gamma^2\,\hbar^3\,T_2\,x_4^2\,y_1\,y_2 - e^{2\,\hbar\,t_1}\,\gamma^2\,\hbar^3\,x_4^2\,y_2^2\Big)\,\epsilon + O[\epsilon]^2\Big]$$

```
ŒQU,1[R1,2 R3,4 R5,6 // m3,1→1 // m5,2→2 // m6,4→4]
```

$$\mathbb{E}_{QU}\Big[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, \{y_4, a_4, x_4\}_4, \frac{1}{\gamma}\left(-\hbar a_2 t_1 - \hbar a_4 t_1 - \hbar a_4 t_2 + \gamma \hbar x_2 y_1 + \gamma \hbar x_4 y_1 + e^{\hbar t_1} \gamma \hbar x_4 y_2\right),$$
$$1 + \frac{1}{4\gamma}\left(4\hbar a_1 a_2 + 4\hbar a_1 a_4 + 4\hbar a_2 a_4 - 4\gamma \hbar^2 a_4 x_2 y_1 - \gamma^2 \hbar^3 x_2^2 y_1^2 - \gamma^2 \hbar^3 x_4^2 y_1^2 - 4 e^{\hbar t_1} \gamma \hbar^2 a_1 x_4 y_2 + 4 e^{\hbar t_1} \gamma^2 \hbar^3 x_2 x_4 y_1 y_2 - e^{2\hbar t_1} \gamma^2 \hbar^3 x_4^2 y_2^2\right) \epsilon + O[\epsilon]^2\Big]$$

```
ŒQU,1[R1,2 R3,4 R5,6 // m1,3→1 // m2,5→2 // m4,6→4] ≡ ŒQU,1[R1,2 R3,4 R5,6 // m3,1→1 // m5,2→2 // m6,4→4]
```

$$\hbar\left(e^{\hbar t_2} - T_2\right) x_4 y_1 == 0 \,\&\&\, \epsilon\hbar\left(e^{\hbar t_2} - T_2\right) x_4 y_1 \left(8 a_2 + \gamma\hbar\left(-4 x_2 y_1 + x_4\left(\left(e^{\hbar t_2} + T_2\right) y_1 - 4 e^{\hbar t_1} y_2\right)\right)\right) == 0$$

The antipode on exponentials in QU.

Computing $S(e^{\xi x})$: If $S(e^{\xi x}) = \mathbb{O}(ax : F e^{-\xi x})$,

then $F_{\xi=0} = 1$ and $\mathbb{O}(ax : (\partial_\xi F - x F) e^{-\xi x}) = \partial_\xi S(e^{\xi x}) = S(x e^{\xi x}) =$
$S(e^{\xi x}) S(x) = \mathbb{O}(ax : F e^{-\xi x})(-e^{\hbar\epsilon a} x) = \mathbb{O}(a x a_2 x_2 : -x_2 F e^{-\xi x + \hbar\epsilon a_2})$, and that's an ODE for F.

SxF

In[∘]:=
```
SxF[0] = 1;
SxF[k_] := SxF[k] = Module[{fs, bs, F, rhs, at0, atξ},
    fs = Flatten@Table[f_{i,j}[ξ], {i, 0, 2k}, {j, 0, 2k - i}];
    F = SxF[k - 1] + ϵ^k fs.(bs = fs /. f_{i_,j_}[ξ] :> a^i x^j);
    rhs = Normal@Last@Cord[ŒQU[{a1, x1, a2, x2}1, -ξ x1, (F /. {a → a1, x → x1})
          Series[-x2 e^(ħ ϵ a2), {ϵ, 0, k}]] /. ξ → ħ ξ] /. {ξ → ħ^-1 ξ, a1 → a, x1 → x};
    at0 = (# == 0) & /@ Flatten@CoefficientList[F - 1 /. ξ → 0, {a, x}];
    atξ = (# == 0) & /@ Flatten@CoefficientList[(∂_ξ F) - x F - rhs, {a, x}];
    F /. DSolve[And @@ (at0 ⋃ atξ), fs, ξ][[1]]
   ];
```

In[∘]:=
```
Timing@Block[{$p = 8, $k = 3}, {
   Collect[SxF[$k], {ϵ, a}],
   HL@Simp[S1@QU1@𝕆QU[{x}, SS[e^(ħ ξ x)]] - QU1@𝕆QU[{a, x}, SS[e^(-ħ ξ x) (SxF[$k] /. ξ → ħ ξ)]]]
  }]
```

Out[∘]=
$$\Big\{2.76563, \Big\{1 + \epsilon\left(-a x \xi \hbar - \frac{1}{2} x^2 \gamma \xi^2 \hbar\right) + \epsilon^2\left(\frac{1}{4} x^2 \gamma^2 \xi^2 \hbar^2 - \frac{1}{2} x^3 \gamma^2 \xi^3 \hbar^2 + \frac{1}{8} x^4 \gamma^2 \xi^4 \hbar^2 + a^2\left(-\frac{1}{2} x \xi \hbar^2 + \frac{1}{2} x^2 \xi^2 \hbar^2\right) + a\left(-x^2 \gamma \xi^2 \hbar^2 + \frac{1}{2} x^3 \gamma \xi^3 \hbar^2\right)\right) +$$
$$\epsilon^3\left(-\frac{1}{12} x^2 \gamma^3 \xi^2 \hbar^3 + \frac{2}{3} x^3 \gamma^3 \xi^3 \hbar^3 - \frac{19}{24} x^4 \gamma^3 \xi^4 \hbar^3 + \frac{1}{4} x^5 \gamma^3 \xi^5 \hbar^3 - \frac{1}{48} x^6 \gamma^3 \xi^6 \hbar^3 + a^3\left(-\frac{1}{6} x \xi \hbar^3 + \frac{1}{2} x^2 \xi^2 \hbar^3 - \frac{1}{6} x^3 \xi^3 \hbar^3\right) + a^2\left(-x^2 \gamma \xi^2 \hbar^3 + \frac{5}{4} x^3 \gamma \xi^3 \hbar^3 - \frac{1}{4} x^4 \gamma \xi^4 \hbar^3\right) + a\left(\frac{1}{2} x^2 \gamma^2 \xi^2 \hbar^3 - \frac{7}{4} x^3 \gamma^2 \xi^3 \hbar^3 + x^4 \gamma^2 \xi^4 \hbar^3 - \frac{1}{8} x^5 \gamma^2 \xi^5 \hbar^3\right)\right), 0\Big\}\Big\}$$

Computing $S(e^{\eta y})$: If $S(e^{\eta y}) = \mathbb{O}(ya : F e^{-T^{-1}\eta y})$,

then $F_{\eta=0} = 1$ and $\mathbb{O}(ya : (\partial_\eta F - T^{-1} y F) e^{-T^{-1}\eta y}) = \partial_\eta S(e^{\eta y}) = S(y e^{\eta y}) = S(e^{\eta y}) S(y) =$

$$\mathbb{O}\left(\text{ya}: F e^{-T^{-1}\eta y}\right)\left(-e^{\hbar\epsilon a}\,T^{-1} y\right) = \mathbb{O}\left(\text{yaa}_2\, y_2: -T^{-1} y_2\, F e^{-T^{-1}\eta y + \hbar\epsilon a_2}\right),$$ and that's an ODE for F.

SyF

In[∘]:=
```
SyF[0] = 1;
SyF[k_] := SyF[k] = Module[{fs, bs, F, rhs, at0, atη},
    fs = Flatten@Table[f_{i,j}[η], {i, 0, 2 k}, {j, 0, 2 k - i}];
    F = SyF[k - 1] + ϵ^k fs.(bs = fs /. f_{i_,j_}[η] :→ y^i a^j);
    rhs = Normal@Last@Cord[𝔼_QU[{y1, a1, a2, y2}1, -T^-1 η y1, (F /. {a → a1, y → y1})
            Series[-y2 T^-1 e^(ħ ϵ a2), {ϵ, 0, k}]] /. η → ħ η] /. {η → ħ^-1 η, a1 → a, y1 → y};
    at0 = (# == 0) & /@ Flatten@CoefficientList[F - 1 /. η → 0, {a, y}];
    atη = (# == 0) & /@ Flatten@CoefficientList[(∂_η F) - T^-1 y F - rhs, {a, y}];
    F /. DSolve[And @@ (at0 ⋃ atη), fs, η][[1]]
  ];
```

In[∘]:=
```
Timing@Block[{$p = 8, $k = 3}, {
    Collect[SyF[$k], {ϵ, a}],
    HL@Simp[S1@QU1@𝕆_QU[{y}, SS[e^(ħ η y)]] - QU1@𝕆_QU[{y, a}, SS[e^(-ħ η y/T) (SyF[$k] /. η → ħ η)]]]
  }]
```

Out[∘]=

$$\Big\{10.5469,$$

$$\Big\{1+\epsilon\left(-\frac{a\,y\,\eta\,\hbar}{T}+\frac{y\,\gamma\,\eta\,\hbar}{T}-\frac{y^2\,\gamma\,\eta^2\,\hbar}{2\,T^2}\right)+\epsilon^2\left(-\frac{y\,\gamma^2\,\eta\,\hbar^2}{2\,T}+\frac{7\,y^2\,\gamma^2\,\eta^2\,\hbar^2}{4\,T^2}-\frac{y^3\,\gamma^2\,\eta^3\,\hbar^2}{T^3}+\frac{y^4\,\gamma^2\,\eta^4\,\hbar^2}{8\,T^4}+\right.$$

$$\left.a^2\left(-\frac{y\,\eta\,\hbar^2}{2\,T}+\frac{y^2\,\eta^2\,\hbar^2}{2\,T^2}\right)+a\left(\frac{y\,\gamma\,\eta\,\hbar^2}{T}-\frac{2\,y^2\,\gamma\,\eta^2\,\hbar^2}{T^2}+\frac{y^3\,\gamma\,\eta^3\,\hbar^2}{2\,T^3}\right)\right)+$$

$$\epsilon^3\left(\frac{y\,\gamma^3\,\eta\,\hbar^3}{6\,T}-\frac{25\,y^2\,\gamma^3\,\eta^2\,\hbar^3}{12\,T^2}+\frac{23\,y^3\,\gamma^3\,\eta^3\,\hbar^3}{6\,T^3}-\frac{49\,y^4\,\gamma^3\,\eta^4\,\hbar^3}{24\,T^4}+\frac{3\,y^5\,\gamma^3\,\eta^5\,\hbar^3}{8\,T^5}-\frac{y^6\,\gamma^3\,\eta^6\,\hbar^3}{48\,T^6}+\right.$$

$$a^3\left(-\frac{y\,\eta\,\hbar^3}{6\,T}+\frac{y^2\,\eta^2\,\hbar^3}{2\,T^2}-\frac{y^3\,\eta^3\,\hbar^3}{6\,T^3}\right)+a^2\left(\frac{y\,\gamma\,\eta\,\hbar^3}{2\,T}-\frac{5\,y^2\,\gamma\,\eta^2\,\hbar^3}{2\,T^2}+\frac{7\,y^3\,\gamma\,\eta^3\,\hbar^3}{4\,T^3}-\frac{y^4\,\gamma\,\eta^4\,\hbar^3}{4\,T^4}\right)+$$

$$\left.a\left(-\frac{y\,\gamma^2\,\eta\,\hbar^3}{2\,T}+\frac{4\,y^2\,\gamma^2\,\eta^2\,\hbar^3}{T^2}-\frac{19\,y^3\,\gamma^2\,\eta^3\,\hbar^3}{4\,T^3}+\frac{3\,y^4\,\gamma^2\,\eta^4\,\hbar^3}{2\,T^4}-\frac{y^5\,\gamma^2\,\eta^5\,\hbar^3}{8\,T^5}\right)\right),\,0\Big\}\Big\}$$

Next task: Define $S\Lambda_{U,k}[\eta, \alpha, \xi, \delta]$, whose value is an $\mathbb{E}_U[\{y_1, a_1, x_1\}_1,\ Q,\ P+0_k]$ such that

U@SΛ_{U,k}[η, α, ξ, δ] == S1@U@𝔼_U[{y1, a1, x1}1, η y1 + α a1 + ξ x1 + δ x1 y1, 0_k].

Alternative Algorithms

AltLogos

In[∘]:=
```
λ_{alt,k_}[CU] := If[k == 0, 1, Module[{eq, d, b, c, so},
    eq = ρ@e^(ξ x_CU).ρ@e^(η y_CU) == ρ@e^(d y_CU).ρ@e^(c (t 1_CU - 2 ϵ a_CU)).ρ@e^(b x_CU);
    {so} = Solve[Thread[Flatten /@ eq], {d, b, c}] /. C@1 → 0;
    Series[e^(-η y - ξ x + η ξ t + c t + d y - 2 ϵ c a + b x) /. so, {ϵ, 0, k}]]];
```

```
{λalt,2[CU], HL@Simplify@Normal[λalt,2[CU] == Last[ΛCU,2[{ξ, η}, {x, y}]]]}
```

$$\left\{1+\left(2\,a\,\eta\,\xi-y\,\gamma\,\eta^2\,\xi-x\,\gamma\,\eta\,\xi^2+\frac{1}{2}\,t\,\gamma\,\eta^2\,\xi^2\right)\epsilon+\frac{1}{2}\left(\left(2\,a\,\eta\,\xi-y\,\gamma\,\eta^2\,\xi-x\,\gamma\,\eta\,\xi^2+\frac{1}{2}\,t\,\gamma\,\eta^2\,\xi^2\right)^2+2\left(-a\,\gamma\,\eta^2\,\xi^2+y\,\gamma^2\,\eta^3\,\xi^2+x\,\gamma^2\,\eta^2\,\xi^3-\frac{1}{3}\,t\,\gamma^2\,\eta^3\,\xi^3\right)\right)\epsilon^2+O[\epsilon]^3,\ \text{True}\right\}$$