Pensieve header: A unified verification notebook for the sl_2-portfolio project; continues pensieve://Projects/PPSA/nb/Verification.pdf.

Continues pensieve://2017-06/ and pensieve://2017-08/.

Prolog

```
Go;
```

```
wdir = SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\SL2Portfolio"];
NotebookOpen[wdir <> "\\MakeSnips.nb"];
```

```
HL[ℰ_] := Style[ℰ, Background → Yellow];
```

Initialization / Utilities

The “degree carrier / filtration parameter” is ℏ, and all “coupling constants” are proportional to it.

TD

```
$p = 2; $k = 1;
If[$k == 0, ϵ = 0, ϵ /: ϵ^k_ /; k > $k := 0];
SetAttributes[{SS, SST}, HoldAll];
TRule = {T_i_ → e^(ℏ t_i), T → e^(ℏ t)}; qrule = q → e^(γ ϵ ℏ);
SS[ℰ_] := Block[{ℏ, ϵ}, (* Shielded Series *)
   Collect[Normal@Series[ℰ, {ℏ, 0, $p}], ℏ, Together] ];
SST[ℰ_] :=
  Block[{ℏ, ϵ}, Collect[Normal@Series[ℰ /. TRule, {ℏ, 0, $p}], ℏ, Together] ];
Simp[ℰ_, op_] := Collect[ℰ, _CU | _QU, op];
Simp[ℰ_] := Simp[ℰ, Collect[Normal@Series[#, {ℏ, 0, $p}], ℏ, Expand] &];
SimpT[ℰ_] :=
  Collect[ℰ, _CU | _QU, Collect[Normal@Series[# /. TRule, {ℏ, 0, $p}], ℏ, Expand] &];
```

Differential polynomials (DP):

Utils

```
DP_{α_→D_x_, β_→D_y_}[P_][λ_] :=
 Total[CoefficientRules[P, {α, β}] /. ({m_, n_} → c_) :→ c D[λ, {x, m}, {y, n}]]
```

DeclareAlgebra

QLImplementation

```
Unprotect[NonCommutativeMultiply]; Attributes[NonCommutativeMultiply] = {};
(NCM = NonCommutativeMultiply)[x_] := x;
NCM[x_, y_, z__] := (x ** y) ** z;
0 ** _ = _ ** 0 = 0;
(x_Plus) ** y_ := (# ** y) & /@ x; x_ ** (y_Plus) := (x ** #) & /@ y;
B[x_, x_] = 0; B[x_, y_] := x ** y - y ** x;
```

QLImplementation

```
DeclareAlgebra[U_Symbol, opts__Rule] := Module[{gp, sr, g, cp, M, CE, pow, k = 0,
    gs = Generators /. {opts}, cs = Centrals /. {opts} },
   (#_U = U@#) & /@ gs;
   gp = Alternatives @@ gs; gp = gp | gp_; (* gens *)
   sr = Flatten@Table[{g → ++k, g_i_ → {i, k}}, {g, gs}]; (* sorting → *)
   cp = Alternatives @@ cs; (* cents *)
   SetAttributes[M, HoldRest]; M[0, _] = 0; M[a_, x_] := a x;
   CE[ℰ_] := Collect[ℰ, _U, (Expand[#] /. ħ^d_ /; d > $p → 0) &];
   U_i_[ℰ_] := ℰ /. {t : cp :→ t_i, u_U :→ Replace[u, x_ :→ x_i, 1]};
   U_i_[NCM[]] = pow[ℰ_, 0] = U@{} = 1_U = U[];
   B[U@(x_)_i_, U@(y_)_i_] := B[U@x_i, U@y_i] = U_i@B[U@x, U@y];
   B[U@(x_)_i_, U@(y_)_j_] /; i =!= j := 0;
   B[U@y_, U@x_] := CE[-B[U@x, U@y]];
   x_ ** (c_. 1_U) := CE[c x]; (c_. 1_U) ** x_ := CE[c x];
   (*(a_.*x_U)**(b_.*y_U) := CE[M[a b,x**y]];*)
   (a_. U[xx___, x_]) ** (b_. U[y_, yy___]) := If[OrderedQ[{x, y} /. sr],
     CE@M[a b, U[xx, x, y, yy]],
     U@xx ** CE@M[a b, U@y ** U@x + B[U@x, U@y]] ** U@yy
    ];
   U@{c_. * (l : gp)^n_, r___} /; FreeQ[c, gp] := CE[c U@Table[l, {n}] ** U@{r}];
   U@{c_. * l : gp, r___} := CE[c U[l] ** U@{r}];
   U@{c_, r___} /; FreeQ[c, gp] := CE[c U@{r}];
   U@{l_Plus, r___} := CE[U@{#, r} & /@ l];
   U@{l_, r___} := U@{Expand[l], r};
   U[ℰ_NonCommutativeMultiply] := U /@ ℰ;
   O_U[specs___, poly_] := Module[{sp, null, vs, us},
     sp = Replace[{specs}, l_List :→ l_null, {1}];
     vs = Join @@ (First /@ sp);
     us = Join @@ (sp /. l_s_ :→ (l /. x_i_ :→ x_s));
     CE[Total[
        CoefficientRules[poly, vs] /. (p_ → c_) :→ c U@(us^p)
       ]] /. x_null :→ x];
   pow[ℰ_, n_] := pow[ℰ, n - 1] ** ℰ;
   S_U[ℰ_, ss___Rule] := CE@Total[
      CoefficientRules[ℰ, First /@ {ss}] /.
       (p_ → c_) :→ c NCM @@ MapThread[pow, {Last /@ {ss}, p}]];
   m_j_→k_[c_. * u_U] := CE[((c /. (t : cp)_j → t_k) DeleteCases[u, _j|k]) **
      U @@ Cases[u, w_j :→ w_k] ** U @@ Cases[u, _k]];
   S_i_[c_. * u_U] := CE[((c /. S_i[U, Centrals]) DeleteCases[u, _i]) **
      U_i[NCM @@ Reverse@Cases[u, x_i :→ S@U@x]]] ]
```

DeclareMorphism

QLImplementation

```
DeclareMorphism[m_, U_ → V_, ongs_List, oncs_List: {}] := (
  Replace[ongs, (g_ → img_) ⧴ (m[U[g]] = img), {1}];
  m[1_U] = 1_V;
  m[U[g_i_]] := V_i[m[U@g]];
  m[U[vs__]] := NCM@@ (m /@ U /@ {vs});
  m[ℰ_] := Simp[ℰ /. oncs /. u_U ⧴ m[u]]; )
```

Meta-Operations

QLImplementation

```
m_{j_→j_} = Identity;
m_{j_→k_}[ℰ_Plus] := Simp[m_{j→k} /@ ℰ];
m_{is___,i_,j_→k_}[ℰ_] := m_{j→k}@m_{is,i→j}@ℰ;
S_i_[ℰ_Plus] := Simp[S_i /@ ℰ];
```

Implementing CU = $\mathcal{U}(sl_2^{\gamma\epsilon})$

CU

```
DeclareAlgebra[CU, Generators → {y, a, x}, Centrals → {t}];
B[a_CU, y_CU] = -γ y_CU; B[x_CU, a_CU] = -γ x_CU;
B[x_CU, y_CU] = 2 ϵ a_CU - t 1_CU;
(S@CU@y = -y_CU; S@a_CU = -a_CU; S@x_CU = -x_CU;)
S_i_[CU, Centrals] = {t_i → -t_i};
```

Verifying associativity on triples of generators:

```
With[{bas = CU /@ {y, a, x}},
 Table[z1 ** (z2 ** z3) - (z1 ** z2) ** z3 // Simp // HL,
  {z1, bas}, {z2, bas}, {z3, bas} ] ]
```

```
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying associativity on a “random” triple:

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
   (rhs = (z1 ** z2) ** z3 // Simp) // Short,
   z1 ** (z2 ** z3) - rhs // Simp // HL
  }] // Timing
```

```
{1.28125,
 {(28 t^2 γ^4 + 116 t γ^5 ϵ) CU[y, y, y, x, x] + <<21>> + CU[y, y, y, y, y, a, a, a, a, x, x, x, x], 0}}
```

Verifying that *S* is an anti-homomorphism on CU:

```
With[{bas = CU /@ {y1, a1, x1}},
 Table[HL@Simp[S1[z1 ** z2] - S1[z2] ** S1[z1]],
  {z1, bas}, {z2, bas} ] ]
```

```
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

Verifying the involutivity of S on products of triples:

```
With[{bas = CU /@ {y1, a1, x1}},
 Table[HL@Simp[z1 ** z2 ** z3 - S1@S1[z1 ** z2 ** z3]],
  {z1, bas}, {z2, bas}, {z3, bas} ] ]
```

```
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying relabeling:

```
t1 t3 CU[y1, a1, x2] + t1 t1 CU[y1, a2, x2] // m1→3
```

$CU[a_2, x_2, y_3]\, t_3^2 + CU[x_2, y_3, a_3]\, t_3^2$

Verifying meta-associativity:

```
Module[{z, u},
 Table[u = CU[z⟦1⟧1, z⟦2⟧2, z⟦3⟧3]; z → HL[m1,3→3@m2,3→3@u == m2,3→3@m1,2→2@u],
  {z, Tuples[{y, a, x}, 3]}]]
```

```
{{y, y, y} → True, {y, y, a} → True, {y, y, x} → True, {y, a, y} → True,
 {y, a, a} → True, {y, a, x} → True, {y, x, y} → True, {y, x, a} → True,
 {y, x, x} → True, {a, y, y} → True, {a, y, a} → True, {a, y, x} → True, {a, a, y} → True,
 {a, a, a} → True, {a, a, x} → True, {a, x, y} → True, {a, x, a} → True, {a, x, x} → True,
 {x, y, y} → True, {x, y, a} → True, {x, y, x} → True, {x, a, y} → True, {x, a, a} → True,
 {x, a, x} → True, {x, x, y} → True, {x, x, a} → True, {x, x, x} → True}
```

Implementing QU = $\mathcal{U}_q(sl_2^{\gamma\epsilon})$

Aside

```
Series[(1 - T ⅇ^(-2 e a h)) / h, {a, 0, 3}]
```

Aside

$$\frac{1-T}{h} + 2\,e\,T\,a - 2\left(e^2\,h\,T\right)a^2 + \frac{4}{3}\,e^3\,h^2\,T\,a^3 + O[a]^4$$

QU

```
DeclareAlgebra[QU, Generators → {y, a, x}, Centrals → {t, T}];
B[aQU, yQU] = -γ yQU; B[xQU, aQU] = -γ QU@x;
B[xQU, yQU] = SS[q - 1 /. qrule /. e → ϵ] QU@{y, x} + 𝕆QU[{a}, SS[(1 - T ⅇ^(-2 ϵ a ℏ)) / ℏ]];
(S@yQU = 𝕆QU[{a, y}, SS[-T^-1 ⅇ^(ℏ ϵ a) y]]; S@aQU = -aQU; S@xQU = 𝕆QU[{a, x}, SS[-ⅇ^(ℏ ϵ a) x]];)
Si_[QU, Centrals] = {ti → -ti, Ti → Ti^-1};
```

```
With[{bas = QU /@ {y, a, x}}, Table[{z1, z2} → Simp[z1 ** z2 - z2 ** z1], {z1, bas}, {z2, bas}] ]
```

$\{\{\{QU[y], QU[y]\} \to 0, \{QU[y], QU[a]\} \to \gamma\, QU[y],$
$\{QU[y], QU[x]\} \to \frac{(-1+T)\, QU[]}{\hbar} - 2 T \epsilon\, QU[a] - \gamma \epsilon \hbar\, QU[y, x]\},$
$\{\{QU[a], QU[y]\} \to -\gamma\, QU[y], \{QU[a], QU[a]\} \to 0, \{QU[a], QU[x]\} \to \gamma\, QU[x]\},$
$\{\{QU[x], QU[y]\} \to \frac{(1-T)\, QU[]}{\hbar} + 2 T \epsilon\, QU[a] + \gamma \epsilon \hbar\, QU[y, x],$
$\{QU[x], QU[a]\} \to -\gamma\, QU[x], \{QU[x], QU[x]\} \to 0\}\}$

Verifying associativity on triples of generators:

```
With[{bas = QU /@ {y, a, x}},
 Table[HL[z1 ** (z2 ** z3) - (z1 ** z2) ** z3] // Simp,
  {z1, bas}, {z2, bas}, {z3, bas} ] ]
```

{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}

Verifying associativity on a “random” triple (~34 secs @ $p=5, $k=2):

```
With[{z1 = QU[y, y, a, a, x, x], z2 = QU[y, a, x], z3 = QU[y, y, a, x]}, {
   (rhs = (z1 ** z2) ** z3 // Simp) // Short,
   HL[z1 ** (z2 ** z3) - rhs // Simp ]
  }] // Timing
```

$\{4.57813, \{\left(\frac{28\gamma^4 - 56 T \gamma^4 + 28 T^2 \gamma^4}{\hbar^2} + \frac{82\gamma^5 \epsilon - 280 \ll 3 \gg + 198 T^2 \gamma^5 \epsilon}{\hbar}\right) QU[y, y, y, x, x] +$
$\ll 18 \gg + (1 + 8\gamma \epsilon \hbar)\, QU[\ll 1 \gg], 0\}\}$

Verifying that *S* is an anti-homomorphism on QU:

```
With[{bas = QU /@ {y1, a1, x1}},
 Table[{z1, z2} → HL@Simp[S1[z1 ** z2] - S1[z2] ** S1[z1]],
  {z1, bas}, {z2, bas} ] ]
```

$\{\{\{QU[y_1], QU[y_1]\} \to 0, \{QU[y_1], QU[a_1]\} \to 0, \{QU[y_1], QU[x_1]\} \to 0\},$
$\{\{QU[a_1], QU[y_1]\} \to 0, \{QU[a_1], QU[a_1]\} \to 0, \{QU[a_1], QU[x_1]\} \to 0\},$
$\{\{QU[x_1], QU[y_1]\} \to 0, \{QU[x_1], QU[a_1]\} \to 0, \{QU[x_1], QU[x_1]\} \to 0\}\}$

Verifying that $\mathrm{Lim}_{\hbar\to 0}$ QU = CU using a “random” product (~23 secs @ $p=5, $k=2):

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
   Short[lhs = z1 ** (z2 ** z3)],
   Short[rhs = (QU @@ z1) ** ((QU @@ z2) ** (QU @@ z3))],
   Expand[Limit[rhs /. TRule ⋃ {QU → CU}, ℏ → 0] - lhs] // HL
  }] // Timing
```

$\{14.7969, \{28 t^2 \gamma^4\, CU[y, y, y, x, x] +$
$116 t \gamma^5 \epsilon\, CU[y, y, y, x, x] + \ll 44 \gg + CU[y, y, y, y, y, a, a, a, a, x, x, x, x],$
$2\left(\frac{\gamma^4}{\hbar^2} - \frac{2 T \gamma^4}{\hbar^2} + \frac{T^2\gamma^4}{\hbar^2} + \frac{\gamma^5\epsilon}{\hbar} - \frac{2 T \gamma^5 \epsilon}{\hbar} + \frac{T^2 \gamma^5 \epsilon}{\hbar}\right) QU[y, y, y, x, x] +$
$\ll 209 \gg + (1 + 8\gamma\epsilon\hbar)\, QU[y, \ll 11 \gg, x], 0\}\}$

Implementing θ

theta

```
DeclareMorphism[Cθ, CU → CU, {y → -x_CU, a → -a_CU, x → -y_CU}, {t → -t, T → T^-1}];
DeclareMorphism[Qθ, QU → QU, {y → 𝕆_QU[{a, x}, SS[-T^(-1/2) ℯ^(ℏ ϵ a) x]],
  a → -a_QU, x → 𝕆_QU[{a, y}, SS[-T^(-1/2) ℯ^(ℏ ϵ a) y]]}, {t → -t, T → T^-1}]
```

Verifying involutivity on CU:

```
With[{bas = CU /@ {y, a, x}},
 Table[z → Cθ[z] → HL[Cθ[Cθ[z]]], {z, bas}]]
```

{CU[y] → −CU[x] → CU[y], CU[a] → −CU[a] → CU[a], CU[x] → −CU[y] → CU[x]}

Verifying that θ is a multiplicative homomorphism on CU:

```
With[{bas = CU /@ {y, a, x}},
 Table[Cθ[z1 ** z2] - Cθ[z1] ** Cθ[z2] // HL, {z1, bas}, {z2, bas}]]
```

{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

Verifying involutivity on QU:

```
With[{bas = QU /@ {y, a, x}},
 Table[z → Qθ[z] → HL[Simp[Qθ[Qθ[z]], PowerExpand]], {z, bas}]]
```

$$\left\{QU[y] \to -\frac{QU[x]}{\sqrt{T}} - \frac{\epsilon\hbar\, QU[a, x]}{\sqrt{T}} \to QU[y],\ QU[a] \to -QU[a] \to QU[a],\right.$$
$$\left. QU[x] \to \left(-\frac{1}{\sqrt{T}} + \frac{\gamma\epsilon\hbar}{\sqrt{T}}\right) QU[y] - \frac{\epsilon\hbar\, QU[y, a]}{\sqrt{T}} \to QU[x]\right\}$$

Verifying that θ is a multiplicative homomorphism on QU:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL[Simp[Qθ[z1 ** z2] - Qθ[z1] ** Qθ[z2]]], {z1, bas}, {z2, bas}]]
```

{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}

The Asymmetric Dequantizator

Following pensieve://People/VanDerVeen/Dequant1.pdf.

ADeq

$$\mathbb{AD}\$f = \gamma\left(\left(\mathrm{Cosh}\left[\hbar\left(a\,e + \frac{\gamma\,e}{2} - \frac{t}{2}\right)\right] - \mathrm{Cosh}\left[\hbar\sqrt{\left(\frac{t-\gamma\,e}{2}\right)^2 + e\,\omega}\right]\right)\Big/\left(\hbar\, e^{\hbar\,((a+\gamma)\,e - t/2)}\,\mathrm{Sinh}\left[\frac{\gamma\,e\,\hbar}{2}\right]\left(a^2\,e + a\,\gamma\,e - a\,t - \omega\right)\right)\right);$$

Scaling behaviour of A𝔻$f:

```
HL@Simplify[A𝔻$f == ((A𝔻$f /. γ → 1) /. {e → γ e, a → γ^-1 a, ω → γ^-1 ω})]
```

True

```
HL@FullSimplify[
  A𝔻$f == ((A𝔻$f /. γ → 1) /. {ℏ → γ^2 ℏ, e → e / γ, a → a / γ, t → γ^-2 t, ω → γ^-3 ω})]
```

True

ADeq

```
A𝔻$ω = γ CU[y, x] + ϵ CU[a, a] - (t - γ ϵ) CU[a];
```

ADeq

```
DeclareMorphism[A𝔻, QU → CU,
 {a → a_CU, x → CU@x, y → 𝕊_CU[SS[A𝔻$f] /. e → ϵ, a → a_CU, ω → A𝔻$ω] ** y_CU}]
```

Verifying that the asymmetric dequantizator is a homomorphism:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL[SimpT[A𝔻[z1 ** z2] - A𝔻[z1] ** A𝔻[z2]]], {z1, bas}, {z2, bas}] ]
```

```
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

The Symmetric Dequantizator

Following pensieve://People/VanDerVeen/Dequant1.pdf.

SDeq

$$\mathbb{SD}\$g = \sqrt{\left(\left(2\gamma\left(\mathrm{Cosh}\left[\frac{\hbar}{2}\sqrt{t^2+\gamma^2 e^2+4 e \varpi}\right]-\mathrm{Cosh}\left[\frac{t-e\gamma-2ea}{2/\hbar}\right]\right)\right)\Big/\left(\mathrm{Sinh}\left[\frac{\gamma e \hbar}{2}\right]\left(t\,(2a+\gamma)-2a\,(a+\gamma)\,e+2\varpi\right)\hbar\right)\right)};$$

Verify agreement with the formulas in pensieve://People/VanDerVeen/Dequant1.pdf:

```
{SD$P = (Cosh[ℏ ((e-t)/2 + e a)] - Cosh[ℏ Sqrt[(t^2+e^2)/4 + e ϖ]]) / (ℏ Sinh[(-e ℏ)/2] (ϖ - e a^2 + (t - e) a + t/2)),
 Simplify[SD$P == (SD$P /. {a → -a - 1, t → -t})] // HL,
 PowerExpand@Simplify[(SD$P /. {ℏ → γ^2 ℏ, e → e/γ, a → a/γ, t → γ^-2 t, ϖ → γ^-3 ϖ}) ==
    SD$g (SD$g /. {a → -a - γ, t → -t})] // HL,
 SD$Q = Simplify[SD$P /. {a → c - 1/2}],
 Simplify[SD$Q == (SD$Q /. {c → -c, t → -t})] // HL,
 FullSimplify[SD$g == FullSimplify[
    Sqrt[SD$Q] /. c → a + 1/2 /. {ℏ → γ^2 ℏ, e → e/γ, a → a/γ, t → γ^-2 t, ϖ → γ^-3 ϖ}]] // HL
}
```

$\{-\left(\left(\left(\left(\mathrm{Cosh}\left[\left(a\,e+\frac{e-t}{2}\right)\hbar\right]-\mathrm{Cosh}\left[\sqrt{\frac{1}{4}\left(e^2+t^2\right)+e\,\varpi}\;\hbar\right]\right)\mathrm{Csch}\left[\frac{e\,\hbar}{2}\right]\right)\Big/\left(\left(-a^2\,e+\frac{t}{2}+a\,(-e+t)+\varpi\right)\hbar\right)\right)\right)$, True, True,

$\left(4\left(-\mathrm{Cosh}\left[\frac{1}{2}\sqrt{e^2+t^2+4\,e\,\varpi}\;\hbar\right]+\mathrm{Cosh}\left[c\,e\,\hbar-\frac{t\,\hbar}{2}\right]\right)\mathrm{Csch}\left[\frac{e\,\hbar}{2}\right]\right)\Big/\left(\left(\left(-1+4\,c^2\right)e-4\,(c\,t+\varpi)\right)\hbar\right)$,
True, True}

SDeq

```
SD$f = Simplify[e^(ℏ (t/2-e a)) (SD$g /. {a → -a, t → -t})];
```

SDeq

```
SD$ϖ = γ CU[y, x] + ϵ CU[a, a] - (t - γ ϵ) CU[a] - t γ 1_CU / 2;
```

SDeq

```
DeclareMorphism[SD, QU → CU, {a → a_CU,
  x → S_CU[SS[SD$f] /. e → ϵ, a → a_CU, ϖ → SD$ϖ] ** x_CU,
  y → S_CU[SS[SD$g] /. e → ϵ, a → a_CU, ϖ → SD$ϖ] ** y_CU }]
```

Verifying the θ-symmetry:

```
Table[HL@SimpT[Cθ[SD[z]] == SD[Qθ[z]]], {z, QU /@ {y, a, x}}]
```

{True, True, True}

Verifying that the symmetric dequantizator is a homomorphism:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL@SimpT[SD[z1 ** z2] - SD[z1] ** SD[z2]], {z1, bas}, {z2, bas}] ]
```

{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}

The representation ρ

rho

```
ρ@y_CU = ρ@y_QU = (0 0; ϵ 0); ρ@a_CU = ρ@a_QU = (γ 0; 0 0);
ρ@x_CU = (0 γ; 0 0); ρ@x_QU = SS@(0 (1 - e^(-γ ϵ ℏ)) / (ϵ ℏ); 0 0);
ρ[e^ℰ_] := MatrixExp[ρ[ℰ]];
ρ[ℰ_] :=
  (ℰ /. {t → γ ϵ, T → e^(ℏ γ ϵ)} /. (U : CU | QU) [u___] :→ Fold[Dot, (1 0; 0 1), ρ /@ U /@ {u}])
```

Verifying that ρ represents CU and QU:

```
Table[ρ[z1 ** z2] == ρ[z1].ρ[z2] // SS // HL,
 {U, {CU, QU}}, {z1, U /@ {y, a, x}}, {z2, U /@ {y, a, x}} ]
```

```
{{{True, True, True}, {True, True, True}, {True, True, True}},
 {{True, True, True}, {True, True, True}, {True, True, True}}}
```

Œ and the logoi Λ

Logoi from Pensieve://Talks/Toulouse-1705/DogmaDemo.nb and from Pensieve://Talks/Sydney-1708/ExtraDetails@@.nb.

CdsO

```
CU@Œ_CU[specs___, Q_, P_] := 𝕆_CU[specs, SS[e^Q P]];
QU@Œ_QU[specs___, Q_, P_] := 𝕆_QU[specs, SS[e^Q P]];
```

```
HL[ρ[e^(ξ CU@x)].ρ[e^(α CU@a)] == ρ[e^(α CU@a)].ρ[e^(e^(-γ α) ξ CU@x)]]
```

```
True
```

Logos

```
Λ_U_[{ξ_, α_}, {x, a}] := Œ_U[{a, x}, α a + e^(-γ α) ξ x, 1];
Λ_U_[{α_, η_}, {a, y}] := Œ_U[{y, a}, α a + e^(-γ α) η y, 1];
```

```
{Λ_#[{ξ, α}, {x, a}], lhs = #@Œ_#[{x, a}, ℏ (ξ x + α a), 1],
   HL[lhs == #@Λ_#[ℏ {ξ, α}, {x, a}]]} & /@ {CU, QU}
```

$\{\{Œ_{CU}[\{a, x\}, a\,\alpha + e^{-\alpha\gamma} x\,\xi, 1],$

$CU[] + \alpha\hbar\, CU[a] + (\xi\hbar - \alpha\gamma\xi\hbar^2)\, CU[x] + \frac{1}{2}\alpha^2\hbar^2\, CU[a, a] + \alpha\xi\hbar^2\, CU[a, x] + \frac{1}{2}\xi^2\hbar^2\, CU[x, x],$

True$\}, \{Œ_{QU}[\{a, x\}, a\,\alpha + e^{-\alpha\gamma} x\,\xi, 1], QU[] + \alpha\hbar\, QU[a] + (\xi\hbar - \alpha\gamma\xi\hbar^2)\, QU[x] +$

$\frac{1}{2}\alpha^2\hbar^2\, QU[a, a] + \alpha\xi\hbar^2\, QU[a, x] + \frac{1}{2}\xi^2\hbar^2\, QU[x, x],$ True$\}\}$

```
{Λ#[{α, η}, {a, y}], lhs = #@Œ#[{a, y}, ℏ (η y + α a), 1],
   HL[lhs == #@Λ#[ℏ {α, η}, {a, y}]]} & /@ {CU, QU}
```

$\{\{\mathbb{E}_{CU}[\{y, a\}, a\,\alpha + e^{-\alpha\gamma}\,y\,\eta, 1],$
$CU[\,] + \alpha\,\hbar\,CU[a] + (\eta\,\hbar - \alpha\,\gamma\,\eta\,\hbar^2)\,CU[y] + \frac{1}{2}\alpha^2\hbar^2\,CU[a, a] + \alpha\,\eta\,\hbar^2\,CU[y, a] + \frac{1}{2}\eta^2\hbar^2\,CU[y, y],$
$\text{True}\}, \{\mathbb{E}_{QU}[\{y, a\}, a\,\alpha + e^{-\alpha\gamma}\,y\,\eta, 1], QU[\,] + \alpha\,\hbar\,QU[a] + (\eta\,\hbar - \alpha\,\gamma\,\eta\,\hbar^2)\,QU[y] +$
$\frac{1}{2}\alpha^2\hbar^2\,QU[a, a] + \alpha\,\eta\,\hbar^2\,QU[y, a] + \frac{1}{2}\eta^2\hbar^2\,QU[y, y], \text{True}\}\}$

Goal. In either U, compute $F = e^{-\eta y} e^{\xi x} e^{\eta y} e^{-\xi x}$. First compute $G = e^{\xi x} y e^{-\xi x}$, a finite sum. Now F satisfies the ODE $\partial_\eta F = \partial_\eta(e^{-\eta y} e^{\eta G}) = -yF + FG$ with initial conditions $F(\eta = 0) = 1$. So we set it up and solve:

```
If[$k > 0, With[{U = CU},
  Module[{G, F, fs, bs, e, b, es, sol},
   G = Echo@Simp[Table[ξ^k / k!, {k, 0, $k + 1}].NestList[Simp[B[x_U, #]] &, y_U, $k + 1]];
   fs = Echo@Flatten@Table[f_{l,i,j,k}[η], {l, 0, $k}, {i, 0, 1}, {j, 0, 1}, {k, 0, 1}];
   F = Echo[fs.(bs = fs /. f_{l_,i_,j_,k_}[η] :> ϵ^l U@{y^i, a^j, x^k})];
   es = Flatten[
     Table[Coefficient[e, b] == 0, {e, {F - 1_U /. η → 0, F ** G - y_U ** F - ∂_η F}}, {b, bs}]];
   sol = Echo@First[F /. DSolve[es, fs, η]];
   Echo[sol /. {e- → 1, U → Times}];
   Collect[sol /. {e- → 1, U → Times}, ϵ, Simplify]
  ]]]
```

» $-t\,\xi\,CU[\,] + 2\,\epsilon\,\xi\,CU[a] - \gamma\,\epsilon\,\xi^2\,CU[x] + CU[y]$

» $\{f_{0,0,0,0}[\eta], f_{1,0,0,0}[\eta], f_{1,0,0,1}[\eta], f_{1,0,1,0}[\eta], f_{1,0,1,1}[\eta], f_{1,1,0,0}[\eta], f_{1,1,0,1}[\eta], f_{1,1,1,0}[\eta], f_{1,1,1,1}[\eta]\}$

» $CU[\,]\,f_{0,0,0,0}[\eta] + \epsilon\,CU[\,]\,f_{1,0,0,0}[\eta] + \epsilon\,CU[x]\,f_{1,0,0,1}[\eta] + \epsilon\,CU[a]\,f_{1,0,1,0}[\eta] + \epsilon\,CU[a, x]\,f_{1,0,1,1}[\eta] + \epsilon\,CU[y]\,f_{1,1,0,0}[\eta] + \epsilon\,CU[y, x]\,f_{1,1,0,1}[\eta] + \epsilon\,CU[y, a]\,f_{1,1,1,0}[\eta] + \epsilon\,CU[y, a, x]\,f_{1,1,1,1}[\eta]$

» $e^{-t\,\eta\,\xi}\,CU[\,] + \frac{1}{2}e^{-t\,\eta\,\xi}\,t\,\gamma\,\epsilon\,\eta^2\,\xi^2\,CU[\,] + 2\,e^{-t\,\eta\,\xi}\,\epsilon\,\eta\,\xi\,CU[a] - e^{-t\,\eta\,\xi}\,\gamma\,\epsilon\,\eta\,\xi^2\,CU[x] - e^{-t\,\eta\,\xi}\,\gamma\,\epsilon\,\eta^2\,\xi\,CU[y]$

» $1 + 2\,a\,\epsilon\,\eta\,\xi - y\,\gamma\,\epsilon\,\eta^2\,\xi - x\,\gamma\,\epsilon\,\eta\,\xi^2 + \frac{1}{2}t\,\gamma\,\epsilon\,\eta^2\,\xi^2$

$1 + \frac{1}{2}\epsilon\,\eta\,\xi\,(4\,a + \gamma\,(-2\,y\,\eta - 2\,x\,\xi + t\,\eta\,\xi))$

Logos

```
ΛU_[{ξ1_, η1_}, {x, y}] :=
  ΛU[{ξ1, η1}, {x, y}] = Module[{ξ, η, G, F, fs, f, bs, eq, b, eqs},
    G = Simp[Table[ξ^k / k!, {k, 0, $k + 1}].NestList[Simp[B[xU, #]] &, yU, $k + 1]];
    fs = Flatten@Table[f_{l,i,j,k}[η], {l, 0, $k}, {i, 0, 1}, {j, 0, 1}, {k, 0, 1}];
    F = fs.(bs = fs /. f_{l_,i_,j_,k_}[η] :> e^l U@{y^i, a^j, x^k} /. e → ϵ);
    eqs = Flatten[Table[Coefficient[eq, b] == 0,
       {eq, {F - 1U /. η → 0, F ** G - yU ** F - ∂η F}}, {b, bs}]];
    F = F /. DSolve[eqs, fs, η][[1]];
    𝔼U[{y, a, x},
      ξ x + η y + (U /. {CU → -t η ξ, QU → η ξ (1 - T) / ħ}),
      F /. {ϵ → 1, U → Times}
     ] /. {ξ → ξ1, η → η1}];
```

```
{ΛCU[{ξ, η}, {x, y}], lhs = CU@𝔼CU[{x, y}, ħ (ξ x + η y), 1],
 HL[lhs == CU@ΛCU[ħ {ξ, η}, {x, y}]]}
```

$\{\mathbb{E}_{CU}[\{y, a, x\}, y\,\eta + x\,\xi - t\,\eta\,\xi, 1 + 2\,a\,\epsilon\,\eta\,\xi - y\,\gamma\,\epsilon\,\eta^2\,\xi - x\,\gamma\,\epsilon\,\eta\,\xi^2 + \frac{1}{2}\,t\,\gamma\,\epsilon\,\eta^2\,\xi^2],$
$(1 - t\,\eta\,\xi\,\hbar^2)\,CU[\,] + 2\,\epsilon\,\eta\,\xi\,\hbar^2\,CU[a] + \xi\,\hbar\,CU[x] + \eta\,\hbar\,CU[y] +$
$\frac{1}{2}\,\xi^2\,\hbar^2\,CU[x, x] + \eta\,\xi\,\hbar^2\,CU[y, x] + \frac{1}{2}\,\eta^2\,\hbar^2\,CU[y, y], \text{True}\}$

```
{ΛQU[{ξ, η}, {x, y}], lhs = QU@𝔼QU[{x, y}, ħ (ξ x + η y), 1],
 HL@SimpT[lhs == QU@ΛQU[ħ {ξ, η}, {x, y}]]}
```

$\{\mathbb{E}_{QU}[\{y, a, x\}, y\,\eta + x\,\xi + \frac{(1 - T)\,\eta\,\xi}{\hbar}, 1 + 2\,a\,T\,\epsilon\,\eta\,\xi - \frac{1}{2}\,(-1 + 3\,T)\,y\,\gamma\,\epsilon\,\eta^2\,\xi -$
$\frac{1}{2}\,(-1 + 3\,T)\,x\,\gamma\,\epsilon\,\eta\,\xi^2 + \frac{(-1 + T)\,(-1 + 3\,T)\,\gamma\,\epsilon\,\eta^2\,\xi^2}{4\,\hbar} + x\,y\,\gamma\,\epsilon\,\eta\,\xi\,\hbar],$
$(1 + \eta\,\xi\,\hbar - T\,\eta\,\xi\,\hbar)\,QU[\,] + 2\,T\,\epsilon\,\eta\,\xi\,\hbar^2\,QU[a] + \xi\,\hbar\,QU[x] + \eta\,\hbar\,QU[y] +$
$\frac{1}{2}\,\xi^2\,\hbar^2\,QU[x, x] + \eta\,\xi\,\hbar^2\,QU[y, x] + \frac{1}{2}\,\eta^2\,\hbar^2\,QU[y, y], \text{True}\}$

```
{tt = Last[ΛCU[{ξ, η}, {x, y}]], Normal@Series[Log[tt], {ϵ, 0, $k}]}
```

$\{1 + 2\,a\,\epsilon\,\eta\,\xi - y\,\gamma\,\epsilon\,\eta^2\,\xi - x\,\gamma\,\epsilon\,\eta\,\xi^2 + \frac{1}{2}\,t\,\gamma\,\epsilon\,\eta^2\,\xi^2, \epsilon\,\left(2\,a\,\eta\,\xi - y\,\gamma\,\eta^2\,\xi - x\,\gamma\,\eta\,\xi^2 + \frac{1}{2}\,t\,\gamma\,\eta^2\,\xi^2\right)\}$

```
{tt = Last[ΛQU[{ξ, η}, {x, y}]], Normal@Series[Log[tt], {ϵ, 0, $k}]}
```

$\{1 + 2\,a\,T\,\epsilon\,\eta\,\xi - \frac{1}{2}\,(-1 + 3\,T)\,y\,\gamma\,\epsilon\,\eta^2\,\xi - \frac{1}{2}\,(-1 + 3\,T)\,x\,\gamma\,\epsilon\,\eta\,\xi^2 +$
$\frac{(-1 + T)\,(-1 + 3\,T)\,\gamma\,\epsilon\,\eta^2\,\xi^2}{4\,\hbar} + x\,y\,\gamma\,\epsilon\,\eta\,\xi\,\hbar, \frac{1}{4\,\hbar}\epsilon\,(\gamma\,\eta^2\,\xi^2 - 4\,T\,\gamma\,\eta^2\,\xi^2 + 3\,T^2\,\gamma\,\eta^2\,\xi^2 +$
$8\,a\,T\,\eta\,\xi\,\hbar + 2\,y\,\gamma\,\eta^2\,\xi\,\hbar - 6\,T\,y\,\gamma\,\eta^2\,\xi\,\hbar + 2\,x\,\gamma\,\eta\,\xi^2\,\hbar - 6\,T\,x\,\gamma\,\eta\,\xi^2\,\hbar + 4\,x\,y\,\gamma\,\eta\,\xi\,\hbar^2)\}$

Logos

```
Simp[ℭU_[specs___, Q_, P_]] :=
  ℭU[specs, ExpandNumerator@Together[Q], Collect[P, ϵ, ExpandNumerator@*Together]];
ΛU_[{υ1_, ω1_, δ_}, {u_, w_}] := Simp@Module[{υ, ω, yax, q, p, Q, d},
   {yax, q, p} = List @@ ΛU[{υ, ω}, {u, w}];
   ℭU[yax, Q = (υ u + ω w + δ u w + d υ ω) / (1 - d δ),
     Expand[(1 - d δ)^-1 ⅇ^-Q DP_{υ→D_u, ω→D_w}[p][ⅇ^Q]]] /. {d → ∂_{υ,ω} q} /. {υ → υ1, ω → ω1}]
```

```
{ΛCU[{ξ, η, δ}, {x, y}], lhs = CU@ℭCU[{x, y}, ℏ (ξ x + η y + δ x y), 1],
 HL[lhs == CU@ΛCU[ℏ {ξ, η, δ}, {x, y}]]}
```

$\Big\{\mathbb{C}_{CU}\big[\{y, a, x\}, \frac{x y \delta + y \eta + x \xi - t \eta \xi}{1 + t \delta}, \frac{1}{1 + t \delta} +$

$\frac{1}{2 (1 + t \delta)^5} \epsilon \big(4 a \delta + 12 a t \delta^2 + 4 a x y \delta^2 + 2 t \gamma \delta^2 - 8 x y \gamma \delta^2 + 12 a t^2 \delta^3 + 8 a t x y \delta^3 + 4 t^2 \gamma \delta^3 -$
$12 t x y \gamma \delta^3 - 4 x^2 y^2 \gamma \delta^3 + 4 a t^3 \delta^4 + 4 a t^2 x y \delta^4 + 2 t^3 \gamma \delta^4 - 4 t^2 x y \gamma \delta^4 - 3 t x^2 y^2 \gamma \delta^4 +$
$4 a y \delta \eta - 4 y \gamma \delta \eta + 8 a t y \delta^2 \eta - 4 t y \gamma \delta^2 \eta - 6 x y^2 \gamma \delta^2 \eta + 4 a t^2 y \delta^3 \eta - 4 t x y^2 \gamma \delta^3 \eta -$
$2 y^2 \gamma \delta \eta^2 - t y^2 \gamma \delta^2 \eta^2 + 4 a x \delta \xi - 4 x \gamma \delta \xi + 8 a t x \delta^2 \xi - 4 t x \gamma \delta^2 \xi - 6 x^2 y \gamma \delta^2 \xi +$
$4 a t^2 x \delta^3 \xi - 4 t x^2 y \gamma \delta^3 \xi + 4 a \eta \xi + 8 a t \delta \eta \xi + 4 t \gamma \delta \eta \xi - 8 x y \gamma \delta \eta \xi + 4 a t^2 \delta^2 \eta \xi +$
$4 t^2 \gamma \delta^2 \eta \xi - 4 t x y \gamma \delta^2 \eta \xi - 2 y \gamma \eta^2 \xi - 2 x^2 \gamma \delta \xi^2 - t x^2 \gamma \delta^2 \xi^2 - 2 x \gamma \eta \xi^2 + t \gamma \eta^2 \xi^2\big)\big],$

$(1 - t \delta \hbar + t^2 \delta^2 \hbar^2 + t \gamma \delta^2 \epsilon \hbar^2 - t \eta \xi \hbar^2)\, CU[\,] + (2 \delta \epsilon \hbar - 4 t \delta^2 \epsilon \hbar^2 + 2 \epsilon \eta \xi \hbar^2)$
$CU[a] +$
$(\xi \hbar - 2 t \delta \xi \hbar^2 - 2 \gamma \delta \epsilon \xi \hbar^2)\, CU[x] +$
$(\eta \hbar - 2 t \delta \eta \hbar^2 - 2 \gamma \delta \epsilon \eta \hbar^2)\, CU[y] +$
$4 \delta \epsilon \xi \hbar^2\, CU[a, x] +$
$\frac{1}{2} \xi^2 \hbar^2\, CU[x, x] +$
$4 \delta \epsilon \eta \hbar^2\, CU[y, a] +$
$(\delta \hbar - 2 t \delta^2 \hbar^2 - 4 \gamma \delta^2 \epsilon \hbar^2 + \eta \xi \hbar^2)\, CU[y, x] +$
$\frac{1}{2} \eta^2 \hbar^2\, CU[y, y] +$
$4 \delta^2 \epsilon \hbar^2\, CU[y, a, x] +$
$\delta \xi \hbar^2\, CU[y, x, x] +$
$\delta \eta \hbar^2\, CU[y, y, x] +$
$\frac{1}{2} \delta^2 \hbar^2\, CU[y, y, x, x]$, True$\Big\}$

```
{Λ_QU[{ξ, η, δ}, {x, y}], lhs = QU@𝔈_QU[{x, y}, ℏ (ξ x + η y + δ x y), 1],
 HL@SimpT[lhs == QU@Λ_QU[ℏ {ξ, η, δ}, {x, y}]]}
```

$$\Big\{\mathbb{E}_{QU}\Big[\{y, a, x\},\ \frac{\eta\,\xi - T\,\eta\,\xi + x\,y\,\delta\,\hbar + y\,\eta\,\hbar + x\,\xi\,\hbar}{-\delta + T\,\delta + \hbar},$$

$$\frac{\hbar}{-\delta + T\,\delta + \hbar} + \frac{1}{4\,(-\delta + T\,\delta + \hbar)^5}\,\epsilon\,\big(-8\,a\,T\,\delta^4\,\hbar^2 + 24\,a\,T^2\,\delta^4\,\hbar^2 - 24\,a\,T^3\,\delta^4\,\hbar^2 + 8\,a\,T^4\,\delta^4\,\hbar^2 + 2\,\gamma\,\delta^4\,\hbar^2 -$$
$$12\,T\,\gamma\,\delta^4\,\hbar^2 + 24\,T^2\,\gamma\,\delta^4\,\hbar^2 - 20\,T^3\,\gamma\,\delta^4\,\hbar^2 + 6\,T^4\,\gamma\,\delta^4\,\hbar^2 + 24\,a\,T\,\delta^3\,\hbar^3 - 48\,a\,T^2\,\delta^3\,\hbar^3 +$$
$$24\,a\,T^3\,\delta^3\,\hbar^3 - 4\,\gamma\,\delta^3\,\hbar^3 + 20\,T\,\gamma\,\delta^3\,\hbar^3 - 28\,T^2\,\gamma\,\delta^3\,\hbar^3 + 12\,T^3\,\gamma\,\delta^3\,\hbar^3 + 8\,a\,T\,x\,y\,\delta^4\,\hbar^3 -$$
$$16\,a\,T^2\,x\,y\,\delta^4\,\hbar^3 + 8\,a\,T^3\,x\,y\,\delta^4\,\hbar^3 - 8\,T\,x\,y\,\gamma\,\delta^4\,\hbar^3 + 16\,T^2\,x\,y\,\gamma\,\delta^4\,\hbar^3 - 8\,T^3\,x\,y\,\gamma\,\delta^4\,\hbar^3 +$$
$$8\,a\,T\,y\,\delta^3\,\eta\,\hbar^3 - 16\,a\,T^2\,y\,\delta^3\,\eta\,\hbar^3 + 8\,a\,T^3\,y\,\delta^3\,\eta\,\hbar^3 + 8\,a\,T\,x\,\delta^3\,\xi\,\hbar^3 - 16\,a\,T^2\,x\,\delta^3\,\xi\,\hbar^3 +$$
$$8\,a\,T^3\,x\,\delta^3\,\xi\,\hbar^3 + 8\,a\,T\,\delta^2\,\eta\,\xi\,\hbar^3 - 16\,a\,T^2\,\delta^2\,\eta\,\xi\,\hbar^3 + 8\,a\,T^3\,\delta^2\,\eta\,\xi\,\hbar^3 - 4\,\gamma\,\delta^2\,\eta\,\xi\,\hbar^3 +$$
$$20\,T\,\gamma\,\delta^2\,\eta\,\xi\,\hbar^3 - 28\,T^2\,\gamma\,\delta^2\,\eta\,\xi\,\hbar^3 + 12\,T^3\,\gamma\,\delta^2\,\eta\,\xi\,\hbar^3 - 24\,a\,T\,\delta^2\,\hbar^4 + 24\,a\,T^2\,\delta^2\,\hbar^4 +$$
$$2\,\gamma\,\delta^2\,\hbar^4 - 8\,T\,\gamma\,\delta^2\,\hbar^4 + 6\,T^2\,\gamma\,\delta^2\,\hbar^4 - 16\,a\,T\,x\,y\,\delta^3\,\hbar^4 + 16\,a\,T^2\,x\,y\,\delta^3\,\hbar^4 + 24\,T\,x\,y\,\gamma\,\delta^3\,\hbar^4 -$$
$$24\,T^2\,x\,y\,\gamma\,\delta^3\,\hbar^4 + x^2\,y^2\,\gamma\,\delta^4\,\hbar^4 + 4\,T\,x^2\,y^2\,\gamma\,\delta^4\,\hbar^4 - 5\,T^2\,x^2\,y^2\,\gamma\,\delta^4\,\hbar^4 - 16\,a\,T\,y\,\delta^2\,\eta\,\hbar^4 +$$
$$16\,a\,T^2\,y\,\delta^2\,\eta\,\hbar^4 - 4\,y\,\gamma\,\delta^2\,\eta\,\hbar^4 + 16\,T\,y\,\gamma\,\delta^2\,\eta\,\hbar^4 - 12\,T^2\,y\,\gamma\,\delta^2\,\eta\,\hbar^4 + 8\,T\,x\,y^2\,\gamma\,\delta^3\,\eta\,\hbar^4 -$$
$$8\,T^2\,x\,y^2\,\gamma\,\delta^3\,\eta\,\hbar^4 - y^2\,\gamma\,\delta^2\,\eta^2\,\hbar^4 + 4\,T\,y^2\,\gamma\,\delta^2\,\eta^2\,\hbar^4 - 3\,T^2\,y^2\,\gamma\,\delta^2\,\eta^2\,\hbar^4 - 16\,a\,T\,x\,\delta^2\,\xi\,\hbar^4 +$$
$$16\,a\,T^2\,x\,\delta^2\,\xi\,\hbar^4 - 4\,x\,\gamma\,\delta^2\,\xi\,\hbar^4 + 16\,T\,x\,\gamma\,\delta^2\,\xi\,\hbar^4 - 12\,T^2\,x\,\gamma\,\delta^2\,\xi\,\hbar^4 + 8\,T\,x^2\,y\,\gamma\,\delta^3\,\xi\,\hbar^4 -$$
$$8\,T^2\,x^2\,y\,\gamma\,\delta^3\,\xi\,\hbar^4 - 16\,a\,T\,\delta\,\eta\,\xi\,\hbar^4 + 16\,a\,T^2\,\delta\,\eta\,\xi\,\hbar^4 + 4\,\gamma\,\delta\,\eta\,\xi\,\hbar^4 - 16\,T\,\gamma\,\delta\,\eta\,\xi\,\hbar^4 +$$
$$12\,T^2\,\gamma\,\delta\,\eta\,\xi\,\hbar^4 + 8\,T\,x\,y\,\gamma\,\delta^2\,\eta\,\xi\,\hbar^4 - 8\,T^2\,x\,y\,\gamma\,\delta^2\,\eta\,\xi\,\hbar^4 - x^2\,\gamma\,\delta^2\,\xi^2\,\hbar^4 + 4\,T\,x^2\,\gamma\,\delta^2\,\xi^2\,\hbar^4 -$$
$$3\,T^2\,x^2\,\gamma\,\delta^2\,\xi^2\,\hbar^4 + \gamma\,\eta^2\,\xi^2\,\hbar^4 - 4\,T\,\gamma\,\eta^2\,\xi^2\,\hbar^4 + 3\,T^2\,\gamma\,\eta^2\,\xi^2\,\hbar^4 + 8\,a\,T\,\delta\,\hbar^5 + 8\,a\,T\,x\,y\,\delta^2\,\hbar^5 -$$
$$4\,x\,y\,\gamma\,\delta^2\,\hbar^5 - 12\,T\,x\,y\,\gamma\,\delta^2\,\hbar^5 - 4\,x^2\,y^2\,\gamma\,\delta^3\,\hbar^5 - 4\,T\,x^2\,y^2\,\gamma\,\delta^3\,\hbar^5 + 8\,a\,T\,y\,\delta\,\eta\,\hbar^5 + 4\,y\,\gamma\,\delta\,\eta\,\hbar^5 -$$
$$12\,T\,y\,\gamma\,\delta\,\eta\,\hbar^5 - 2\,x\,y^2\,\gamma\,\delta^2\,\eta\,\hbar^5 - 10\,T\,x\,y^2\,\gamma\,\delta^2\,\eta\,\hbar^5 + 2\,y^2\,\gamma\,\delta\,\eta^2\,\hbar^5 - 6\,T\,y^2\,\gamma\,\delta\,\eta^2\,\hbar^5 +$$
$$8\,a\,T\,x\,\delta\,\xi\,\hbar^5 + 4\,x\,\gamma\,\delta\,\xi\,\hbar^5 - 12\,T\,x\,\gamma\,\delta\,\xi\,\hbar^5 - 2\,x^2\,y\,\gamma\,\delta^2\,\xi\,\hbar^5 - 10\,T\,x^2\,y\,\gamma\,\delta^2\,\xi\,\hbar^5 + 8\,a\,T\,\eta\,\xi\,\hbar^5 -$$
$$16\,T\,x\,y\,\gamma\,\delta\,\eta\,\xi\,\hbar^5 + 2\,y\,\gamma\,\eta^2\,\xi\,\hbar^5 - 6\,T\,y\,\gamma\,\eta^2\,\xi\,\hbar^5 + 2\,x^2\,\gamma\,\delta\,\xi^2\,\hbar^5 - 6\,T\,x^2\,\gamma\,\delta\,\xi^2\,\hbar^5 + 2\,x\,\gamma\,\eta\,\xi^2\,\hbar^5 -$$
$$6\,T\,x\,\gamma\,\eta\,\xi^2\,\hbar^5 + 4\,x\,y\,\gamma\,\delta\,\hbar^6 + 4\,x^2\,y^2\,\gamma\,\delta^2\,\hbar^6 + 4\,x\,y^2\,\gamma\,\delta\,\eta\,\hbar^6 + 4\,x^2\,y\,\gamma\,\delta\,\xi\,\hbar^6 + 4\,x\,y\,\gamma\,\eta\,\xi\,\hbar^6\big)\Big],$$

$$\Big(1 + \delta - T\,\delta + \delta^2 - 2\,T\,\delta^2 + T^2\,\delta^2 + \frac{1}{2}\,\gamma\,\delta^2\,\epsilon\,\hbar - 2\,T\,\gamma\,\delta^2\,\epsilon\,\hbar + \frac{3}{2}\,T^2\,\gamma\,\delta^2\,\epsilon\,\hbar + \eta\,\xi\,\hbar - T\,\eta\,\xi\,\hbar\Big)\ QU[\,] +$$
$$\big(2\,T\,\delta\,\epsilon\,\hbar + 4\,T\,\delta^2\,\epsilon\,\hbar - 4\,T^2\,\delta^2\,\epsilon\,\hbar + 2\,T\,\epsilon\,\eta\,\xi\,\hbar^2\big)\ QU[a] +$$
$$\big(\xi\,\hbar + 2\,\delta\,\xi\,\hbar - 2\,T\,\delta\,\xi\,\hbar + \gamma\,\delta\,\epsilon\,\xi\,\hbar^2 - 3\,T\,\gamma\,\delta\,\epsilon\,\xi\,\hbar^2\big)\ QU[x] +$$
$$\big(\eta\,\hbar + 2\,\delta\,\eta\,\hbar - 2\,T\,\delta\,\eta\,\hbar + \gamma\,\delta\,\epsilon\,\eta\,\hbar^2 - 3\,T\,\gamma\,\delta\,\epsilon\,\eta\,\hbar^2\big)\ QU[y] +$$
$$4\,T\,\delta\,\epsilon\,\xi\,\hbar^2\,QU[a, x] + \frac{1}{2}\,\xi^2\,\hbar^2\,QU[x, x] +$$
$$4\,T\,\delta\,\epsilon\,\eta\,\hbar^2\,QU[y, a] +$$
$$\big(\delta\,\hbar + 2\,\delta^2\,\hbar - 2\,T\,\delta^2\,\hbar + \gamma\,\delta\,\epsilon\,\hbar^2 + 4\,\gamma\,\delta^2\,\epsilon\,\hbar^2 - 8\,T\,\gamma\,\delta^2\,\epsilon\,\hbar^2 + \eta\,\xi\,\hbar^2\big)\ QU[y, x] +$$
$$\frac{1}{2}\,\eta^2\,\hbar^2\,QU[y, y] + 4\,T\,\delta^2\,\epsilon\,\hbar^2\,QU[y, a, x] +$$
$$\delta\,\xi\,\hbar^2\,QU[y, x, x] +$$
$$\delta\,\eta\,\hbar^2\,QU[y, y, x] +$$
$$\frac{1}{2}\,\delta^2\,\hbar^2\,QU[y, y, x, x],\ \text{True}\Big\}$$

```
{tt = Last[ΛCU[{ξ, η, δ}, {x, y}]], Normal@Series[Log[tt], {ϵ, 0, $k}]}
```

$$
\Big\{ \frac{1}{1+t\,\delta} + \frac{1}{2\,(1+t\,\delta)^5}\,\epsilon\,\big(4\,a\,\delta + 12\,a\,t\,\delta^2 + 4\,a\,x\,y\,\delta^2 + 2\,t\,\gamma\,\delta^2 - 8\,x\,y\,\gamma\,\delta^2 + 12\,a\,t^2\,\delta^3 + 8\,a\,t\,x\,y\,\delta^3 + 4\,t^2\,\gamma\,\delta^3 -
$$
$$
12\,t\,x\,y\,\gamma\,\delta^3 - 4\,x^2\,y^2\,\gamma\,\delta^3 + 4\,a\,t^3\,\delta^4 + 4\,a\,t^2\,x\,y\,\delta^4 + 2\,t^3\,\gamma\,\delta^4 - 4\,t^2\,x\,y\,\gamma\,\delta^4 - 3\,t\,x^2\,y^2\,\gamma\,\delta^4 +
$$
$$
4\,a\,y\,\delta\,\eta - 4\,y\,\gamma\,\delta\,\eta + 8\,a\,t\,y\,\delta^2\,\eta - 4\,t\,y\,\gamma\,\delta^2\,\eta - 6\,x\,y^2\,\gamma\,\delta^2\,\eta + 4\,a\,t^2\,y\,\delta^3\,\eta - 4\,t\,x\,y^2\,\gamma\,\delta^3\,\eta -
$$
$$
2\,y^2\,\gamma\,\delta\,\eta^2 - t\,y^2\,\gamma\,\delta^2\,\eta^2 + 4\,a\,x\,\delta\,\xi - 4\,x\,\gamma\,\delta\,\xi + 8\,a\,t\,x\,\delta^2\,\xi - 4\,t\,x\,\gamma\,\delta^2\,\xi - 6\,x^2\,y\,\gamma\,\delta^2\,\xi +
$$
$$
4\,a\,t^2\,x\,\delta^3\,\xi - 4\,t\,x^2\,y\,\gamma\,\delta^3\,\xi + 4\,a\,\eta\,\xi + 8\,a\,t\,\delta\,\eta\,\xi + 4\,t\,\gamma\,\delta\,\eta\,\xi - 8\,x\,y\,\gamma\,\delta\,\eta\,\xi + 4\,a\,t^2\,\delta^2\,\eta\,\xi +
$$
$$
4\,t^2\,\gamma\,\delta^2\,\eta\,\xi - 4\,t\,x\,y\,\gamma\,\delta^2\,\eta\,\xi - 2\,y\,\gamma\,\eta^2\,\xi - 2\,x^2\,\gamma\,\delta\,\xi^2 - t\,x^2\,\gamma\,\delta^2\,\xi^2 - 2\,x\,\gamma\,\eta\,\xi^2 + t\,\gamma\,\eta^2\,\xi^2\big),
$$
$$
\frac{1}{2\,(1+t\,\delta)^4}\,\epsilon\,\big(4\,a\,\delta + 12\,a\,t\,\delta^2 + 4\,a\,x\,y\,\delta^2 + 2\,t\,\gamma\,\delta^2 - 8\,x\,y\,\gamma\,\delta^2 + 12\,a\,t^2\,\delta^3 + 8\,a\,t\,x\,y\,\delta^3 +
$$
$$
4\,t^2\,\gamma\,\delta^3 - 12\,t\,x\,y\,\gamma\,\delta^3 - 4\,x^2\,y^2\,\gamma\,\delta^3 + 4\,a\,t^3\,\delta^4 + 4\,a\,t^2\,x\,y\,\delta^4 + 2\,t^3\,\gamma\,\delta^4 - 4\,t^2\,x\,y\,\gamma\,\delta^4 -
$$
$$
3\,t\,x^2\,y^2\,\gamma\,\delta^4 + 4\,a\,y\,\delta\,\eta - 4\,y\,\gamma\,\delta\,\eta + 8\,a\,t\,y\,\delta^2\,\eta - 4\,t\,y\,\gamma\,\delta^2\,\eta - 6\,x\,y^2\,\gamma\,\delta^2\,\eta +
$$
$$
4\,a\,t^2\,y\,\delta^3\,\eta - 4\,t\,x\,y^2\,\gamma\,\delta^3\,\eta - 2\,y^2\,\gamma\,\delta\,\eta^2 - t\,y^2\,\gamma\,\delta^2\,\eta^2 + 4\,a\,x\,\delta\,\xi - 4\,x\,\gamma\,\delta\,\xi +
$$
$$
8\,a\,t\,x\,\delta^2\,\xi - 4\,t\,x\,\gamma\,\delta^2\,\xi - 6\,x^2\,y\,\gamma\,\delta^2\,\xi + 4\,a\,t^2\,x\,\delta^3\,\xi - 4\,t\,x^2\,y\,\gamma\,\delta^3\,\xi + 4\,a\,\eta\,\xi +
$$
$$
8\,a\,t\,\delta\,\eta\,\xi + 4\,t\,\gamma\,\delta\,\eta\,\xi - 8\,x\,y\,\gamma\,\delta\,\eta\,\xi + 4\,a\,t^2\,\delta^2\,\eta\,\xi + 4\,t^2\,\gamma\,\delta^2\,\eta\,\xi - 4\,t\,x\,y\,\gamma\,\delta^2\,\eta\,\xi -
$$
$$
2\,y\,\gamma\,\eta^2\,\xi - 2\,x^2\,\gamma\,\delta\,\xi^2 - t\,x^2\,\gamma\,\delta^2\,\xi^2 - 2\,x\,\gamma\,\eta\,\xi^2 + t\,\gamma\,\eta^2\,\xi^2\big) + \mathrm{Log}\Big[\frac{1}{1+t\,\delta}\Big]\Big\}
$$

```
{tt = Last[Λ_QU[{ξ, η, δ}, {x, y}]], Normal@Series[Log[tt], {ϵ, 0, $k}]}
```

$$\Big\{\frac{\hbar}{-\delta+T\delta+\hbar}+$$

$$\frac{1}{4\,(-\delta+T\delta+\hbar)^5}\,\epsilon\,\Big(-8\,a\,T\,\delta^4\hbar^2+24\,a\,T^2\delta^4\hbar^2-24\,a\,T^3\delta^4\hbar^2+8\,a\,T^4\delta^4\hbar^2+2\,\gamma\,\delta^4\hbar^2-12\,T\,\gamma\,\delta^4\hbar^2+$$
$$24\,T^2\gamma\,\delta^4\hbar^2-20\,T^3\gamma\,\delta^4\hbar^2+6\,T^4\gamma\,\delta^4\hbar^2+24\,a\,T\,\delta^3\hbar^3-48\,a\,T^2\delta^3\hbar^3+24\,a\,T^3\delta^3\hbar^3-$$
$$4\,\gamma\,\delta^3\hbar^3+20\,T\,\gamma\,\delta^3\hbar^3-28\,T^2\gamma\,\delta^3\hbar^3+12\,T^3\gamma\,\delta^3\hbar^3+8\,a\,T\,x\,y\,\delta^4\hbar^3-16\,a\,T^2x\,y\,\delta^4\hbar^3+$$
$$8\,a\,T^3x\,y\,\delta^4\hbar^3-8\,T\,x\,y\,\gamma\,\delta^4\hbar^3+16\,T^2x\,y\,\gamma\,\delta^4\hbar^3-8\,T^3x\,y\,\gamma\,\delta^4\hbar^3+8\,a\,T\,y\,\delta^3\eta\,\hbar^3-$$
$$16\,a\,T^2y\,\delta^3\eta\,\hbar^3+8\,a\,T^3y\,\delta^3\eta\,\hbar^3+8\,a\,T\,x\,\delta^3\xi\,\hbar^3-16\,a\,T^2x\,\delta^3\xi\,\hbar^3+8\,a\,T^3x\,\delta^3\xi\,\hbar^3+$$
$$8\,a\,T\,\delta^2\eta\,\xi\,\hbar^3-16\,a\,T^2\delta^2\eta\,\xi\,\hbar^3+8\,a\,T^3\delta^2\eta\,\xi\,\hbar^3-4\,\gamma\,\delta^2\eta\,\xi\,\hbar^3+20\,T\,\gamma\,\delta^2\eta\,\xi\,\hbar^3-$$
$$28\,T^2\gamma\,\delta^2\eta\,\xi\,\hbar^3+12\,T^3\gamma\,\delta^2\eta\,\xi\,\hbar^3-24\,a\,T\,\delta^2\hbar^4+24\,a\,T^2\delta^2\hbar^4+2\,\gamma\,\delta^2\hbar^4-8\,T\,\gamma\,\delta^2\hbar^4+$$
$$6\,T^2\gamma\,\delta^2\hbar^4-16\,a\,T\,x\,y\,\delta^3\hbar^4+16\,a\,T^2x\,y\,\delta^3\hbar^4+24\,T\,x\,y\,\gamma\,\delta^3\hbar^4-24\,T^2x\,y\,\gamma\,\delta^3\hbar^4+$$
$$x^2y^2\gamma\,\delta^4\hbar^4+4\,T\,x^2y^2\gamma\,\delta^4\hbar^4-5\,T^2x^2y^2\gamma\,\delta^4\hbar^4-16\,a\,T\,y\,\delta^2\eta\,\hbar^4+16\,a\,T^2y\,\delta^2\eta\,\hbar^4-$$
$$4\,y\,\gamma\,\delta^2\eta\,\hbar^4+16\,T\,y\,\gamma\,\delta^2\eta\,\hbar^4-12\,T^2y\,\gamma\,\delta^2\eta\,\hbar^4+8\,T\,x\,y^2\gamma\,\delta^3\eta\,\hbar^4-8\,T^2x\,y^2\gamma\,\delta^3\eta\,\hbar^4-$$
$$y^2\gamma\,\delta^2\eta^2\hbar^4+4\,T\,y^2\gamma\,\delta^2\eta^2\hbar^4-3\,T^2y^2\gamma\,\delta^2\eta^2\hbar^4-16\,a\,T\,x\,\delta^2\xi\,\hbar^4+16\,a\,T^2x\,\delta^2\xi\,\hbar^4-$$
$$4\,x\,\gamma\,\delta^2\xi\,\hbar^4+16\,T\,x\,\gamma\,\delta^2\xi\,\hbar^4-12\,T^2x\,\gamma\,\delta^2\xi\,\hbar^4+8\,T\,x^2y\,\gamma\,\delta^3\xi\,\hbar^4-8\,T^2x^2y\,\gamma\,\delta^3\xi\,\hbar^4-$$
$$16\,a\,T\,\delta\,\eta\,\xi\,\hbar^4+16\,a\,T^2\delta\,\eta\,\xi\,\hbar^4+4\,\gamma\,\delta\,\eta\,\xi\,\hbar^4-16\,T\,\gamma\,\delta\,\eta\,\xi\,\hbar^4+12\,T^2\gamma\,\delta\,\eta\,\xi\,\hbar^4+$$
$$8\,T\,x\,y\,\gamma\,\delta^2\eta\,\xi\,\hbar^4-8\,T^2x\,y\,\gamma\,\delta^2\eta\,\xi\,\hbar^4-x^2\gamma\,\delta^2\xi^2\hbar^4+4\,T\,x^2\gamma\,\delta^2\xi^2\hbar^4-3\,T^2x^2\gamma\,\delta^2\xi^2\hbar^4+$$
$$\gamma\,\eta^2\xi^2\hbar^4-4\,T\,\gamma\,\eta^2\xi^2\hbar^4+3\,T^2\gamma\,\eta^2\xi^2\hbar^4+8\,a\,T\,\delta\,\hbar^5+8\,a\,T\,x\,y\,\delta^2\hbar^5-4\,x\,y\,\gamma\,\delta^2\hbar^5-$$
$$12\,T\,x\,y\,\gamma\,\delta^2\hbar^5-4\,x^2y^2\gamma\,\delta^3\hbar^5-4\,T\,x^2y^2\gamma\,\delta^3\hbar^5+8\,a\,T\,y\,\delta\,\eta\,\hbar^5+4\,y\,\gamma\,\delta\,\eta\,\hbar^5-$$
$$12\,T\,y\,\gamma\,\delta\,\eta\,\hbar^5-2\,x\,y^2\gamma\,\delta^2\eta\,\hbar^5-10\,T\,x\,y^2\gamma\,\delta^2\eta\,\hbar^5+2\,y^2\gamma\,\delta\,\eta^2\hbar^5-6\,T\,y^2\gamma\,\delta\,\eta^2\hbar^5+$$
$$8\,a\,T\,x\,\delta\,\xi\,\hbar^5+4\,x\,\gamma\,\delta\,\xi\,\hbar^5-12\,T\,x\,\gamma\,\delta\,\xi\,\hbar^5-2\,x^2y\,\gamma\,\delta^2\xi\,\hbar^5-10\,T\,x^2y\,\gamma\,\delta^2\xi\,\hbar^5+8\,a\,T\,\eta\,\xi\,\hbar^5-$$
$$16\,T\,x\,y\,\gamma\,\delta\,\eta\,\xi\,\hbar^5+2\,y\,\gamma\,\eta^2\xi\,\hbar^5-6\,T\,y\,\gamma\,\eta^2\xi\,\hbar^5+2\,x^2\gamma\,\delta\,\xi^2\hbar^5-6\,T\,x^2\gamma\,\delta\,\xi^2\hbar^5+2\,x\,\gamma\,\eta\,\xi^2\hbar^5-$$
$$6\,T\,x\,\gamma\,\eta\,\xi^2\hbar^5+4\,x\,y\,\gamma\,\delta\,\hbar^6+4\,x^2y^2\gamma\,\delta^2\hbar^6+4\,x\,y^2\gamma\,\delta\,\eta\,\hbar^6+4\,x^2y\,\gamma\,\delta\,\xi\,\hbar^6+4\,x\,y\,\gamma\,\eta\,\xi\,\hbar^6\Big),$$

$$\frac{1}{4\,(-\delta+T\delta+\hbar)^4}\,\epsilon\,\Big(-8\,a\,T\,\delta^4\hbar+24\,a\,T^2\delta^4\hbar-24\,a\,T^3\delta^4\hbar+8\,a\,T^4\delta^4\hbar+2\,\gamma\,\delta^4\hbar-12\,T\,\gamma\,\delta^4\hbar+$$
$$24\,T^2\gamma\,\delta^4\hbar-20\,T^3\gamma\,\delta^4\hbar+6\,T^4\gamma\,\delta^4\hbar+24\,a\,T\,\delta^3\hbar^2-48\,a\,T^2\delta^3\hbar^2+24\,a\,T^3\delta^3\hbar^2-4\,\gamma\,\delta^3\hbar^2+$$
$$20\,T\,\gamma\,\delta^3\hbar^2-28\,T^2\gamma\,\delta^3\hbar^2+12\,T^3\gamma\,\delta^3\hbar^2+8\,a\,T\,x\,y\,\delta^4\hbar^2-16\,a\,T^2x\,y\,\delta^4\hbar^2+8\,a\,T^3x\,y\,\delta^4\hbar^2-$$
$$8\,T\,x\,y\,\gamma\,\delta^4\hbar^2+16\,T^2x\,y\,\gamma\,\delta^4\hbar^2-8\,T^3x\,y\,\gamma\,\delta^4\hbar^2+8\,a\,T\,y\,\delta^3\eta\,\hbar^2-16\,a\,T^2y\,\delta^3\eta\,\hbar^2+$$
$$8\,a\,T^3y\,\delta^3\eta\,\hbar^2+8\,a\,T\,x\,\delta^3\xi\,\hbar^2-16\,a\,T^2x\,\delta^3\xi\,\hbar^2+8\,a\,T^3x\,\delta^3\xi\,\hbar^2+8\,a\,T\,\delta^2\eta\,\xi\,\hbar^2-$$
$$16\,a\,T^2\delta^2\eta\,\xi\,\hbar^2+8\,a\,T^3\delta^2\eta\,\xi\,\hbar^2-4\,\gamma\,\delta^2\eta\,\xi\,\hbar^2+20\,T\,\gamma\,\delta^2\eta\,\xi\,\hbar^2-28\,T^2\gamma\,\delta^2\eta\,\xi\,\hbar^2+$$
$$12\,T^3\gamma\,\delta^2\eta\,\xi\,\hbar^2-24\,a\,T\,\delta^2\hbar^3+24\,a\,T^2\delta^2\hbar^3+2\,\gamma\,\delta^2\hbar^3-8\,T\,\gamma\,\delta^2\hbar^3+6\,T^2\gamma\,\delta^2\hbar^3-$$
$$16\,a\,T\,x\,y\,\delta^3\hbar^3+16\,a\,T^2x\,y\,\delta^3\hbar^3+24\,T\,x\,y\,\gamma\,\delta^3\hbar^3-24\,T^2x\,y\,\gamma\,\delta^3\hbar^3+x^2y^2\gamma\,\delta^4\hbar^3+$$
$$4\,T\,x^2y^2\gamma\,\delta^4\hbar^3-5\,T^2x^2y^2\gamma\,\delta^4\hbar^3-16\,a\,T\,y\,\delta^2\eta\,\hbar^3+16\,a\,T^2y\,\delta^2\eta\,\hbar^3-4\,y\,\gamma\,\delta^2\eta\,\hbar^3+$$
$$16\,T\,y\,\gamma\,\delta^2\eta\,\hbar^3-12\,T^2y\,\gamma\,\delta^2\eta\,\hbar^3+8\,T\,x\,y^2\gamma\,\delta^3\eta\,\hbar^3-8\,T^2x\,y^2\gamma\,\delta^3\eta\,\hbar^3-y^2\gamma\,\delta^2\eta^2\hbar^3+$$
$$4\,T\,y^2\gamma\,\delta^2\eta^2\hbar^3-3\,T^2y^2\gamma\,\delta^2\eta^2\hbar^3-16\,a\,T\,x\,\delta^2\xi\,\hbar^3+16\,a\,T^2x\,\delta^2\xi\,\hbar^3-4\,x\,\gamma\,\delta^2\xi\,\hbar^3+$$
$$16\,T\,x\,\gamma\,\delta^2\xi\,\hbar^3-12\,T^2x\,\gamma\,\delta^2\xi\,\hbar^3+8\,T\,x^2y\,\gamma\,\delta^3\xi\,\hbar^3-8\,T^2x^2y\,\gamma\,\delta^3\xi\,\hbar^3-16\,a\,T\,\delta\,\eta\,\xi\,\hbar^3+$$
$$16\,a\,T^2\delta\,\eta\,\xi\,\hbar^3+4\,\gamma\,\delta\,\eta\,\xi\,\hbar^3-16\,T\,\gamma\,\delta\,\eta\,\xi\,\hbar^3+12\,T^2\gamma\,\delta\,\eta\,\xi\,\hbar^3+8\,T\,x\,y\,\gamma\,\delta^2\eta\,\xi\,\hbar^3-$$
$$8\,T^2x\,y\,\gamma\,\delta^2\eta\,\xi\,\hbar^3-x^2\gamma\,\delta^2\xi^2\hbar^3+4\,T\,x^2\gamma\,\delta^2\xi^2\hbar^3-3\,T^2x^2\gamma\,\delta^2\xi^2\hbar^3+\gamma\,\eta^2\xi^2\hbar^3-$$
$$4\,T\,\gamma\,\eta^2\xi^2\hbar^3+3\,T^2\gamma\,\eta^2\xi^2\hbar^3+8\,a\,T\,\delta\,\hbar^4+8\,a\,T\,x\,y\,\delta^2\hbar^4-4\,x\,y\,\gamma\,\delta^2\hbar^4-12\,T\,x\,y\,\gamma\,\delta^2\hbar^4-$$
$$4\,x^2y^2\gamma\,\delta^3\hbar^4-4\,T\,x^2y^2\gamma\,\delta^3\hbar^4+8\,a\,T\,y\,\delta\,\eta\,\hbar^4+4\,y\,\gamma\,\delta\,\eta\,\hbar^4-12\,T\,y\,\gamma\,\delta\,\eta\,\hbar^4-2\,x\,y^2\gamma\,\delta^2\eta\,\hbar^4-$$
$$10\,T\,x\,y^2\gamma\,\delta^2\eta\,\hbar^4+2\,y^2\gamma\,\delta\,\eta^2\hbar^4-6\,T\,y^2\gamma\,\delta\,\eta^2\hbar^4+8\,a\,T\,x\,\delta\,\xi\,\hbar^4+4\,x\,\gamma\,\delta\,\xi\,\hbar^4-12\,T\,x\,\gamma\,\delta\,\xi\,\hbar^4-$$
$$2\,x^2y\,\gamma\,\delta^2\xi\,\hbar^4-10\,T\,x^2y\,\gamma\,\delta^2\xi\,\hbar^4+8\,a\,T\,\eta\,\xi\,\hbar^4-16\,T\,x\,y\,\gamma\,\delta\,\eta\,\xi\,\hbar^4+2\,y\,\gamma\,\eta^2\xi\,\hbar^4-$$
$$6\,T\,y\,\gamma\,\eta^2\xi\,\hbar^4+2\,x^2\gamma\,\delta\,\xi^2\hbar^4-6\,T\,x^2\gamma\,\delta\,\xi^2\hbar^4+2\,x\,\gamma\,\eta\,\xi^2\hbar^4-6\,T\,x\,\gamma\,\eta\,\xi^2\hbar^4+4\,x\,y\,\gamma\,\delta\,\hbar^5+$$
$$4\,x^2y^2\gamma\,\delta^2\hbar^5+4\,x\,y^2\gamma\,\delta\,\eta\,\hbar^5+4\,x^2y\,\gamma\,\delta\,\xi\,\hbar^5+4\,x\,y\,\gamma\,\eta\,\xi\,\hbar^5\Big)+\mathrm{Log}\Big[\frac{\hbar}{-\delta+T\delta+\hbar}\Big]\Big\}$$

Reorderings with Rord

Rord

```
Rord_{u_i_,w_j_→k_}[𝔈_U_[L___, {l___, u_i_, w_j_, r___}_s_, R___, Q_, P_]] :=
 Simp@Module[{υ, ω, δ, Λ1, yax, q, p, δ1 = ∂_{u_i,w_j} Q},
   {yax, q, p} = List @@ If[δ1 === 0, Λ_U[{υ, ω}, {u, w}], Λ_U[{υ, ω, δ}, {u, w}]] /.
     {y → y_k, a → a_k, x → x_k, t → t_s, T → T_s};
   𝔈_U[L, {l, Sequence @@ yax, r}_s, R, q + (Q /. u_i | w_j → 0), 𝕖^-q DP_{u_i→D_υ, w_j→D_ω}[P][p 𝕖^q]] /.
    {υ → ∂_{u_i} Q /. w_j → 0, ω → ∂_{w_j} Q /. u_i → 0, δ → δ1}]
```

```
With[{c⊙ = 𝔈_CU[{y1, x1}_1, {x2, a2, y2}_2, ℏ t1 a2 + ℏ t1^-1 (𝕖^t1 - 1) y1 x2, 1 + ϵ x1 y2]},
 {Short[rhs = c⊙ // Rord_{x2,a2→3}, 3], HL[CU[c⊙] == CU[rhs]]}]
```

$\left\{\mathfrak{E}_{CU}\left[\{y_1, x_1\}_1, \{a_3, x_3, y_2\}_2, \frac{e^{-\gamma\hbar t_1}\left(e^{\gamma\hbar t_1}\hbar a_3 t_1^2 - \hbar x_3 y_1 + e^{t_1}\hbar x_3 y_1\right)}{t_1}, 1 + \epsilon x_1 y_2\right], \text{True}\right\}$

```
With[{c⊙ = 𝔈_CU[{y1, a1, x1}_1, {x2, a2, y2}_2,
    ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    1 + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 {Short[rhs = c⊙ // Rord_{x2,a2→3}, 3], HL[CU[c⊙] == CU[rhs]]}]
```

$\{\mathfrak{E}_{CU}[\{y_1, a_1, x_1\}_1, \{\ll 1 \gg\}_2, \ll 1 \gg \ll 1 \gg,$
$1 + e^{-\gamma\hbar(l_{12} t_1 + l_{22} t_2)} \epsilon \left(e^{\gamma\hbar(l_{12} t_1 + l_{22} t_2)} a_1 l_1 + e^{\gamma\hbar(l_{12} t_1 + l_{22} t_2)} a_3 l_2 + e^{\gamma\hbar(l_{12} t_1 + l_{22} t_2)} p_{11} x_1 y_1 +\right.$
$\left. p_{21} x_3 y_1 + e^{\ll 1 \gg} p_{12} x_1 y_2 + p_{22} x_3 y_2 - \gamma\hbar l_2 x_3 y_1 \gamma_{21} - \gamma\hbar l_2 x_3 y_2 \gamma_{22}\right)], \text{True}\}$

```
With[{q⊙ = 𝔈_QU[{y1, a1, x1}_1, {x2, a2, y2}_2,
    ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    1 + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 {Short[rhs = q⊙ // Rord_{x2,a2→3}, 3], HL[QU[q⊙] == QU[rhs]]}]
```

$\{\mathfrak{E}_{QU}[\{y_1, a_1, x_1\}_1, \{\ll 1 \gg\}_2, \ll 1 \gg \ll 1 \gg,$
$1 + e^{-\gamma\hbar(l_{12} t_1 + l_{22} t_2)} \epsilon \left(e^{\gamma\hbar(l_{12} t_1 + l_{22} t_2)} a_1 l_1 + e^{\gamma\hbar(l_{12} t_1 + l_{22} t_2)} a_3 l_2 + e^{\gamma\hbar(l_{12} t_1 + l_{22} t_2)} p_{11} x_1 y_1 +\right.$
$\left. p_{21} x_3 y_1 + e^{\ll 1 \gg} p_{12} x_1 y_2 + p_{22} x_3 y_2 - \gamma\hbar l_2 x_3 y_1 \gamma_{21} - \gamma\hbar l_2 x_3 y_2 \gamma_{22}\right)], \text{True}\}$

```
With[{q⊙ = 𝔈_QU[{y1, a1, x1}_1, {x2, a2, y2}_2,
    ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    1 + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 {Short[rhs = q⊙ // Rord_{a2,y2→3}, 3], HL[QU[q⊙] == QU[rhs]]}]
```

$\{\mathfrak{E}_{QU}[\{y_1, a_1, x_1\}_1, \{\ll 1 \gg\}_2, \ll 1 \gg \ll 1 \gg,$
$1 + e^{-\gamma\hbar(l_{12} t_1 + l_{22} t_2)} \epsilon \left(e^{\gamma\hbar(l_{12} t_1 + l_{22} t_2)} a_1 l_1 + e^{\gamma\hbar(l_{12} t_1 + l_{22} t_2)} a_3 l_2 + e^{\gamma\hbar(l_{12} t_1 + l_{22} t_2)} p_{11} x_1 y_1 +\right.$
$\left. e^{\ll 1 \gg} p_{21} x_2 y_1 + p_{12} x_1 y_3 + p_{22} x_2 y_3 - \gamma\hbar l_2 x_1 y_3 \gamma_{12} - \gamma\hbar l_2 x_2 y_3 \gamma_{22}\right)], \text{True}\}$

```
With[{qΦ = ℭQU[{x1, y1}1, {x2, a2, y2}2,
    ℏ (l12 t1 a2 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    1 + ϵ (l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 {Short[rhs = qΦ // Rord_{x1,y1→3}, 5], HL@SimpT[QU[qΦ] == QU[rhs]]}]
```

$\{\mathfrak{C}_{QU}[\{y_3, a_3, x_3\}_1, \{x_2, a_2, y_2\}_2, \frac{\hbar\, a_2\, l_{12}\, t_1 + \ll 16 \gg + \hbar\, T_1\, x_2\, y_2\, \gamma_{11}\, \gamma_{22}}{1 - \gamma_{11} + T_1\, \gamma_{11}},$

$\frac{1}{1 - \gamma_{11} + T_1 \gamma_{11}} + (\epsilon\, (4\, \hbar\, a_2\, l_2 + 4\, p_{11} - 4\, p_{11}\, T_1 + 4\, \hbar\, p_{22}\, x_2\, y_2 + \ll 339 \gg + \gamma\, \hbar^4\, x_2^2\, y_2^2\, \gamma_{12}^2\, \gamma_{21}^2 -$

$4\, \gamma\, \hbar^4\, T_1\, x_2^2\, y_2^2\, \gamma_{12}^2\, \gamma_{21}^2 + 3\, \gamma\, \hbar^4\, T_1^2\, x_2^2\, y_2^2\, \gamma_{12}^2\, \gamma_{21}^2)) \,/\, (4\, \hbar\, (1 - \gamma_{11} + T_1\, \gamma_{11})^5)], \text{True}\}$

R in QU.

Faddeev-Quesne's formula:

Faddeev

$$\mathbb{e}_{q\_,k\_}[x\_] := \mathbb{e}\verb|^|\left(\sum_{j=1}^{k} \frac{(1-q)^j x^j}{j\,(1-q^j)}\right);\ \mathbb{e}_{q\_}[x\_] := \mathbb{e}_{q,\$k}[x]$$

```
Table[Together@SeriesCoefficient[𝕖q,5[x], {x, 0, n}], {n, 0, 5}]
```

$\left\{1, 1, \frac{1}{1+q}, \frac{1}{(1+q)\,(1+q+q^2)}, \frac{1}{(1+q)^2\,(1+q^2)\,(1+q+q^2)},\right.$
$\left. 1 \big/ \left((1+q)^2\,(1+q^2)\,(1+q+q^2)\,(1+q+q^2+q^3+q^4)\right)\right\}$

```
Table[HL@FunctionExpand[QFactorial[n, q] SeriesCoefficient[𝕖q,5[x], {x, 0, n}]], {n, 0, 5}]
```

{1, 1, 1, 1, 1, 1}

R

$$QU[R_{i\_,j\_}] := \mathbb{O}_{QU}\left[\{y_1, a_1\}_i, \{a_2, x_2\}_j, SS\left[\mathbb{e}^{\hbar\, b_1\, a_2}\, \mathbb{e}_{q/.qrule/.e\to\epsilon}[\hbar\, y_1\, x_2]\ /.\ b_1 \to \gamma^{-1}\,(\epsilon\, a_1 - t_i)\right]\right];$$
$$QU\left[R^{-1}_{i\_,j\_}\right] := S_j@QU[R_{i,j}];$$

```
QU[R3,4] // Short
```

$QU[\,] + \frac{\epsilon\, \hbar\, QU[a_3, a_4]}{\gamma} + \hbar\, QU[y_3, x_4] + \frac{\ll 1 \gg}{\gamma} +$

$\ll 1 \gg - \frac{\ll 1 \gg}{\gamma} - \frac{\epsilon \ll 3 \gg}{\gamma^2} - \frac{\hbar^2\, QU[y_3, a_4, x_4]\, t_3}{\gamma} + \frac{\hbar^2\, QU[a_4, a_4]\, t_3^2}{2\, \gamma^2}$

Verifying R2 (~2 secs @ $p=4, $k=2):

$QU\left[R_{1,2} ** R^{-1}_{1,2}\right]$ // Simp // HL // Timing

{0.0625, QU[]}

Verifying R3 (~156 secs @ $p=4, $k=2):

```
{Short[lhs = QU[R1,2 ** R1,3 ** R2,3]], HL@SimpT[lhs - QU[R2,3 ** R1,3 ** R1,2]]} // Timing
```

$\{0.1875, \{QU[] + \frac{\epsilon \hbar QU[a_1, a_2]}{\gamma} + \ll 81 \gg + QU[y_1, a_3, x_3] \left(-\frac{\hbar^2 t_2}{\gamma} + \frac{\hbar^2 t_2 T_2}{\gamma}\right), 0\}\}$

R in Œ$_{QU}$.

RinOE

```
ŒQU[Ri_,j_] := ŒQU[{yi, ai, xi}i, {yj, aj, xj}j, -ħ γ^-1 ti aj + ħ yi xj, Normal@Series[
    e^(ħ γ^-1 ti aj - ħ yi xj) (e^(ħ bi aj) eq[ħ yi xj] /. bi → γ^-1 (e ai - ti)) /. qrule, {e, 0, $k}] /. e → ϵ]
```

```
ŒQU[R1,2]
```

$Œ_{QU}[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, -\frac{\hbar a_2 t_1}{\gamma} + \hbar x_2 y_1, 1 + \frac{\epsilon \hbar a_1 a_2}{\gamma}]$

𝔼

𝔼[L, Q, P] means $e^{\hbar(L+Q)} P$, where L is linear in the a's, Q is a combination of $x_i y_j$, and P is a perturbation polynomial. It should be interpreted via C𝕆[𝔼[...], $\{x_1, a_1, y_1\}_i$, ...] (with some default for direct interpretation), or likewise via Q𝕆[𝔼[...], $\{x_1, a_1, y_1\}_i$, ...]. In themselves, C𝕆 and Q𝕆 should have an interpretation in CU/QU by casting.

Alternative Algorithms

```
λalt[CU] := If[$k == 0, 1, Module[{eq, d, b, c, so},
    eq = ρ@e^(ξ xCU).ρ@e^(η yCU) == ρ@e^(d yCU).ρ@e^(c (t 1CU - 2 ϵ aCU)).ρ@e^(b xCU);
    {so} = Solve[Thread[Flatten /@ eq], {d, b, c}] /. C@1 → 0;
    Normal@Series[e^(-η y - ξ x + η ξ t + c t + d y - 2 ϵ c a + b x) /. so, {ϵ, 0, $k}]]];
```

```
{λalt[CU], HL@Simplify[λalt[CU] == Last[ΛCU[{ξ, η}, {x, y}]]]}
```

$\{1 + \epsilon \left(2 a \eta \xi - y \gamma \eta^2 \xi - x \gamma \eta \xi^2 + \frac{1}{2} t \gamma \eta^2 \xi^2\right), \text{True}\}$