Pensieve header: A unified verification notebook for the sl_2-portfolio project; continues pensieve://Projects/PPSA/nb/Verification.pdf.

Continues pensieve://2017-06/ and pensieve://2017-08/.

Prolog

```
Go;
```

```
wdir = SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\SL2Portfolio"];
NotebookOpen[wdir <> "\\MakeSnips.nb"];
```

```
HL[ℰ_] := Style[ℰ, Background → Yellow];
```

Initialization / Utilities

The “degree carrier / filtration parameter” is ℏ, and all “coupling constants” are proportional to it.

TD

```
$p = 3; $k = 1; (* $k can't be ∞ at least because of Faddeev-Quesne. *)
If[$k == 0, ϵ = 0, ϵ /: ϵ^k_ /; k > $k := 0]; (* $k=0 fails in Series[..{ϵ,...}] *)
SetAttributes[{SS, SST}, HoldAll];
TRule = {T_i_ → e^(ℏ t_i), T → e^(ℏ t)};
SS[ℰ_] := Block[{ℏ, ϵ}, (* Shielded Series *)
   Collect[Normal@Series[ℰ, {ℏ, 0, $p}], ℏ, Together] ];
SST[ℰ_] :=
  Block[{ℏ, ϵ}, Collect[Normal@Series[ℰ /. TRule, {ℏ, 0, $p}], ℏ, Together] ];
Simp[ℰ_, op_] := Collect[ℰ, _CU | _QU, op];
Simp[ℰ_] := Simp[ℰ, Collect[Normal@Series[#, {ℏ, 0, $p}], ℏ, Expand] &];
SimpT[ℰ_] :=
  Collect[ℰ, _CU | _QU, Collect[Normal@Series[# /. TRule, {ℏ, 0, $p}], ℏ, Expand] &];
```

Differential polynomials (DP):

Utils

```
DP_{α_→D_x_, β_→D_y_}[P_][λ_] :=
 Total[CoefficientRules[P, {α, β}] /. ({m_, n_} → c_) :→ c D[λ, {x, m}, {y, n}]]
```

DeclareAlgebra

QLImplementation

```
Unprotect[NonCommutativeMultiply]; Attributes[NonCommutativeMultiply] = {};
(NCM = NonCommutativeMultiply)[x_] := x;
NCM[x_, y_, z__] := (x ** y) ** z;
0 ** _ = _ ** 0 = 0;
(x_Plus) ** y_ := (# ** y) & /@ x; x_ ** (y_Plus) := (x ** #) & /@ y;
B[x_, x_] = 0; B[x_, y_] := x ** y - y ** x;
```

QLImplementation

```
DeclareAlgebra[U_Symbol, opts__Rule] := Module[{gp, sr, g, cp, CE, pow, k = 0,
    gs = Generators /. {opts}, cs = Centrals /. {opts} },
   (#_U = U@#) & /@ gs;
   gp = Alternatives @@ gs; gp = gp | gp_; (* gens *)
   sr = Flatten@Table[{g → ++k, g_i_ → {i, k}}, {g, gs}]; (* sorting → *)
   cp = Alternatives @@ cs; (* cents *)
   CE[ℰ_] := Collect[ℰ, _U, (Expand[#] /. ħ^d_ /; d > $p :→ 0) &];
   U_i_[ℰ_] := ℰ /. {t : cp :→ t_i, u_U :→ Replace[u, x_ :→ x_i, 1]};
   U_i_[NCM[]] = pow[ℰ_, 0] = U@{} = 1_U = U[];
   B[U@(x_)_i_, U@(y_)_i_] := B[U@x_i, U@y_i] = U_i@B[U@x, U@y];
   B[U@(x_)_i_, U@(y_)_j_] /; i =!= j := 0;
   B[U@y_, U@x_] := CE[-B[U@x, U@y]];
   x_ ** 1_U := x; 1_U ** x_ := x;
   (a_. * x_U) ** (b_. * y_U) := If[a b === 0, 0, CE[a b (x ** y)]];
   U[xx___, x_] ** U[y_, yy___] := If[OrderedQ[{x, y} /. sr],
     U[xx, x, y, yy], U@xx ** (U@y ** U@x + B[U@x, U@y]) ** U@yy];
   U@{c_. * (l : gp)^n_, r___} /; FreeQ[c, gp] := CE[c U@Table[l, {n}] ** U@{r}];
   U@{c_. * l : gp, r___} := CE[c U[l] ** U@{r}];
   U@{c_, r___} /; FreeQ[c, gp] := CE[c U@{r}];
   U@{l_Plus, r___} := CE[U@{#, r} & /@ l];
   U@{l_, r___} := U@{Expand[l], r};
   U[ℰ_NonCommutativeMultiply] := U /@ ℰ;
   𝕆_U[specs___, poly_] := Module[{sp, null, vs, us},
     sp = Replace[{specs}, l_List :→ l_null, {1}];
     vs = Join @@ (First /@ sp);
     us = Join @@ (sp /. l_s_ :→ (l /. x_i_ :→ x_s));
     CE[Total[
        CoefficientRules[poly, vs] /. (p_ → c_) :→ c U@(us^p)
       ]] /. x_null :→ x];
   pow[ℰ_, n_] := pow[ℰ, n - 1] ** ℰ;
   𝕊_U[ℰ_, ss___Rule] := CE@Total[
     CoefficientRules[ℰ, First /@ {ss}] /.
      (p_ → c_) :→ c NCM @@ MapThread[pow, {Last /@ {ss}, p}]];
   m_j_→k_[c_. * u_U] := CE[(c /. ((#_j → #_k) & /@ cs))
     DeleteCases[u, _(j|k)] ** U @@ Cases[u, w_j :→ w_k] ** U @@ Cases[u, _k]];
   S_i_[c_. * u_U] := CE[(c /. S_i[U, Centrals])
     DeleteCases[u, _i] ** U_i[NCM @@ Reverse@Cases[u, x_i :→ S@U@x]]] ]
```

DeclareMorphism

QLImplementation

```
DeclareMorphism[m_, U_ → V_, ongs_List, oncs_List: {}] := (
  Replace[ongs, (g_ → img_) ⧴ (m[U[g]] = img), {1}];
  m[1_U] = 1_V;
  m[U[g_i_]] := V_i[m[U@g]];
  m[U[vs__]] := NCM@@ (m /@ U /@ {vs});
  m[ℰ_] := Simp[ℰ /. oncs /. u_U ⧴ m[u]];)
```

Meta-Operations

QLImplementation

```
m_{j_→j_} = Identity;
m_{j_→k_}[ℰ_Plus] := Simp[m_{j→k} /@ ℰ];
m_{is___,i_,j_→k_}[ℰ_] := m_{j→k}@m_{is,i→j}@ℰ;
S_i_[ℰ_Plus] := Simp[S_i /@ ℰ];
```

Implementing CU = $\mathcal{U}(sl_2^{\gamma\epsilon})$

CU

```
DeclareAlgebra[CU, Generators → {y, a, x}, Centrals → {t}];
B[a_CU, y_CU] = -γ y_CU; B[x_CU, a_CU] = -γ x_CU;
B[x_CU, y_CU] = 2 ϵ a_CU - t 1_CU;
(S@CU@y = -y_CU; S@a_CU = -a_CU; S@x_CU = -x_CU;)
S_i_[CU, Centrals] = {t_i → -t_i};
```

Verifying associativity on triples of generators:

```
With[{bas = CU /@ {y, a, x}},
 Table[z1 ** (z2 ** z3) - (z1 ** z2) ** z3 // Simp // HL,
  {z1, bas}, {z2, bas}, {z3, bas} ] ]
```

```
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying associativity on a “random” triple:

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
   (rhs = (z1 ** z2) ** z3 // Simp) // Short,
   z1 ** (z2 ** z3) - rhs // Simp // HL
  }] // Timing
```

```
{0.765625,
 {(28 t^2 γ^4 + 116 t γ^5 ϵ) CU[y, y, y, x, x] + <<21>> + CU[y, y, y, y, y, a, a, a, a, x, x, x, x], 0}}
```

Verifying that S is an anti-homomorphism on CU:

```
With[{bas = CU /@ {y₁, a₁, x₁}},
 Table[HL@Simp[S₁[z1 ** z2] - S₁[z2] ** S₁[z1]],
  {z1, bas}, {z2, bas} ] ]
```

```
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

Verifying the involutivity of S on products of triples:

```
With[{bas = CU /@ {y₁, a₁, x₁}},
 Table[HL@Simp[z1 ** z2 ** z3 - S₁@S₁[z1 ** z2 ** z3]],
  {z1, bas}, {z2, bas}, {z3, bas} ] ]
```

```
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying relabeling:

```
t₁ t₃ CU[y₁, a₁, x₂] + t₁ t₁ CU[y₁, a₂, x₂] // m₁→₃
```

$CU[a_2, x_2, y_3]\, t_3^2 + CU[x_2, y_3, a_3]\, t_3^2$

Verifying meta-associativity:

```
Module[{z, u},
 Table[u = CU[z⟦1⟧₁, z⟦2⟧₂, z⟦3⟧₃]; z → HL[m₁,₃→₃@m₂,₃→₃@u == m₂,₃→₃@m₁,₂→₂@u],
  {z, Tuples[{y, a, x}, 3]}]]
```

```
{{y, y, y} → True, {y, y, a} → True, {y, y, x} → True, {y, a, y} → True,
 {y, a, a} → True, {y, a, x} → True, {y, x, y} → True, {y, x, a} → True,
 {y, x, x} → True, {a, y, y} → True, {a, y, a} → True, {a, y, x} → True, {a, a, y} → True,
 {a, a, a} → True, {a, a, x} → True, {a, x, y} → True, {a, x, a} → True, {a, x, x} → True,
 {x, y, y} → True, {x, y, a} → True, {x, y, x} → True, {x, a, y} → True, {x, a, a} → True,
 {x, a, x} → True, {x, x, y} → True, {x, x, a} → True, {x, x, x} → True}
```

Implementing QU = $\mathcal{U}_q(sl_2^{\gamma\epsilon})$

Aside

```
Series[(1 - T ⅇ^(-2 e a h)) / h, {a, 0, 3}]
```

Aside

$$\frac{1-T}{h} + 2\,e\,T\,a - 2\,(e^2\,h\,T)\,a^2 + \frac{4}{3}\,e^3\,h^2\,T\,a^3 + O[a]^4$$

QU

```
DeclareAlgebra[QU, Generators → {y, a, x}, Centrals → {t, T}];
q = SS[ⅇ^(γ ϵ ℏ)];
B[a_QU, y_QU] = -γ y_QU; B[x_QU, a_QU] = -γ QU@x;
B[x_QU, y_QU] = (q - 1) QU@{y, x} + 𝕆_QU[{a}, SS[(1 - T ⅇ^(-2 ϵ a ℏ)) / ℏ]];
(S@y_QU = 𝕆_QU[{a, y}, SS[-T^-1 ⅇ^(ℏ ϵ a) y]]; S@a_QU = -a_QU; S@x_QU = 𝕆_QU[{a, x}, SS[-ⅇ^(ℏ ϵ a) x]];)
S_i_[QU, Centrals] = {t_i → -t_i, T_i → T_i^-1};
```

```
With[{bas = QU /@ {y, a, x}}, Table[{z1, z2} → Simp[z1 ** z2 - z2 ** z1], {z1, bas}, {z2, bas}] ]
```

$$\left\{\left\{\{QU[y], QU[y]\} \to 0, \{QU[y], QU[a]\} \to \gamma\, QU[y], \{QU[y], QU[x]\} \to \frac{(-1+T)\, QU[]}{\hbar} - 2 T \epsilon\, QU[a] - \gamma \epsilon \hbar\, QU[y, x]\right\}, \left\{\{QU[a], QU[y]\} \to -\gamma\, QU[y], \{QU[a], QU[a]\} \to 0, \{QU[a], QU[x]\} \to \gamma\, QU[x]\right\}, \left\{\{QU[x], QU[y]\} \to \frac{(1-T)\, QU[]}{\hbar} + 2 T \epsilon\, QU[a] + \gamma \epsilon \hbar\, QU[y, x], \{QU[x], QU[a]\} \to -\gamma\, QU[x], \{QU[x], QU[x]\} \to 0\right\}\right\}$$

Verifying associativity on triples of generators:

```
With[{bas = QU /@ {y, a, x}},
 Table[HL[z1 ** (z2 ** z3) - (z1 ** z2) ** z3] // Simp,
  {z1, bas}, {z2, bas}, {z3, bas} ] ]
```

{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}

Verifying associativity on a “random” triple (~34 secs @ $p=5, $k=2):

```
With[{z1 = QU[y, y, a, a, x, x], z2 = QU[y, a, x], z3 = QU[y, y, a, x]}, {
   (rhs = (z1 ** z2) ** z3 // Simp) // Short,
   HL[z1 ** (z2 ** z3) - rhs // Simp]
  }] // Timing
```

$$\left\{6.3125, \left\{\left(\frac{28\gamma^4 - 56 T\gamma^4 + 28 T^2\gamma^4}{\hbar^2} + \frac{82\gamma^5\epsilon - 280 \ll 3 \gg + 198 T^2\gamma^5\epsilon}{\hbar}\right) QU[y, y, y, x, x] + \ll 18 \gg + (1 + 8\gamma\epsilon\hbar)\, QU[\ll 1 \gg], 0\right\}\right\}$$

Verifying that *S* is an anti-homomorphism on QU:

```
With[{bas = QU /@ {y₁, a₁, x₁}},
 Table[{z1, z2} → HL@Simp[S₁[z1 ** z2] - S₁[z2] ** S₁[z1]],
  {z1, bas}, {z2, bas} ] ]
```

$$\{\{\{QU[y_1], QU[y_1]\} \to 0, \{QU[y_1], QU[a_1]\} \to 0, \{QU[y_1], QU[x_1]\} \to 0\}, \{\{QU[a_1], QU[y_1]\} \to 0, \{QU[a_1], QU[a_1]\} \to 0, \{QU[a_1], QU[x_1]\} \to 0\}, \{\{QU[x_1], QU[y_1]\} \to 0, \{QU[x_1], QU[a_1]\} \to 0, \{QU[x_1], QU[x_1]\} \to 0\}\}$$

Verifying that $\mathrm{Lim}_{\hbar\to 0}$ QU = CU using a “random” product (~23 secs @ $p=5, $k=2):

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
   Short[lhs = z1 ** (z2 ** z3)],
   Short[rhs = (QU @@ z1) ** ((QU @@ z2) ** (QU @@ z3))],
   Expand[Limit[rhs /. TRule ⋃ {QU → CU}, ℏ → 0] - lhs] // HL
  }] // Timing
```

$$\left\{10.5313, \left\{48 t\gamma^5\epsilon\, CU[y, y, y, x, x] + \ll 77 \gg + CU[y, y, y, y, y, a, a, a, a, x, x, x, x], 2\left(\frac{4\gamma^5\epsilon}{\hbar} - \frac{8 T\gamma^5\epsilon}{\hbar} + \frac{4 T^2\gamma^5\epsilon}{\hbar}\right) QU[y, y, y, x, x] + \ll 217 \gg + 8\gamma\epsilon\hbar\, QU[y, y, y, y, y, a, a, a, a, x, x, x, x], 0\right\}\right\}$$

Implementing θ

theta

```
DeclareMorphism[Cθ, CU → CU, {y → -x_CU, a → -a_CU, x → -y_CU}, {t → -t, T → T^-1}];
DeclareMorphism[Qθ, QU → QU, {y → 𝕆_QU[{a, x}, SS[-T^(-1/2) ℯ^(ℏ ϵ a) x]],
  a → -a_QU, x → 𝕆_QU[{a, y}, SS[-T^(-1/2) ℯ^(ℏ ϵ a) y]]}, {t → -t, T → T^-1}]
```

Verifying involutivity on CU:

```
With[{bas = CU /@ {y, a, x}},
 Table[z → Cθ[z] → HL[Cθ[Cθ[z]]], {z, bas}] ]
```

{CU[y] → −CU[x] → CU[y], CU[a] → −CU[a] → CU[a], CU[x] → −CU[y] → CU[x]}

Verifying that θ is a multiplicative homomorphism on CU:

```
With[{bas = CU /@ {y, a, x}},
 Table[Cθ[z1 ** z2] - Cθ[z1] ** Cθ[z2] // HL, {z1, bas}, {z2, bas}] ]
```

{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

Verifying involutivity on QU:

```
With[{bas = QU /@ {y, a, x}},
 Table[z → Qθ[z] → HL[Simp[Qθ[Qθ[z]], PowerExpand]], {z, bas}] ]
```

$\left\{QU[y] \to -\frac{QU[x]}{\sqrt{T}} - \frac{\epsilon\hbar\, QU[a, x]}{\sqrt{T}} \to QU[y],\ QU[a] \to -QU[a] \to QU[a],\right.$

$\left. QU[x] \to \left(-\frac{1}{\sqrt{T}} + \frac{\gamma\epsilon\hbar}{\sqrt{T}}\right) QU[y] - \frac{\epsilon\hbar\, QU[y, a]}{\sqrt{T}} \to QU[x]\right\}$

Verifying that θ is a multiplicative homomorphism on QU:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL[Simp[Qθ[z1 ** z2] - Qθ[z1] ** Qθ[z2]]], {z1, bas}, {z2, bas}] ]
```

{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}

The Asymmetric Dequantizator

Following pensieve://People/VanDerVeen/Dequant1.pdf.

ADeq

$$\mathbb{AD}\$f = \gamma\left(\left(\mathrm{Cosh}\left[\hbar\left(a\,e + \frac{\gamma\,e}{2} - \frac{t}{2}\right)\right] - \mathrm{Cosh}\left[\hbar\sqrt{\left(\frac{t-\gamma\,e}{2}\right)^2 + e\,\omega}\right]\right) \Big/ \left(\hbar\, e^{\hbar\,((a+\gamma)\,e - t/2)}\,\mathrm{Sinh}\left[\frac{\gamma\,\epsilon\,\hbar}{2}\right]\left(a^2 e + a\,\gamma\,e - a\,t - \omega\right)\right)\right);$$

Scaling behaviour of A𝔻$f:

```
HL@Simplify[A𝔻$f == ((A𝔻$f /. γ → 1) /. {e → γ e, a → γ^-1 a, ω → γ^-1 ω})]
```

True

```
HL@FullSimplify[
  A𝔻$f == ((A𝔻$f /. γ → 1) /. {ℏ → γ^2 ℏ, e → e / γ, a → a / γ, t → γ^-2 t, ω → γ^-3 ω})]
```

True

ADeq

```
A𝔻$ω = γ CU[y, x] + ϵ CU[a, a] - (t - γ ϵ) CU[a];
```

ADeq

```
DeclareMorphism[A𝔻, QU → CU,
 {a → a_CU, x → CU@x, y → 𝕊_CU[SS[A𝔻$f] /. e → ϵ, a → a_CU, ω → A𝔻$ω] ** y_CU}]
```

Verifying that the asymmetric dequantizator is a homomorphism:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL[SimpT[A𝔻[z1 ** z2] - A𝔻[z1] ** A𝔻[z2]]], {z1, bas}, {z2, bas}] ]
```

```
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

The Symmetric Dequantizator

Following pensieve://People/VanDerVeen/Dequant1.pdf.

SDeq

$$S\mathbb{D}\$g = \sqrt{\left(\left(2\gamma\left(\text{Cosh}\left[\frac{\hbar}{2}\sqrt{t^2+\gamma^2 e^2+4 e\varpi}\right]-\text{Cosh}\left[\frac{t-e\gamma-2 e a}{2/\hbar}\right]\right)\right)\Big/\left(\text{Sinh}\left[\frac{\gamma e\hbar}{2}\right]\left(t\,(2a+\gamma)-2a\,(a+\gamma)\,e+2\varpi\right)\hbar\right)\right)};$$

Verify agreement with the formulas in pensieve://People/VanDerVeen/Dequant1.pdf:

```
{SD$P = (Cosh[ħ ((e-t)/2 + e a)] - Cosh[ħ Sqrt[(t^2+e^2)/4 + e ϖ]]) / (ħ Sinh[-e ħ/2] (ϖ - e a^2 + (t - e) a + t/2)),
 Simplify[SD$P == (SD$P /. {a → -a - 1, t → -t})] // HL,
 PowerExpand@Simplify[(SD$P /. {ħ → γ^2 ħ, e → e/γ, a → a/γ, t → γ^-2 t, ϖ → γ^-3 ϖ}) ==
     SD$g (SD$g /. {a → -a - γ, t → -t})] // HL,
 SD$Q = Simplify[SD$P /. {a → c - 1/2}],
 Simplify[SD$Q == (SD$Q /. {c → -c, t → -t})] // HL,
 FullSimplify[SD$g == FullSimplify[
     Sqrt[SD$Q] /. c → a + 1/2 /. {ħ → γ^2 ħ, e → e/γ, a → a/γ, t → γ^-2 t, ϖ → γ^-3 ϖ}]] // HL
}
```

$$\left\{-\left(\left(\left(\text{Cosh}\left[\left(a\,e+\frac{e-t}{2}\right)\hbar\right]-\text{Cosh}\left[\sqrt{\frac{1}{4}\left(e^2+t^2\right)+e\,\varpi}\;\hbar\right]\right)\text{Csch}\left[\frac{e\,\hbar}{2}\right]\right)\Big/\left(\left(-a^2 e+\frac{t}{2}+a\,(-e+t)+\varpi\right)\hbar\right)\right),\ \text{True},\ \text{True},\right.$$

$$\left(4\left(-\text{Cosh}\left[\frac{1}{2}\sqrt{e^2+t^2+4\,e\,\varpi}\;\hbar\right]+\text{Cosh}\left[c\,e\,\hbar-\frac{t\,\hbar}{2}\right]\right)\text{Csch}\left[\frac{e\,\hbar}{2}\right]\right)\Big/\left(\left(\left(-1+4\,c^2\right)e-4\,(c\,t+\varpi)\right)\hbar\right),$$

$$\left.\text{True},\ \text{True}\right\}$$

SDeq

```
SD$f = Simplify[e^(ħ (t/2 - e a)) (SD$g /. {a → -a, t → -t})];
```

SDeq

```
SD$ϖ = γ CU[y, x] + ϵ CU[a, a] - (t - γ ϵ) CU[a] - t γ 1_CU/2;
```

SDeq

```
DeclareMorphism[SD, QU → CU, {a → a_CU,
   x → S_CU[SS[SD$f] /. e → ϵ, a → a_CU, ϖ → SD$ϖ] ** x_CU,
   y → S_CU[SS[SD$g] /. e → ϵ, a → a_CU, ϖ → SD$ϖ] ** y_CU }]
```

Verifying the θ-symmetry:

```
Table[HL@SimpT[Cθ[SD[z]] == SD[Qθ[z]]], {z, QU /@ {y, a, x}}]
```

```
{True, True, True}
```

Verifying that the symmetric dequantizator is a homomorphism:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL@SimpT[SD[z1 ** z2] - SD[z1] ** SD[z2]], {z1, bas}, {z2, bas}] ]
```

```
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

The representation ρ

rho

```
ρ@y_CU = ρ@y_QU = (0 0; ϵ 0); ρ@a_CU = ρ@a_QU = (γ 0; 0 0);
ρ@x_CU = (0 γ; 0 0); ρ@x_QU = SS@(0 (1 - e^(-γ ϵ ħ)) / (ϵ ħ); 0 0);
ρ[e^ℰ_] := MatrixExp[ρ[ℰ]];
ρ[ℰ_] :=
  (ℰ /. {t → γ ϵ, T → e^(ħ γ ϵ)} /. (U : CU | QU) [u___] ⧴ Fold[Dot, (1 0; 0 1), ρ /@ U /@ {u}])
```

Verifying that ρ represents CU and QU:

```
Table[ρ[z1 ** z2] == ρ[z1].ρ[z2] // SS // HL,
 {U, {CU, QU}}, {z1, U /@ {y, a, x}}, {z2, U /@ {y, a, x}} ]
```

```
{{{True, True, True}, {True, True, True}, {True, True, True}},
 {{True, True, True}, {True, True, True}, {True, True, True}}}
```

Œ and the logoi Λ

Logoi from Pensieve://Talks/Toulouse-1705/DogmaDemo.nb and from Pensieve://Talks/Sydney-1708/ExtraDetails@@.nb.

CdsO

```
CU@Œ_CU[specs___, Q_, P_] := 𝕆_CU[specs, SS[e^(ħ Q) (P /. ϵ → ħ ϵ)]];
QU@Œ_QU[specs___, Q_, P_] := 𝕆_QU[specs, SS[e^(ħ Q) (P /. ϵ → ħ ϵ)]];
```

```
HL[ρ[e^(ξ CU@x)].ρ[e^(α CU@a)] == ρ[e^(α CU@a)].ρ[e^(e^(-γ α) ξ CU@x)]]
```

```
True
```

Logos

```
Λ_U_[{ξ_, α_}, {x, a}] := Œ_U[{a, x}, α a + e^(-γ α) ξ x, 1];
Λ_U_[{α_, η_}, {a, y}] := Œ_U[{y, a}, α a + e^(-γ α) η y, 1];
```

```
{Λ#[{ξ, α}, {x, a}], lhs = #@Œ#[{x, a}, ξ x + α a, 1], HL[lhs == #@Λ#[{ξ, α}, {x, a}]]} & /@
 {CU, QU}
```

$\{\{\text{Œ}_{CU}[\{a, x\}, a\,\alpha + e^{-\alpha\gamma} x\,\xi, 1], CU[\,] + \alpha\hbar\,CU[a] + \left(\xi\hbar - \alpha\gamma\xi\hbar^2 + \frac{1}{2}\alpha^2\gamma^2\xi\hbar^3\right) CU[x] + \frac{1}{2}\alpha^2\hbar^2 CU[a, a] + (\alpha\xi\hbar^2 - \alpha^2\gamma\xi\hbar^3)\,CU[a, x] + \left(\frac{\xi^2\hbar^2}{2} - \alpha\gamma\xi^2\hbar^3\right) CU[x, x] + \frac{1}{6}\alpha^3\hbar^3 CU[a, a, a] + \frac{1}{2}\alpha^2\xi\hbar^3 CU[a, a, x] + \frac{1}{2}\alpha\xi^2\hbar^3 CU[a, x, x] + \frac{1}{6}\xi^3\hbar^3 CU[x, x, x],$

$CU[\,] + \alpha\hbar\,CU[a] + \left(\xi\hbar - \alpha\gamma\xi\hbar^2 + \frac{1}{2}\alpha^2\gamma^2\xi\hbar^3\right) CU[x] + \frac{1}{2}\alpha^2\hbar^2 CU[a, a] + (\alpha\xi\hbar^2 - \alpha^2\gamma\xi\hbar^3)\,CU[a, x] + \left(\frac{\xi^2\hbar^2}{2} - \alpha\gamma\xi^2\hbar^3\right) CU[x, x] + \frac{1}{6}\alpha^3\hbar^3 CU[a, a, a] + \frac{1}{2}\alpha^2\xi\hbar^3 CU[a, a, x] + \frac{1}{2}\alpha\xi^2\hbar^3 CU[a, x, x] + \frac{1}{6}\xi^3\hbar^3 CU[x, x, x] == CU[\,] + \alpha\hbar\,CU[a] + e^{-\alpha\gamma}\xi\hbar\,CU[x] + \frac{1}{2}\alpha^2\hbar^2 CU[a, a] + e^{-\alpha\gamma}\alpha\xi\hbar^2 CU[a, x] + \frac{1}{2}e^{-2\alpha\gamma}\xi^2\hbar^2 CU[x, x] + \frac{1}{6}\alpha^3\hbar^3 CU[a, a, a] + \frac{1}{2}e^{-\alpha\gamma}\alpha^2\xi\hbar^3 CU[a, a, x] + \frac{1}{2}e^{-2\alpha\gamma}\alpha\xi^2\hbar^3 CU[a, x, x] + \frac{1}{6}e^{-3\alpha\gamma}\xi^3\hbar^3 CU[x, x, x]\},$

$\{\text{Œ}_{QU}[\{a, x\}, a\,\alpha + e^{-\alpha\gamma} x\,\xi, 1], QU[\,] + \alpha\hbar\,QU[a] + \left(\xi\hbar - \alpha\gamma\xi\hbar^2 + \frac{1}{2}\alpha^2\gamma^2\xi\hbar^3\right) QU[x] + \frac{1}{2}\alpha^2\hbar^2 QU[a, a] + (\alpha\xi\hbar^2 - \alpha^2\gamma\xi\hbar^3)\,QU[a, x] + \left(\frac{\xi^2\hbar^2}{2} - \alpha\gamma\xi^2\hbar^3\right) QU[x, x] + \frac{1}{6}\alpha^3\hbar^3 QU[a, a, a] + \frac{1}{2}\alpha^2\xi\hbar^3 QU[a, a, x] + \frac{1}{2}\alpha\xi^2\hbar^3 QU[a, x, x] + \frac{1}{6}\xi^3\hbar^3 QU[x, x, x],$

$QU[\,] + \alpha\hbar\,QU[a] + \left(\xi\hbar - \alpha\gamma\xi\hbar^2 + \frac{1}{2}\alpha^2\gamma^2\xi\hbar^3\right) QU[x] + \frac{1}{2}\alpha^2\hbar^2 QU[a, a] + (\alpha\xi\hbar^2 - \alpha^2\gamma\xi\hbar^3)\,QU[a, x] + \left(\frac{\xi^2\hbar^2}{2} - \alpha\gamma\xi^2\hbar^3\right) QU[x, x] + \frac{1}{6}\alpha^3\hbar^3 QU[a, a, a] + \frac{1}{2}\alpha^2\xi\hbar^3 QU[a, a, x] + \frac{1}{2}\alpha\xi^2\hbar^3 QU[a, x, x] + \frac{1}{6}\xi^3\hbar^3 QU[x, x, x] == QU[\,] + \alpha\hbar\,QU[a] + e^{-\alpha\gamma}\xi\hbar\,QU[x] + \frac{1}{2}\alpha^2\hbar^2 QU[a, a] + e^{-\alpha\gamma}\alpha\xi\hbar^2 QU[a, x] + \frac{1}{2}e^{-2\alpha\gamma}\xi^2\hbar^2 QU[x, x] + \frac{1}{6}\alpha^3\hbar^3 QU[a, a, a] + \frac{1}{2}e^{-\alpha\gamma}\alpha^2\xi\hbar^3 QU[a, a, x] + \frac{1}{2}e^{-2\alpha\gamma}\alpha\xi^2\hbar^3 QU[a, x, x] + \frac{1}{6}e^{-3\alpha\gamma}\xi^3\hbar^3 QU[x, x, x]\}\}$

```
{Λ#[{α, η}, {a, y}], lhs = #@Œ#[{a, y}, η y + α a, 1], HL[lhs == #@Λ#[{α, η}, {a, y}]]} & /@
 {CU, QU}
```

$$\{\{\text{Œ}_{CU}[\{y, a\}, a\,\alpha + e^{-\alpha\gamma} y\,\eta, 1], CU[\,] + \alpha\hbar\,CU[a] + \left(\eta\hbar - \alpha\gamma\eta\hbar^2 + \frac{1}{2}\alpha^2\gamma^2\eta\hbar^3\right) CU[y] + \frac{1}{2}\alpha^2\hbar^2\,CU[a, a] + \left(\alpha\eta\hbar^2 - \alpha^2\gamma\eta\hbar^3\right) CU[y, a] + \left(\frac{\eta^2\hbar^2}{2} - \alpha\gamma\eta^2\hbar^3\right) CU[y, y] + \frac{1}{6}\alpha^3\hbar^3\,CU[a, a, a] + \frac{1}{2}\alpha^2\eta\hbar^3\,CU[y, a, a] + \frac{1}{2}\alpha\eta^2\hbar^3\,CU[y, y, a] + \frac{1}{6}\eta^3\hbar^3\,CU[y, y, y],$$

$$CU[\,] + \alpha\hbar\,CU[a] + \left(\eta\hbar - \alpha\gamma\eta\hbar^2 + \frac{1}{2}\alpha^2\gamma^2\eta\hbar^3\right) CU[y] + \frac{1}{2}\alpha^2\hbar^2\,CU[a, a] + \left(\alpha\eta\hbar^2 - \alpha^2\gamma\eta\hbar^3\right) CU[y, a] + \left(\frac{\eta^2\hbar^2}{2} - \alpha\gamma\eta^2\hbar^3\right) CU[y, y] + \frac{1}{6}\alpha^3\hbar^3\,CU[a, a, a] + \frac{1}{2}\alpha^2\eta\hbar^3\,CU[y, a, a] + \frac{1}{2}\alpha\eta^2\hbar^3\,CU[y, y, a] + \frac{1}{6}\eta^3\hbar^3\,CU[y, y, y] ==$$
$$CU[\,] + \alpha\hbar\,CU[a] + e^{-\alpha\gamma}\eta\hbar\,CU[y] + \frac{1}{2}\alpha^2\hbar^2\,CU[a, a] + e^{-\alpha\gamma}\alpha\eta\hbar^2\,CU[y, a] + \frac{1}{2}e^{-2\alpha\gamma}\eta^2\hbar^2\,CU[y, y] + \frac{1}{6}\alpha^3\hbar^3\,CU[a, a, a] + \frac{1}{2}e^{-\alpha\gamma}\alpha^2\eta\hbar^3\,CU[y, a, a] + \frac{1}{2}e^{-2\alpha\gamma}\alpha\eta^2\hbar^3\,CU[y, y, a] + \frac{1}{6}e^{-3\alpha\gamma}\eta^3\hbar^3\,CU[y, y, y]\},$$

$$\{\text{Œ}_{QU}[\{y, a\}, a\,\alpha + e^{-\alpha\gamma} y\,\eta, 1], QU[\,] + \alpha\hbar\,QU[a] + \left(\eta\hbar - \alpha\gamma\eta\hbar^2 + \frac{1}{2}\alpha^2\gamma^2\eta\hbar^3\right) QU[y] + \frac{1}{2}\alpha^2\hbar^2\,QU[a, a] + \left(\alpha\eta\hbar^2 - \alpha^2\gamma\eta\hbar^3\right) QU[y, a] + \left(\frac{\eta^2\hbar^2}{2} - \alpha\gamma\eta^2\hbar^3\right) QU[y, y] + \frac{1}{6}\alpha^3\hbar^3\,QU[a, a, a] + \frac{1}{2}\alpha^2\eta\hbar^3\,QU[y, a, a] + \frac{1}{2}\alpha\eta^2\hbar^3\,QU[y, y, a] + \frac{1}{6}\eta^3\hbar^3\,QU[y, y, y],$$

$$QU[\,] + \alpha\hbar\,QU[a] + \left(\eta\hbar - \alpha\gamma\eta\hbar^2 + \frac{1}{2}\alpha^2\gamma^2\eta\hbar^3\right) QU[y] + \frac{1}{2}\alpha^2\hbar^2\,QU[a, a] + \left(\alpha\eta\hbar^2 - \alpha^2\gamma\eta\hbar^3\right) QU[y, a] + \left(\frac{\eta^2\hbar^2}{2} - \alpha\gamma\eta^2\hbar^3\right) QU[y, y] + \frac{1}{6}\alpha^3\hbar^3\,QU[a, a, a] + \frac{1}{2}\alpha^2\eta\hbar^3\,QU[y, a, a] + \frac{1}{2}\alpha\eta^2\hbar^3\,QU[y, y, a] + \frac{1}{6}\eta^3\hbar^3\,QU[y, y, y] ==$$
$$QU[\,] + \alpha\hbar\,QU[a] + e^{-\alpha\gamma}\eta\hbar\,QU[y] + \frac{1}{2}\alpha^2\hbar^2\,QU[a, a] + e^{-\alpha\gamma}\alpha\eta\hbar^2\,QU[y, a] + \frac{1}{2}e^{-2\alpha\gamma}\eta^2\hbar^2\,QU[y, y] + \frac{1}{6}\alpha^3\hbar^3\,QU[a, a, a] + \frac{1}{2}e^{-\alpha\gamma}\alpha^2\eta\hbar^3\,QU[y, a, a] + \frac{1}{2}e^{-2\alpha\gamma}\alpha\eta^2\hbar^3\,QU[y, y, a] + \frac{1}{6}e^{-3\alpha\gamma}\eta^3\hbar^3\,QU[y, y, y]\}\}$$

Goal. In either U, compute $F = e^{-\eta y} e^{\xi x} e^{\eta y} e^{-\xi x}$. First compute $G = e^{\xi x}\, y\, e^{-\xi x}$, a finite sum. Now F satisfies the ODE $\partial_\eta F = \partial_\eta\left(e^{-\eta y} e^{\eta G}\right) = -yF + FG$ with initial conditions $F(\eta = 0) = 1$. So we set it up and solve:

```
With[{U = CU},
 Module[{G, F, fs, bs, e, b, es, sol},
  G = Echo@Simp[Table[ξ^k / k!, {k, 0, $k + 1}].NestList[Simp[B[x_U , #]] &, y_U, $k + 1]];
  fs = Echo@Flatten@Table[f_{l,i,j,k}[η], {l, 0, $k}, {i, 0, 1}, {j, 0, 1}, {k, 0, 1}];
  F = Echo[fs.(bs = fs /. f_{l_,i_,j_,k_}[η] :> ϵ^l U@{y^i, a^j, x^k})];
  es = Flatten[
    Table[Coefficient[e, b] == 0, {e, {F - 1_U /. η → 0, F ** G - y_U ** F - ∂_η F}}, {b, bs}]];
  sol = Echo@First[F /. DSolve[es, fs, η]];
  Echo[sol /. {ⅇ- → 1, U → Times}];
  Collect[sol /. {ⅇ- → 1, U → Times}, ϵ, Simplify]
 ]]
```

$-t\,\xi\,CU[\,] + 2\,\epsilon\,\xi\,CU[a] - \gamma\,\epsilon\,\xi^2\,CU[x] + CU[y]$

$\{f_{0,0,0,0}[\eta],\ f_{1,0,0,0}[\eta],\ f_{1,0,0,1}[\eta],\ f_{1,0,1,0}[\eta],\ f_{1,0,1,1}[\eta],\ f_{1,1,0,0}[\eta],\ f_{1,1,0,1}[\eta],\ f_{1,1,1,0}[\eta],\ f_{1,1,1,1}[\eta]\}$

$CU[\,]\,f_{0,0,0,0}[\eta] + \epsilon\,CU[\,]\,f_{1,0,0,0}[\eta] + \epsilon\,CU[x]\,f_{1,0,0,1}[\eta] + \epsilon\,CU[a]\,f_{1,0,1,0}[\eta] + \epsilon\,CU[a, x]\,f_{1,0,1,1}[\eta] + \epsilon\,CU[y]\,f_{1,1,0,0}[\eta] + \epsilon\,CU[y, x]\,f_{1,1,0,1}[\eta] + \epsilon\,CU[y, a]\,f_{1,1,1,0}[\eta] + \epsilon\,CU[y, a, x]\,f_{1,1,1,1}[\eta]$

» $e^{-t\,\eta\,\xi}\,CU[\,] + \frac{1}{2}\,e^{-t\,\eta\,\xi}\,t\,\gamma\,\epsilon\,\eta^2\,\xi^2\,CU[\,] + 2\,e^{-t\,\eta\,\xi}\,\epsilon\,\eta\,\xi\,CU[a] - e^{-t\,\eta\,\xi}\,\gamma\,\epsilon\,\eta\,\xi^2\,CU[x] - e^{-t\,\eta\,\xi}\,\gamma\,\epsilon\,\eta^2\,\xi\,CU[y]$

» $1 + 2\,a\,\epsilon\,\eta\,\xi - y\,\gamma\,\epsilon\,\eta^2\,\xi - x\,\gamma\,\epsilon\,\eta\,\xi^2 + \frac{1}{2}\,t\,\gamma\,\epsilon\,\eta^2\,\xi^2$

$1 + \frac{1}{2}\,\epsilon\,\eta\,\xi\,\left(4\,a + \gamma\,\left(-2\,y\,\eta - 2\,x\,\xi + t\,\eta\,\xi\right)\right)$

Logos

```
Λ_U_[{ξ1_, η1_}, {x, y}] := Λ_U[{ξ1, η1}, {x, y}] = Module[{ξ, η, G, F, fs, f, bs, e, b, es},
   G = Simp[Table[ξ^k / k!, {k, 0, $k + 1}].NestList[Simp[B[x_U , #]] &, y_U, $k + 1]];
   fs = Flatten@Table[f_{l,i,j,k}[η], {l, 0, $k}, {i, 0, 1}, {j, 0, 1}, {k, 0, 1}];
   F = fs.(bs = fs /. f_{l_,i_,j_,k_}[η] :> ϵ^l U@{y^i, a^j, x^k});
   es = Flatten[
     Table[Coefficient[e, b] == 0, {e, {F - 1_U /. η → 0, F ** G - y_U ** F - ∂_η F}}, {b, bs}]];
   F = F /. DSolve[es, fs, η][[1]];
   Œ_U[{y, a, x},
     ξ x + η y + (U /. {CU → -t η ξ, QU → η ξ (1 - T) / ℏ}),
     F /. {ⅇ- → 1, U → Times}
    ] /. {ξ → ξ1, η → η1}];
```

```
{Λ_CU[{ξ, η}, {x, y}], lhs = CU@Œ_CU[{x, y}, ℏ (ξ x + η y), 1],
 HL[lhs == CU@Λ_CU[ℏ {ξ, η}, {x, y}]]}
```

$\{Œ_{CU}[\{y, a, x\},\ y\,\eta + x\,\xi - t\,\eta\,\xi,\ 1 + 2\,a\,\epsilon\,\eta\,\xi - y\,\gamma\,\epsilon\,\eta^2\,\xi - x\,\gamma\,\epsilon\,\eta\,\xi^2 + \frac{1}{2}\,t\,\gamma\,\epsilon\,\eta^2\,\xi^2],$
$\left(1 - t\,\eta\,\xi\,\hbar^2\right)\,CU[\,] + 2\,\epsilon\,\eta\,\xi\,\hbar^2\,CU[a] + \xi\,\hbar\,CU[x] + \eta\,\hbar\,CU[y] + \frac{1}{2}\,\xi^2\,\hbar^2\,CU[x, x] + \eta\,\xi\,\hbar^2\,CU[y, x] + \frac{1}{2}\,\eta^2\,\hbar^2\,CU[y, y],\ \text{True}\}$

```
{Λ_QU[{ξ, η}, {x, y}], lhs = QU@Œ_QU[{x, y}, ℏ (ξ x + η y), 1],
 HL@SimpT[lhs == QU@Λ_QU[ℏ {ξ, η}, {x, y}]]}
```

$\{\mathbb{E}_{QU}[\{y, a, x\}, y\eta + x\xi + \frac{(1-T)\eta\xi}{\hbar}, 1 + 2aT\epsilon\eta\xi - \frac{1}{2}(-1+3T)y\gamma\epsilon\eta^2\xi - \frac{1}{2}(-1+3T)x\gamma\epsilon\eta\xi^2 + \frac{(-1+T)(-1+3T)\gamma\epsilon\eta^2\xi^2}{4\hbar} + xy\gamma\epsilon\eta\xi\hbar],$
$(1+\eta\xi\hbar - T\eta\xi\hbar)QU[] + 2T\epsilon\eta\xi\hbar^2 QU[a] + \xi\hbar QU[x] + \eta\hbar QU[y] + \frac{1}{2}\xi^2\hbar^2 QU[x, x] + \eta\xi\hbar^2 QU[y, x] + \frac{1}{2}\eta^2\hbar^2 QU[y, y], \text{True}\}$

```
{tt = Last[Λ_CU[{ξ, η}, {x, y}]], Normal@Series[Log[tt], {ϵ, 0, $k}]}
```

$\{1 + 2a\epsilon\eta\xi - y\gamma\epsilon\eta^2\xi - x\gamma\epsilon\eta\xi^2 + \frac{1}{2}t\gamma\epsilon\eta^2\xi^2, \epsilon\left(2a\eta\xi - y\gamma\eta^2\xi - x\gamma\eta\xi^2 + \frac{1}{2}t\gamma\eta^2\xi^2\right)\}$

```
{tt = Last[Λ_QU[{ξ, η}, {x, y}]], Normal@Series[Log[tt], {ϵ, 0, $k}]}
```

$\{1 + 2aT\epsilon\eta\xi - \frac{1}{2}(-1+3T)y\gamma\epsilon\eta^2\xi - \frac{1}{2}(-1+3T)x\gamma\epsilon\eta\xi^2 + \frac{(-1+T)(-1+3T)\gamma\epsilon\eta^2\xi^2}{4\hbar} + xy\gamma\epsilon\eta\xi\hbar, \frac{1}{4\hbar}\epsilon\left(\gamma\eta^2\xi^2 - 4T\gamma\eta^2\xi^2 + 3T^2\gamma\eta^2\xi^2 + 8aT\eta\xi\hbar + 2y\gamma\eta^2\xi\hbar - 6Ty\gamma\eta^2\xi\hbar + 2x\gamma\eta\xi^2\hbar - 6Tx\gamma\eta\xi^2\hbar + 4xy\gamma\eta\xi\hbar^2\right)\}$

Logos

```
Simp[Œ_U_[specs___, Q_, P_]] :=
  Œ_U[specs, ExpandNumerator@Together[Q], Collect[P, ϵ, ExpandNumerator@*Together]];
Λ_U_[{υ1_, ω1_, δ_}, {u_, w_}] := Simp@Module[{υ, ω, yax, q, p, Q, d},
   {yax, q, p} = List @@ Λ_U[{υ, ω}, {u, w}];
   Œ_U[yax, Q = (υ u + ω w + δ u w + d υ ω) / (1 - d δ),
     Expand[(1 - d δ)^-1 e^-Q DP_{υ→D_u, ω→D_w}[p][e^Q]]] /. {d → ∂_{υ,ω} q} /. {υ → υ1, ω → ω1}]
```

```
{ΛCU[{ξ, η, δ}, {x, y}], lhs = CU@ℭCU[{x, y}, ℏ (ξ x + η y + δ x y), 1],
 HL[lhs == CU@ΛCU[ℏ {ξ, η, δ}, {x, y}]]}
```

$$\Big\{\mathbb{C}_{CU}\Big[\{y, a, x\}, \frac{x\,y\,\delta + y\,\eta + x\,\xi - t\,\eta\,\xi}{1 + t\,\delta}, \frac{1}{1 + t\,\delta} + \frac{1}{2\,(1 + t\,\delta)^5}\,\epsilon\,\big(4\,a\,\delta + 12\,a\,t\,\delta^2 + 4\,a\,x\,y\,\delta^2 + 2\,t\,\gamma\,\delta^2 - 8\,x\,y\,\gamma\,\delta^2 + 12\,a\,t^2\,\delta^3 + 8\,a\,t\,x\,y\,\delta^3 + 4\,t^2\,\gamma\,\delta^3 - 12\,t\,x\,y\,\gamma\,\delta^3 - 4\,x^2\,y^2\,\gamma\,\delta^3 + 4\,a\,t^3\,\delta^4 + 4\,a\,t^2\,x\,y\,\delta^4 + 2\,t^3\,\gamma\,\delta^4 - 4\,t^2\,x\,y\,\gamma\,\delta^4 - 3\,t\,x^2\,y^2\,\gamma\,\delta^4 + 4\,a\,y\,\delta\,\eta - 4\,y\,\gamma\,\delta\,\eta + 8\,a\,t\,y\,\delta^2\,\eta - 4\,t\,y\,\gamma\,\delta^2\,\eta - 6\,x\,y^2\,\gamma\,\delta^2\,\eta + 4\,a\,t^2\,y\,\delta^3\,\eta - 4\,t\,x\,y^2\,\gamma\,\delta^3\,\eta - 2\,y^2\,\gamma\,\delta\,\eta^2 - t\,y^2\,\gamma\,\delta^2\,\eta^2 + 4\,a\,x\,\delta\,\xi - 4\,x\,\gamma\,\delta\,\xi + 8\,a\,t\,x\,\delta^2\,\xi - 4\,t\,x\,\gamma\,\delta^2\,\xi - 6\,x^2\,y\,\gamma\,\delta^2\,\xi + 4\,a\,t^2\,x\,\delta^3\,\xi - 4\,t\,x^2\,y\,\gamma\,\delta^3\,\xi + 4\,a\,\eta\,\xi + 8\,a\,t\,\delta\,\eta\,\xi + 4\,t\,\gamma\,\delta\,\eta\,\xi - 8\,x\,y\,\gamma\,\delta\,\eta\,\xi + 4\,a\,t^2\,\delta^2\,\eta\,\xi + 4\,t^2\,\gamma\,\delta^2\,\eta\,\xi - 4\,t\,x\,y\,\gamma\,\delta^2\,\eta\,\xi - 2\,y\,\gamma\,\eta^2\,\xi - 2\,x^2\,\gamma\,\delta\,\xi^2 - t\,x^2\,\gamma\,\delta^2\,\xi^2 - 2\,x\,\gamma\,\eta\,\xi^2 + t\,\gamma\,\eta^2\,\xi^2\big)\Big],$$

$$\left(1 - t\,\delta\,\hbar + t^2\,\delta^2\,\hbar^2 + t\,\gamma\,\delta^2\,\epsilon\,\hbar^2 - t\,\eta\,\xi\,\hbar^2\right) CU[\,] + \left(2\,\delta\,\epsilon\,\hbar - 4\,t\,\delta^2\,\epsilon\,\hbar^2 + 2\,\epsilon\,\eta\,\xi\,\hbar^2\right) CU[a] +$$
$$\left(\xi\,\hbar - 2\,t\,\delta\,\xi\,\hbar^2 - 2\,\gamma\,\delta\,\epsilon\,\xi\,\hbar^2\right) CU[x] +$$
$$\left(\eta\,\hbar - 2\,t\,\delta\,\eta\,\hbar^2 - 2\,\gamma\,\delta\,\epsilon\,\eta\,\hbar^2\right) CU[y] +$$
$$4\,\delta\,\epsilon\,\xi\,\hbar^2\,CU[a, x] +$$
$$\frac{1}{2}\,\xi^2\,\hbar^2\,CU[x, x] +$$
$$4\,\delta\,\epsilon\,\eta\,\hbar^2\,CU[y, a] +$$
$$\left(\delta\,\hbar - 2\,t\,\delta^2\,\hbar^2 - 4\,\gamma\,\delta^2\,\epsilon\,\hbar^2 + \eta\,\xi\,\hbar^2\right) CU[y, x] +$$
$$\frac{1}{2}\,\eta^2\,\hbar^2\,CU[y, y] +$$
$$4\,\delta^2\,\epsilon\,\hbar^2\,CU[y, a, x] +$$
$$\delta\,\xi\,\hbar^2\,CU[y, x, x] +$$
$$\delta\,\eta\,\hbar^2\,CU[y, y, x] +$$
$$\frac{1}{2}\,\delta^2\,\hbar^2\,CU[y, y, x, x],\ \text{True}\Big\}$$

```
{Λ_QU[{ξ, η, δ}, {x, y}], lhs = QU@Œ_QU[{x, y}, ℏ (ξ x + η y + δ x y), 1],
 HL@SimpT[lhs == QU@Λ_QU[ℏ {ξ, η, δ}, {x, y}]]}
```

$$\Big\{Œ_{QU}\Big[\{y, a, x\}, \frac{\eta\,\xi - T\,\eta\,\xi + x\,y\,\delta\,\hbar + y\,\eta\,\hbar + x\,\xi\,\hbar}{-\delta + T\,\delta + \hbar},$$

$$\frac{\hbar}{-\delta + T\,\delta + \hbar} + \frac{1}{4\,(-\delta + T\,\delta + \hbar)^5}\,\epsilon\,\Big(-8\,a\,T\,\delta^4\,\hbar^2 + 24\,a\,T^2\,\delta^4\,\hbar^2 - 24\,a\,T^3\,\delta^4\,\hbar^2 + 8\,a\,T^4\,\delta^4\,\hbar^2 + 2\,\gamma\,\delta^4\,\hbar^2 -$$
$$12\,T\,\gamma\,\delta^4\,\hbar^2 + 24\,T^2\,\gamma\,\delta^4\,\hbar^2 - 20\,T^3\,\gamma\,\delta^4\,\hbar^2 + 6\,T^4\,\gamma\,\delta^4\,\hbar^2 + 24\,a\,T\,\delta^3\,\hbar^3 - 48\,a\,T^2\,\delta^3\,\hbar^3 +$$
$$24\,a\,T^3\,\delta^3\,\hbar^3 - 4\,\gamma\,\delta^3\,\hbar^3 + 20\,T\,\gamma\,\delta^3\,\hbar^3 - 28\,T^2\,\gamma\,\delta^3\,\hbar^3 + 12\,T^3\,\gamma\,\delta^3\,\hbar^3 + 8\,a\,T\,x\,y\,\delta^4\,\hbar^3 -$$
$$16\,a\,T^2\,x\,y\,\delta^4\,\hbar^3 + 8\,a\,T^3\,x\,y\,\delta^4\,\hbar^3 - 8\,T\,x\,y\,\gamma\,\delta^4\,\hbar^3 + 16\,T^2\,x\,y\,\gamma\,\delta^4\,\hbar^3 - 8\,T^3\,x\,y\,\gamma\,\delta^4\,\hbar^3 +$$
$$8\,a\,T\,y\,\delta^3\,\eta\,\hbar^3 - 16\,a\,T^2\,y\,\delta^3\,\eta\,\hbar^3 + 8\,a\,T^3\,y\,\delta^3\,\eta\,\hbar^3 + 8\,a\,T\,x\,\delta^3\,\xi\,\hbar^3 - 16\,a\,T^2\,x\,\delta^3\,\xi\,\hbar^3 +$$
$$8\,a\,T^3\,x\,\delta^3\,\xi\,\hbar^3 + 8\,a\,T\,\delta^2\,\eta\,\xi\,\hbar^3 - 16\,a\,T^2\,\delta^2\,\eta\,\xi\,\hbar^3 + 8\,a\,T^3\,\delta^2\,\eta\,\xi\,\hbar^3 - 4\,\gamma\,\delta^2\,\eta\,\xi\,\hbar^3 +$$
$$20\,T\,\gamma\,\delta^2\,\eta\,\xi\,\hbar^3 - 28\,T^2\,\gamma\,\delta^2\,\eta\,\xi\,\hbar^3 + 12\,T^3\,\gamma\,\delta^2\,\eta\,\xi\,\hbar^3 - 24\,a\,T\,\delta^2\,\hbar^4 + 24\,a\,T^2\,\delta^2\,\hbar^4 +$$
$$2\,\gamma\,\delta^2\,\hbar^4 - 8\,T\,\gamma\,\delta^2\,\hbar^4 + 6\,T^2\,\gamma\,\delta^2\,\hbar^4 - 16\,a\,T\,x\,y\,\delta^3\,\hbar^4 + 16\,a\,T^2\,x\,y\,\delta^3\,\hbar^4 + 24\,T\,x\,y\,\gamma\,\delta^3\,\hbar^4 -$$
$$24\,T^2\,x\,y\,\gamma\,\delta^3\,\hbar^4 + x^2\,y^2\,\gamma\,\delta^4\,\hbar^4 + 4\,T\,x^2\,y^2\,\gamma\,\delta^4\,\hbar^4 - 5\,T^2\,x^2\,y^2\,\gamma\,\delta^4\,\hbar^4 - 16\,a\,T\,y\,\delta^2\,\eta\,\hbar^4 +$$
$$16\,a\,T^2\,y\,\delta^2\,\eta\,\hbar^4 - 4\,y\,\gamma\,\delta^2\,\eta\,\hbar^4 + 16\,T\,y\,\gamma\,\delta^2\,\eta\,\hbar^4 - 12\,T^2\,y\,\gamma\,\delta^2\,\eta\,\hbar^4 + 8\,T\,x\,y^2\,\gamma\,\delta^3\,\eta\,\hbar^4 -$$
$$8\,T^2\,x\,y^2\,\gamma\,\delta^3\,\eta\,\hbar^4 - y^2\,\gamma\,\delta^2\,\eta^2\,\hbar^4 + 4\,T\,y^2\,\gamma\,\delta^2\,\eta^2\,\hbar^4 - 3\,T^2\,y^2\,\gamma\,\delta^2\,\eta^2\,\hbar^4 - 16\,a\,T\,x\,\delta^2\,\xi\,\hbar^4 +$$
$$16\,a\,T^2\,x\,\delta^2\,\xi\,\hbar^4 - 4\,x\,\gamma\,\delta^2\,\xi\,\hbar^4 + 16\,T\,x\,\gamma\,\delta^2\,\xi\,\hbar^4 - 12\,T^2\,x\,\gamma\,\delta^2\,\xi\,\hbar^4 + 8\,T\,x^2\,y\,\gamma\,\delta^3\,\xi\,\hbar^4 -$$
$$8\,T^2\,x^2\,y\,\gamma\,\delta^3\,\xi\,\hbar^4 - 16\,a\,T\,\delta\,\eta\,\xi\,\hbar^4 + 16\,a\,T^2\,\delta\,\eta\,\xi\,\hbar^4 + 4\,\gamma\,\delta\,\eta\,\xi\,\hbar^4 - 16\,T\,\gamma\,\delta\,\eta\,\xi\,\hbar^4 +$$
$$12\,T^2\,\gamma\,\delta\,\eta\,\xi\,\hbar^4 + 8\,T\,x\,y\,\gamma\,\delta^2\,\eta\,\xi\,\hbar^4 - 8\,T^2\,x\,y\,\gamma\,\delta^2\,\eta\,\xi\,\hbar^4 - x^2\,\gamma\,\delta^2\,\xi^2\,\hbar^4 + 4\,T\,x^2\,\gamma\,\delta^2\,\xi^2\,\hbar^4 -$$
$$3\,T^2\,x^2\,\gamma\,\delta^2\,\xi^2\,\hbar^4 + \gamma\,\eta^2\,\xi^2\,\hbar^4 - 4\,T\,\gamma\,\eta^2\,\xi^2\,\hbar^4 + 3\,T^2\,\gamma\,\eta^2\,\xi^2\,\hbar^4 + 8\,a\,T\,\delta\,\hbar^5 + 8\,a\,T\,x\,y\,\delta^2\,\hbar^5 -$$
$$4\,x\,y\,\gamma\,\delta^2\,\hbar^5 - 12\,T\,x\,y\,\gamma\,\delta^2\,\hbar^5 - 4\,x^2\,y^2\,\gamma\,\delta^3\,\hbar^5 - 4\,T\,x^2\,y^2\,\gamma\,\delta^3\,\hbar^5 + 8\,a\,T\,y\,\delta\,\eta\,\hbar^5 + 4\,y\,\gamma\,\delta\,\eta\,\hbar^5 -$$
$$12\,T\,y\,\gamma\,\delta\,\eta\,\hbar^5 - 2\,x\,y^2\,\gamma\,\delta^2\,\eta\,\hbar^5 - 10\,T\,x\,y^2\,\gamma\,\delta^2\,\eta\,\hbar^5 + 2\,y^2\,\gamma\,\delta\,\eta^2\,\hbar^5 - 6\,T\,y^2\,\gamma\,\delta\,\eta^2\,\hbar^5 +$$
$$8\,a\,T\,x\,\delta\,\xi\,\hbar^5 + 4\,x\,\gamma\,\delta\,\xi\,\hbar^5 - 12\,T\,x\,\gamma\,\delta\,\xi\,\hbar^5 - 2\,x^2\,y\,\gamma\,\delta^2\,\xi\,\hbar^5 - 10\,T\,x^2\,y\,\gamma\,\delta^2\,\xi\,\hbar^5 + 8\,a\,T\,\eta\,\xi\,\hbar^5 -$$
$$16\,T\,x\,y\,\gamma\,\delta\,\eta\,\xi\,\hbar^5 + 2\,y\,\gamma\,\eta^2\,\xi\,\hbar^5 - 6\,T\,y\,\gamma\,\eta^2\,\xi\,\hbar^5 + 2\,x^2\,\gamma\,\delta\,\xi^2\,\hbar^5 - 6\,T\,x^2\,\gamma\,\delta\,\xi^2\,\hbar^5 + 2\,x\,\gamma\,\eta\,\xi^2\,\hbar^5 -$$
$$6\,T\,x\,\gamma\,\eta\,\xi^2\,\hbar^5 + 4\,x\,y\,\gamma\,\delta\,\hbar^6 + 4\,x^2\,y^2\,\gamma\,\delta^2\,\hbar^6 + 4\,x\,y^2\,\gamma\,\delta\,\eta\,\hbar^6 + 4\,x^2\,y\,\gamma\,\delta\,\xi\,\hbar^6 + 4\,x\,y\,\gamma\,\eta\,\xi\,\hbar^6\Big)\Big],$$

$$\Big(1 + \delta - T\,\delta + \delta^2 - 2\,T\,\delta^2 + T^2\,\delta^2 + \frac{1}{2}\,\gamma\,\delta^2\,\epsilon\,\hbar - 2\,T\,\gamma\,\delta^2\,\epsilon\,\hbar + \frac{3}{2}\,T^2\,\gamma\,\delta^2\,\epsilon\,\hbar + \eta\,\xi\,\hbar - T\,\eta\,\xi\,\hbar\Big)\,QU[\,] +$$
$$\big(2\,T\,\delta\,\epsilon\,\hbar + 4\,T\,\delta^2\,\epsilon\,\hbar - 4\,T^2\,\delta^2\,\epsilon\,\hbar + 2\,T\,\epsilon\,\eta\,\xi\,\hbar^2\big)\,QU[a] +$$
$$\big(\xi\,\hbar + 2\,\delta\,\xi\,\hbar - 2\,T\,\delta\,\xi\,\hbar + \gamma\,\delta\,\epsilon\,\xi\,\hbar^2 - 3\,T\,\gamma\,\delta\,\epsilon\,\xi\,\hbar^2\big)\,QU[x] +$$
$$\big(\eta\,\hbar + 2\,\delta\,\eta\,\hbar - 2\,T\,\delta\,\eta\,\hbar + \gamma\,\delta\,\epsilon\,\eta\,\hbar^2 - 3\,T\,\gamma\,\delta\,\epsilon\,\eta\,\hbar^2\big)\,QU[y] +$$
$$4\,T\,\delta\,\epsilon\,\xi\,\hbar^2\,QU[a, x] + \frac{1}{2}\,\xi^2\,\hbar^2\,QU[x, x] +$$
$$4\,T\,\delta\,\epsilon\,\eta\,\hbar^2\,QU[y, a] +$$
$$\big(\delta\,\hbar + 2\,\delta^2\,\hbar - 2\,T\,\delta^2\,\hbar + \gamma\,\delta\,\epsilon\,\hbar^2 + 4\,\gamma\,\delta^2\,\epsilon\,\hbar^2 - 8\,T\,\gamma\,\delta^2\,\epsilon\,\hbar^2 + \eta\,\xi\,\hbar^2\big)\,QU[y, x] +$$
$$\frac{1}{2}\,\eta^2\,\hbar^2\,QU[y, y] + 4\,T\,\delta^2\,\epsilon\,\hbar^2\,QU[y, a, x] +$$
$$\delta\,\xi\,\hbar^2\,QU[y, x, x] +$$
$$\delta\,\eta\,\hbar^2\,QU[y, y, x] +$$
$$\frac{1}{2}\,\delta^2\,\hbar^2\,QU[y, y, x, x],\ \text{True}\Big\}$$

```
{tt = Last[ΛCU[{ξ, η, δ}, {x, y}]], Normal@Series[Log[tt], {ϵ, 0, $k}]}
```

$$\left\{\frac{1}{1+t\,\delta}+\frac{1}{2\,(1+t\,\delta)^5}\,\epsilon\,\left(4\,a\,\delta+12\,a\,t\,\delta^2+4\,a\,x\,y\,\delta^2+2\,t\,\gamma\,\delta^2-8\,x\,y\,\gamma\,\delta^2+12\,a\,t^2\,\delta^3+8\,a\,t\,x\,y\,\delta^3+4\,t^2\,\gamma\,\delta^3-12\,t\,x\,y\,\gamma\,\delta^3-4\,x^2\,y^2\,\gamma\,\delta^3+4\,a\,t^3\,\delta^4+4\,a\,t^2\,x\,y\,\delta^4+2\,t^3\,\gamma\,\delta^4-4\,t^2\,x\,y\,\gamma\,\delta^4-3\,t\,x^2\,y^2\,\gamma\,\delta^4+4\,a\,y\,\delta\,\eta-4\,y\,\gamma\,\delta\,\eta+8\,a\,t\,y\,\delta^2\,\eta-4\,t\,y\,\gamma\,\delta^2\,\eta-6\,x\,y^2\,\gamma\,\delta^2\,\eta+4\,a\,t^2\,y\,\delta^3\,\eta-4\,t\,x\,y^2\,\gamma\,\delta^3\,\eta-2\,y^2\,\gamma\,\delta\,\eta^2-t\,y^2\,\gamma\,\delta^2\,\eta^2+4\,a\,x\,\delta\,\xi-4\,x\,\gamma\,\delta\,\xi+8\,a\,t\,x\,\delta^2\,\xi-4\,t\,x\,\gamma\,\delta^2\,\xi-6\,x^2\,y\,\gamma\,\delta^2\,\xi+4\,a\,t^2\,x\,\delta^3\,\xi-4\,t\,x^2\,y\,\gamma\,\delta^3\,\xi+4\,a\,\eta\,\xi+8\,a\,t\,\delta\,\eta\,\xi+4\,t\,\gamma\,\delta\,\eta\,\xi-8\,x\,y\,\gamma\,\delta\,\eta\,\xi+4\,a\,t^2\,\delta^2\,\eta\,\xi+4\,t^2\,\gamma\,\delta^2\,\eta\,\xi-4\,t\,x\,y\,\gamma\,\delta^2\,\eta\,\xi-2\,y\,\gamma\,\eta^2\,\xi-2\,x^2\,\gamma\,\delta\,\xi^2-t\,x^2\,\gamma\,\delta^2\,\xi^2-2\,x\,\gamma\,\eta\,\xi^2+t\,\gamma\,\eta^2\,\xi^2\right),\right.$$

$$\left.\frac{1}{2\,(1+t\,\delta)^4}\,\epsilon\,\left(4\,a\,\delta+12\,a\,t\,\delta^2+4\,a\,x\,y\,\delta^2+2\,t\,\gamma\,\delta^2-8\,x\,y\,\gamma\,\delta^2+12\,a\,t^2\,\delta^3+8\,a\,t\,x\,y\,\delta^3+4\,t^2\,\gamma\,\delta^3-12\,t\,x\,y\,\gamma\,\delta^3-4\,x^2\,y^2\,\gamma\,\delta^3+4\,a\,t^3\,\delta^4+4\,a\,t^2\,x\,y\,\delta^4+2\,t^3\,\gamma\,\delta^4-4\,t^2\,x\,y\,\gamma\,\delta^4-3\,t\,x^2\,y^2\,\gamma\,\delta^4+4\,a\,y\,\delta\,\eta-4\,y\,\gamma\,\delta\,\eta+8\,a\,t\,y\,\delta^2\,\eta-4\,t\,y\,\gamma\,\delta^2\,\eta-6\,x\,y^2\,\gamma\,\delta^2\,\eta+4\,a\,t^2\,y\,\delta^3\,\eta-4\,t\,x\,y^2\,\gamma\,\delta^3\,\eta-2\,y^2\,\gamma\,\delta\,\eta^2-t\,y^2\,\gamma\,\delta^2\,\eta^2+4\,a\,x\,\delta\,\xi-4\,x\,\gamma\,\delta\,\xi+8\,a\,t\,x\,\delta^2\,\xi-4\,t\,x\,\gamma\,\delta^2\,\xi-6\,x^2\,y\,\gamma\,\delta^2\,\xi+4\,a\,t^2\,x\,\delta^3\,\xi-4\,t\,x^2\,y\,\gamma\,\delta^3\,\xi+4\,a\,\eta\,\xi+8\,a\,t\,\delta\,\eta\,\xi+4\,t\,\gamma\,\delta\,\eta\,\xi-8\,x\,y\,\gamma\,\delta\,\eta\,\xi+4\,a\,t^2\,\delta^2\,\eta\,\xi+4\,t^2\,\gamma\,\delta^2\,\eta\,\xi-4\,t\,x\,y\,\gamma\,\delta^2\,\eta\,\xi-2\,y\,\gamma\,\eta^2\,\xi-2\,x^2\,\gamma\,\delta\,\xi^2-t\,x^2\,\gamma\,\delta^2\,\xi^2-2\,x\,\gamma\,\eta\,\xi^2+t\,\gamma\,\eta^2\,\xi^2\right)+\mathrm{Log}\left[\frac{1}{1+t\,\delta}\right]\right\}$$

```
{tt = Last[Λ_QU[{ξ, η, δ}, {x, y}]], Normal@Series[Log[tt], {ϵ, 0, $k}]}
```

$$\Big\{\frac{\hbar}{-\delta+T\delta+\hbar}+$$
$$\frac{1}{4\,(-\delta+T\delta+\hbar)^5}\,\epsilon\,\Big(-8aT\delta^4\hbar^2+24aT^2\delta^4\hbar^2-24aT^3\delta^4\hbar^2+8aT^4\delta^4\hbar^2+2\gamma\delta^4\hbar^2-12T\gamma\delta^4\hbar^2+$$
$$24T^2\gamma\delta^4\hbar^2-20T^3\gamma\delta^4\hbar^2+6T^4\gamma\delta^4\hbar^2+24aT\delta^3\hbar^3-48aT^2\delta^3\hbar^3+24aT^3\delta^3\hbar^3-$$
$$4\gamma\delta^3\hbar^3+20T\gamma\delta^3\hbar^3-28T^2\gamma\delta^3\hbar^3+12T^3\gamma\delta^3\hbar^3+8aTxy\delta^4\hbar^3-16aT^2xy\delta^4\hbar^3+$$
$$8aT^3xy\delta^4\hbar^3-8Txy\gamma\delta^4\hbar^3+16T^2xy\gamma\delta^4\hbar^3-8T^3xy\gamma\delta^4\hbar^3+8aTy\delta^3\eta\hbar^3-$$
$$16aT^2y\delta^3\eta\hbar^3+8aT^3y\delta^3\eta\hbar^3+8aTx\delta^3\xi\hbar^3-16aT^2x\delta^3\xi\hbar^3+8aT^3x\delta^3\xi\hbar^3+$$
$$8aT\delta^2\eta\xi\hbar^3-16aT^2\delta^2\eta\xi\hbar^3+8aT^3\delta^2\eta\xi\hbar^3-4\gamma\delta^2\eta\xi\hbar^3+20T\gamma\delta^2\eta\xi\hbar^3-$$
$$28T^2\gamma\delta^2\eta\xi\hbar^3+12T^3\gamma\delta^2\eta\xi\hbar^3-24aT\delta^2\hbar^4+24aT^2\delta^2\hbar^4+2\gamma\delta^2\hbar^4-8T\gamma\delta^2\hbar^4+$$
$$6T^2\gamma\delta^2\hbar^4-16aTxy\delta^3\hbar^4+16aT^2xy\delta^3\hbar^4+24Txy\gamma\delta^3\hbar^4-24T^2xy\gamma\delta^3\hbar^4+$$
$$x^2y^2\gamma\delta^4\hbar^4+4Tx^2y^2\gamma\delta^4\hbar^4-5T^2x^2y^2\gamma\delta^4\hbar^4-16aTy\delta^2\eta\hbar^4+16aT^2y\delta^2\eta\hbar^4-$$
$$4y\gamma\delta^2\eta\hbar^4+16Ty\gamma\delta^2\eta\hbar^4-12T^2y\gamma\delta^2\eta\hbar^4+8Txy^2\gamma\delta^3\eta\hbar^4-8T^2xy^2\gamma\delta^3\eta\hbar^4-$$
$$y^2\gamma\delta^2\eta^2\hbar^4+4Ty^2\gamma\delta^2\eta^2\hbar^4-3T^2y^2\gamma\delta^2\eta^2\hbar^4-16aTx\delta^2\xi\hbar^4+16aT^2x\delta^2\xi\hbar^4-$$
$$4x\gamma\delta^2\xi\hbar^4+16Tx\gamma\delta^2\xi\hbar^4-12T^2x\gamma\delta^2\xi\hbar^4+8Tx^2y\gamma\delta^3\xi\hbar^4-8T^2x^2y\gamma\delta^3\xi\hbar^4-$$
$$16aT\delta\eta\xi\hbar^4+16aT^2\delta\eta\xi\hbar^4+4\gamma\delta\eta\xi\hbar^4-16T\gamma\delta\eta\xi\hbar^4+12T^2\gamma\delta\eta\xi\hbar^4+$$
$$8Txy\gamma\delta^2\eta\xi\hbar^4-8T^2xy\gamma\delta^2\eta\xi\hbar^4-x^2\gamma\delta^2\xi^2\hbar^4+4Tx^2\gamma\delta^2\xi^2\hbar^4-3T^2x^2\gamma\delta^2\xi^2\hbar^4+$$
$$\gamma\eta^2\xi^2\hbar^4-4T\gamma\eta^2\xi^2\hbar^4+3T^2\gamma\eta^2\xi^2\hbar^4+8aT\delta\hbar^5+8aTxy\delta^2\hbar^5-4xy\gamma\delta^2\hbar^5-$$
$$12Txy\gamma\delta^2\hbar^5-4x^2y^2\gamma\delta^3\hbar^5-4Tx^2y^2\gamma\delta^3\hbar^5+8aTy\delta\eta\hbar^5+4y\gamma\delta\eta\hbar^5-$$
$$12Ty\gamma\delta\eta\hbar^5-2xy^2\gamma\delta^2\eta\hbar^5-10Txy^2\gamma\delta^2\eta\hbar^5+2y^2\gamma\delta\eta^2\hbar^5-6Ty^2\gamma\delta\eta^2\hbar^5+$$
$$8aTx\delta\xi\hbar^5+4x\gamma\delta\xi\hbar^5-12Tx\gamma\delta\xi\hbar^5-2x^2y\gamma\delta^2\xi\hbar^5-10Tx^2y\gamma\delta^2\xi\hbar^5+8aT\eta\xi\hbar^5-$$
$$16Txy\gamma\delta\eta\xi\hbar^5+2y\gamma\eta^2\xi\hbar^5-6Ty\gamma\eta^2\xi\hbar^5+2x^2\gamma\delta\xi^2\hbar^5-6Tx^2\gamma\delta\xi^2\hbar^5+2x\gamma\eta\xi^2\hbar^5-$$
$$6Tx\gamma\eta\xi^2\hbar^5+4xy\gamma\delta\hbar^6+4x^2y^2\gamma\delta^2\hbar^6+4xy^2\gamma\delta\eta\hbar^6+4x^2y\gamma\delta\xi\hbar^6+4xy\gamma\eta\xi\hbar^6\Big),$$

$$\frac{1}{4\,(-\delta+T\delta+\hbar)^4}\,\epsilon\,\Big(-8aT\delta^4\hbar+24aT^2\delta^4\hbar-24aT^3\delta^4\hbar+8aT^4\delta^4\hbar+2\gamma\delta^4\hbar-12T\gamma\delta^4\hbar+$$
$$24T^2\gamma\delta^4\hbar-20T^3\gamma\delta^4\hbar+6T^4\gamma\delta^4\hbar+24aT\delta^3\hbar^2-48aT^2\delta^3\hbar^2+24aT^3\delta^3\hbar^2-4\gamma\delta^3\hbar^2+$$
$$20T\gamma\delta^3\hbar^2-28T^2\gamma\delta^3\hbar^2+12T^3\gamma\delta^3\hbar^2+8aTxy\delta^4\hbar^2-16aT^2xy\delta^4\hbar^2+8aT^3xy\delta^4\hbar^2-$$
$$8Txy\gamma\delta^4\hbar^2+16T^2xy\gamma\delta^4\hbar^2-8T^3xy\gamma\delta^4\hbar^2+8aTy\delta^3\eta\hbar^2-16aT^2y\delta^3\eta\hbar^2+$$
$$8aT^3y\delta^3\eta\hbar^2+8aTx\delta^3\xi\hbar^2-16aT^2x\delta^3\xi\hbar^2+8aT^3x\delta^3\xi\hbar^2+8aT\delta^2\eta\xi\hbar^2-$$
$$16aT^2\delta^2\eta\xi\hbar^2+8aT^3\delta^2\eta\xi\hbar^2-4\gamma\delta^2\eta\xi\hbar^2+20T\gamma\delta^2\eta\xi\hbar^2-28T^2\gamma\delta^2\eta\xi\hbar^2+$$
$$12T^3\gamma\delta^2\eta\xi\hbar^2-24aT\delta^2\hbar^3+24aT^2\delta^2\hbar^3+2\gamma\delta^2\hbar^3-8T\gamma\delta^2\hbar^3+6T^2\gamma\delta^2\hbar^3-$$
$$16aTxy\delta^3\hbar^3+16aT^2xy\delta^3\hbar^3+24Txy\gamma\delta^3\hbar^3-24T^2xy\gamma\delta^3\hbar^3+x^2y^2\gamma\delta^4\hbar^3+$$
$$4Tx^2y^2\gamma\delta^4\hbar^3-5T^2x^2y^2\gamma\delta^4\hbar^3-16aTy\delta^2\eta\hbar^3+16aT^2y\delta^2\eta\hbar^3-4y\gamma\delta^2\eta\hbar^3+$$
$$16Ty\gamma\delta^2\eta\hbar^3-12T^2y\gamma\delta^2\eta\hbar^3+8Txy^2\gamma\delta^3\eta\hbar^3-8T^2xy^2\gamma\delta^3\eta\hbar^3-y^2\gamma\delta^2\eta^2\hbar^3+$$
$$4Ty^2\gamma\delta^2\eta^2\hbar^3-3T^2y^2\gamma\delta^2\eta^2\hbar^3-16aTx\delta^2\xi\hbar^3+16aT^2x\delta^2\xi\hbar^3-4x\gamma\delta^2\xi\hbar^3+$$
$$16Tx\gamma\delta^2\xi\hbar^3-12T^2x\gamma\delta^2\xi\hbar^3+8Tx^2y\gamma\delta^3\xi\hbar^3-8T^2x^2y\gamma\delta^3\xi\hbar^3-16aT\delta\eta\xi\hbar^3+$$
$$16aT^2\delta\eta\xi\hbar^3+4\gamma\delta\eta\xi\hbar^3-16T\gamma\delta\eta\xi\hbar^3+12T^2\gamma\delta\eta\xi\hbar^3+8Txy\gamma\delta^2\eta\xi\hbar^3-$$
$$8T^2xy\gamma\delta^2\eta\xi\hbar^3-x^2\gamma\delta^2\xi^2\hbar^3+4Tx^2\gamma\delta^2\xi^2\hbar^3-3T^2x^2\gamma\delta^2\xi^2\hbar^3+\gamma\eta^2\xi^2\hbar^3-$$
$$4T\gamma\eta^2\xi^2\hbar^3+3T^2\gamma\eta^2\xi^2\hbar^3+8aT\delta\hbar^4+8aTxy\delta^2\hbar^4-4xy\gamma\delta^2\hbar^4-12Txy\gamma\delta^2\hbar^4-$$
$$4x^2y^2\gamma\delta^3\hbar^4-4Tx^2y^2\gamma\delta^3\hbar^4+8aTy\delta\eta\hbar^4+4y\gamma\delta\eta\hbar^4-12Ty\gamma\delta\eta\hbar^4-2xy^2\gamma\delta^2\eta\hbar^4-$$
$$10Txy^2\gamma\delta^2\eta\hbar^4+2y^2\gamma\delta\eta^2\hbar^4-6Ty^2\gamma\delta\eta^2\hbar^4+8aTx\delta\xi\hbar^4+4x\gamma\delta\xi\hbar^4-12Tx\gamma\delta\xi\hbar^4-$$
$$2x^2y\gamma\delta^2\xi\hbar^4-10Tx^2y\gamma\delta^2\xi\hbar^4+8aT\eta\xi\hbar^4-16Txy\gamma\delta\eta\xi\hbar^4+2y\gamma\eta^2\xi\hbar^4-$$
$$6Ty\gamma\eta^2\xi\hbar^4+2x^2\gamma\delta\xi^2\hbar^4-6Tx^2\gamma\delta\xi^2\hbar^4+2x\gamma\eta\xi^2\hbar^4-6Tx\gamma\eta\xi^2\hbar^4+4xy\gamma\delta\hbar^5+$$
$$4x^2y^2\gamma\delta^2\hbar^5+4xy^2\gamma\delta\eta\hbar^5+4x^2y\gamma\delta\xi\hbar^5+4xy\gamma\eta\xi\hbar^5\Big)+\mathrm{Log}\Big[\frac{\hbar}{-\delta+T\delta+\hbar}\Big]\Big\}$$

Reorderings with Rord

Rord

```
Rord_{u_i_,w_j_→k_}[𝔼_U_[L___, {l___, u_i_, w_j_, r___}_s_, R___, Q_, P_]] :=
 Simp@Module[{υ, ω, δ, Λ1, yax, q, p, δ1 = ∂_{u_i,w_j} Q},
   {yax, q, p} = List @@ If[δ1 === 0, Λ_U[{υ, ω}, {u, w}], Λ_U[{υ, ω, δ}, {u, w}]] /.
     {y → y_k, a → a_k, x → x_k, t → t_s, T → T_s};
   𝔼_U[L, {l, Sequence @@ yax, r}_s, R, q + (Q /. u_i | w_j → 0), e^{-q} DP_{u_i→D_υ, w_j→D_ω}[P][p e^q]] /.
    {υ → ∂_{u_i} Q /. w_j → 0, ω → ∂_{w_j} Q /. u_i → 0, δ → δ1}]
```

```
With[{c⊙ = 𝔼_CU[{y1, x1}_1, {x2, a2, y2}_2, ℏ t1 a2 + ℏ t1^-1 (e^t1 - 1) y1 x2, 1 + ϵ x1 y2]},
 {Short[rhs = c⊙ // Rord_{x2,a2→3}, 3], HL[CU[c⊙] ≡ CU[rhs]]}]
```

$$\left\{\mathbb{E}_{CU}\left[\{y_1, x_1\}_1, \{a_3, x_3, y_2\}_2, \frac{e^{-\gamma\hbar t_1}\left(e^{\gamma\hbar t_1}\hbar a_3 t_1^2 - \hbar x_3 y_1 + e^{t_1}\hbar x_3 y_1\right)}{t_1}, 1 + \epsilon x_1 y_2\right], \text{True}\right\}$$

```
With[{c⊙ = 𝔼_CU[{y1, a1, x1}_1, {x2, a2, y2}_2,
    ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    1 + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 {Short[rhs = c⊙ // Rord_{x2,a2→3}, 3], HL[CU[c⊙] ≡ CU[rhs]]}]
```

$$\{\mathbb{E}_{CU}[\{y_1, a_1, x_1\}_1, \{\ll 1 \gg\}_2, \ll 1 \gg \ll 1 \gg, 1 + e^{-\gamma\hbar(l_{12}t_1+l_{22}t_2)}\epsilon\left(e^{\gamma\hbar(l_{12}t_1+l_{22}t_2)} a_1 l_1 + e^{\gamma\hbar(l_{12}t_1+l_{22}t_2)} a_3 l_2 + e^{\gamma\hbar(l_{12}t_1+l_{22}t_2)} p_{11} x_1 y_1 + p_{21} x_3 y_1 + e^{\ll 1 \gg} p_{12} x_1 y_2 + p_{22} x_3 y_2 - \gamma\hbar l_2 x_3 y_1 \gamma_{21} - \gamma\hbar l_2 x_3 y_2 \gamma_{22}\right)], \text{True}\}$$

```
With[{q⊙ = 𝔼_QU[{y1, a1, x1}_1, {x2, a2, y2}_2,
    ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    1 + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 {Short[rhs = q⊙ // Rord_{x2,a2→3}, 3], HL[QU[q⊙] ≡ QU[rhs]]}]
```

$$\{\mathbb{E}_{QU}[\{y_1, a_1, x_1\}_1, \{\ll 1 \gg\}_2, \ll 1 \gg \ll 1 \gg, 1 + e^{-\gamma\hbar(l_{12}t_1+l_{22}t_2)}\epsilon\left(e^{\gamma\hbar(l_{12}t_1+l_{22}t_2)} a_1 l_1 + e^{\gamma\hbar(l_{12}t_1+l_{22}t_2)} a_3 l_2 + e^{\gamma\hbar(l_{12}t_1+l_{22}t_2)} p_{11} x_1 y_1 + p_{21} x_3 y_1 + e^{\ll 1 \gg} p_{12} x_1 y_2 + p_{22} x_3 y_2 - \gamma\hbar l_2 x_3 y_1 \gamma_{21} - \gamma\hbar l_2 x_3 y_2 \gamma_{22}\right)], \text{True}\}$$

```
With[{q⊙ = 𝔼_QU[{y1, a1, x1}_1, {x2, a2, y2}_2,
    ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    1 + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 {Short[rhs = q⊙ // Rord_{a2,y2→3}, 3], HL[QU[q⊙] ≡ QU[rhs]]}]
```

$$\{\mathbb{E}_{QU}[\{y_1, a_1, x_1\}_1, \{\ll 1 \gg\}_2, \ll 1 \gg \ll 1 \gg, 1 + e^{-\gamma\hbar(l_{12}t_1+l_{22}t_2)}\epsilon\left(e^{\gamma\hbar(l_{12}t_1+l_{22}t_2)} a_1 l_1 + e^{\gamma\hbar(l_{12}t_1+l_{22}t_2)} a_3 l_2 + e^{\gamma\hbar(l_{12}t_1+l_{22}t_2)} p_{11} x_1 y_1 + e^{\ll 1 \gg} p_{21} x_2 y_1 + p_{12} x_1 y_3 + p_{22} x_2 y_3 - \gamma\hbar l_2 x_1 y_3 \gamma_{12} - \gamma\hbar l_2 x_2 y_3 \gamma_{22}\right)], \text{True}\}$$

```
With[{qΦ = 𝔼QU[{x1, y1}1, {x2, a2, y2}2,
    ℏ (l12 t1 a2 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    1 + ϵ (l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 {Short[rhs = qΦ // Rord_{x1,y1→3}, 5], HL@SimpT[QU[qΦ] == QU[rhs]]}]
```

$\{\mathbb{E}_{QU}[\{y_3, a_3, x_3\}_1, \{x_2, a_2, y_2\}_2, \frac{\hbar a_2 l_{12} t_1 + \ll 16 \gg + \hbar T_1 x_2 y_2 \gamma_{11} \gamma_{22}}{1 - \gamma_{11} + T_1 \gamma_{11}},$
$\frac{1}{1 - \gamma_{11} + T_1 \gamma_{11}} + (\epsilon (4 \hbar a_2 l_2 + 4 p_{11} - 4 p_{11} T_1 + 4 \hbar p_{22} x_2 y_2 + \ll 339 \gg + \gamma \hbar^4 x_2^2 y_2^2 \gamma_{12}^2 \gamma_{21}^2 -$
$4 \gamma \hbar^4 T_1 x_2^2 y_2^2 \gamma_{12}^2 \gamma_{21}^2 + 3 \gamma \hbar^4 T_1^2 x_2^2 y_2^2 \gamma_{12}^2 \gamma_{21}^2)) / (4 \hbar (1 - \gamma_{11} + T_1 \gamma_{11})^5)], \text{True}\}$

R in QU.

Faddeev-Quesne's formula:

Faddeev

```
𝕖_{q_,k_}[x_] := 𝕖^(Σ_{j=1}^{k} ((1 - q)^j x^j)/(j (1 - q^j))); 𝕖_{q_}[x_] := 𝕖_{q,$k}[x]
```

```
Table[Together@SeriesCoefficient[𝕖_{ρ,5}[x], {x, 0, n}], {n, 0, 5}]
```

$\{1, 1, \frac{1}{1+\rho}, \frac{1}{(1+\rho)(1+\rho+\rho^2)}, \frac{1}{(1+\rho)^2 (1+\rho^2)(1+\rho+\rho^2)},$
$1 / ((1+\rho)^2 (1+\rho^2)(1+\rho+\rho^2)(1+\rho+\rho^2+\rho^3+\rho^4))\}$

```
Table[HL@FunctionExpand[QFactorial[n, ρ] SeriesCoefficient[𝕖_{ρ,5}[x], {x, 0, n}]], {n, 0, 5}]
```

{1, 1, 1, 1, 1, 1}

R

```
QU[R_{i_,j_}] := 𝕆QU[{y1, a1}i, {a2, x2}j, SS[𝕖^(ℏ b1 a2) 𝕖q[ℏ y1 x2] /. b1 → γ^-1 (ϵ a1 - ti)]];
QU[R^-1_{i_,j_}] := Sj@QU[R_{i,j}];
```

```
QU[R3,4] // Short
```

$QU[] + \frac{\epsilon \hbar QU[a_3, a_4]}{\gamma} + \hbar QU[y_3, x_4] + \frac{\ll 1 \gg}{\gamma} +$
$\ll 1 \gg - \frac{\ll 1 \gg}{\gamma} - \frac{\epsilon \ll 3 \gg}{\gamma^2} - \frac{\hbar^2 QU[y_3, a_4, x_4] t_3}{\gamma} + \frac{\hbar^2 QU[a_4, a_4] t_3^2}{2 \gamma^2}$

Verifying R2 (~2 secs @ $p=4, $k=2):

```
QU[R1,2 ** R^-1_{1,2}] // Simp // HL // Timing
```

{0.0625, QU[]}

Verifying R3 (~156 secs @ $p=4, $k=2):

```
{Short[lhs = QU[R1,2 ** R1,3 ** R2,3]], HL@SimpT[lhs - QU[R2,3 ** R1,3 ** R1,2]]} // Timing
```

$\left\{0.34375, \left\{QU[\,] + \frac{\epsilon\,\hbar\,QU[a_1, a_2]}{\gamma} + \ll 85 \gg + QU[y_1, y_1, x_3, x_3]\left(\frac{\hbar^2}{2} - \hbar^2 T_2 + \frac{1}{2}\hbar^2 T_2^2\right), 0\right\}\right\}$

R in $Œ_{QU}$.

RinOE

```
ŒQU[Ri_,j_] := ŒQU[{yi, ai, xi}i, {yj, aj, xj}j, -ħ γ^-1 ti aj + ħ yi xj,
  Normal@Series[e^(ħ γ^-1 ti aj - ħ yi xj) (e^(ħ bi aj) eq[ħ yi xj] /. bi → γ^-1 (ϵ ai - ti)), {ϵ, 0, $k}]]
```

```
ŒQU[R1,2]
```

$Œ_{QU}\left[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, -\frac{\hbar\,a_2\,t_1}{\gamma} + \hbar\,x_2\,y_1,\right.$
$\left. 1 + \epsilon\left(\frac{\hbar\,a_1\,a_2}{\gamma} - \frac{1}{4}\gamma\,\hbar^3\,x_2^2\,y_1^2\right) + \frac{1}{2}\epsilon^2\left(\frac{\hbar\,a_1\,a_2}{\gamma} - \frac{1}{4}\gamma\,\hbar^3\,x_2^2\,y_1^2\right)^2\right]$

𝔼

𝔼[*L*, *Q*, *P*] means $e^{\hbar(L+Q)}P$, where *L* is linear in the *a*'s, *Q* is a combination of $x_i\,y_j$, and *P* is a perturbation polynomial. It should be interpreted via C𝕆[𝔼[...], $\{x_1, a_1, y_1\}_i$, ...] (with some default for direct interpretation), or likewise via Q𝕆[𝔼[...], $\{x_1, a_1, y_1\}_i$, ...]. In themselves, C𝕆 and Q𝕆 should have an interpretation in CU/QU by casting.

Alternative Algorithms

```
λalt[CU] := Module[{eq, d, b, c, so},
   eq = ρ@e^(ξ xCU).ρ@e^(η yCU) == ρ@e^(d yCU).ρ@e^(c (t 1CU - 2 ϵ aCU)).ρ@e^(b xCU);
   {so} = Solve[Thread[Flatten /@ eq], {d, b, c}] /. C@1 → 0;
   Normal@Series[e^(-η y - ξ x + η ξ t + c t + d y - 2 ϵ c a + b x) /. so, {ϵ, 0, $k}]];
```

```
{λalt[CU], HL@Simplify[λalt[CU] == Last[ΛCU[{ξ, η}, {x, y}]]]}
```

$\left\{1 + \epsilon\left(2\,a\,\eta\,\xi - y\,\gamma\,\eta^2\,\xi - x\,\gamma\,\eta\,\xi^2 + \frac{1}{2}\,t\,\gamma\,\eta^2\,\xi^2\right), \text{True}\right\}$