Pensieve header: A unified verification testing suite for the sl_2-portfolio project, Uxi version. Continues pensieve://Projects/SL2Portfolio/nb/Verification.pdf.

Also continues pensieve://Projects/PPSA/nb/Verification.pdf and pensieve://2017-06/ and pensieve://2017-08/.

Prolog

In[ ]:=
```
wdir = SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\SL2Portfolio"];
<< "SL2PortfolioProgram.m"
```

In[ ]:=
```
$p = 2; $k = 1; $U = QU;
```

In[ ]:=
```
HL[ℰ_] := Style[ℰ, Background → Yellow];
```

DocileQ

In[ ]:= $\mathrm{DQ} \,/@\, \{\epsilon^2\, x\, y\, a_2,\ \epsilon^2\, x^2\, y^3\}$

Out[ ]= {True, False}

Initialization / Utilities

$\mathrm{HL}\left[\mathrm{DP}_{x\to D_\xi,\,y\to D_\eta}\left[x^2\, y^3\right]\left[e^{\delta\,\xi\,\eta}\right] == 6\, e^{\delta\,\eta\,\xi}\,\delta^3\,\xi + 6\, e^{\delta\,\eta\,\xi}\,\delta^4\,\eta\,\xi^2 + e^{\delta\,\eta\,\xi}\,\delta^5\,\eta^2\,\xi^3\right]$

True

$\mathrm{SP}_{\{\xi\to x\}}\left[\left(\xi^2+\xi+3\right)\left(x^5\, e^x + 7\, x\right) + 99\, a\right]$

$7 + 99\, a + 21\, x + 20\, e^x\, x^3 + 15\, e^x\, x^4 + 5\, e^x\, x^5$

$\mathrm{SP}_{\{\xi\to x,\,\eta\to y\}}\left[\left(\xi^2+\xi+3+2\,\xi\,\eta\right)\left(x^5\, e^x + 7\, x\right) + 99\, a + e^{\delta\, x\, y}\,\xi\,\eta\right]$

$7 + 99\, a + 21\, x + 20\, e^x\, x^3 + 15\, e^x\, x^4 + 5\, e^x\, x^5 + e^{x\, y\,\delta}\,\delta + e^{x\, y\,\delta}\, x\, y\,\delta^2$

Implementing CU = $\mathcal{U}(sl_2^{\gamma\epsilon})$

Verify σ and Δ! Also Generalize Δ to $\Delta_{i,j_1,j_2,\ldots}$.

Verifying associativity on triples of generators:

```
With[{bas = CU /@ {y, a, x}},
 Table[z1 ** (z2 ** z3) - (z1 ** z2) ** z3 // Simp // HL,
  {z1, bas}, {z2, bas}, {z3, bas}] ]
```

{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}

Verifying associativity on a “random” triple:

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
    (rhs = (z1 ** z2) ** z3 // Simp) // Short,
    z1 ** (z2 ** z3) - rhs // Simp // HL
  }] // Timing
```

{1.32813,
 {$(28 t^2 \gamma^4 + 116 t \gamma^5 \epsilon)$ CU[y, y, y, x, x] + <<21>> + CU[y, y, y, y, y, a, a, a, a, x, x, x, x], 0}}

Implementing QU = $\mathcal{U}_q(sl_2^{\gamma\epsilon})$

In[∘]:=
```
HL /@ DQ /@ Series[{(1 - T ⅇ^(-2 ϵ a ℏ)) / ℏ, ⅇ^(ℏ ϵ a)}, {ϵ, 0, 5}]
```

Out[∘]= {True, True}

```
With[{bas = QU /@ {y, a, x}}, Table[{z1, z2} → Simp[z1 ** z2 - z2 ** z1], {z1, bas}, {z2, bas}] ]
```

{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → γ QU[y],
  {QU[y], QU[x]} → $\frac{(-1 + T)\, \text{QU}[]}{\hbar} - 2 T \epsilon$ QU[a] $- \gamma \epsilon \hbar$ QU[y, x]},
 {{QU[a], QU[y]} → $-\gamma$ QU[y], {QU[a], QU[a]} → 0, {QU[a], QU[x]} → γ QU[x]},
 {{QU[x], QU[y]} → $\frac{(1 - T)\, \text{QU}[]}{\hbar} + 2 T \epsilon$ QU[a] $+ \gamma \epsilon \hbar$ QU[y, x],
  {QU[x], QU[a]} → $-\gamma$ QU[x], {QU[x], QU[x]} → 0}}

Verifying associativity on triples of generators:

```
With[{bas = QU /@ {y, a, x}},
 Table[HL[z1 ** (z2 ** z3) - (z1 ** z2) ** z3] // Simp,
  {z1, bas}, {z2, bas}, {z3, bas} ] ]
```

{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}

Verifying associativity on a “random” triple (~34 secs @ $p=5, $k=2):

```
With[{z1 = QU[y, y, a, a, x, x], z2 = QU[y, a, x], z3 = QU[y, y, a, x]}, {
    (rhs = (z1 ** z2) ** z3 // Simp) // Short,
    HL[z1 ** (z2 ** z3) - rhs // Simp ]
  }] // Timing
```

{3.78125, {$\left(\frac{28 \gamma^4 - 56 T \gamma^4 + 28 T^2 \gamma^4}{\hbar^2} + \frac{82 \gamma^5 \epsilon - 280 T \ll 1 \gg \epsilon + 198 T^2 \gamma^5 \epsilon}{\hbar}\right)$ QU[y, y, y, x, x] +
  <<18>> + $(1 + 8 \gamma \epsilon \hbar)$ QU[y, <<11>>, x], 0}}

Verifying that $\text{Lim}_{\hbar \to 0}$ QU = CU using a “random” product (~23 secs @ $p=5, $k=2):

```
With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
   Short[lhs = z1 ** (z2 ** z3)],
   Short[rhs = (QU @@ z1) ** ((QU @@ z2) ** (QU @@ z3))],
   Expand[Limit[rhs /. T2t ⋃ {QU → CU}, ℏ → 0] - lhs] // HL
  }] // Timing
```

```
{10.125, {28 t^2 γ^4 CU[y, y, y, x, x] +
   116 t γ^5 ∈ CU[y, y, y, x, x] + <<44>> + CU[y, y, y, y, y, a, a, a, a, x, x, x, x],
  2 (γ^4/ℏ^2 - 2 T γ^4/ℏ^2 + T^2 γ^4/ℏ^2 + γ^5 ∈/ℏ - 2 T γ^5 ∈/ℏ + T^2 γ^5 ∈/ℏ) QU[y, y, y, x, x] +
   <<209>> + (1 + 8 γ ∈ ℏ) QU[y, y, y, <<7>>, x, x, x], 0}}
```

Verifying σ, m, S, and Δ.

Verifying $\sigma_{i\to j,k\to l}$:

```
In[∘]:= CU@x₁ + CU@x₂ // σ₁→₃,₂→₄
Out[∘]= CU[x₃] + CU[x₄]
```

Verifying relabeling using m:

```
In[∘]:= t₁ t₃ CU[y₁, a₁, x₂] + t₁ t₁ CU[y₁, a₂, x₂] // m₁→₃
Out[∘]= CU[a₂, x₂, y₃] t₃² + CU[x₂, y₃, a₃] t₃²
```

Verifying the meta-associativity of m:

```
In[∘]:= Module[{z, u},
         Table[u = CU[z〚1〛₁, z〚2〛₂, z〚3〛₃]; HL[m₁,₃→₃@m₂,₃→₃@u == m₂,₃→₃@m₁,₂→₂@u],
          {z, Tuples[{y, a, x}, 3]}, {U, {CU, QU}}]]
Out[∘]= {{True, True}, {True, True}, {True, True}, {True, True}, {True, True},
         {True, True}, {True, True}, {True, True}, {True, True}, {True, True},
         {True, True}, {True, True}, {True, True}, {True, True}, {True, True}, {True, True},
         {True, True}, {True, True}, {True, True}, {True, True}, {True, True}, {True, True},
         {True, True}, {True, True}, {True, True}, {True, True}, {True, True}}
```

Verifying the involutivity of S on CU on products of triples:

```
In[∘]:= With[{bas = CU /@ {y₁, a₁, x₁}},
         Table[HL@Simp[z1 ** z2 ** z3 - S₁@S₁[z1 ** z2 ** z3]],
          {z1, bas}, {z2, bas}, {z3, bas}] ]
Out[∘]= {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
         {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying that S is an anti-homomorphism on CU/QU:

```
In[∘]:= With[{bas = U /@ {y₁, a₁, x₁}},
         Table[HL@Simp[S₁[z1 ** z2] - S₁[z2] ** S₁[z1]],
          {z1, bas}, {z2, bas} , {U, {CU, QU}}] ]
Out[∘]= {{{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}}
```

Verifying the co-associativity of Δ:

```
In[•]:= Block[{bas = U /@ {y1, a1, x1}},
          Table[(z1 ** z2 ** z3 // Δ1→1,2 // Δ2→2,3) − (z1 ** z2 ** z3 // Δ1→1,3 // Δ1→1,2) // Simp // HL,
           {z1, bas}, {z2, bas}, {z3, bas} , {U, {CU, QU}}] ]

Out[•]= {{{{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}},
         {{{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}},
         {{{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}}}
```

Verifying *S*-Δ compatibility:

```
In[•]:= Block[{bas = U /@ {y1, a1, x1}},
          Table[z1 ** z2 ** z3 // Δ1→1,2 // Si // m1,2→1 // Simp // HL,
           {U, {CU, QU}}, {i, 2}, {z1, bas}, {z2, bas}, {z3, bas}] ]

Out[•]= {{{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
            {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}, {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
            {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}},
          {{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
            {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}, {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
            {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}}}
```

Verifying *S*-Δ compatibility for opposite *m*, only for CU:

```
In[•]:= Block[{bas = CU /@ {y1, a1, x1}},
          Table[z1 ** z2 ** z3 // Δ1→1,2 // Si // m2,1→1 // Simp // HL,
           {i, 2}, {z1, bas}, {z2, bas}, {z3, bas} ]]

Out[•]= {{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
           {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}, {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
           {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}}
```

Verifying *m*-Δ compatibility:

```
In[•]:= Block[{bas1 = U /@ {y1, a1, x1}, bas2 = U /@ {y2, a2, x2}},
          Table[(z1 ** z2 ** z3 ** z4 // m1,2→1 // Δ1→1,2) −
              (z1 ** z2 ** z3 ** z4 // Δ1→3,4 // Δ2→5,6 // m3,5→1 // m4,6→2) // Simp // HL,
           {U, {CU, QU}}, {z1, bas1}, {z2, bas1}, {z3, bas2}, {z4, bas2} ] ]

Out[•]= {{{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
            {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}},
           {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
            {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}, {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
            {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}},
          {{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
            {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}, {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
            {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}},
           {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
            {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}}}
```

Implementing θ

Verifying involutivity on CU:

```
With[{bas = CU /@ {y, a, x}},
 Table[z → Cθ[z] → HL[Cθ[Cθ[z]]], {z, bas}]]
```

{CU[y] → −CU[x] → CU[y], CU[a] → −CU[a] → CU[a], CU[x] → −CU[y] → CU[x]}

Verifying that θ is a multiplicative homomorphism on CU:

```
With[{bas = CU /@ {y, a, x}},
 Table[Cθ[z1 ** z2] - Cθ[z1] ** Cθ[z2] // HL, {z1, bas}, {z2, bas}]]
```

{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

Verifying involutivity on QU:

```
With[{bas = QU /@ {y, a, x}},
 Table[z → Qθ[z] → HL[Simp[Qθ[Qθ[z]], PowerExpand]], {z, bas}]]
```

$$\left\{QU[y] \to -\frac{QU[x]}{\sqrt{T}} - \frac{\epsilon\hbar\, QU[a, x]}{\sqrt{T}} \to QU[y],\ QU[a] \to -QU[a] \to QU[a],\right.$$
$$\left. QU[x] \to \left(-\frac{1}{\sqrt{T}} + \frac{\gamma\epsilon\hbar}{\sqrt{T}}\right) QU[y] - \frac{\epsilon\hbar\, QU[y, a]}{\sqrt{T}} \to QU[x]\right\}$$

Verifying that θ is a multiplicative homomorphism on QU:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL[Simp[Qθ[z1 ** z2] - Qθ[z1] ** Qθ[z2]]], {z1, bas}, {z2, bas}]]
```

{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}

The Asymmetric Dequantizator

Following pensieve://People/VanDerVeen/Dequant1.pdf.

Docility of A𝔻$f:

```
In[]:= HL@DQ@Block[{$p = 4}, Collect[SS@A𝔻$f /. ω → a1, ϵ]]
Out[]= True
```

Scaling behaviour of A𝔻$f:

```
HL@Simplify[A𝔻$f == ((A𝔻$f /. γ → 1) /. {ϵ → γ ϵ, a → γ^-1 a, ω → γ^-1 ω})]
```

True

```
HL@FullSimplify[
  A𝔻$f == ((A𝔻$f /. γ → 1) /. {ℏ → γ^2 ℏ, ϵ → ϵ / γ, a → a / γ, t → γ^-2 t, ω → γ^-3 ω})]
```

True

Verifying that the asymmetric dequantizator is a homomorphism:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL[SimpT[AD[z1 ** z2] - AD[z1] ** AD[z2]]], {z1, bas}, {z2, bas}]]
```

```
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

The Symmetric Dequantizator

Following pensieve://People/VanDerVeen/Dequant1.pdf.

Verify agreement with the formulas in pensieve://People/VanDerVeen/Dequant1.pdf:

In[∘]:=

$$\{\mathbb{SD\$P} = \frac{\text{Cosh}\left[\hbar\left(\frac{\epsilon-t}{2}+\epsilon\,a\right)\right]-\text{Cosh}\left[\hbar\sqrt{\frac{t^2+\epsilon^2}{4}+\epsilon\,\varpi}\right]}{\hbar\,\text{Sinh}\left[\frac{-\epsilon\,\hbar}{2}\right]\left(\varpi-\epsilon\,a^2+(t-\epsilon)\,a+t/2\right)},$$

```
 Simplify[SD$P == (SD$P /. {a → -a - 1, t → -t})] // HL,
 PowerExpand@Simplify[(SD$P /. {ℏ → γ^2 ℏ, ϵ → ϵ/γ, a → a/γ, t → γ^-2 t, ϖ → γ^-3 ϖ}) ==
    SD$g (SD$g /. {a → -a - γ, t → -t})] // HL,
 SD$Q = Simplify[SD$P /. {a → c - 1/2}],
 Simplify[SD$Q == (SD$Q /. {c → -c, t → -t})] // HL,
 FullSimplify[SD$g == FullSimplify[
    √SD$Q /. c → a + 1/2 /. {ℏ → γ^2 ℏ, ϵ → ϵ/γ, a → a/γ, t → γ^-2 t, ϖ → γ^-3 ϖ}]] // HL,
 HL@DQ@Block[{$p = 4}, Collect[SS@SD$g /. ω → a1, ϵ]],
 HL@DQ@Block[{$p = 4}, Collect[SS@SD$f /. ω → a1, ϵ]]
}
```

Out[∘]=

$$\left\{-\left(\left(\left(\text{Cosh}\left[\left(a\,\epsilon+\frac{1}{2}(-t+\epsilon)\right)\hbar\right]-\text{Cosh}\left[\sqrt{\frac{1}{4}\left(t^2+\epsilon^2\right)+\epsilon\,\varpi}\;\hbar\right]\right)\text{Csch}\left[\frac{\epsilon\,\hbar}{2}\right]\right)\Big/\left(\left(\frac{t}{2}+a\,(t-\epsilon)-a^2\,\epsilon+\varpi\right)\hbar\right)\right),\ \text{True},\ \text{True},\right.$$

$$\left.-\left(\left(4\left(\text{Cosh}\left[\frac{1}{2}(t-2\,c\,\epsilon)\,\hbar\right]-\text{Cosh}\left[\frac{1}{2}\sqrt{t^2+\epsilon^2+4\,\epsilon\,\varpi}\;\hbar\right]\right)\text{Csch}\left[\frac{\epsilon\,\hbar}{2}\right]\right)\Big/\left(\left(4\,c\,t+\epsilon-4\,c^2\,\epsilon+4\,\varpi\right)\hbar\right)\right),\ \text{True},\ \text{True},\ \text{True},\ \text{True}\right\}$$

Verifying the θ-symmetry:

```
Table[HL@SimpT[Cθ[SD[z]] == SD[Qθ[z]]], {z, QU /@ {y, a, x}}]
```

```
{True, True, True}
```

Verifying that the symmetric dequantizator is a homomorphism:

```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL@SimpT[S𝔻[z1 ** z2] - S𝔻[z1] ** S𝔻[z2]], {z1, bas}, {z2, bas}] ]
```

```
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

The representation ρ

Verifying that ρ represents CU and QU:

```
Table[HL[SS[ρ[z1 ** z2] == ρ[z1].ρ[z2]] /. ϵ^k_· /; k > $k → 0],
 {U, {CU, QU}}, {z1, U /@ {y, a, x}}, {z2, U /@ {y, a, x}} ]
```

```
{{{True, True, True}, {True, True, True}, {True, True, True}},
 {{True, True, True}, {True, True, True}, {True, True, True}}}
```

Commuting $e^{\alpha a}$ with $e^{\xi x}$:

```
Table[HL[ρ[𝕖^(ξ U@x)].ρ[𝕖^(α U@a)] == ρ[𝕖^(α U@a)].ρ[𝕖^(𝕖^(-γα) ξ U@x)]], {U, {CU, QU}}]
```

```
{True, True}
```

Œ and the logoi Λ

Logoi from Pensieve://Talks/Toulouse-1705/DogmaDemo.nb and from Pensieve://Talks/Sydney-1708/ExtraDetails@@.nb.

```
Table[
 {Λ_{U,1}[{α, β}, {u, u}],
  lhs = U@Œ_U[{u_1, u_2}, ℏ (α u_1 + β u_2), 1], HL[lhs == U@Λ_{U,1}[ℏ {α, β}, {u, u}]]},
 {U, {CU, QU}}, {u, {y, a, x}}]
```

$\{\{\{Œ_{CU}[\{y\}, y\,(\alpha+\beta), 1+O[\epsilon]^2],$
$\quad CU[\,] + (\alpha\hbar+\beta\hbar)\,CU[y] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right) CU[y, y], True\},$
$\{Œ_{CU}[\{a\}, a\,(\alpha+\beta), 1+O[\epsilon]^2], CU[\,] + (\alpha\hbar+\beta\hbar)\,CU[a] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right) CU[a, a],$
$True\}, \{Œ_{CU}[\{x\}, x\,(\alpha+\beta), 1+O[\epsilon]^2],$
$\quad CU[\,] + (\alpha\hbar+\beta\hbar)\,CU[x] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right) CU[x, x], True\}\},$
$\{\{Œ_{QU}[\{y\}, y\,(\alpha+\beta), 1+O[\epsilon]^2], QU[\,] + (\alpha\hbar+\beta\hbar)\,QU[y] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right) QU[y, y],$
$True\}, \{Œ_{QU}[\{a\}, a\,(\alpha+\beta), 1+O[\epsilon]^2], QU[\,] + (\alpha\hbar+\beta\hbar)\,QU[a] +$
$\left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right) QU[a, a], True\}, \{Œ_{QU}[\{x\}, x\,(\alpha+\beta), 1+O[\epsilon]^2],$
$QU[\,] + (\alpha\hbar+\beta\hbar)\,QU[x] + \left(\frac{\alpha^2\hbar^2}{2}+\alpha\beta\hbar^2+\frac{\beta^2\hbar^2}{2}\right) QU[x, x], True\}\}\}$

```
{Λ#,1[{ξ, α}, {x, a}], lhs = #@Œ#[{x, a}, ℏ (ξ x + α a), 1],
   HL[lhs == #@Λ#,1[ℏ {ξ, α}, {x, a}]]} & /@ {CU, QU}
```

$\{\{Œ_{CU}[\{a, x\}, a\,\alpha + e^{-\alpha\gamma} x\,\xi, 1 + O[\epsilon]^2],$
$CU[] + \alpha\hbar\, CU[a] + (\xi\hbar - \alpha\gamma\xi\hbar^2)\, CU[x] + \frac{1}{2}\alpha^2\hbar^2 CU[a, a] + \alpha\xi\hbar^2 CU[a, x] + \frac{1}{2}\xi^2\hbar^2 CU[x, x],$
$True\}, \{Œ_{QU}[\{a, x\}, a\,\alpha + e^{-\alpha\gamma} x\,\xi, 1 + O[\epsilon]^2], QU[] + \alpha\hbar\, QU[a] +$
$(\xi\hbar - \alpha\gamma\xi\hbar^2)\, QU[x] + \frac{1}{2}\alpha^2\hbar^2 QU[a, a] + \alpha\xi\hbar^2 QU[a, x] + \frac{1}{2}\xi^2\hbar^2 QU[x, x], True\}\}$

```
{Λ#,2[{α, η}, {a, y}], lhs = #@Œ#[{a, y}, ℏ (η y + α a), 1],
   HL[lhs == #@Λ#,2[ℏ {α, η}, {a, y}]]} & /@ {CU, QU}
```

$\{\{Œ_{CU}[\{y, a\}, a\,\alpha + e^{-\alpha\gamma} y\,\eta, 1 + O[\epsilon]^3],$
$CU[] + \alpha\hbar\, CU[a] + (\eta\hbar - \alpha\gamma\eta\hbar^2)\, CU[y] + \frac{1}{2}\alpha^2\hbar^2 CU[a, a] + \alpha\eta\hbar^2 CU[y, a] + \frac{1}{2}\eta^2\hbar^2 CU[y, y],$
$True\}, \{Œ_{QU}[\{y, a\}, a\,\alpha + e^{-\alpha\gamma} y\,\eta, 1 + O[\epsilon]^3], QU[] + \alpha\hbar\, QU[a] +$
$(\eta\hbar - \alpha\gamma\eta\hbar^2)\, QU[y] + \frac{1}{2}\alpha^2\hbar^2 QU[a, a] + \alpha\eta\hbar^2 QU[y, a] + \frac{1}{2}\eta^2\hbar^2 QU[y, y], True\}\}$

In[ ]:=
```
Timing@Λ_QU,2[{ξ, η}, {x, y}]
```

Out[ ]=
$$\Big\{1.64063, Œ_{QU}\Big[\{y, a, x\}, y\,\eta + x\,\xi + \frac{(1 - T)\,\eta\,\xi}{\hbar}, 1 + \frac{1}{4\hbar}\eta\,\xi\,(\gamma\eta\xi - 4T\gamma\eta\xi + 3T^2\gamma\eta\xi + 8aT\hbar + 2y\gamma\eta\hbar - 6Ty\gamma\eta\hbar + 2x\gamma\xi\hbar - 6Tx\gamma\xi\hbar + 4xy\gamma\hbar^2)\,\epsilon +$$
$$\Big(-aTy\gamma\eta^2\xi\,(-\eta\xi + 3T\eta\xi - 3\hbar) - aTx\gamma\eta\xi^2\,(-\eta\xi + 3T\eta\xi - 3\hbar) + 2a^2T\eta\xi\,(T\eta\xi - \hbar) +$$
$$2aTxy\gamma\eta^2\xi^2\hbar - \frac{1}{2}xy^2\gamma^2\eta^2\xi\,(-\eta\xi + 3T\eta\xi - \hbar)\,\hbar - \frac{1}{2}x^2y\gamma^2\eta\xi^2\,(-\eta\xi + 3T\eta\xi - \hbar)\,\hbar +$$
$$\frac{1}{2}x^2y^2\gamma^2\eta^2\xi^2\hbar^2 + \frac{1}{24}y^2\gamma^2\eta^3\xi\,(3\eta\xi - 18T\eta\xi + 27T^2\eta\xi + 4\hbar - 28T\hbar) + \frac{1}{24}x^2\gamma^2\eta\xi^3$$
$$(3\eta\xi - 18T\eta\xi + 27T^2\eta\xi + 4\hbar - 28T\hbar) + \frac{1}{2\hbar}aT\gamma\eta^2\xi^2\,(\eta\xi - 4T\eta\xi + 3T^2\eta\xi + 4\hbar - 6T\hbar) +$$
$$\frac{1}{4}xy\gamma^2\eta\xi\,(2\eta^2\xi^2 - 10T\eta^2\xi^2 + 12T^2\eta^2\xi^2 + 5\eta\xi\hbar - 21T\eta\xi\hbar + 2\hbar^2) -$$
$$\frac{1}{24\hbar}y\gamma^2\eta^2\xi\,(-3\eta^2\xi^2 + 21T\eta^2\xi^2 - 45T^2\eta^2\xi^2 + 27T^3\eta^2\xi^2 - 10\eta\xi\hbar +$$
$$68T\eta\xi\hbar - 82T^2\eta\xi\hbar - 6\hbar^2 + 30T\hbar^2) - \frac{1}{24\hbar}x\gamma^2\eta\xi^2\,(-3\eta^2\xi^2 + 21T\eta^2\xi^2 -$$
$$45T^2\eta^2\xi^2 + 27T^3\eta^2\xi^2 - 10\eta\xi\hbar + 68T\eta\xi\hbar - 82T^2\eta\xi\hbar - 6\hbar^2 + 30T\hbar^2) +$$
$$\frac{1}{288\hbar^2}(-1 + T)\,\gamma^2\eta^2\xi^2\,(-9\eta^2\xi^2 + 63T\eta^2\xi^2 - 135T^2\eta^2\xi^2 + 81T^3\eta^2\xi^2 -$$
$$40\eta\xi\hbar + 272T\eta\xi\hbar - 328T^2\eta\xi\hbar - 36\hbar^2 + 180T\hbar^2)\Big)\,\epsilon^2 + O[\epsilon]^3\Big]\Big\}$$

```
{Λ_{CU,1}[{ξ, η}, {x, y}], lhs = CU@Œ_{CU}[{x, y}, ℏ (ξ x + η y), 1],
 HL[lhs == CU@Λ_{CU,1}[ℏ {ξ, η}, {x, y}]]}
```

$$\left\{Œ_{CU}\left[\{y, a, x\}, y\,\eta + x\,\xi - t\,\eta\,\xi, 1 + \frac{1}{2}\eta\,\xi\,\left(4\,a - 2\,y\,\gamma\,\eta - 2\,x\,\gamma\,\xi + t\,\gamma\,\eta\,\xi\right)\,\epsilon + O[\epsilon]^2\right],\right.$$
$$\left(1 - t\,\eta\,\xi\,\hbar^2\right)\,CU[\,] + 2\,\epsilon\,\eta\,\xi\,\hbar^2\,CU[a] + \xi\,\hbar\,CU[x] + \eta\,\hbar\,CU[y] +$$
$$\left.\frac{1}{2}\xi^2\,\hbar^2\,CU[x, x] + \eta\,\xi\,\hbar^2\,CU[y, x] + \frac{1}{2}\eta^2\,\hbar^2\,CU[y, y], \text{True}\right\}$$

In[ ]:=
```
{Λ_{QU,1}[{ξ, η}, {x, y}], lhs = QU@Œ_{QU}[{x, y}, ℏ (ξ x + η y), 1],
 HL@SimpT[lhs == QU@Λ_{QU,1}[ℏ {ξ, η}, {x, y}]]}
```

Out[ ]=
$$\left\{Œ_{QU}\left[\{y, a, x\}, y\,\eta + x\,\xi + \frac{\left(1 - T\right)\,\eta\,\xi}{\hbar}, 1 + \frac{1}{4\,\hbar}\right.\right.$$
$$\eta\,\xi\,\left(\gamma\,\eta\,\xi - 4\,T\,\gamma\,\eta\,\xi + 3\,T^2\,\gamma\,\eta\,\xi + 8\,a\,T\,\hbar + 2\,y\,\gamma\,\eta\,\hbar - 6\,T\,y\,\gamma\,\eta\,\hbar + 2\,x\,\gamma\,\xi\,\hbar - 6\,T\,x\,\gamma\,\xi\,\hbar + 4\,x\,y\,\gamma\,\hbar^2\right)\,\epsilon +$$
$$\left.O[\epsilon]^2\right], \left(1 + \eta\,\xi\,\hbar - T\,\eta\,\xi\,\hbar\right)\,QU[\,] + 2\,T\,\epsilon\,\eta\,\xi\,\hbar^2\,QU[a] + \xi\,\hbar\,QU[x] +$$
$$\left.\eta\,\hbar\,QU[y] + \frac{1}{2}\xi^2\,\hbar^2\,QU[x, x] + \eta\,\xi\,\hbar^2\,QU[y, x] + \frac{1}{2}\eta^2\,\hbar^2\,QU[y, y], \text{True}\right\}$$

```
{tt = Last[Λ_{CU,2}[{ξ, η}, {x, y}]], Log[tt],
  Exponent[Normal@Log[tt] /. {ξ → ℏ ξ, η → ℏ η, x → ℏ x, y → ℏ y}, ℏ]} // Expand
```

$$\left\{1 + \left(2\,a\,\eta\,\xi - y\,\gamma\,\eta^2\,\xi - x\,\gamma\,\eta\,\xi^2 + \frac{1}{2}t\,\gamma\,\eta^2\,\xi^2\right)\,\epsilon +\right.$$
$$\left(2\,a^2\,\eta^2\,\xi^2 - a\,\gamma\,\eta^2\,\xi^2 - 2\,a\,y\,\gamma\,\eta^3\,\xi^2 + y\,\gamma^2\,\eta^3\,\xi^2 + \frac{1}{2}y^2\,\gamma^2\,\eta^4\,\xi^2 - 2\,a\,x\,\gamma\,\eta^2\,\xi^3 + x\,\gamma^2\,\eta^2\,\xi^3 + a\,t\,\gamma\,\eta^3\,\xi^3 -\right.$$
$$\left.\frac{1}{3}t\,\gamma^2\,\eta^3\,\xi^3 + x\,y\,\gamma^2\,\eta^3\,\xi^3 - \frac{1}{2}t\,y\,\gamma^2\,\eta^4\,\xi^3 + \frac{1}{2}x^2\,\gamma^2\,\eta^2\,\xi^4 - \frac{1}{2}t\,x\,\gamma^2\,\eta^3\,\xi^4 + \frac{1}{8}t^2\,\gamma^2\,\eta^4\,\xi^4\right)\,\epsilon^2 + O[\epsilon]^3,$$
$$\left(2\,a\,\eta\,\xi - y\,\gamma\,\eta^2\,\xi - x\,\gamma\,\eta\,\xi^2 + \frac{1}{2}t\,\gamma\,\eta^2\,\xi^2\right)\,\epsilon + \left(-a\,\gamma\,\eta^2\,\xi^2 + y\,\gamma^2\,\eta^3\,\xi^2 + x\,\gamma^2\,\eta^2\,\xi^3 - \frac{1}{3}t\,\gamma^2\,\eta^3\,\xi^3\right)\,\epsilon^2 +$$
$$\left.O[\epsilon]^3, 6\right\}$$

```
{tt = Last[Λ_{QU,2}[{ξ, η}, {x, y}]], Log[tt],
  Exponent[Normal@Log[tt] /. {ξ → d ξ, η → d η, x → d x, y → d y}, d]} // Expand
```

$$\Big\{1+\Big(2\,a\,T\,\eta\,\xi+\frac{1}{2}\,y\,\gamma\,\eta^2\,\xi-\frac{3}{2}\,T\,y\,\gamma\,\eta^2\,\xi+\frac{1}{2}\,x\,\gamma\,\eta\,\xi^2-\frac{3}{2}\,T\,x\,\gamma\,\eta\,\xi^2+\frac{\gamma\,\eta^2\,\xi^2}{4\,\hbar}-\frac{T\,\gamma\,\eta^2\,\xi^2}{\hbar}+\frac{3\,T^2\,\gamma\,\eta^2\,\xi^2}{4\,\hbar}+x\,y\,\gamma\,\eta\,\xi\,\hbar\Big)\,\epsilon+$$

$$\Big(2\,a^2\,T^2\,\eta^2\,\xi^2+2\,a\,T\,\gamma\,\eta^2\,\xi^2-3\,a\,T^2\,\gamma\,\eta^2\,\xi^2+\frac{1}{8}\,\gamma^2\,\eta^2\,\xi^2-\frac{3}{4}\,T\,\gamma^2\,\eta^2\,\xi^2+\frac{5}{8}\,T^2\,\gamma^2\,\eta^2\,\xi^2+$$

$$a\,T\,y\,\gamma\,\eta^3\,\xi^2-3\,a\,T^2\,y\,\gamma\,\eta^3\,\xi^2+\frac{5}{12}\,y\,\gamma^2\,\eta^3\,\xi^2-\frac{17}{6}\,T\,y\,\gamma^2\,\eta^3\,\xi^2+\frac{41}{12}\,T^2\,y\,\gamma^2\,\eta^3\,\xi^2+\frac{1}{8}\,y^2\,\gamma^2\,\eta^4\,\xi^2-$$

$$\frac{3}{4}\,T\,y^2\,\gamma^2\,\eta^4\,\xi^2+\frac{9}{8}\,T^2\,y^2\,\gamma^2\,\eta^4\,\xi^2+a\,T\,x\,\gamma\,\eta^2\,\xi^3-3\,a\,T^2\,x\,\gamma\,\eta^2\,\xi^3+\frac{5}{12}\,x\,\gamma^2\,\eta^2\,\xi^3-\frac{17}{6}\,T\,x\,\gamma^2\,\eta^2\,\xi^3+$$

$$\frac{41}{12}\,T^2\,x\,\gamma^2\,\eta^2\,\xi^3+\frac{1}{2}\,x\,y\,\gamma^2\,\eta^3\,\xi^3-\frac{5}{2}\,T\,x\,y\,\gamma^2\,\eta^3\,\xi^3+3\,T^2\,x\,y\,\gamma^2\,\eta^3\,\xi^3+\frac{1}{8}\,x^2\,\gamma^2\,\eta^2\,\xi^4-$$

$$\frac{3}{4}\,T\,x^2\,\gamma^2\,\eta^2\,\xi^4+\frac{9}{8}\,T^2\,x^2\,\gamma^2\,\eta^2\,\xi^4+\frac{\gamma^2\,\eta^4\,\xi^4}{32\,\hbar^2}-\frac{T\,\gamma^2\,\eta^4\,\xi^4}{4\,\hbar^2}+\frac{11\,T^2\,\gamma^2\,\eta^4\,\xi^4}{16\,\hbar^2}-\frac{3\,T^3\,\gamma^2\,\eta^4\,\xi^4}{4\,\hbar^2}+$$

$$\frac{9\,T^4\,\gamma^2\,\eta^4\,\xi^4}{32\,\hbar^2}+\frac{a\,T\,\gamma\,\eta^3\,\xi^3}{2\,\hbar}-\frac{2\,a\,T^2\,\gamma\,\eta^3\,\xi^3}{\hbar}+\frac{3\,a\,T^3\,\gamma\,\eta^3\,\xi^3}{2\,\hbar}+\frac{5\,\gamma^2\,\eta^3\,\xi^3}{36\,\hbar}-\frac{13\,T\,\gamma^2\,\eta^3\,\xi^3}{12\,\hbar}+$$

$$\frac{25\,T^2\,\gamma^2\,\eta^3\,\xi^3}{12\,\hbar}-\frac{41\,T^3\,\gamma^2\,\eta^3\,\xi^3}{36\,\hbar}+\frac{y\,\gamma^2\,\eta^4\,\xi^3}{8\,\hbar}-\frac{7\,T\,y\,\gamma^2\,\eta^4\,\xi^3}{8\,\hbar}+\frac{15\,T^2\,y\,\gamma^2\,\eta^4\,\xi^3}{8\,\hbar}-\frac{9\,T^3\,y\,\gamma^2\,\eta^4\,\xi^3}{8\,\hbar}+$$

$$\frac{x\,\gamma^2\,\eta^3\,\xi^4}{8\,\hbar}-\frac{7\,T\,x\,\gamma^2\,\eta^3\,\xi^4}{8\,\hbar}+\frac{15\,T^2\,x\,\gamma^2\,\eta^3\,\xi^4}{8\,\hbar}-\frac{9\,T^3\,x\,\gamma^2\,\eta^3\,\xi^4}{8\,\hbar}-2\,a^2\,T\,\eta\,\xi\,\hbar+3\,a\,T\,y\,\gamma\,\eta^2\,\xi\,\hbar+$$

$$\frac{1}{4}\,y\,\gamma^2\,\eta^2\,\xi\,\hbar-\frac{5}{4}\,T\,y\,\gamma^2\,\eta^2\,\xi\,\hbar+\frac{1}{6}\,y^2\,\gamma^2\,\eta^3\,\xi\,\hbar-\frac{7}{6}\,T\,y^2\,\gamma^2\,\eta^3\,\xi\,\hbar+3\,a\,T\,x\,\gamma\,\eta\,\xi^2\,\hbar+\frac{1}{4}\,x\,\gamma^2\,\eta\,\xi^2\,\hbar-$$

$$\frac{5}{4}\,T\,x\,\gamma^2\,\eta\,\xi^2\,\hbar+2\,a\,T\,x\,y\,\gamma\,\eta^2\,\xi^2\,\hbar+\frac{5}{4}\,x\,y\,\gamma^2\,\eta^2\,\xi^2\,\hbar-\frac{21}{4}\,T\,x\,y\,\gamma^2\,\eta^2\,\xi^2\,\hbar+\frac{1}{2}\,x\,y^2\,\gamma^2\,\eta^3\,\xi^2\,\hbar-$$

$$\frac{3}{2}\,T\,x\,y^2\,\gamma^2\,\eta^3\,\xi^2\,\hbar+\frac{1}{6}\,x^2\,\gamma^2\,\eta\,\xi^3\,\hbar-\frac{7}{6}\,T\,x^2\,\gamma^2\,\eta\,\xi^3\,\hbar+\frac{1}{2}\,x^2\,y\,\gamma^2\,\eta^2\,\xi^3\,\hbar-\frac{3}{2}\,T\,x^2\,y\,\gamma^2\,\eta^2\,\xi^3\,\hbar+$$

$$\frac{1}{2}\,x\,y\,\gamma^2\,\eta\,\xi\,\hbar^2+\frac{1}{2}\,x\,y^2\,\gamma^2\,\eta^2\,\xi\,\hbar^2+\frac{1}{2}\,x^2\,y\,\gamma^2\,\eta\,\xi^2\,\hbar^2+\frac{1}{2}\,x^2\,y^2\,\gamma^2\,\eta^2\,\xi^2\,\hbar^2\Big)\,\epsilon^2+O[\epsilon]^3,$$

$$\Big(2\,a\,T\,\eta\,\xi+\frac{1}{2}\,y\,\gamma\,\eta^2\,\xi-\frac{3}{2}\,T\,y\,\gamma\,\eta^2\,\xi+\frac{1}{2}\,x\,\gamma\,\eta\,\xi^2-\frac{3}{2}\,T\,x\,\gamma\,\eta\,\xi^2+\frac{\gamma\,\eta^2\,\xi^2}{4\,\hbar}-$$

$$\frac{T\,\gamma\,\eta^2\,\xi^2}{\hbar}+\frac{3\,T^2\,\gamma\,\eta^2\,\xi^2}{4\,\hbar}+x\,y\,\gamma\,\eta\,\xi\,\hbar\Big)\,\epsilon+$$

$$\Big(2\,a\,T\,\gamma\,\eta^2\,\xi^2-3\,a\,T^2\,\gamma\,\eta^2\,\xi^2+\frac{1}{8}\,\gamma^2\,\eta^2\,\xi^2-\frac{3}{4}\,T\,\gamma^2\,\eta^2\,\xi^2+\frac{5}{8}\,T^2\,\gamma^2\,\eta^2\,\xi^2+\frac{5}{12}\,y\,\gamma^2\,\eta^3\,\xi^2-$$

$$\frac{17}{6}\,T\,y\,\gamma^2\,\eta^3\,\xi^2+\frac{41}{12}\,T^2\,y\,\gamma^2\,\eta^3\,\xi^2+\frac{5}{12}\,x\,\gamma^2\,\eta^2\,\xi^3-\frac{17}{6}\,T\,x\,\gamma^2\,\eta^2\,\xi^3+\frac{41}{12}\,T^2\,x\,\gamma^2\,\eta^2\,\xi^3+$$

$$\frac{5\,\gamma^2\,\eta^3\,\xi^3}{36\,\hbar}-\frac{13\,T\,\gamma^2\,\eta^3\,\xi^3}{12\,\hbar}+\frac{25\,T^2\,\gamma^2\,\eta^3\,\xi^3}{12\,\hbar}-\frac{41\,T^3\,\gamma^2\,\eta^3\,\xi^3}{36\,\hbar}-2\,a^2\,T\,\eta\,\xi\,\hbar+3\,a\,T\,y\,\gamma\,\eta^2\,\xi\,\hbar+$$

$$\frac{1}{4}\,y\,\gamma^2\,\eta^2\,\xi\,\hbar-\frac{5}{4}\,T\,y\,\gamma^2\,\eta^2\,\xi\,\hbar+\frac{1}{6}\,y^2\,\gamma^2\,\eta^3\,\xi\,\hbar-\frac{7}{6}\,T\,y^2\,\gamma^2\,\eta^3\,\xi\,\hbar+3\,a\,T\,x\,\gamma\,\eta\,\xi^2\,\hbar+$$

$$\frac{1}{4}\,x\,\gamma^2\,\eta\,\xi^2\,\hbar-\frac{5}{4}\,T\,x\,\gamma^2\,\eta\,\xi^2\,\hbar+\frac{5}{4}\,x\,y\,\gamma^2\,\eta^2\,\xi^2\,\hbar-\frac{21}{4}\,T\,x\,y\,\gamma^2\,\eta^2\,\xi^2\,\hbar+\frac{1}{6}\,x^2\,\gamma^2\,\eta\,\xi^3\,\hbar-$$

$$\frac{7}{6}\,T\,x^2\,\gamma^2\,\eta\,\xi^3\,\hbar+\frac{1}{2}\,x\,y\,\gamma^2\,\eta\,\xi\,\hbar^2+\frac{1}{2}\,x\,y^2\,\gamma^2\,\eta^2\,\xi\,\hbar^2+\frac{1}{2}\,x^2\,y\,\gamma^2\,\eta\,\xi^2\,\hbar^2\Big)\,\epsilon^2+O[\epsilon]^3,\ 6\Big\}$$

```
Block[{$p = 4, $k = 1},
 {Λ_{CU,$k}[ℏ {ξ, η, δ}, {x, y}],
  Short[lhs = CU@Œ_CU[{x, y}, ℏ (ξ x + η y + δ x y), 1_$k], 5],
   HL@Simp[lhs - CU@Λ_{CU,$k}[ℏ {ξ, η, δ}, {x, y}]]}
]
```

$$\Big\{Œ_{CU}[\{y, a, x\}, \frac{x y \delta \hbar + y \eta \hbar + x \xi \hbar - t \eta \xi \hbar^2}{1 + t \delta \hbar},$$

$$\frac{1}{1 + t \delta \hbar} + \big((4 a \delta \hbar + 12 a t \delta^2 \hbar^2 + 4 a x y \delta^2 \hbar^2 + 2 t \gamma \delta^2 \hbar^2 - 8 x y \gamma \delta^2 \hbar^2 + 4 a y \delta \eta \hbar^2 - 4 y \gamma \delta \eta \hbar^2 + 4 a x \delta \xi \hbar^2 - 4 x \gamma \delta \xi \hbar^2 + 4 a \eta \xi \hbar^2 + 12 a t^2 \delta^3 \hbar^3 + 8 a t x y \delta^3 \hbar^3 + 4 t^2 \gamma \delta^3 \hbar^3 - 12 t x y \gamma \delta^3 \hbar^3 - 4 x^2 y^2 \gamma \delta^3 \hbar^3 + 8 a t y \delta^2 \eta \hbar^3 - 4 t y \gamma \delta^2 \eta \hbar^3 - 6 x y^2 \gamma \delta^2 \eta \hbar^3 - 2 y^2 \gamma \delta \eta^2 \hbar^3 + 8 a t x \delta^2 \xi \hbar^3 - 4 t x \gamma \delta^2 \xi \hbar^3 - 6 x^2 y \gamma \delta^2 \xi \hbar^3 + 8 a t \delta \eta \xi \hbar^3 + 4 t \gamma \delta \eta \xi \hbar^3 - 8 x y \gamma \delta \eta \xi \hbar^3 - 2 y \gamma \eta^2 \xi \hbar^3 - 2 x^2 \gamma \delta \xi^2 \hbar^3 - 2 x \gamma \eta \xi^2 \hbar^3 + 4 a t^3 \delta^4 \hbar^4 + 4 a t^2 x y \delta^4 \hbar^4 + 2 t^3 \gamma \delta^4 \hbar^4 - 4 t^2 x y \gamma \delta^4 \hbar^4 - 3 t x^2 y^2 \gamma \delta^4 \hbar^4 + 4 a t^2 y \delta^3 \eta \hbar^4 - 4 t x y^2 \gamma \delta^3 \eta \hbar^4 - t y^2 \gamma \delta^2 \eta^2 \hbar^4 + 4 a t^2 x \delta^3 \xi \hbar^4 - 4 t x^2 y \gamma \delta^3 \xi \hbar^4 + 4 a t^2 \delta^2 \eta \xi \hbar^4 + 4 t^2 \gamma \delta^2 \eta \xi \hbar^4 - 4 t x y \gamma \delta^2 \eta \xi \hbar^4 - t x^2 \gamma \delta^2 \xi^2 \hbar^4 + t \gamma \eta^2 \xi^2 \hbar^4) \epsilon\big) \big/ \left(2 + 10 t \delta \hbar + 20 t^2 \delta^2 \hbar^2 + 20 t^3 \delta^3 \hbar^3 + 10 t^4 \delta^4 \hbar^4 + 2 t^5 \delta^5 \hbar^5\right) + O[\epsilon]^2\Big],$$

$$\Big(1 - t \delta \hbar + t^2 \delta^2 \hbar^2 + t \gamma \delta^2 \epsilon \hbar^2 - t \eta \xi \hbar^2 - t^3 \delta^3 \hbar^3 - 3 t^2 \gamma \delta^3 \epsilon \hbar^3 + 2 t^2 \delta \eta \xi \hbar^3 + 2 t \gamma \delta \epsilon \eta \xi \hbar^3 + t^4 \delta^4 \hbar^4 + 6 t^3 \gamma \delta^4 \epsilon \hbar^4 - 3 t^3 \delta^2 \eta \xi \hbar^4 - 9 t^2 \gamma \delta^2 \epsilon \eta \xi \hbar^4 + \frac{1}{2} t^2 \eta^2 \xi^2 \hbar^4 + \frac{1}{2} t \gamma \epsilon \eta^2 \xi^2 \hbar^4\Big) CU[\,] +$$

$$\left(2 \delta \epsilon \hbar - 4 t \delta^2 \epsilon \hbar^2 + 2 \epsilon \eta \xi \hbar^2 + 6 t^2 \delta^3 \epsilon \hbar^3 - 8 t \delta \epsilon \eta \xi \hbar^3 - 8 t^3 \delta^4 \epsilon \hbar^4 + 18 t^2 \delta^2 \epsilon \eta \xi \hbar^4 - 2 t \epsilon \eta^2 \xi^2 \hbar^4\right) CU[a] +$$

$$\ll 37 \gg + \frac{1}{6} \delta^3 \eta \hbar^4 CU[y, y, y, y, x, x, x] + \frac{1}{24} \delta^4 \hbar^4 CU[y, y, y, y, x, x, x, x], 0\Big\}$$

```
{Λ_{QU,2}[{ξ, η, δ}, {x, y}], lhs = QU@Œ_QU[{x, y}, ℏ (ξ x + η y + δ x y), 1],
 HL@SimpT[lhs == QU@Λ_{QU,1}[ℏ {ξ, η, δ}, {x, y}]]}
```

$$\Big\{Œ_{QU}[\{y, a, x\}, \frac{\cdots 1 \cdots}{\cdots 1 \cdots},$$

$$\frac{\hbar}{-\delta + T \delta + \hbar} + \big((-8 a T \delta^4 \hbar^2 + 24 a T^2 \delta^4 \hbar^2 - 24 a T^3 \delta^4 \hbar^2 + 8 a T^4 \delta^4 \hbar^2 + \cdots 149 \cdots + 4 x^2 y^2 \gamma \delta^2 \hbar^6 + 4 x y^2 \gamma \delta \eta \hbar^6 + 4 x^2 y \gamma \delta \xi \hbar^6 + 4 x y \gamma \eta \xi \hbar^6) \epsilon\big) \big/ \left(-4 \delta^5 + 20 T \delta^5 - 40 T^2 \delta^5 + 40 T^3 \delta^5 - 20 T^4 \delta^5 + 4 T^5 \delta^5 + \cdots 12 \cdots + 40 T^3 \delta^3 \hbar^2 + 40 \delta^2 \hbar^3 - 80 T \delta^2 \hbar^3 + 40 T^2 \delta^2 \hbar^3 - 20 \delta \hbar^4 + 20 T \delta \hbar^4 + 4 \hbar^5\right) + \frac{(\cdots 1 \cdots)\ \cdots 1 \cdots}{\cdots 1 \cdots} + O[\epsilon]^3\Big], \cdots 1 \cdots, \text{True}\Big\}$$

large output | show less | show more | show all | set size limit...

```
{tt = ComposeSeries[(1 + t δ) Last[Λ_CU,2[{ξ, η, δ}, {x, y}]], (1 + t δ)^4 ϵ + O[ϵ]^18];
  Together@Log[tt],
  Exponent[Normal@Together@Log[tt] /. {ξ → d ξ, η → d η, x → d x, y → d y}, d],
  Exponent[Normal@Together@Log[tt] /. {x → d x, y → d y}, d]
 } // Expand
```

$$\Big\{\Big(2 a \delta + 6 a t \delta^2 + 2 a x y \delta^2 + t \gamma \delta^2 - 4 x y \gamma \delta^2 + 6 a t^2 \delta^3 + 4 a t x y \delta^3 + 2 t^2 \gamma \delta^3 - 6 t x y \gamma \delta^3 -$$
$$2 x^2 y^2 \gamma \delta^3 + 2 a t^3 \delta^4 + 2 a t^2 x y \delta^4 + t^3 \gamma \delta^4 - 2 t^2 x y \gamma \delta^4 - \frac{3}{2} t x^2 y^2 \gamma \delta^4 + 2 a y \delta \eta -$$
$$2 y \gamma \delta \eta + 4 a t y \delta^2 \eta - 2 t y \gamma \delta^2 \eta - 3 x y^2 \gamma \delta^2 \eta + 2 a t^2 y \delta^3 \eta - 2 t x y^2 \gamma \delta^3 \eta -$$
$$y^2 \gamma \delta \eta^2 - \frac{1}{2} t y^2 \gamma \delta^2 \eta^2 + 2 a x \delta \xi - 2 x \gamma \delta \xi + 4 a t x \delta^2 \xi - 2 t x \gamma \delta^2 \xi - 3 x^2 y \gamma \delta^2 \xi +$$
$$2 a t^2 x \delta^3 \xi - 2 t x^2 y \gamma \delta^3 \xi + 2 a \eta \xi + 4 a t \delta \eta \xi + 2 t \gamma \delta \eta \xi - 4 x y \gamma \delta \eta \xi + 2 a t^2 \delta^2 \eta \xi +$$
$$2 t^2 \gamma \delta^2 \eta \xi - 2 t x y \gamma \delta^2 \eta \xi - y \gamma \eta^2 \xi - x^2 \gamma \delta \xi^2 - \frac{1}{2} t x^2 \gamma \delta^2 \xi^2 - x \gamma \eta \xi^2 + \frac{1}{2} t \gamma \eta^2 \xi^2\Big) \epsilon +$$
$$\Big(2 a^2 \delta^2 - 2 a \gamma \delta^2 + 12 a^2 t \delta^3 + 4 a^2 x y \delta^3 - 8 a t \gamma \delta^3 - 20 a x y \gamma \delta^3 - 2 t \gamma^2 \delta^3 + 18 x y \gamma^2 \delta^3 +$$
$$30 a^2 t^2 \delta^4 + 20 a^2 t x y \delta^4 - 10 a t^2 \gamma \delta^4 - 88 a t x y \gamma \delta^4 - 13 a x^2 y^2 \gamma \delta^4 - \frac{15}{2} t^2 \gamma^2 \delta^4 +$$
$$64 t x y \gamma^2 \delta^4 + 34 x^2 y^2 \gamma^2 \delta^4 + 40 a^2 t^3 \delta^5 + 40 a^2 t^2 x y \delta^5 - 152 a t^2 x y \gamma \delta^5 - 48 a t x^2 y^2 \gamma \delta^5 -$$
$$10 t^3 \gamma^2 \delta^5 + 86 t^2 x y \gamma^2 \delta^5 + 107 t x^2 y^2 \gamma^2 \delta^5 + 11 x^3 y^3 \gamma^2 \delta^5 + 30 a^2 t^4 \delta^6 + 40 a^2 t^3 x y \delta^6 +$$
$$10 a t^4 \gamma \delta^6 - 128 a t^3 x y \gamma \delta^6 - 66 a t^2 x^2 y^2 \gamma \delta^6 - 5 t^4 \gamma^2 \delta^6 + 54 t^3 x y \gamma^2 \delta^6 + \frac{247}{2} t^2 x^2 y^2 \gamma^2 \delta^6 +$$
$$\frac{80}{3} t x^3 y^3 \gamma^2 \delta^6 + 12 a^2 t^5 \delta^7 + 20 a^2 t^4 x y \delta^7 + 8 a t^5 \gamma \delta^7 - 52 a t^4 x y \gamma \delta^7 - 40 a t^3 x^2 y^2 \gamma \delta^7 +$$
$$16 t^4 x y \gamma^2 \delta^7 + 62 t^3 x^2 y^2 \gamma^2 \delta^7 + \frac{64}{3} t^2 x^3 y^3 \gamma^2 \delta^7 + 2 a^2 t^6 \delta^8 + 4 a^2 t^5 x y \delta^8 + 2 a t^6 \gamma \delta^8 -$$
$$8 a t^5 x y \gamma \delta^8 - 9 a t^4 x^2 y^2 \gamma \delta^8 + \frac{1}{2} t^6 \gamma^2 \delta^8 + 2 t^5 x y \gamma^2 \delta^8 + \frac{23}{2} t^4 x^2 y^2 \gamma^2 \delta^8 + \frac{17}{3} t^3 x^3 y^3 \gamma^2 \delta^8 +$$
$$4 a^2 y \delta^2 \eta - 12 a y \gamma \delta^2 \eta + 6 y \gamma^2 \delta^2 \eta + 20 a^2 t y \delta^3 \eta - 48 a t y \gamma \delta^3 \eta - 20 a x y^2 \gamma \delta^3 \eta +$$
$$14 t y \gamma^2 \delta^3 \eta + 40 x y^2 \gamma^2 \delta^3 \eta + 40 a^2 t^2 y \delta^4 \eta - 72 a t^2 y \gamma \delta^4 \eta - 72 a t x y^2 \gamma \delta^4 \eta + 6 t^2 y \gamma^2 \delta^4 \eta +$$
$$115 t x y^2 \gamma^2 \delta^4 \eta + 23 x^2 y^3 \gamma^2 \delta^4 \eta + 40 a^2 t^3 y \delta^5 \eta - 48 a t^3 y \gamma \delta^5 \eta - 96 a t^2 x y^2 \gamma \delta^5 \eta -$$
$$6 t^3 y \gamma^2 \delta^5 \eta + 118 t^2 x y^2 \gamma^2 \delta^5 \eta + 53 t x^2 y^3 \gamma^2 \delta^5 \eta + 20 a^2 t^4 y \delta^6 \eta - 12 a t^4 y \gamma \delta^6 \eta -$$
$$56 a t^3 x y^2 \gamma \delta^6 \eta - 4 t^4 y \gamma^2 \delta^6 \eta + 51 t^3 x y^2 \gamma^2 \delta^6 \eta + 40 t^2 x^2 y^3 \gamma^2 \delta^6 \eta + 4 a^2 t^5 y \delta^7 \eta -$$
$$12 a t^4 x y^2 \gamma \delta^7 \eta + 8 t^4 x y^2 \gamma^2 \delta^7 \eta + 10 t^3 x^2 y^3 \gamma^2 \delta^7 \eta - 7 a y^2 \gamma \delta^2 \eta^2 + 10 y^2 \gamma^2 \delta^2 \eta^2 -$$
$$24 a t y^2 \gamma \delta^3 \eta^2 + 24 t y^2 \gamma^2 \delta^3 \eta^2 + 15 x y^3 \gamma^2 \delta^3 \eta^2 - 30 a t^2 y^2 \gamma \delta^4 \eta^2 + \frac{37}{2} t^2 y^2 \gamma^2 \delta^4 \eta^2 +$$
$$32 t x y^3 \gamma^2 \delta^4 \eta^2 - 16 a t^3 y^2 \gamma \delta^5 \eta^2 + 5 t^3 y^2 \gamma^2 \delta^5 \eta^2 + 22 t^2 x y^3 \gamma^2 \delta^5 \eta^2 - 3 a t^4 y^2 \gamma \delta^6 \eta^2 +$$
$$\frac{1}{2} t^4 y^2 \gamma^2 \delta^6 \eta^2 + 5 t^3 x y^3 \gamma^2 \delta^6 \eta^2 + 3 y^3 \gamma^2 \delta^2 \eta^3 + \frac{17}{3} t y^3 \gamma^2 \delta^3 \eta^3 + \frac{10}{3} t^2 y^3 \gamma^2 \delta^4 \eta^3 +$$
$$\frac{2}{3} t^3 y^3 \gamma^2 \delta^5 \eta^3 + 4 a^2 x \delta^2 \xi - 12 a x \gamma \delta^2 \xi + 6 x \gamma^2 \delta^2 \xi + 20 a^2 t x \delta^3 \xi - 48 a t x \gamma \delta^3 \xi -$$
$$20 a x^2 y \gamma \delta^3 \xi + 14 t x \gamma^2 \delta^3 \xi + 40 x^2 y \gamma^2 \delta^3 \xi + 40 a^2 t^2 x \delta^4 \xi - 72 a t^2 x \gamma \delta^4 \xi - 72 a t x^2 y \gamma \delta^4 \xi +$$
$$6 t^2 x \gamma^2 \delta^4 \xi + 115 t x^2 y \gamma^2 \delta^4 \xi + 23 x^3 y^2 \gamma^2 \delta^4 \xi + 40 a^2 t^3 x \delta^5 \xi - 48 a t^3 x \gamma \delta^5 \xi -$$
$$96 a t^2 x^2 y \gamma \delta^5 \xi - 6 t^3 x \gamma^2 \delta^5 \xi + 118 t^2 x^2 y \gamma^2 \delta^5 \xi + 53 t x^3 y^2 \gamma^2 \delta^5 \xi + 20 a^2 t^4 x \delta^6 \xi -$$
$$12 a t^4 x \gamma \delta^6 \xi - 56 a t^3 x^2 y \gamma \delta^6 \xi - 4 t^4 x \gamma^2 \delta^6 \xi + 51 t^3 x^2 y \gamma^2 \delta^6 \xi + 40 t^2 x^3 y^2 \gamma^2 \delta^6 \xi +$$
$$4 a^2 t^5 x \delta^7 \xi - 12 a t^4 x^2 y \gamma \delta^7 \xi + 8 t^4 x^2 y \gamma^2 \delta^7 \xi + 10 t^3 x^3 y^2 \gamma^2 \delta^7 \xi + 4 a^2 \delta \eta \xi - 4 a \gamma \delta \eta \xi +$$
$$20 a^2 t \delta^2 \eta \xi - 8 a t \gamma \delta^2 \eta \xi - 28 a x y \gamma \delta^2 \eta \xi - 6 t \gamma^2 \delta^2 \eta \xi + 38 x y \gamma^2 \delta^2 \eta \xi + 40 a^2 t^2 \delta^3 \eta \xi +$$

$$8\,a\,t^2\gamma\,\delta^3\eta\,\xi - 96\,a\,t\,x\,y\,\gamma\,\delta^3\eta\,\xi - 14\,t^2\gamma^2\delta^3\eta\,\xi + 88\,t\,x\,y\,\gamma^2\delta^3\eta\,\xi + 44\,x^2y^2\gamma^2\delta^3\eta\,\xi +$$
$$40\,a^2t^3\delta^4\eta\,\xi + 32\,a\,t^3\gamma\,\delta^4\eta\,\xi - 120\,a\,t^2x\,y\,\gamma\,\delta^4\eta\,\xi - 6\,t^3\gamma^2\delta^4\eta\,\xi + 62\,t^2x\,y\,\gamma^2\delta^4\eta\,\xi +$$
$$93\,t\,x^2y^2\gamma^2\delta^4\eta\,\xi + 20\,a^2t^4\delta^5\eta\,\xi + 28\,a\,t^4\gamma\,\delta^5\eta\,\xi - 64\,a\,t^3x\,y\,\gamma\,\delta^5\eta\,\xi + 6\,t^4\gamma^2\delta^5\eta\,\xi +$$
$$12\,t^3x\,y\,\gamma^2\delta^5\eta\,\xi + 63\,t^2x^2y^2\gamma^2\delta^5\eta\,\xi + 4\,a^2t^5\delta^6\eta\,\xi + 8\,a\,t^5\gamma\,\delta^6\eta\,\xi - 12\,a\,t^4x\,y\,\gamma\,\delta^6\eta\,\xi +$$
$$4\,t^5\gamma^2\delta^6\eta\,\xi + 14\,t^3x^2y^2\gamma^2\delta^6\eta\,\xi - 8\,a\,y\,\gamma\,\delta\,\eta^2\xi + 6\,y\,\gamma^2\delta\,\eta^2\xi - 24\,a\,t\,y\,\gamma\,\delta^2\eta^2\xi +$$
$$5\,t\,y\,\gamma^2\delta^2\eta^2\xi + 25\,x\,y^2\gamma^2\delta^2\eta^2\xi - 24\,a\,t^2y\,\gamma\,\delta^3\eta^2\xi - 8\,t^2y\,\gamma^2\delta^3\eta^2\xi + 45\,t\,x\,y^2\gamma^2\delta^3\eta^2\xi -$$
$$8\,a\,t^3y\,\gamma\,\delta^4\eta^2\xi - 7\,t^3y\,\gamma^2\delta^4\eta^2\xi + 24\,t^2x\,y^2\gamma^2\delta^4\eta^2\xi + 4\,t^3x\,y^2\gamma^2\delta^5\eta^2\xi + 4\,y^2\gamma^2\delta\,\eta^3\xi +$$
$$5\,t\,y^2\gamma^2\delta^2\eta^3\xi + t^2y^2\gamma^2\delta^3\eta^3\xi - 7\,a\,x^2\gamma\,\delta^2\xi^2 + 10\,x^2\gamma^2\delta^2\xi^2 - 24\,a\,t\,x^2\gamma\,\delta^3\xi^2 +$$
$$24\,t\,x^2\gamma^2\delta^3\xi^2 + 15\,x^3y\,\gamma^2\delta^3\xi^2 - 30\,a\,t^2x^2\gamma\,\delta^4\xi^2 + \frac{37}{2}t^2x^2\gamma^2\delta^4\xi^2 + 32\,t\,x^3y\,\gamma^2\delta^4\xi^2 -$$
$$16\,a\,t^3x^2\gamma\,\delta^5\xi^2 + 5\,t^3x^2\gamma^2\delta^5\xi^2 + 22\,t^2x^3y\,\gamma^2\delta^5\xi^2 - 3\,a\,t^4x^2\gamma\,\delta^6\xi^2 + \frac{1}{2}t^4x^2\gamma^2\delta^6\xi^2 +$$
$$5\,t^3x^3y\,\gamma^2\delta^6\xi^2 - 8\,a\,x\,\gamma\,\delta\,\eta\,\xi^2 + 6\,x\,\gamma^2\delta\,\eta\,\xi^2 - 24\,a\,t\,x\,\gamma\,\delta^2\eta\,\xi^2 + 5\,t\,x\,\gamma^2\delta^2\eta\,\xi^2 +$$
$$25\,x^2y\,\gamma^2\delta^2\eta\,\xi^2 - 24\,a\,t^2x\,\gamma\,\delta^3\eta\,\xi^2 - 8\,t^2x\,\gamma^2\delta^3\eta\,\xi^2 + 45\,t\,x^2y\,\gamma^2\delta^3\eta\,\xi^2 - 8\,a\,t^3x\,\gamma\,\delta^4\eta\,\xi^2 -$$
$$7\,t^3x\,\gamma^2\delta^4\eta\,\xi^2 + 24\,t^2x^2y\,\gamma^2\delta^4\eta\,\xi^2 + 4\,t^3x^2y\,\gamma^2\delta^5\eta\,\xi^2 - a\,\gamma\,\eta^2\xi^2 - 3\,t\,\gamma^2\delta\,\eta^2\xi^2 +$$
$$11\,x\,y\,\gamma^2\delta\,\eta^2\xi^2 + 6\,a\,t^2\gamma\,\delta^2\eta^2\xi^2 - \frac{5}{2}t^2\gamma^2\delta^2\eta^2\xi^2 + 12\,t\,x\,y\,\gamma^2\delta^2\eta^2\xi^2 + 8\,a\,t^3\gamma\,\delta^3\eta^2\xi^2 +$$
$$4\,t^3\gamma^2\delta^3\eta^2\xi^2 + 3\,a\,t^4\gamma\,\delta^4\eta^2\xi^2 + \frac{7}{2}t^4\gamma^2\delta^4\eta^2\xi^2 - t^3x\,y\,\gamma^2\delta^4\eta^2\xi^2 + y\,\gamma^2\eta^3\xi^2 -$$
$$t\,y\,\gamma^2\delta\,\eta^3\xi^2 - 2\,t^2y\,\gamma^2\delta^2\eta^3\xi^2 + 3\,x^3\gamma^2\delta^2\xi^3 + \frac{17}{3}t\,x^3\gamma^2\delta^3\xi^3 + \frac{10}{3}t^2x^3\gamma^2\delta^4\xi^3 +$$
$$\frac{2}{3}t^3x^3\gamma^2\delta^5\xi^3 + 4\,x^2\gamma^2\delta\,\eta\,\xi^3 + 5\,t\,x^2\gamma^2\delta^2\eta\,\xi^3 + t^2x^2\gamma^2\delta^3\eta\,\xi^3 + x\,\gamma^2\eta^2\xi^3 - t\,x\,\gamma^2\delta\,\eta^2\xi^3 -$$
$$2\,t^2x\,\gamma^2\delta^2\eta^2\xi^3 - \frac{1}{3}t\,\gamma^2\eta^3\xi^3 + \frac{1}{3}t^2\gamma^2\delta\,\eta^3\xi^3 + \frac{2}{3}t^3\gamma^2\delta^2\eta^3\xi^3\Big)\,\epsilon^2 + O[\epsilon]^3,\ 6,\ 6\Big\}$$

```
{tt = Last[Λ_{QU,2}[{ξ, η, δ}, {x, y}]];
  Log[tt],
  Exponent[Normal@Together@Log[tt] /. {ξ → d ξ, η → d η, x → d x, y → d y}, d]} // Expand
```

$$\Big\{\text{Log}\Big[\frac{\hbar}{-\delta+T\,\delta+\hbar}\Big] + \Big(\frac{2\,a\,T\,\delta^5\,\hbar}{(-\delta+T\,\delta+\hbar)^5} - \frac{8\,a\,T^2\,\delta^5\,\hbar}{(-\delta+T\,\delta+\hbar)^5} + \frac{12\,a\,T^3\,\delta^5\,\hbar}{(-\delta+T\,\delta+\hbar)^5} -$$
$$\frac{8\,a\,T^4\,\delta^5\,\hbar}{(-\delta+T\,\delta+\hbar)^5} + \cdots 267 \cdots + \frac{x^2\,y^2\,\gamma\,\delta^2\,\hbar^6}{(-\delta+T\,\delta+\hbar)^5} + \frac{x\,y^2\,\gamma\,\delta\,\eta\,\hbar^6}{(-\delta+T\,\delta+\hbar)^5} + \frac{x^2\,y\,\gamma\,\delta\,\xi\,\hbar^6}{(-\delta+T\,\delta+\hbar)^5} + \frac{x\,y\,\gamma\,\eta\,\xi\,\hbar^6}{(-\delta+T\,\delta+\hbar)^5}\Big)\,\epsilon +$$
$$\Big(-\frac{32\,a^2\,T^2\,\delta^{10}\,\hbar^2}{(\cdots 1 \cdots)^2} + \cdots 8307 \cdots + \cdots 1 \cdots + \frac{144\,x^2\,y^2\,\gamma^2\,\eta^2\,\xi^2\,\hbar^{11}}{\cdots 1 \cdots}\Big)\,\epsilon^2 + O[\epsilon]^3,\ 6\Big\}$$

large output | show less | show more | show all | set size limit...

tSW

In[∘]:=
```
HL@Simp[𝕆_{$U}[{y_3, a_3, x_3}_3, tSW_{xy,1,2→3} /. {ξ_1 → ℏ ξ_1, η_2 → ℏ η_2} /. T2t] -
   SS[𝕆_{$U}[{x_3, y_3}_3, SS[ⅇ^(ℏ (ξ_1 x_3+η_2 y_3))]] /. T2t]]
```

Out[∘]= 0

In[∘]:=
```
HL@Simp[
  𝕆_{$U}[{y_1, a_1, x_1}_1, tSW_{xa,1,1→1} /. {ξ_1 → ℏ ξ_1, α_1 → ℏ α_1}] - SS[𝕆_{$U}[{x_1, a_1}_1, SS[ⅇ^(ℏ (ξ_1 x_1+α_1 a_1))]]]]
```

Out[∘]= 0

In[∘]:=
```
HL@Simp[
  𝕆$U[{y1, a1, x1}1, tSW_{ay,1,1→1} /. {η1 → ℏ η1, α1 → ℏ α1}] - SS[𝕆$U[{a1, y1}1, SS[e^(ℏ (η1 y1+α1 a1))]]]]
```
Out[∘]= 0

Reorderings with Rord

```
With[{c𝕠 = ℭCU[{y1, x1}1, {x2, a2, y2}2, ℏ t1 a2 + ℏ t1^-1 (e^t1 - 1) y1 x2, 12 + ϵ x1 y2]},
 {Short[rhs = c𝕠 // Rord_{x2,a2→3}, 3], HL[CU[c𝕠] == CU[rhs]]}]
```

$\left\{\mathfrak{C}_{CU}\left[\{y_1, x_1\}_1, \{a_3, x_3, y_2\}_2, \frac{e^{-\gamma\hbar t_1}\left(e^{\gamma\hbar t_1}\hbar a_3 t_1^2 - \hbar x_3 y_1 + e^{t_1}\hbar x_3 y_1\right)}{t_1}, 1 + x_1 y_2 \epsilon + O[\epsilon]^3\right], \text{True}\right\}$

```
With[{c𝕠 = ℭCU[{y1, a1, a2}1, {x2, x1, y2}2,
    ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    12 + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 {Short[rhs = c𝕠 // Rord_{a1,a2→3} // Rord_{x2,x1→4}, 3], HL[CU[c𝕠] == CU[rhs]]}]
```

$\{\mathfrak{C}_{CU}[\{y_1, a_3\}_1, \{x_4, y_2\}_2,$
$\hbar a_3 l_{11} t_1 + \hbar a_3 l_{12} t_1 + \hbar a_3 l_{21} t_2 + \hbar a_3 l_{22} t_2 + \hbar x_4 y_1 \gamma_{11} + \hbar x_4 y_2 \gamma_{12} + \hbar x_4 y_1 \gamma_{21} + \hbar x_4 y_2 \gamma_{22},$
$1 + (a_3 l_1 + a_3 l_2 + p_{11} x_4 y_1 + p_{21} x_4 y_1 + p_{12} x_4 y_2 + p_{22} x_4 y_2)\,\epsilon + O[\epsilon]^3], \text{True}\}$

```
ℏ a3 l11 t1 + ℏ a3 l12 t1 + ℏ a3 l21 t2 + ℏ a3 l22 t2 +
  ℏ x4 y1 γ11 + ℏ x4 y2 γ12 + ℏ x4 y1 γ21 + ℏ x4 y2 γ22 // Simplify
```

$\hbar\left(a_3\left(l_{11} t_1 + l_{12} t_1 + (l_{21} + l_{22}) t_2\right) + x_4\left(y_1(\gamma_{11} + \gamma_{21}) + y_2(\gamma_{12} + \gamma_{22})\right)\right)$

```
With[{c𝕠 = ℭCU[{y1, a1, x1}1, {x2, a2, y2}2,
    ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    12 + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 {Short[rhs = c𝕠 // Rord_{x2,a2→3}, 3], HL[CU[c𝕠] == CU[rhs]]}]
```

$\{\mathfrak{C}_{CU}[\{y_1, a_1, x_1\}_1, \ll 1 \gg_2, \ll 1 \gg \ll 1 \gg,$
$1 + e^{-\gamma\hbar l_{12} t_1 - \gamma\hbar l_{22} t_2}\left(e^{\gamma\hbar l_{12} t_1 + \gamma\hbar l_{22} t_2} a_1 l_1 + e^{\gamma\hbar l_{12} t_1 + \gamma\hbar l_{22} t_2} a_3 l_2 + e^{\gamma\hbar l_{12} t_1 + \gamma\hbar l_{22} t_2} p_{11} x_1 y_1 + p_{21} x_3 y_1 +\right.$
$\left.e^{\ll 1 \gg + \ll 1 \gg} p_{12} x_1 y_2 + p_{22} x_3 y_2 - \gamma\hbar l_2 x_3 y_1 \gamma_{21} - \gamma\hbar l_2 x_3 y_2 \gamma_{22}\right)\epsilon + O[\epsilon]^3], \text{True}\}$

```
With[{q𝕠 = ℭQU[{y1, a1, x1}1, {x2, a2, y2}2,
    ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    12 + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 {Short[rhs = q𝕠 // Rord_{x2,a2→3}, 3], HL[QU[q𝕠] == QU[rhs]]}]
```

$\{\mathfrak{C}_{QU}[\{y_1, a_1, x_1\}_1, \ll 1 \gg_2, \ll 1 \gg \ll 1 \gg,$
$1 + e^{-\gamma\hbar l_{12} t_1 - \gamma\hbar l_{22} t_2}\left(e^{\gamma\hbar l_{12} t_1 + \gamma\hbar l_{22} t_2} a_1 l_1 + e^{\gamma\hbar l_{12} t_1 + \gamma\hbar l_{22} t_2} a_3 l_2 + e^{\gamma\hbar l_{12} t_1 + \gamma\hbar l_{22} t_2} p_{11} x_1 y_1 + p_{21} x_3 y_1 +\right.$
$\left.e^{\ll 1 \gg + \ll 1 \gg} p_{12} x_1 y_2 + p_{22} x_3 y_2 - \gamma\hbar l_2 x_3 y_1 \gamma_{21} - \gamma\hbar l_2 x_3 y_2 \gamma_{22}\right)\epsilon + O[\epsilon]^3], \text{True}\}$

```
With[{q𝕠 = ℭQU[{y1, a1, x1}1, {x2, a2, y2}2,
    ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    12 + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 {Short[rhs = q𝕠 // Rord_{a2,y2→3}, 3], HL[QU[q𝕠] == QU[rhs]]}]
```

$\{\ll 1 \gg, \text{True}\}$

$$\mathbf{Timing@With}\Big[\Big\{\mathrm{q}\circ=\mathbb{E}_{QU}\big[\{x_1,y_1\}_1,\ \{x_2,a_2,y_2\}_2,\ \hbar\left(l_{12}\,t_1\,a_2+l_{22}\,t_2\,a_2+\gamma_{11}\,x_1\,y_1+\gamma_{12}\,x_1\,y_2+\gamma_{21}\,x_2\,y_1+\gamma_{22}\,x_2\,y_2\right),\ \theta_2+\epsilon\left(l_2\,a_2+p_{11}\,x_1\,y_1+p_{12}\,x_1\,y_2+p_{21}\,x_2\,y_1+p_{22}\,x_2\,y_2\right)\big]\Big\},\ \{\mathbf{Short}[\mathrm{rhs}=\mathrm{q}\circ\ //\ \mathbf{Rord}_{x_1,y_1\to 3},\ 5]\}\Big]$$

$$\Big\{116.156,\ \Big\{\mathbb{E}_{QU}\Big[\{y_3,a_3,x_3\}_1,\ \ll 1\gg_2,\ \frac{\ll 1\gg}{1-\ll 1\gg+\ll 1\gg},\ \Big(\big(\hbar\,a_2\,l_2+p_{11}-p_{11}\,T_1+\hbar\,p_{22}\,x_2\,y_2+\hbar\,p_{12}\,x_3\,y_2+\ll 46\gg+2\,\hbar\,p_{12}\,T_1\,x_2\,y_2\,\gamma_{11}\,\gamma_{21}-\hbar\,p_{12}\,T_1^2\,x_2\,y_2\,\gamma_{11}\,\gamma_{21}+\hbar\,p_{11}\,x_2\,y_2\,\gamma_{12}\,\gamma_{21}-2\,\hbar\,p_{11}\,T_1\,x_2\,y_2\,\gamma_{12}\,\gamma_{21}+\hbar\,p_{11}\,T_1^2\,x_2\,y_2\,\gamma_{12}\,\gamma_{21}\big)\,\epsilon\Big)\Big/\left(\hbar-3\,\hbar\,\gamma_{11}+3\,\hbar\,T_1\,\gamma_{11}+3\,\hbar\,\gamma_{11}^2-6\,\hbar\,T_1\,\gamma_{11}^2+3\,\hbar\,T_1^2\,\gamma_{11}^2-\hbar\,\gamma_{11}^3+3\,\hbar\,T_1\,\gamma_{11}^3-3\,\hbar\,T_1^2\,\gamma_{11}^3+\hbar\,T_1^3\,\gamma_{11}^3\right)+\Big(\big(8\,a_3\,p_{11}\,T_1+\ll 1\gg+\ll 2726\gg+3\,\gamma\ll 6\gg\gamma_{21}^3\big)\ll 1\gg\Big)\Big/\left(4-28\,\gamma_{11}+\ll 48\gg+4\,T_1^7\,\gamma_{11}^7\right)+O[\epsilon]^3\Big]\Big\}\Big\}$$

$$\mathbf{Timing@With}\Big[\Big\{\mathrm{q}\circ=\mathbb{E}_{QU}\big[\{x_1,y_1\}_1,\ \{x_2,a_2,y_2\}_2,\ \hbar\left(l_{12}\,t_1\,a_2+l_{22}\,t_2\,a_2+\gamma_{11}\,x_1\,y_1+\gamma_{12}\,x_1\,y_2+\gamma_{21}\,x_2\,y_1+\gamma_{22}\,x_2\,y_2\right),\ 1_2+\epsilon\left(l_2\,a_2+p_{11}\,x_1\,y_1+p_{12}\,x_1\,y_2+p_{21}\,x_2\,y_1+p_{22}\,x_2\,y_2\right)\big]\Big\},\ \{\mathbf{Short}[\mathrm{rhs}=\mathrm{q}\circ\ //\ \mathbf{Rord}_{x_1,y_1\to 3},\ 5],\ \mathbf{HL@SimpT}[\mathbf{QU}[\mathrm{q}\circ]==\mathbf{QU}[\mathrm{rhs}]]\}\Big]$$

$$\Big\{388.922,\ \Big\{\mathbb{E}_{QU}\Big[\{y_3,a_3,x_3\}_1,\ \{\ll 1\gg\}_2,\ \frac{\ll 1\gg}{\ll 1\gg},\ \frac{1}{1-\gamma_{11}+T_1\,\gamma_{11}}+\Big(\big(4\,\hbar\,a_2\,l_2+4\,p_{11}-4\,p_{11}\,T_1+4\,\hbar\,p_{22}\,x_2\,y_2+\ll 339\gg+\gamma\,\hbar^4\,x_2^2\,y_2^2\,\gamma_{12}^2\,\gamma_{21}^2-4\,\gamma\,\hbar^4\,T_1\,x_2^2\,y_2^2\,\gamma_{12}^2\,\gamma_{21}^2+3\,\gamma\,\hbar^4\,T_1^2\,x_2^2\,y_2^2\,\gamma_{12}^2\,\gamma_{21}^2\big)\,\epsilon\Big)\Big/\left(4\,\hbar-20\,\hbar\,\gamma_{11}+20\,\hbar\,T_1\,\gamma_{11}+40\,\hbar\,\gamma_{11}^2-80\,\hbar\,T_1\,\gamma_{11}^2+40\,\hbar\,T_1^2\,\gamma_{11}^2-40\,\hbar\,\gamma_{11}^3+\ll 13\gg+20\,\hbar\,T_1\,\gamma_{11}^5-40\,\hbar\,T_1^2\,\gamma_{11}^5+40\,\hbar\,T_1^3\,\gamma_{11}^5-20\,\hbar\,T_1^4\,\gamma_{11}^5+4\,\hbar\,T_1^5\,\gamma_{11}^5\right)+\frac{\left(576\,a_3\,p_{11}\,T_1+\ll 8073\gg+\ll 1\gg\right)\ll 1\gg}{\ll 79\gg+288\,T_1^9\,\gamma_{11}^9}+O[\epsilon]^3\Big],\ \mathrm{True}\Big\}\Big\}$$

$$\mathbf{Timing@With}\Big[\Big\{\mathrm{q}\circ=\mathbb{E}_{QU}\big[\{x_1,y_1\}_1,\ \{x_2,a_2,y_2\}_2,\ \hbar\left(l_{12}\,t_1\,a_2+l_{22}\,t_2\,a_2+\gamma_{11}\,x_1\,y_1+\gamma_{12}\,x_1\,y_2+\gamma_{21}\,x_2\,y_1+\gamma_{22}\,x_2\,y_2\right),\ 1_2+\epsilon\left(l_2\,a_2+p_{11}\,x_1\,y_1+p_{12}\,x_1\,y_2+p_{21}\,x_2\,y_1+p_{22}\,x_2\,y_2\right)\big]\Big\},\ \{\mathbf{Short}[\mathrm{rhs}=\mathrm{q}\circ\ //\ \mathbf{Rord}_{x_1,y_1\to 1},\ 5],\ \mathbf{HL@SimpT}[\mathbf{QU}[\mathrm{q}\circ]==\mathbf{QU}[\mathrm{rhs}]]\}\Big]$$

$$\Big\{336.781,\ \Big\{\mathbb{E}_{QU}\Big[\{y_1,a_1,x_1\}_1,\ \{\ll 1\gg\}_2,\ \frac{\ll 1\gg}{\ll 1\gg},\ \frac{1}{1-\gamma_{11}+T_1\,\gamma_{11}}+\Big(\big(4\,\hbar\,a_2\,l_2+4\,p_{11}-4\,p_{11}\,T_1+4\,\hbar\,p_{11}\,x_1\,y_1+4\,\hbar\,p_{21}\,x_2\,y_1+\ll 338\gg+\gamma\,\hbar^4\,x_2^2\,y_2^2\,\gamma_{12}^2\,\gamma_{21}^2-4\,\gamma\,\hbar^4\,T_1\,x_2^2\,y_2^2\,\gamma_{12}^2\,\gamma_{21}^2+3\,\gamma\,\hbar^4\,T_1^2\,x_2^2\,y_2^2\,\gamma_{12}^2\,\gamma_{21}^2\big)\,\epsilon\Big)\Big/\left(4\,\hbar-20\,\hbar\,\gamma_{11}+20\,\hbar\,T_1\,\gamma_{11}+40\,\hbar\,\gamma_{11}^2-80\,\hbar\,T_1\,\gamma_{11}^2+40\,\hbar\,T_1^2\,\gamma_{11}^2-40\,\hbar\,\gamma_{11}^3+\ll 10\gg+20\,\hbar\,T_1^4\,\gamma_{11}^4-4\,\hbar\,\gamma_{11}^5+20\,\hbar\,T_1\,\gamma_{11}^5-40\,\hbar\,T_1^2\,\gamma_{11}^5+40\,\hbar\,T_1^3\,\gamma_{11}^5-20\,\hbar\,T_1^4\,\gamma_{11}^5+4\,\hbar\,T_1^5\,\gamma_{11}^5\right)+\frac{\left(576\,a_1\,p_{11}\,T_1+\ll 8073\gg+\ll 1\gg\right)\ll 1\gg}{\ll 79\gg+288\,T_1^9\,\gamma_{11}^9}+O[\epsilon]^3\Big],\ \mathrm{True}\Big\}\Big\}$$

Canonical ordering with Cord

$$\mathbf{Cord@}\mathbb{E}_{CU}[\{x_1,y_1\}_1,\ 0,\ \theta_1+x_1\,y_1]$$

$$\mathbb{E}_{CU}\left[\{y_1,a_1,x_1\}_1,\ 0,\ \left(-t_1+x_1\,y_1\right)+2\,a_1\,\epsilon+O[\epsilon]^2\right]$$

```
Block[{$p = 4, $k = 0, c𝕠 = 𝔈CU[{y1, a1, x1, x2, a2, y2}1,
    ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    10 + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 Timing@{Short[Cord[c𝕠], 8], HL@Simp[CU[c𝕠] - CU[Cord[c𝕠]]]}]
```

```
{4.53125,
 {𝔈CU[{y1, a1, x1}1, (e^(γ ℏ l11 t1+2 γ ℏ l12 t1+γ ℏ l21 t2+2 γ ℏ l22 t2) ℏ a1 l11 t1 + e^(γ ℏ l11 t1+2 γ ℏ l12 t1+γ ℏ l21 t2+2 γ ℏ l22 t2)
      ℏ a1 l12 t1 + <<12>> + e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ^2 a1 l22 t1 t2 γ22 + ℏ x1 y1 γ22) /
   (e^(γ ℏ l11 t1+2 γ ℏ l12 t1+γ ℏ l21 t2+2 γ ℏ l22 t2) + e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ t1 γ12 +
     e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ t1 γ22), e^(γ ℏ l12 t1+γ ℏ l22 t2) / (e^(γ ℏ l12 t1+γ ℏ l22 t2) + ℏ t1 γ12 + ℏ t1 γ22) + O[ϵ]^1], 0}}
```

```
Block[{$p = 4, $k = 1, c𝕠 = 𝔈CU[{y1, a1, x1, x2, a2, y2}1,
    ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    11 + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 Timing@{Short[Cord[c𝕠], 8], HL@Simp[CU[c𝕠] - CU[Cord[c𝕠]]]}]
```

```
{81.2656, {𝔈CU[{y1, a1, x1}1, (e^(γ ℏ l11 t1+2 γ ℏ l12 t1+γ ℏ l21 t2+2 γ ℏ l22 t2) ℏ a1 l11 t1 + <<14>> + ℏ x1 y1 γ22) /
    (e^(γ ℏ l11 t1+2 γ ℏ l12 t1+γ ℏ l21 t2+2 γ ℏ l22 t2) + e^(γ ℏ l11 t1+<<2>>+γ ℏ <<1>> t2) ℏ t1 γ12 +
      e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ t1 γ22),
   e^(γ ℏ l12 t1+γ ℏ l22 t2) / (e^(γ ℏ l12 t1+γ ℏ l22 t2) + ℏ t1 γ12 + ℏ t1 γ22) + ((2 e^(2 γ ℏ l11 t1+6 γ ℏ l12 t1+2 γ ℏ l21 t2+6 γ ℏ l22 t2) a1 l1 + <<419>>) ϵ) /
    (2 e^(2 γ ℏ l11 t1+6 γ ℏ l12 t1+2 γ ℏ l21 t2+6 γ ℏ l22 t2) + 10 e^(2 γ ℏ l11 t1+5 γ ℏ l12 t1+2 γ ℏ l21 t2+5 γ ℏ l22 t2) ℏ t1 γ12 +
      <<18>> + 2 e^(2 γ ℏ l11 t1+γ ℏ l12 t1+2 γ ℏ l21 t2+γ ℏ l22 t2) ℏ^5 t1^5 γ22^5) + O[ϵ]^2], 0}}
```

```
With[{q𝕠 = 𝔈QU[{y1, a1, x1, x2, a2, y2}1,
    ℏ (l11 t1 a1 + l12 t1 a2 + l21 t2 a1 + l22 t2 a2 + γ11 x1 y1 + γ12 x1 y2 + γ21 x2 y1 + γ22 x2 y2),
    10 + ϵ (l1 a1 + l2 a2 + p11 x1 y1 + p12 x1 y2 + p21 x2 y1 + p22 x2 y2)]},
 Cord[
  q𝕠]]
```

```
𝔈QU[{y1, a1, x1}1,
 (e^(γ ℏ l11 t1+2 γ ℏ l12 t1+γ ℏ l21 t2+2 γ ℏ l22 t2) ℏ a1 l11 t1 + e^(γ ℏ l11 t1+2 γ ℏ l12 t1+γ ℏ l21 t2+2 γ ℏ l22 t2) ℏ a1 l12 t1 +
    e^(γ ℏ l11 t1+2 γ ℏ l12 t1+γ ℏ l21 t2+2 γ ℏ l22 t2) ℏ a1 l21 t2 + e^(γ ℏ l11 t1+2 γ ℏ l12 t1+γ ℏ l21 t2+2 γ ℏ l22 t2) ℏ a1 l22 t2 +
    e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ x1 y1 γ11 - e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ a1 l11 t1 γ12 -
    e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ a1 l12 t1 γ12 - e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ a1 l21 t2 γ12 -
    e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ a1 l22 t2 γ12 + e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ a1 l11 t1 T1 γ12 +
    e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ a1 l12 t1 T1 γ12 + e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ a1 l21 t2 T1 γ12 +
    e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ a1 l22 t2 T1 γ12 + ℏ x1 y1 γ12 +
    e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ x1 y1 γ21 - e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ a1 l11 t1 γ22 -
    e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ a1 l12 t1 γ22 - e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ a1 l21 t2 γ22 -
    e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ a1 l22 t2 γ22 + e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ a1 l11 t1 T1 γ22 +
    e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ a1 l12 t1 T1 γ22 + e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ a1 l21 t2 T1 γ22 +
    e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) ℏ a1 l22 t2 T1 γ22 + ℏ x1 y1 γ22) /
  (e^(γ ℏ l11 t1+2 γ ℏ l12 t1+γ ℏ l21 t2+2 γ ℏ l22 t2) - e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) γ12 + e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2)
     T1 γ12 - e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) γ22 + e^(γ ℏ l11 t1+γ ℏ l12 t1+γ ℏ l21 t2+γ ℏ l22 t2) T1 γ22),
 e^(γ ℏ l12 t1+γ ℏ l22 t2) / (e^(γ ℏ l12 t1+γ ℏ l22 t2) - γ12 + T1 γ12 - γ22 + T1 γ22) + O[ϵ]^1]
```

Stitching Œ's.

```
c⊕ = Œ_CU[{y1, a1, x1}1, {y2, a2, x2}2,
    {y3, a3, x3}3, ℏ Sum[l_{10 i+j} t_i a_j + γ_{10 i+j} y_i x_j, {i, 3}, {j, 3}], 1_2];
{c⊕ // m_{3→4}, HL@Simp[CU[m_{3→4}[c⊕]] - m_{3→4}[CU[c⊕]]]}
```

$\{Œ_{CU}[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, \{y_4, a_4, x_4\}_4, \hbar (a_1 l_{11} t_1 + a_2 l_{12} t_1 + a_4 l_{13} t_1 + a_1 l_{21} t_2 + a_2 l_{22} t_2 + a_4 l_{23} t_2 + a_1 l_{31} t_4 + a_2 l_{32} t_4 + a_4 l_{33} t_4 + x_1 y_1 \gamma_{11} + x_2 y_1 \gamma_{12} + x_4 y_1 \gamma_{13} + x_1 y_2 \gamma_{21} + x_2 y_2 \gamma_{22} + x_4 y_2 \gamma_{23} + x_1 y_4 \gamma_{31} + x_2 y_4 \gamma_{32} + x_4 y_4 \gamma_{33}), 1 + O[\epsilon]^3], 0\}$

Verifying that *m* commutes with evaluation, in CU:

```
c⊕ = Œ_CU[{y1, a1, x1}1, {y2, a2, x2}2,
    {y3, a3, x3}3, ℏ Sum[l_{10 i+j} t_i a_j + γ_{10 i+j} y_i x_j, {i, 3}, {j, 3}], 1_2];
Timing@{c⊕ // m_{2→3}, HL@Simp[CU[m_{2→3}[c⊕]] - m_{2→3}[CU[c⊕]]]}
```

$\{513.453, \{Œ_{CU}[\{y_1, a_1, x_1\}_1, \{y_3, a_3, x_3\}_3, \frac{\ll 1 \gg}{e^{\ll 1 \gg} \ll 1 \gg \ll 1 \gg}, \frac{1}{1+\hbar t_3 \gamma_{32}} + ((4 e^{2\gamma\hbar l_{12} t_1 + \ll 4 \gg + 2\gamma\hbar l_{33} t_3} \hbar^2 a_3 x_1 y_1 \gamma_{12} \gamma_{31} - 2 \ll 7 \gg \gamma_{31} + \ll 154 \gg) \epsilon) / (2 e^{2\gamma\hbar l_{12} t_1 + 2\gamma\hbar l_{13} t_1 + \ll 3 \gg + 2\gamma\hbar l_{33} t_3} + 10 e^{\ll 1 \gg} \hbar t_3 \gamma_{32} + \ll 2 \gg + \ll 1 \gg + 2 e^{2\gamma\hbar l_{12} t_1 + \ll 4 \gg + \ll 1 \gg} \hbar^5 t_3^5 \gamma_{32}^5) + \frac{(\ll 1 \gg) \epsilon^2}{\ll 1 \gg} + O[\epsilon]^3], 0\}\}$

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Verifying that *m* commutes with evaluation, in QU:

```
q⊕ = Œ_QU[{y1, a1, x1}1, {y2, a2, x2}2,
    {y3, a3, x3}3, ℏ Sum[l_{10 i+j} t_i a_j + γ_{10 i+j} y_i x_j, {i, 3}, {j, 3}], 1_2];
Timing@{q⊕ // m_{2→3}, HL@SimpT[QU[m_{2→3}[q⊕]] - m_{2→3}[QU[q⊕]]]}
```

$\{7831.47, \{Œ_{QU}[\{y_1, a_1, x_1\}_1, \{y_3, a_3, x_3\}_3, \frac{\ll 1 \gg}{\ll 1 \gg}, \frac{1}{1-\gamma_{32}+T_3 \gamma_{32}} + ((8 e^{2\gamma\hbar l_{12} t_1 + \ll 4 \gg + 2\gamma\hbar l_{33} t_3} \hbar^2 a_3 T_3 x_1 y_1 \gamma_{12} \gamma_{31} + 4 \ll 8 \gg \gamma_{31} + \ll 371 \gg) \epsilon) / (4 e^{2\gamma\hbar l_{12} t_1 + 2\gamma\hbar l_{13} t_1 + \ll 3 \gg + 2\gamma\hbar l_{33} t_3} - 20 e^{\ll 1 \gg} \gamma_{32} + \ll 26 \gg + 4 e^{2\gamma\hbar l_{12} t_1 + \ll 4 \gg + \ll 1 \gg} T_3^5 \gamma_{32}^5) + \frac{(\ll 1 \gg) \epsilon^2}{\ll 79 \gg + \ll 1 \gg} + O[\epsilon]^3], 0\}\}$

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Verifying meta-associativity in CU:

```
c⊕ = Œ_CU[{y1, a1, x1}1, {y2, a2, x2}2,
    {y3, a3, x3}3, ℏ Sum[λ_{10 i+j} t_i a_j + γ_{10 i+j} y_i x_j, {i, 3}, {j, 3}], 1_0];
Timing@HL[(lhs = c⊕ // m_{1,2→1} // m_{1,3→1}) ≡ (rhs = c⊕ // m_{2,3→2} // m_{1,2→1})]
```

{41.9219, True}

$$\mathbf{c\Phi = \text{Œ}_{CU}[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, \{y_3, a_3, x_3\}_3, \hbar\, Sum[\lambda_{10\,i+j}\, t_i\, a_j + \gamma_{10\,i+j}\, y_i\, x_j, \{i, 3\}, \{j, 3\}], 1_1];}$$
$$\mathbf{Timing@HL\left[\left(lhs = c\Phi\ //\ m_{1,2\to 1}\ //\ m_{1,3\to 1}\right) \equiv \left(rhs = c\Phi\ //\ m_{2,3\to 2}\ //\ m_{1,2\to 1}\right)\right]}$$

$$\{30119.8, \text{True}\}$$

$$\mathbf{c\Phi = \text{Œ}_{CU}[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, \hbar\, Sum[l_{10\,i+j}\, t_i\, a_j + \gamma_{10\,i+j}\, y_i\, x_j, \{i, 2\}, \{j, 2\}], 1_1];}$$
$$\mathbf{Short\left[Simplify\ /@\ \left(cexample = c\Phi\ //\ m_{1\to 2}\right), 12\right]}$$
$$\mathbf{Short\left[Simplify\ /@\ \left(qexample = \left(q\Phi = c\Phi\ /.\ CU \to QU\right)\ //\ m_{1\to 2}\right), 12\right]}$$

$$\text{Œ}_{CU}\Big[\{y_2, a_2, x_2\}_2, \hbar\, a_2 \left(l_{11} + l_{12} + l_{21} + l_{22}\right) t_2 + \frac{1}{1 + \hbar\, t_2\, \gamma_{21}} e^{-\gamma\hbar\,(l_{11}+l_{12}+l_{21}+l_{22})\, t_2}\, \hbar\, x_2\, y_2 \Big(\gamma_{21} + e^{\gamma\hbar\,(l_{11}+l_{12}+l_{21}+l_{22})\, t_2}\, \gamma_{12} \left(1 + \hbar\, t_2\, \gamma_{21}\right) + e^{\gamma\hbar\,(l_{12}+l_{22})\, t_2}\, \gamma_{22} + \gamma_{11} \left(e^{\gamma\hbar\,(l_{11}+l_{21})\, t_2} - e^{\gamma\hbar\,(l_{11}+l_{12}+l_{21}+l_{22})\, t_2}\, \hbar\, t_2\, \gamma_{22}\right)\Big),$$
$$\frac{1}{1 + \hbar\, t_2\, \gamma_{21}} + \frac{1}{2\left(1 + \hbar\, t_2\, \gamma_{21}\right)^5} e^{-2\gamma\hbar\,(l_{11}+l_{12}+l_{21}+l_{22})\, t_2}\, \hbar \Big(4\, a_2 \left(1 + \hbar\, t_2\, \gamma_{21}\right)^2 \Big(e^{\gamma\hbar\,(l_{11}+l_{12}+l_{21}+l_{22})\, t_2}\, \hbar \left(e^{\gamma\hbar\,(l_{11}+l_{12}+l_{21}+l_{22})\, t_2}\, t_2 + x_2\, y_2\right) \gamma_{21}^2 + e^{2\gamma\hbar\,(l_{11}+l_{12}+l_{21}+l_{22})\, t_2}\, \hbar\, x_2\, y_2\, \gamma_{11}\, \gamma_{22} + \gamma_{21} \Big(e^{2\gamma\hbar\,(l_{11}+l_{12}+l_{21}+l_{22})\, t_2} + \hbar\, x_2\, y_2 \left(e^{\gamma\hbar\,(2 l_{11}+l_{12}+2 l_{21}+l_{22})\, t_2}\, \gamma_{11} + e^{\gamma\hbar\,(l_{11}+2 l_{12}+l_{21}+2 l_{22})\, t_2}\, \gamma_{22}\right)\Big)\Big) - \gamma\hbar \Big(-2\, e^{2\gamma\hbar\,(l_{11}+l_{12}+l_{21}+l_{22})\, t_2}\, t_2\, \gamma_{21}^2 \left(1 + \hbar\, t_2\, \gamma_{21}\right)^2 + 4 \ll 5 \gg \left(\ll 1 \gg\right) + \hbar \ll 4 \gg \Big(3\, \hbar\, t_2\, \gamma_{21}^2 + 2\, e^{\gamma\hbar\,(l_{12}+l_{\ll 2 \gg})\, t_2}\, \gamma_{22} + \gamma_{21} \left(4 + e^{\gamma \ll 3 \gg}\, \hbar\, t_2\, \gamma_{22}\right) + e^{\gamma\hbar\,(l_{11}+l_{21})\, t_2}\, \gamma_{11} \left(2 + \hbar\, t_2 \left(\gamma_{21} - e^{\gamma\hbar\,(l_{12}+l_{22})\, t_2}\, \gamma_{22}\right)\right)\Big)\Big)\Big)\, \epsilon + O[\epsilon]^2\Big]$$

$$\text{Œ}_{QU}\Big[\{y_2, a_2, x_2\}_2, \hbar\, a_2 \left(l_{11} + l_{12} + l_{21} + l_{22}\right) t_2 + \frac{1}{1 + \left(-1 + T_2\right) \gamma_{21}} e^{-\gamma\hbar\,(l_{11}+l_{12}+l_{21}+l_{22})\, t_2}\, \hbar\, x_2\, y_2 \Big(\gamma_{21} + e^{\gamma\hbar\,(l_{11}+l_{12}+l_{21}+l_{22})\, t_2}\, \gamma_{12} \left(1 + \left(-1 + T_2\right) \gamma_{21}\right) + e^{\gamma\hbar\,(l_{12}+l_{22})\, t_2}\, \gamma_{22} + \gamma_{11} \left(e^{\gamma\hbar\,(l_{11}+l_{21})\, t_2} - e^{\gamma\hbar\,(l_{11}+l_{12}+l_{21}+l_{22})\, t_2} \left(-1 + T_2\right) \gamma_{22}\right)\Big),$$
$$\frac{1}{1 + \left(-1 + T_2\right) \gamma_{21}} + \frac{1}{4\left(1 + \left(-1 + T_2\right) \gamma_{21}\right)^5} e^{-2\gamma\hbar\,(l_{11}+l_{12}+l_{21}+l_{22})\, t_2}\, \hbar \Big(8\, a_2\, T_2 \left(1 + \left(-1 + T_2\right) \gamma_{21}\right)^2 \Big(e^{\gamma\hbar\,(l_{11}+l_{12}+l_{21}+l_{22})\, t_2} \left(-e^{\gamma\hbar\,(l_{11}+l_{12}+l_{21}+l_{22})\, t_2} + e^{\gamma\hbar\,(l_{11}+l_{12}+l_{21}+l_{22})\, t_2}\, T_2 + \hbar\, x_2\, y_2\right) \gamma_{21}^2 + e^{2\gamma\hbar\,(l_{11}+l_{12}+l_{21}+l_{22})\, t_2}\, \hbar\, x_2\, y_2\, \gamma_{11}\, \gamma_{22} + \gamma_{21} \Big(e^{2\gamma\hbar\,(l_{11}+l_{12}+l_{21}+l_{22})\, t_2} + \hbar\, x_2\, y_2 \left(e^{\gamma\hbar\,(2 l_{11}+l_{12}+2 l_{21}+l_{22})\, t_2}\, \gamma_{11} + e^{\gamma\hbar\,(l_{11}+2 l_{12}+l_{21}+2 l_{22})\, t_2}\, \gamma_{22}\right)\Big)\Big) + \gamma \Big(2\, e^{2\gamma\hbar\,(l_{11}+l_{12}+l_{21}+l_{22})\, t_2} \left(1 - 4\, T_2 + 3\, T_2^2\right) \gamma_{21}^2 \left(1 + \left(-1 + T_2\right) \gamma_{21}\right)^2 + 4\, e^{\gamma\hbar\,(l_{11}+l_{12}+l_{21}+l_{22})\, t_2}\, \hbar\, x_2\, y_2\, \gamma_{21} \left(1 + \left(-1 + T_2\right) \gamma_{21}\right) \left(\left(\ll 1 \gg\right) - \ll 1 \gg\right)\Big)\Big)\, \epsilon + O[\epsilon]^2\Big]$$

R in QU.

```
Table[Series[𝕖_{q_ℏ,k}[x], {ϵ, 0, 4}], {k, 0, 5}] // Column
```

$$e^x$$

$$e^x - \frac{1}{4}\left(e^x x^2 \gamma \hbar\right)\epsilon + \frac{1}{32} e^x x^4 \gamma^2 \hbar^2 \epsilon^2 - \frac{1}{384}\left(e^x x^2\left(-8+x^4\right)\gamma^3\hbar^3\right)\epsilon^3 + \frac{e^x x^4\left(-32+x^4\right)\gamma^4\hbar^4\epsilon^4}{6144} + O[\epsilon]^5$$

$$e^x - \frac{1}{4}\left(e^x x^2 \gamma \hbar\right)\epsilon + \frac{1}{288} e^x x^3\left(32+9x\right)\gamma^2\hbar^2\epsilon^2 - \frac{\left(e^x x^2\left(-24+32x^3+3x^4\right)\gamma^3\hbar^3\right)\epsilon^3}{1152} + \frac{e^x x^3\left(-4608-864x+1024x^3+576x^4+27x^5\right)\gamma^4\hbar^4\epsilon^4}{165888} + O[\epsilon]^5$$

$$e^x - \frac{1}{4}\left(e^x x^2 \gamma \hbar\right)\epsilon + \frac{1}{288} e^x x^3\left(32+9x\right)\gamma^2\hbar^2\epsilon^2 - \frac{\left(e^x x^2\left(-24+72x^2+32x^3+3x^4\right)\gamma^3\hbar^3\right)\epsilon^3}{1152} + \frac{e^x x^3\left(-4608-864x+3616x^3+576x^4+27x^5\right)\gamma^4\hbar^4\epsilon^4}{165888} + O[\epsilon]^5$$

$$e^x - \frac{1}{4}\left(e^x x^2 \gamma \hbar\right)\epsilon + \frac{1}{288} e^x x^3\left(32+9x\right)\gamma^2\hbar^2\epsilon^2 - \frac{\left(e^x x^2\left(-24+72x^2+32x^3+3x^4\right)\gamma^3\hbar^3\right)\epsilon^3}{1152} + \frac{1}{4147200} e^x x^3\left(-115200-21600x+165888x^2+90400x^3+14400x^4+675x^5\right)\gamma^4\hbar^4\epsilon^4 + O[\epsilon]^5$$

$$e^x - \frac{1}{4}\left(e^x x^2 \gamma \hbar\right)\epsilon + \frac{1}{288} e^x x^3\left(32+9x\right)\gamma^2\hbar^2\epsilon^2 - \frac{\left(e^x x^2\left(-24+72x^2+32x^3+3x^4\right)\gamma^3\hbar^3\right)\epsilon^3}{1152} + \frac{1}{4147200} e^x x^3\left(-115200-21600x+165888x^2+90400x^3+14400x^4+675x^5\right)\gamma^4\hbar^4\epsilon^4 + O[\epsilon]^5$$

```
Table[Together@SeriesCoefficient[𝕖_{q,5}[x], {x, 0, n}], {n, 0, 5}]
```

$$\left\{1, 1, \frac{1}{1+q}, \frac{1}{(1+q)\left(1+q+q^2\right)}, \frac{1}{(1+q)^2\left(1+q^2\right)\left(1+q+q^2\right)}, 1\Big/\left((1+q)^2\left(1+q^2\right)\left(1+q+q^2\right)\left(1+q+q^2+q^3+q^4\right)\right)\right\}$$

```
Table[HL@FunctionExpand[QFactorial[n, q] SeriesCoefficient[𝕖_{q,5}[x], {x, 0, n}]], {n, 0, 5}]
```

{1, 1, 1, 1, 1, 1}

```
QU[R_{3,4}] // Short
```

$$\text{QU}[\,] + \frac{\epsilon\hbar\,\text{QU}[a_3, a_4]}{\gamma} + \hbar\,\text{QU}[y_3, x_4] + \frac{\epsilon \ll 1 \gg \ll 1 \gg}{\gamma} + \frac{1}{2}\ll 1 \gg \ll 1 \gg - \frac{\ll 1 \gg}{\gamma} - \frac{\epsilon\hbar^2 \ll 1 \gg t_3}{\gamma^2} - \frac{\hbar^2\,\text{QU}[y_3, a_4, x_4]\,t_3}{\gamma} + \frac{\hbar^2\,\text{QU}[a_4, a_4]\,t_3^2}{2\gamma^2}$$

Verifying R2 (~2 secs @ $p=4, $k=2):

```
QU[R_{1,2} ** R^{-1}_{1,2}] // Simp // HL // Timing
```

{0.078125, QU[]}

Verifying R3 (~156 secs @ $p=4, $k=2):

```
{Short[lhs = QU[R_{1,2} ** R_{1,3} ** R_{2,3}]], HL@SimpT[lhs - QU[R_{2,3} ** R_{1,3} ** R_{1,2}]]} // Timing
```

$$\left\{0.203125, \left\{\text{QU}[\,] + \frac{\epsilon\hbar\,\text{QU}[a_1, a_2]}{\gamma} + \frac{\epsilon\hbar\,\text{QU}[a_1, a_3]}{\gamma} + \ll 73 \gg + 2\epsilon\hbar^2\,\text{QU}[y_1, a_2, x_3]\,T_2 + \text{QU}[y_1, x_3]\,(\hbar - \hbar\,T_2), 0\right\}\right\}$$

R in $Œ_{QU}$.

{Œ$_{QU,1}$[R$_{1,2}$], Œ$_{QU,2}$[R$_{1,2}$]}

$$\left\{Œ_{QU}\left[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, -\frac{\hbar a_2 t_1}{\gamma} + \hbar x_2 y_1, 1 + \left(\frac{\hbar a_1 a_2}{\gamma} - \frac{1}{4}\gamma \hbar^3 x_2^2 y_1^2\right) \epsilon + O[\epsilon]^2\right],\right.$$
$$Œ_{QU}\left[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, -\frac{\hbar a_2 t_1}{\gamma} + \hbar x_2 y_1, 1 + \left(\frac{\hbar a_1 a_2}{\gamma} - \frac{1}{4}\gamma \hbar^3 x_2^2 y_1^2\right) \epsilon +\right.$$
$$\left.\left.\frac{1}{288\gamma^2}\left(144\hbar^2 a_1^2 a_2^2 - 72\gamma^2 \hbar^4 a_1 a_2 x_2^2 y_1^2 + 32\gamma^4 \hbar^5 x_2^3 y_1^3 + 9\gamma^4 \hbar^6 x_2^4 y_1^4\right)\epsilon^2 + O[\epsilon]^3\right]\right\}$$

The morphism $Œ_{U,k}$.

Œ$_{QU,1}$[R$_{1,2}$ R$_{3,4}$ R$_{5,6}$]

$$Œ_{QU}\left[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, \{y_3, a_3, x_3\}_3, \{y_4, a_4, x_4\}_4,\right.$$
$$\{y_5, a_5, x_5\}_5, \{y_6, a_6, x_6\}_6, -\frac{\hbar a_2 t_1}{\gamma} - \frac{\hbar a_4 t_3}{\gamma} - \frac{\hbar a_6 t_5}{\gamma} + \hbar x_2 y_1 + \hbar x_4 y_3 + \hbar x_6 y_5,$$
$$\left.1 + \left(\frac{\hbar a_1 a_2}{\gamma} + \frac{\hbar a_3 a_4}{\gamma} + \frac{\hbar a_5 a_6}{\gamma} - \frac{1}{4}\gamma\hbar^3 x_2^2 y_1^2 - \frac{1}{4}\gamma\hbar^3 x_4^2 y_3^2 - \frac{1}{4}\gamma\hbar^3 x_6^2 y_5^2\right)\epsilon + O[\epsilon]^2\right]$$

Œ$_{QU,1}$[R$_{1,2}$ R$_{3,4}$ R$_{5,6}$ // m$_{1,3\to1}$ // m$_{2,5\to2}$ // m$_{4,6\to4}$]

$$Œ_{QU}\left[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, \{y_4, a_4, x_4\}_4, \frac{1}{\gamma}\right.$$
$$\left(-\hbar a_2 t_1 - \hbar a_4 t_1 - \hbar a_4 t_2 + \gamma\hbar x_2 y_1 + \gamma\hbar x_4 y_1 + e^{\hbar t_2}\gamma\hbar x_4 y_1 - \gamma\hbar T_2 x_4 y_1 + e^{\hbar t_1}\gamma\hbar x_4 y_2\right),$$
$$1 + \frac{1}{4\gamma}\left(4\hbar a_1 a_2 + 4\hbar a_1 a_4 + 4\hbar a_2 a_4 - 4\gamma\hbar^2 a_4 x_2 y_1 - 8 e^{\hbar t_2}\gamma\hbar^2 a_2 x_4 y_1 +\right.$$
$$8\gamma\hbar^2 a_2 T_2 x_4 y_1 - \gamma^2\hbar^3 x_2^2 y_1^2 + 4 e^{\hbar t_2}\gamma^2\hbar^3 x_2 x_4 y_1^2 - 4\gamma^2\hbar^3 T_2 x_2 x_4 y_1^2 - \gamma^2\hbar^3 x_4^2 y_1^2 -$$
$$e^{2\hbar t_2}\gamma^2\hbar^3 x_4^2 y_1^2 + \gamma^2\hbar^3 T_2^2 x_4^2 y_1^2 - 4 e^{\hbar t_1}\gamma\hbar^2 a_1 x_4 y_2 + 4 e^{\hbar t_1}\gamma^2\hbar^3 x_2 x_4 y_1 y_2 +$$
$$\left.\left.4 e^{\hbar t_1 + \hbar t_2}\gamma^2\hbar^3 x_4^2 y_1 y_2 - 4 e^{\hbar t_1}\gamma^2\hbar^3 T_2 x_4^2 y_1 y_2 - e^{2\hbar t_1}\gamma^2\hbar^3 x_4^2 y_2^2\right)\epsilon + O[\epsilon]^2\right]$$

Œ$_{QU,1}$[R$_{1,2}$ R$_{3,4}$ R$_{5,6}$ // m$_{3,1\to1}$ // m$_{5,2\to2}$ // m$_{6,4\to4}$]

$$Œ_{QU}\left[\{y_1, a_1, x_1\}_1, \{y_2, a_2, x_2\}_2, \{y_4, a_4, x_4\}_4, \frac{1}{\gamma}\right.$$
$$\left(-\hbar a_2 t_1 - \hbar a_4 t_1 - \hbar a_4 t_2 + \gamma\hbar x_2 y_1 + \gamma\hbar x_4 y_1 + e^{\hbar t_1}\gamma\hbar x_4 y_2\right),$$
$$1 + \frac{1}{4\gamma}\left(4\hbar a_1 a_2 + 4\hbar a_1 a_4 + 4\hbar a_2 a_4 - 4\gamma\hbar^2 a_4 x_2 y_1 - \gamma^2\hbar^3 x_2^2 y_1^2 - \gamma^2\hbar^3 x_4^2 y_1^2 -\right.$$
$$\left.\left.4 e^{\hbar t_1}\gamma\hbar^2 a_1 x_4 y_2 + 4 e^{\hbar t_1}\gamma^2\hbar^3 x_2 x_4 y_1 y_2 - e^{2\hbar t_1}\gamma^2\hbar^3 x_4^2 y_2^2\right)\epsilon + O[\epsilon]^2\right]$$

Œ$_{QU,1}$[R$_{1,2}$ R$_{3,4}$ R$_{5,6}$ // m$_{1,3\to1}$ // m$_{2,5\to2}$ // m$_{4,6\to4}$] ≡ Œ$_{QU,1}$[R$_{1,2}$ R$_{3,4}$ R$_{5,6}$ // m$_{3,1\to1}$ // m$_{5,2\to2}$ // m$_{6,4\to4}$]

$$\hbar\left(e^{\hbar t_2} - T_2\right) x_4 y_1 == 0 \,\&\&\, \epsilon\hbar\left(e^{\hbar t_2} - T_2\right) x_4 y_1 \left(8 a_2 + \gamma\hbar\left(-4 x_2 y_1 + x_4\left(\left(e^{\hbar t_2} + T_2\right) y_1 - 4 e^{\hbar t_1} y_2\right)\right)\right) == 0$$

Exponentials as needed.

In[∘]:=
```
Block[{$p = 2, $k = 2}, TableForm[StringSplit[
  "y|a|x|Cθ@y_CU|Cθ@a_CU|Cθ@x_CU|Qθ@y_QU|Qθ@a_QU|Qθ@x_QU|A𝔻@y_QU|A𝔻@a_QU|A𝔻@x_QU|S𝔻@y_QU|S𝔻@a_QU|S𝔻@x_QU|S@y_CU|S@a_CU|S@x_CU|S@y_QU|S@a_QU|S@x_QU|Δ@y_CU|Δ@a_CU|Δ@x_CU|Δ@y_QU|Δ@a_QU|Δ@x_QU",
  "|"] /. s_String ↦
  {s, Normal@Simplify@Series[ToExpression[s] /. CU | QU → Times, {ϵ, 0, $k}]}]]
```

Out[∘]//TableForm=

y	y
a	a
x	x
$C\theta@y_{CU}$	$-x$
$C\theta@a_{CU}$	$-a$
$C\theta@x_{CU}$	$-y$
$Q\theta@y_{QU}$	$-\frac{x}{\sqrt{T}}-\frac{a\,x\,\epsilon\,\hbar}{\sqrt{T}}-\frac{a^2\,x\,\epsilon^2\,\hbar^2}{2\sqrt{T}}$
$Q\theta@a_{QU}$	$-a$
$Q\theta@x_{QU}$	$-\frac{y}{\sqrt{T}}+\frac{y\,(-a+\gamma)\,\epsilon\,\hbar}{\sqrt{T}}-\frac{y\,(a-\gamma)^2\,\epsilon^2\,\hbar^2}{2\sqrt{T}}$
$A\mathbb{D}@y_{QU}$	$\frac{2}{3}a^2\,y\,\epsilon^2\,\hbar^2+\frac{1}{6}y\left(6+3\,t\,\hbar+t^2\,\hbar^2\right)+\frac{1}{12}y\,\epsilon\,\hbar\left(x\,y\,\gamma\,\hbar-4\,a\left(3+2\,t\,\hbar\right)\right)$
$A\mathbb{D}@a_{QU}$	a
$A\mathbb{D}@x_{QU}$	x
$S\mathbb{D}@y_{QU}$	$y+\frac{1}{48}t^2\,y\,\hbar^2+\frac{1}{24}y\left(-2\,a\,t+x\,y\,\gamma\right)\epsilon\,\hbar^2+\frac{1}{12}a^2\,y\,\epsilon^2\,\hbar^2$
$S\mathbb{D}@a_{QU}$	a
$S\mathbb{D}@x_{QU}$	$\frac{7}{12}a^2\,x\,\epsilon^2\,\hbar^2+x\left(1+\frac{t\,\hbar}{2}+\frac{7\,t^2\,\hbar^2}{48}\right)+\frac{1}{24}x\,\epsilon\,\hbar\left(x\,y\,\gamma\,\hbar-2\,a\left(12+7\,t\,\hbar\right)\right)$
$S@y_{CU}$	$-y$
$S@a_{CU}$	$-a$
$S@x_{CU}$	$-x$
$S@y_{QU}$	$-\frac{y}{T}+\frac{y\,(-a+\gamma)\,\epsilon\,\hbar}{T}-\frac{y\,(a-\gamma)^2\,\epsilon^2\,\hbar^2}{2\,T}$
$S@a_{QU}$	$-a$
$S@x_{QU}$	$-x-a\,x\,\epsilon\,\hbar-\frac{1}{2}a^2\,x\,\epsilon^2\,\hbar^2$
$\Delta@y_{CU}$	y_1+y_2
$\Delta@a_{CU}$	a_1+a_2
$\Delta@x_{CU}$	x_1+x_2
$\Delta@y_{QU}$	$y_1+T_1\,y_2-\epsilon\,\hbar\,a_1\,T_1\,y_2+\frac{1}{2}\epsilon^2\,\hbar^2\,a_1^2\,T_1\,y_2$
$\Delta@a_{QU}$	a_1+a_2
$\Delta@x_{QU}$	$x_1+x_2-\epsilon\,\hbar\,a_1\,x_2+\frac{1}{2}\epsilon^2\,\hbar^2\,a_1^2\,x_2$

In[∘]:=
```
Timing@Block[{$p = 4, $k = 2}, {
   s = S₁[QU[y₁]] /. QU → Times,
   exps = Exp_{QU₁,$k}[η, s], (* Warning: wrong unless $p≥$k+1! *)
   HL@Simp[S₁@𝕆_QU[{y₁}₁, SS[e^{ℏ η y₁}]] - 𝕆_QU[{y₁, a₁, x₁}₁, (exps /. η → ℏ η)]]
  }]
```

Out[∘]=
$$\left\{2.90625, \left\{a_1\left(-\frac{\epsilon\hbar}{T_1}+\frac{\gamma\epsilon^2\hbar^2}{T_1}\right)y_1+\left(-\frac{1}{T_1}+\frac{\gamma\epsilon\hbar}{T_1}-\frac{\gamma^2\epsilon^2\hbar^2}{2T_1}\right)y_1-\frac{\epsilon^2\hbar^2a_1^2y_1}{2T_1},\right.\right.$$
$$\mathbb{E}\left[0,-\frac{\eta y_1}{T_1},1+\frac{\left(2\gamma\eta\hbar T_1y_1-2\eta\hbar a_1T_1y_1-\gamma\eta^2\hbar y_1^2\right)\epsilon}{2T_1^2}+\right.$$
$$\left(-\frac{\gamma^2\eta\hbar^2y_1}{2T_1}+\frac{\gamma\eta\hbar^2a_1y_1}{T_1}-\frac{\eta\hbar^2a_1^2y_1}{2T_1}+\frac{7\gamma^2\eta^2\hbar^2y_1^2}{4T_1^2}-\frac{2\gamma\eta^2\hbar^2a_1y_1^2}{T_1^2}+\right.$$
$$\left.\left.\left.\frac{\eta^2\hbar^2a_1^2y_1^2}{2T_1^2}-\frac{\gamma^2\eta^3\hbar^2y_1^3}{T_1^3}+\frac{\gamma\eta^3\hbar^2a_1y_1^3}{2T_1^3}+\frac{\gamma^2\eta^4\hbar^2y_1^4}{8T_1^4}\right)\epsilon^2+O[\epsilon]^3\right],0\right\}\right\}$$

In[∘]:=
```
Timing@Block[{$p = 4, $k = 2}, {
   s = S₁[QU[a₁]] /. QU → Times,
   exps = Exp_{QU₁,$k}[α, s], (* Warning: wrong unless $p≥$k+1! *)
   HL@Simp[S₁@𝕆_QU[{a₁}₁, SS[e^{ℏ α a₁}]] - 𝕆_QU[{y₁, a₁, x₁}₁, exps /. α → ℏ α]]
  }]
```

Out[∘]=
$$\left\{2.17188, \left\{-a_1, \mathbb{E}\left[-\alpha a_1, 0, 1+O[\epsilon]^3\right], 0\right\}\right\}$$

In[∘]:=
```
Timing@Block[{$p = 4, $k = 2}, {
   s = S₁[QU[x₁]] /. QU → Times,
   exps = Exp_{QU₁,$k}[ξ, s], (* Warning: wrong unless $p≥$k+1! *)
   HL@Simp[S₁@𝕆_QU[{x₁}₁, SS[e^{ℏ ξ x₁}]] - 𝕆_QU[{y₁, a₁, x₁}₁, (exps /. ξ → ℏ ξ)]]
  }]
```

Out[∘]=
$$\left\{1.84375, \left\{-x_1-\epsilon\hbar a_1x_1-\frac{1}{2}\epsilon^2\hbar^2a_1^2x_1,\right.\right.$$
$$\mathbb{E}\left[0,-\xi x_1,1+\left(-\xi\hbar a_1x_1-\frac{1}{2}\gamma\xi^2\hbar x_1^2\right)\epsilon+\left(-\frac{1}{2}\xi\hbar^2a_1^2x_1+\frac{1}{4}\gamma^2\xi^2\hbar^2x_1^2-\gamma\xi^2\hbar^2a_1x_1^2+\right.\right.$$
$$\left.\left.\left.\frac{1}{2}\xi^2\hbar^2a_1^2x_1^2-\frac{1}{2}\gamma^2\xi^3\hbar^2x_1^3+\frac{1}{2}\gamma\xi^3\hbar^2a_1x_1^3+\frac{1}{8}\gamma^2\xi^4\hbar^2x_1^4\right)\epsilon^2+O[\epsilon]^3\right],0\right\}\right\}$$

$S(e^{\eta y} e^{\alpha a} e^{\xi x})$

In[ ]:=
```
Timing@Block[{$p = 3, $k = 1}, {
  (sexp = m3,2,1→1[ExpQU1,$k[η, S1[QU[y1]] /. QU → Times] ExpQU2,$k[α, S2[QU[a2]] /. QU → Times]
       ExpQU3,$k[ξ, S3[QU[x3]] /. QU → Times]]) /. u_1 :→ u,
  HL@SimpT[𝕆QU[{y1, a1, x1}1, sexp /. {η → ℏ η, α → ℏ α, ξ → ℏ ξ}] -
     S1@𝕆QU[{y1, a1, x1}1, SS[e^(ℏ (η y1+α a1+ξ x1))]]]
 }]
```

Out[ ]=
$$\Big\{1.92188, \Big\{\mathbb{E}\Big[-a\,\alpha, \frac{e^{\alpha\gamma}\eta\xi - e^{\alpha\gamma}T\eta\xi - e^{\alpha\gamma}y\eta\hbar - e^{\alpha\gamma}Tx\xi\hbar}{T\hbar}, 1 + \frac{1}{4T^2\hbar}\big(-3e^{2\alpha\gamma}\gamma\eta^2\xi^2 + 4e^{2\alpha\gamma}T\gamma\eta^2\xi^2 - e^{2\alpha\gamma}T^2\gamma\eta^2\xi^2 + 8a\,e^{\alpha\gamma}T\eta\xi\hbar - 4e^{\alpha\gamma}T\gamma\eta\xi\hbar + 4e^{\alpha\gamma}T^2\gamma\eta\xi\hbar + 6e^{2\alpha\gamma}y\gamma\eta^2\xi\hbar - 2e^{2\alpha\gamma}Ty\gamma\eta^2\xi\hbar + 6e^{2\alpha\gamma}Tx\gamma\eta\xi^2\hbar - 2e^{2\alpha\gamma}T^2x\gamma\eta\xi^2\hbar - 4a\,e^{\alpha\gamma}Ty\eta\hbar^2 + 4e^{\alpha\gamma}Ty\gamma\eta\hbar^2 - 2e^{2\alpha\gamma}y^2\gamma\eta^2\hbar^2 - 4a\,e^{\alpha\gamma}T^2x\xi\hbar^2 - 4e^{2\alpha\gamma}Txy\gamma\eta\xi\hbar^2 - 2e^{2\alpha\gamma}T^2x^2\gamma\xi^2\hbar^2\big)\epsilon + O[\epsilon]^2\Big], 0\Big\}\Big\}$$

$\Delta_{1\to1,2}(e^{\eta y_1} e^{\alpha a_1} e^{\xi x_1})$

In[ ]:=
```
Timing@Block[{$p = 4, $k = 2}, {
  sexp = m1,3,5→1@m2,4,6→2@Times[ (* Warning: wrong unless $p≥$k+1! *)
       ReplacePart[1 → 0]@ExpQU1,$k[η, Δ1→1,2[QU[y1]] /. QU → Times],
       ReplacePart[2 → 0]@ExpQU3,$k[α, Δ3→3,4[QU[a3]] /. QU → Times],
       ReplacePart[1 → 0]@ExpQU5,$k[ξ, Δ5→5,6[QU[x5]] /. QU → Times]
     ] /. {η → ℏ η, α → ℏ α, ξ → ℏ ξ},
  HL@SimpT[
    𝕆QU[{y1, a1, x1}1, {y2, a2, x2}2, sexp] - Δ1→1,2@𝕆QU[{y1, a1, x1}1, SS[e^(ℏ (η y1+α a1+ξ x1))]]]
 }]
```

Out[ ]=
$$\Big\{14.9063, \Big\{\mathbb{E}\Big[\alpha\hbar a_1 + \alpha\hbar a_2, \xi\hbar x_1 + \xi\hbar x_2 + \eta\hbar y_1 + \eta\hbar T_1 y_2, 1 + \frac{1}{2}\big(-2\xi\hbar^2 a_1 x_2 + \gamma\xi^2\hbar^3 x_1 x_2 - 2\eta\hbar^2 a_1 T_1 y_2 + \gamma\eta^2\hbar^3 T_1 y_1 y_2\big)\epsilon + \frac{1}{24}\big(12\xi\hbar^3 a_1^2 x_2 + 6\gamma^2\xi^2\hbar^4 x_1 x_2 - 12\gamma\xi^2\hbar^4 a_1 x_1 x_2 + 4\gamma^2\xi^3\hbar^5 x_1^2 x_2 + 12\xi^2\hbar^4 a_1^2 x_2^2 + 4\gamma^2\xi^3\hbar^5 x_1 x_2^2 - 12\gamma\xi^3\hbar^5 a_1 x_1 x_2^2 + 3\gamma^2\xi^4\hbar^6 x_1^2 x_2^2 + 12\eta\hbar^3 a_1^2 T_1 y_2 + 24\eta\xi\hbar^4 a_1^2 T_1 x_2 y_2 - 12\gamma\eta\xi^2\hbar^5 a_1 T_1 x_1 x_2 y_2 + 6\gamma^2\eta^2\hbar^4 T_1 y_1 y_2 - 12\gamma\eta^2\hbar^4 a_1 T_1 y_1 y_2 - 12\gamma\eta^2\xi\hbar^5 a_1 T_1 x_2 y_1 y_2 + 6\gamma^2\eta^2\xi^2\hbar^6 T_1 x_1 x_2 y_1 y_2 + 4\gamma^2\eta^3\hbar^5 T_1 y_1^2 y_2 + 12\eta^2\hbar^4 a_1^2 T_1^2 y_2^2 + 4\gamma^2\eta^3\hbar^5 T_1^2 y_1 y_2^2 - 12\gamma\eta^3\hbar^5 a_1 T_1^2 y_1 y_2^2 + 3\gamma^2\eta^4\hbar^6 T_1^2 y_1^2 y_2^2\big)\epsilon^2 + O[\epsilon]^3\Big], 0\Big\}\Big\}$$

Zip and Bind

QZip implements the “Q-level zips” on $\mathbb{E}(L, Q, P) = P e^{L+Q}$. Such zips regard the L variables as scalars.

In[∘]:=
```
Timing@{E0 = 𝔼[0, Sum[a_{10 i+j} x_i ξ_j, {i, 3}, {j, 3}],
    1 + ϵ Sum[f_i[x_1, x_2, x_3] ξ_i, {i, 3}] + ϵ Sum[f_{10 i+j}[x_1, x_2, x_3] ξ_i ξ_j, {i, 3}, {j, 3}]],
  lhs = QZip_{ξ_1,ξ_2}@E0,
  HL[lhs == QZip_{ξ_1}@QZip_{ξ_2}@E0]}
```

Out[∘]=
```
{38.6875, {𝔼[0, … 1 …, 1 + ϵ (ξ_1 … 1 … + … 1 … + … 1 …) +
    ϵ (ξ_1^2 f_11[x_1, x_2, x_3] + … 7 … + ξ_3^2 f_33[x_1, x_2, x_3])], … 1 …, True}}
```
large output | show less | show more | show all | set size limit...

In[∘]:=
```
Timing@{
  Eh = 𝔼[0, h Sum[a_{10 i+j} x_i ξ_j, {i, 3}, {j, 3}],
    1 + ϵ Sum[f_i[x_1, x_2, x_3] ξ_i, {i, 3}] + ϵ Sum[f_{10 i+j}[x_1, x_2, x_3] ξ_i ξ_j, {i, 3}, {j, 3}]],
  lhs = Normal[Eh /. 𝔼[L_, Q_, P_] :> Series[P e^(L+Q), {h, 0, 2}]] // Zip_{ξ_1},
  HL@Simplify[lhs == Normal[QZip_{ξ_1}[Eh] /. 𝔼[L_, Q_, P_] :> Series[P e^(L+Q), {h, 0, 2}]]]}
```

Out[∘]=
```
{18.4375, {𝔼[0, h … 1 …, 1 + ϵ (ξ_1 … 1 … + … 1 … + … 1 …) +
    ϵ (ξ_1^2 f_11[x_1, x_2, x_3] + … 7 … + ξ_3^2 f_33[x_1, x_2, x_3])], … 1 …, True}}
```
large output | show less | show more | show all | set size limit...

LZip implements the “L-level zips” on $\mathbb{E}(L, Q, P) = P e^{L+Q}$. Such zips regard all of $P e^{Q}$ as a single”P”. Here the z’s are t and α and the ζ’s are τ and a.

In[∘]:=
```
Bind_{2}[𝔼[0, ξ (x_1 + x_2), 1], 𝔼[0, ξ_2 (x_2 + x_3), 1]]
```
Out[∘]=
```
𝔼[0, ξ (x_1 + x_2 + x_3), 1]
```

In[∘]:=
```
Bind_{2}[𝔼[0, (ξ_2 + ξ_3) x_2, 1], 𝔼[0, (ξ_1 + ξ_2) x, 1]]
```
Out[∘]=
```
𝔼[0, x (ξ_1 + ξ_2 + ξ_3), 1]
```

In[∘]:=
```
Bind_{1,2}[𝔼[0, (ξ_2 + ξ_3) x_2 + ξ_1 x_1, 1], 𝔼[0, (ξ_1 + ξ_2) x, 1]]
```
Out[∘]=
```
𝔼[0, x (ξ_1 + ξ_2 + ξ_3), 1]
```

An xay → axy → ayx –> yax ≡ xay → xya → yxa → yax test:

In[∘]:=
```
Bind[𝔼[α_1 a_1 + τ_1 t_1, e^(γ α_1) ξ_1 x_1 + η_1 y_1, 1], {1}, 𝔼[τ_1 t_1 + α_1 a_1, ξ_1 x_1 + η_1 y_1 + ξ_1 η_1 t_1, 1]]
```
Out[∘]=
```
𝔼[a_1 α_1 + t_1 τ_1, y_1 η_1 + e^(γ α_1) (x_1 + t_1 η_1) ξ_1, 1]
```

In[∘]:=
```
Column@{Cord[Œ_CU[{x_1, a_1}_1, ξ_1 x_1 + α_1 a_1, 1 + 0_0]],
  Cord[Œ_CU[{x_1, y_1}_1, ξ_1 x_1 + η_1 y_1, 1 + 0_0]],
  Cord[Œ_CU[{a_1, y_1}_1, α_1 a_1 + η_1 y_1, 1 + 0_0]]}
```
Out[∘]=
```
Œ_CU[{a_1, x_1}_1, e^(-γ α_1) (e^(γ α_1) a_1 α_1 + x_1 ξ_1), 1 + O[ϵ]^1]
Œ_CU[{y_1, a_1, x_1}_1, y_1 η_1 + x_1 ξ_1 - t_1 η_1 ξ_1, 1 + O[ϵ]^1]
Œ_CU[{y_1, a_1}_1, e^(-γ α_1) (e^(γ α_1) a_1 α_1 + y_1 η_1), 1 + O[ϵ]^1]
```

```
In[∘]:= {rxa = 𝔼[τ1 t1 + α1 a1, e^(-γ α1) ξ1 x1 + η1 y1, 1];
  rxy = 𝔼[τ1 t1 + α1 a1, ξ1 x1 + η1 y1 - ξ1 η1 t1, 1];
  ray = 𝔼[τ1 t1 + α1 a1, e^(-γ α1) η1 y1 + ξ1 x1, 1];
  lhs = Expand /@ Bind[rxa, {1}, rxy, {1}, ray],
  HL[lhs == Expand /@ Bind[ray, {1}, rxy, {1}, rxa] ]}
```

Out[∘]= $\{\mathbb{E}[a_1 \alpha_1 + t_1 \tau_1,\ e^{-\gamma \alpha_1} y_1 \eta_1 + e^{-\gamma \alpha_1} x_1 \xi_1 - e^{-\gamma \alpha_1} t_1 \eta_1 \xi_1,\ 1],\ \text{True}\}$

```
In[∘]:= Simplify /@ m_{i,j→k}@ℭ_CU[{y_i, a_i, x_i}_i, {y_j, a_j, x_j}_j, η_i y_i + α_i a_i + ξ_i x_i + η_j y_j + α_j a_j + ξ_j x_j, 1 + 0_1]
```

Out[∘]= $ℭ_{CU}[\{y_k, a_k, x_k\}_k,\ a_k (\alpha_i + \alpha_j) + y_k (\eta_i + e^{-\gamma \alpha_i} \eta_j) + e^{-\gamma \alpha_j} x_k \xi_i - t_k \eta_j \xi_i + x_k \xi_j,$
$1 + \frac{1}{2} \eta_j \xi_i (4 a_k - 2 e^{-\gamma \alpha_i} \gamma y_k \eta_j + \gamma (-2 e^{-\gamma \alpha_j} x_k + t_k \eta_j) \xi_i) \epsilon + O[\epsilon]^2]$

Tensorial Representations

Associativity of tm.

```
In[∘]:= Table[Block[{$U = U, $k = kk},
   {lhs = Bind[tm_{1,2→2}, {2}, tm_{2,3→1}];
    {$U, $k} -> HL[lhs ≡ Bind[tm_{2,3→2}, {2}, tm_{1,2→1}]]}
  ], {U, {CU, QU}}, {kk, 0, 1}]
```

Out[∘]= {{{{CU, 0} → True}, {{CU, 1} → True}}, {{{QU, 0} → True}, {{QU, 1} → True}}}

```
In[∘]:= Block[{$U = CU, $k = 2}, Timing@{lhs = Bind[tm_{1,2→2}, {2}, tm_{2,3→1}];
    HL[lhs ≡ Bind[tm_{2,3→2}, {2}, tm_{1,2→1}]]}]
```

Out[∘]= {1.79688, {True}}

tS is an anti-homomorphism for tm.

```
In[∘]:= HL[(tS_1 tS_2) ~ B_{1,2} ~ tm_{1,2→1} ≡ tm_{2,1→1} ~ B_1 ~ tS_1]
```

Out[∘]= True

Testing co-associativity.

```
In[∘]:= HL[tΔ_{1→1,2} ~ B_2 ~ tΔ_{2→2,3} ≡ tΔ_{1→1,3} ~ B_1 ~ tΔ_{1→1,2}]
```

Out[∘]= True

Testing S is an anti-co-homomorphism

```
In[∘]:= HL[tS_1 ~ B_1 ~ tΔ_{1→1,2} ≡ tΔ_{1→2,1} ~ B_{1,2} ~ (tS_1 tS_2)]
```

Out[∘]= True

Testing convolution inverse:

```
In[∘]:= {HL[tΔ_{1→1,2} ~ B_1 ~ tS_1 ~ B_{1,2} ~ tm_{1,2→1} ≡ tη ~ B_{} ~ t𝟙],
  HL[tΔ_{1→1,2} ~ B_2 ~ tS_2 ~ B_{1,2} ~ tm_{1,2→1} ≡ tη ~ B_{} ~ t𝟙]}
```

Out[∘]= {True, True}

Testing R2

In[∘]:= $\mathrm{HL}\left[\left(\overline{\mathrm{tR}}_{1,2}\ \mathrm{tR}_{3,4}\right) \sim \mathrm{B}_{1,2,3,4} \sim \left(\mathrm{tm}_{1,3\to 1}\ \mathrm{tm}_{2,4\to 2}\right) == \mathrm{t}\mathbb{1}\right]$

Out[∘]= True

Testing quasi-triangular axioms

In[∘]:= $\mathrm{HL}\left[\left(\mathrm{t}\Delta_{1\text{->}1,2}\ \mathrm{tR}_{3,4}\right) \sim \mathrm{B}_{1,2,3,4} \sim \left(\mathrm{tm}_{1,3\to 1}\ \mathrm{tm}_{2,4\to 2}\right) \equiv \left(\mathrm{t}\Delta_{1\to 2,1}\ \mathrm{tR}_{3,4}\right) \sim \mathrm{B}_{1,2,3,4} \sim \left(\mathrm{tm}_{3,1\to 1}\ \mathrm{tm}_{4,2\to 2}\right)\right]$

Out[∘]= True

In[∘]:= $\mathrm{HL}\left[\mathrm{tR}_{1,3} \sim \mathrm{B}_1 \sim \mathrm{t}\Delta_{1\to 1,2} \equiv \left(\mathrm{tR}_{1,4}\ \mathrm{tR}_{2,3}\right) \sim \mathrm{B}_{3,4} \sim \mathrm{tm}_{3,4\to 3}\right]$

Out[∘]= True

Testing R3

In[∘]:= $\mathrm{HL}\left[\left(\mathrm{tR}_{2,3}\ \mathrm{tR}_{1,4}\ \mathrm{tR}_{5,6}\right) \sim \mathrm{B}_{\mathrm{Range}@6} \sim \left(\mathrm{tm}_{1,5\to 1}\ \mathrm{tm}_{2,6\to 2}\ \mathrm{tm}_{3,4\to 3}\right) \equiv \left(\mathrm{tR}_{1,2}\ \mathrm{tR}_{5,3}\ \mathrm{tR}_{6,4}\right) \sim \mathrm{B}_{\mathrm{Range}@6} \sim \left(\mathrm{tm}_{1,5\to 1}\ \mathrm{tm}_{2,6\to 2}\ \mathrm{tm}_{3,4\to 3}\right)\right]$

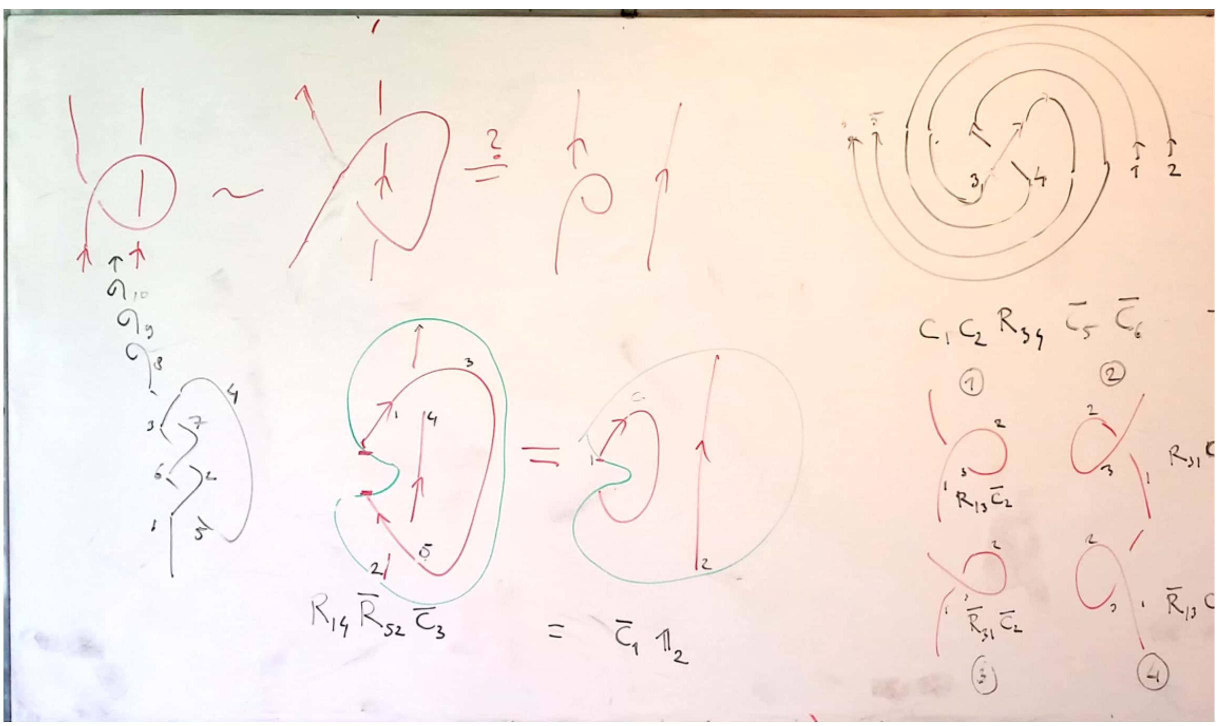
Out[∘]= True

tC is the counterclockwise spinner; $\overline{\mathrm{tC}}$ is its inverse:

In[∘]:= $\mathrm{Block}\left[\{\$k = 1\},\ \mathrm{HL}\left[\left(\mathrm{tC}_1\ \overline{\mathrm{tC}}_2\right) \sim \mathrm{B}_{1,2} \sim \mathrm{tm}_{1,2\to 1} \equiv \mathrm{t}\mathbb{1}\right]\right]$

Out[∘]= True

The 180 419 blackboard :



Cyclic R2 as on 180419 blackboard:

In[∘]:= $\text{Block}[\{\$k = 2\}, \text{HL}[(tR_{1,4}\,\overline{tR}_{5,2}\,\overline{tC}_3) \sim B_{\{1,3,2,4\}} \sim (tm_{1,3\to1}\,tm_{2,4\to2}) \sim B_{1,5} \sim tm_{1,5\to1} \equiv \overline{tC}_1]]\ //\ \text{Timing}$

Out[∘]= {22.0469, True}

Swirl relation as on 180419 blackboard:

In[∘]:= $\text{Block}[\{\$k = 1\}, \text{HL}[tR_{1,2} \equiv (tC_1\,tC_2\,tR_{3,4}\,\overline{tC}_5\,\overline{tC}_6) \sim B_{1,2,3,4} \sim (tm_{1,3\to1}\,tm_{2,4\to2}) \sim B_{1,2,5,6} \sim (tm_{1,5\to1}\,tm_{2,6\to2})]]\ //\ \text{Timing}$

Out[∘]= {0.96875, True}

The Four Kinks

In[∘]:= $\text{Block}[\{\$k = 2, K1, K2, K3, K4\},$
$\text{Column}@\{K1 = (tR_{1,3}\,\overline{tC}_2) \sim B_{1,2} \sim tm_{1,2\to1} \sim B_{1,3} \sim tm_{1,3\to1},\ K2 = (tR_{3,1}\,tC_2) \sim B_{1,2} \sim tm_{1,2\to1} \sim B_{1,3} \sim tm_{1,3\to1},$
$K3 = (\overline{tR}_{3,1}\,\overline{tC}_2) \sim B_{1,2} \sim tm_{1,2\to1} \sim B_{1,3} \sim tm_{1,3\to1},\ K4 = (\overline{tR}_{1,3}\,tC_2) \sim B_{1,2} \sim tm_{1,2\to1} \sim B_{1,3} \sim tm_{1,3\to1},$
$\text{HL}\ /@\ \{$
$K1 \equiv tKink_1,\ K3 \equiv \overline{tKink}_1,$
$K1 \equiv K2,\ K3 \equiv K4,$
$(K1\ (K3 \sim B_{1,2} \sim tm_{1,2\to2})) \sim B_{1,2} \sim tm_{1,2\to1} \equiv t\mathbb{1},$
$K1 \sim B_1 \sim tS_1 \equiv K1,\ K3 \sim B_1 \sim tS_1 \equiv K3\}$
$\}]$

$$\mathbb{E}\Big[-\frac{\hbar a_1 t_1}{\gamma}, \hbar x_1 y_1, \frac{1}{\sqrt{T_1}} + \frac{(4\gamma\hbar a_1 + 4\hbar a_1^2 - \gamma^2\hbar^3 x_1^2 y_1^2)\,\epsilon}{4\gamma\sqrt{T_1}} + \frac{1}{288\gamma^2\sqrt{T_1}}(144\gamma^2\hbar^2 a_1^2 + 288\gamma\hbar^2 a_1^3 + 144\hbar^2 a_1^4 - 72\gamma^3\hbar^4 a_1 x_1^2 y_1^2 - 72\gamma^2\hbar^4 a_1^2 x_1^2 y_1^2 + 32\gamma^4\hbar^5 x_1^3 y_1^3 + 9\gamma^4\hbar^6 x_1^4 y_1^4)\,\epsilon^2 + O[\epsilon]^3\Big]$$

$$\mathbb{E}\Big[-\frac{\hbar a_1 t_1}{\gamma}, \hbar x_1 y_1, \frac{1}{\sqrt{T_1}} + \frac{(4\gamma\hbar a_1 + 4\hbar a_1^2 - \gamma^2\hbar^3 x_1^2 y_1^2)\,\epsilon}{4\gamma\sqrt{T_1}} + \frac{1}{288\gamma^2\sqrt{T_1}}(144\gamma^2\hbar^2 a_1^2 + 288\gamma\hbar^2 a_1^3 + 144\hbar^2 a_1^4 - 72\gamma^3\hbar^4 a_1 x_1^2 y_1^2 - 72\gamma^2\hbar^4 a_1^2 x_1^2 y_1^2 + 32\gamma^4\hbar^5 x_1^3 y_1^3 + 9\gamma^4\hbar^6 x_1^4 y_1^4)\,\epsilon^2 + O[\epsilon]^3\Big]$$

Out[∘]=

$$\mathbb{E}\Big[\frac{\hbar a_1 t_1}{\gamma}, -\frac{\hbar x_1 y_1}{T_1}, \sqrt{T_1} + \frac{(-4\gamma\hbar a_1 T_1^2 - 4\hbar a_1^2 T_1^2 - 8\gamma\hbar^2 a_1 T_1 x_1 y_1 - 3\gamma^2\hbar^3 x_1^2 y_1^2)\,\epsilon}{4\gamma T_1^{3/2}} + \frac{1}{288\gamma^2 T_1^{7/2}}(144\gamma^2\hbar^2 a_1^2 T_1^4 + 288\gamma\hbar^2 a_1^3 T_1^4 + 144\hbar^2 a_1^4 T_1^4 + 576\gamma\hbar^3 a_1^3 T_1^3 x_1 y_1 + 144\gamma^4\hbar^4 T_1^2 x_1^2 y_1^2 - 648\gamma^3\hbar^4 a_1 T_1^2 x_1^2 y_1^2 + 792\gamma^2\hbar^4 a_1^2 T_1^2 x_1^2 y_1^2 - 320\gamma^4\hbar^5 T_1 x_1^3 y_1^3 + 432\gamma^3\hbar^5 a_1 T_1 x_1^3 y_1^3 + 81\gamma^4\hbar^6 x_1^4 y_1^4)\,\epsilon^2 + O[\epsilon]^3\Big]$$

$$\mathbb{E}\Big[\frac{\hbar a_1 t_1}{\gamma}, -\frac{\hbar x_1 y_1}{T_1}, \sqrt{T_1} + \frac{(-4\gamma\hbar a_1 T_1^2 - 4\hbar a_1^2 T_1^2 - 8\gamma\hbar^2 a_1 T_1 x_1 y_1 - 3\gamma^2\hbar^3 x_1^2 y_1^2)\,\epsilon}{4\gamma T_1^{3/2}} + \frac{1}{288\gamma^2 T_1^{7/2}}(144\gamma^2\hbar^2 a_1^2 T_1^4 + 288\gamma\hbar^2 a_1^3 T_1^4 + 144\hbar^2 a_1^4 T_1^4 + 576\gamma\hbar^3 a_1^3 T_1^3 x_1 y_1 + 144\gamma^4\hbar^4 T_1^2 x_1^2 y_1^2 - 648\gamma^3\hbar^4 a_1 T_1^2 x_1^2 y_1^2 + 792\gamma^2\hbar^4 a_1^2 T_1^2 x_1^2 y_1^2 - 320\gamma^4\hbar^5 T_1 x_1^3 y_1^3 + 432\gamma^3\hbar^5 a_1 T_1 x_1^3 y_1^3 + 81\gamma^4\hbar^6 x_1^4 y_1^4)\,\epsilon^2 + O[\epsilon]^3\Big]$$

{True, True, True, True, True, True, True}

Trefoil as on 180419 blackboard:

In[∘]:=
```
Timing[
  Z = tR1,5 tR6,2 tR3,7 tC4 tKink8 tKink9 tKink10;
  Do[Z; Z = Z ~ B1,k ~ tm1,k→1, {k, 2, 10}];
  Z]
```

Out[∘]=

$$\left\{101.953, \mathbb{E}\left[0, 0, \frac{T_1}{1 - T_1 + T_1^2} + \left(\left(-2\hbar a_1 T_1 - \gamma\hbar T_1^2 + 2\hbar a_1 T_1^2 + 2\gamma\hbar T_1^3 - 3\gamma\hbar T_1^4 - 2\hbar a_1 T_1^4 + 2\gamma\hbar T_1^5 + 2\hbar a_1 T_1^5 - 2\gamma\hbar^2 T_1 x_1 y_1 - 2\gamma\hbar^2 T_1^4 x_1 y_1\right)\epsilon\right) \Big/ \left(1 - 3T_1 + 6T_1^2 - 7T_1^3 + 6T_1^4 - 3T_1^5 + T_1^6\right) + O[\epsilon]^2\right]\right\}$$

In[∘]:=

$$\left\{98.53125`, \mathbb{E}\left[0, 0, \frac{T_1}{1 - T_1 + T_1^2} + \left(\left(-2\hbar a_1 T_1 - \gamma\hbar T_1^2 + 2\hbar a_1 T_1^2 + 2\gamma\hbar T_1^3 - 3\gamma\hbar T_1^4 - 2\hbar a_1 T_1^4 + 2\gamma\hbar T_1^5 + 2\hbar a_1 T_1^5 - 2\gamma\hbar^2 T_1 x_1 y_1 - 2\gamma\hbar^2 T_1^4 x_1 y_1\right)\epsilon\right) \Big/ \left(1 - 3T_1 + 6T_1^2 - 7T_1^3 + 6T_1^4 - 3T_1^5 + T_1^6\right) + O[\epsilon]^2\right]\right\}$$

Out[∘]=

$$\left\{98.5313, \mathbb{E}\left[0, 0, \frac{T_1}{1 - T_1 + T_1^2} + \left(\left(-2\hbar a_1 T_1 - \gamma\hbar T_1^2 + 2\hbar a_1 T_1^2 + 2\gamma\hbar T_1^3 - 3\gamma\hbar T_1^4 - 2\hbar a_1 T_1^4 + 2\gamma\hbar T_1^5 + 2\hbar a_1 T_1^5 - 2\gamma\hbar^2 T_1 x_1 y_1 - 2\gamma\hbar^2 T_1^4 x_1 y_1\right)\epsilon\right) \Big/ \left(1 - 3T_1 + 6T_1^2 - 7T_1^3 + 6T_1^4 - 3T_1^5 + T_1^6\right) + O[\epsilon]^2\right]\right\}$$

Alternative Algorithms

In[∘]:=
```
{λalt,2[CU], HL@Simplify@Normal[λalt,2[CU] == Last[ΛCU,2[{ξ, η}, {x, y}]]]}
```

Out[∘]=

$$\left\{1 + \left(2a\eta\xi - y\gamma\eta^2\xi - x\gamma\eta\xi^2 + \frac{1}{2}t\gamma\eta^2\xi^2\right)\epsilon + \frac{1}{2}\left(\left(2a\eta\xi - y\gamma\eta^2\xi - x\gamma\eta\xi^2 + \frac{1}{2}t\gamma\eta^2\xi^2\right)^2 + 2\left(-a\gamma\eta^2\xi^2 + y\gamma^2\eta^3\xi^2 + x\gamma^2\eta^2\xi^3 - \frac{1}{3}t\gamma^2\eta^3\xi^3\right)\right)\epsilon^2 + O[\epsilon]^3, \text{True}\right\}$$