Methodology. If $P_0 := P_{\epsilon=0}$ and inductively $e^{\xi(X\cdot P)} = \mathbb{O}\left(e^{\xi P_0 X} F_k(\xi)\right)$ to ϵ^k, then $F_0 = 1$ and with $F = F_k = F_{k-1} + \epsilon^k \varphi$ we have $F(\xi = 0) = 1$

```
In[]:= SyF[0] = 1;
SyF[k_] := SyF[k] = Module[{fs, bs, F, rhs, at0, atη},
    fs = Flatten@Table[f_{i,j}[η], {i, 0, 2 k}, {j, 0, 2 k - i}];
    F = SyF[k - 1] + ϵ^k fs.(bs = fs /. f_{i_,j_}[η] :> y^i a^j);
    rhs = Normal@Last@Cord[ℭ_QU[{y1, a1, a2, y2}_1, -T^-1 η y1, (F /. {a → a1, y → y1})
          Series[-y2 T^-1 e^(ℏ ϵ a2), {ϵ, 0, k}]] /. η → ℏ η] /. {η → ℏ^-1 η, a1 → a, y1 → y};
    at0 = (# == 0) & /@ Flatten@CoefficientList[F - 1 /. η → 0, {a, y}];
    atη = (# == 0) & /@ Flatten@CoefficientList[(∂_η F) - T^-1 y F - rhs, {a, y}];
    F /. DSolve[And @@ (at0 ⋃ atη), fs, η][[1]]
   ];

In[]:= Exp_{U_,0}[ξ_, (X : (x | y))_i_, P_] := ℭ_U[{y_i, a_i, x_i}_i, Normal@P /. ϵ → 0, 1 + 0_0];

In[]:= {U, X, i, P} = {QU, y, 1, -1/T + (1 (-a + γ) ϵ ℏ)/T - (1 (a - γ)^2 ϵ^2 ℏ^2)/(2 T) /. {y → y1, a → a1, T → T1}}
```

Out[]= $\left\{QU, y, 1, -\frac{1}{T_1} + \frac{\epsilon \hbar (\gamma - a_1)}{T_1} - \frac{\epsilon^2 \hbar^2 (-\gamma + a_1)^2}{2 T_1}\right\}$

```
In[]:= k = 1
```

Out[]= 1

```
In[]:= P0 = Normal@P /. ϵ → 0
```

Out[]= $-\frac{1}{T_1}$

```
In[]:= f = Normal@Last@Exp_{U,k-1}[ξ, X_i, P]
```

Out[]= 1

```
In[]:= φs = Flatten@Table[φ_{j1,j2,j3}[ξ], {j2, 0, 2 k}, {j1, 0, 2 k - j2}, {j3, 0, 2 k - j2 - j1}]
```

Out[]= $\{\varphi_{0,0,0}[\xi], \varphi_{0,0,1}[\xi], \varphi_{0,0,2}[\xi], \varphi_{1,0,0}[\xi], \varphi_{1,0,1}[\xi], \varphi_{2,0,0}[\xi], \varphi_{0,1,0}[\xi], \varphi_{0,1,1}[\xi], \varphi_{1,1,0}[\xi], \varphi_{0,2,0}[\xi]\}$

```
In[]:= F = f + ϵ^k φs.(φs /. φ_{j1_,j2_,j3_}[ξ] :> y_i^j1 a_i^j2 x_i^j3)
```

Out[]= $1 + \epsilon\,(\varphi_{0,0,0}[\xi] + x_1 \varphi_{0,0,1}[\xi] + x_1^2 \varphi_{0,0,2}[\xi] + a_1 \varphi_{0,1,0}[\xi] + a_1 x_1 \varphi_{0,1,1}[\xi] + a_1^2 \varphi_{0,2,0}[\xi] + y_1 \varphi_{1,0,0}[\xi] + x_1 y_1 \varphi_{1,0,1}[\xi] + a_1 y_1 \varphi_{1,1,0}[\xi] + y_1^2 \varphi_{2,0,0}[\xi])$

```
In[]:= m_{i→c}@ℭ_U[{y_i, a_i, x_i}_i, 0, P + 0_k]
```

Out[]= $\mathbb{C}_{QU}\left[\{y_c, a_c, x_c\}_c, 0, -\frac{1}{T_c} + \frac{(\gamma \hbar - \hbar a_c)\epsilon}{T_c} + O[\epsilon]^2\right]$

```
In[]:= ℭ_U[{X_b}_b, 0, X_b + 0_k]
```

Out[]= $\mathbb{C}_{QU}\left[\{y_b\}_b, 0, y_b + O[\epsilon]^2\right]$

In[∘]:= $\text{Œ}_U[\{y_i, a_i, x_i\}_i,\ \xi X_i P0,\ F + \theta_k]$

Out[∘]= $\text{Œ}_{QU}\Big[\{y_1, a_1, x_1\}_1,\ -\frac{\xi y_1}{T_1},$
$$1 + \big(\varphi_{0,0,0}[\xi] + x_1\varphi_{0,0,1}[\xi] + x_1^2\varphi_{0,0,2}[\xi] + a_1\varphi_{0,1,0}[\xi] + a_1x_1\varphi_{0,1,1}[\xi] + a_1^2\varphi_{0,2,0}[\xi] + y_1\varphi_{1,0,0}[\xi] + x_1y_1\varphi_{1,0,1}[\xi] + a_1y_1\varphi_{1,1,0}[\xi] + y_1^2\varphi_{2,0,0}[\xi]\big)\,\epsilon + O[\epsilon]^2\Big]$$

In[∘]:= $\text{Œ}_U[\{y_i, a_i, x_i\}_i,\ \xi X_i P0,\ F + \theta_k]\ \text{Œ}_U[\{X_b\}_b,\ 0,\ X_b + \theta_k]\ m_{i\to c}@\text{Œ}_U[\{y_i, a_i, x_i\}_i,\ 0,\ P + \theta_k]$

Out[∘]= $\text{Œ}_{QU}\Big[\{y_c, a_c, x_c\}_c,\ \{y_b\}_b,\ \{y_1, a_1, x_1\}_1,\ -\frac{\xi y_1}{T_1},$
$$-\frac{y_b}{T_c} + \Bigg(\frac{(\gamma\hbar - \hbar a_c)\,y_b}{T_c} - \frac{1}{T_c}y_b\,\big(\varphi_{0,0,0}[\xi] + x_1\varphi_{0,0,1}[\xi] + x_1^2\varphi_{0,0,2}[\xi] + a_1\varphi_{0,1,0}[\xi] + a_1x_1\varphi_{0,1,1}[\xi] + a_1^2\varphi_{0,2,0}[\xi] + y_1\varphi_{1,0,0}[\xi] + x_1y_1\varphi_{1,0,1}[\xi] + a_1y_1\varphi_{1,1,0}[\xi] + y_1^2\varphi_{2,0,0}[\xi]\big)\Bigg)\,\epsilon + O[\epsilon]^2\Big]$$

In[∘]:= $m_{i,b\to i}[\text{Œ}_U[\{y_i, a_i, x_i\}_i,\ \xi X_i P0,\ F + \theta_k]\ \text{Œ}_U[\{X_b\}_b,\ 0,\ X_b + \theta_k]]$

Out[∘]= $\text{Œ}_{QU}\Big[\{y_1, a_1, x_1\}_1,\ -\frac{\xi y_1}{T_1},$
$$y_1 + \frac{1}{\hbar}\big(\hbar y_1\varphi_{0,0,0}[\xi] + \varphi_{0,0,1}[\xi] - T_1\varphi_{0,0,1}[\xi] + \hbar x_1y_1\varphi_{0,0,1}[\xi] + 2x_1\varphi_{0,0,2}[\xi] - 2T_1x_1\varphi_{0,0,2}[\xi] + \hbar x_1^2y_1\varphi_{0,0,2}[\xi] - \gamma\hbar y_1\varphi_{0,1,0}[\xi] + \hbar a_1y_1\varphi_{0,1,0}[\xi] + a_1\varphi_{0,1,1}[\xi] - a_1T_1\varphi_{0,1,1}[\xi] - \gamma\hbar x_1y_1\varphi_{0,1,1}[\xi] + \hbar a_1x_1y_1\varphi_{0,1,1}[\xi] + \gamma^2\hbar y_1\varphi_{0,2,0}[\xi] - 2\gamma\hbar a_1y_1\varphi_{0,2,0}[\xi] + \hbar a_1^2y_1\varphi_{0,2,0}[\xi] + \hbar y_1^2\varphi_{1,0,0}[\xi] + y_1\varphi_{1,0,1}[\xi] - T_1y_1\varphi_{1,0,1}[\xi] + \hbar x_1y_1^2\varphi_{1,0,1}[\xi] - \gamma\hbar y_1^2\varphi_{1,1,0}[\xi] + \hbar a_1y_1^2\varphi_{1,1,0}[\xi] + \hbar y_1^3\varphi_{2,0,0}[\xi]\big)\,\epsilon + O[\epsilon]^2\Big]$$

In[∘]:= $m_{i,b,c\to i}[\text{Œ}_U[\{y_i, a_i, x_i\}_i,\ \xi X_i P0,\ F + \theta_k]\ \text{Œ}_U[\{X_b\}_b,\ 0,\ X_b + \theta_k]\ m_{i\to c}@\text{Œ}_U[\{y_i, a_i, x_i\}_i,\ 0,\ P + \theta_k]]$

Out[∘]= $\text{Œ}_{QU}\Big[\{y_1, a_1, x_1\}_1,\ -\frac{\xi y_1}{T_1},$
$$-\frac{y_1}{T_1} + \frac{1}{\hbar T_1}\big(\gamma\hbar^2y_1 - \hbar^2a_1y_1 - \hbar y_1\varphi_{0,0,0}[\xi] - \varphi_{0,0,1}[\xi] + T_1\varphi_{0,0,1}[\xi] - \hbar x_1y_1\varphi_{0,0,1}[\xi] - 2x_1\varphi_{0,0,2}[\xi] + 2T_1x_1\varphi_{0,0,2}[\xi] - \hbar x_1^2y_1\varphi_{0,0,2}[\xi] + \gamma\hbar y_1\varphi_{0,1,0}[\xi] - \hbar a_1y_1\varphi_{0,1,0}[\xi] - a_1\varphi_{0,1,1}[\xi] + a_1T_1\varphi_{0,1,1}[\xi] + \gamma\hbar x_1y_1\varphi_{0,1,1}[\xi] - \hbar a_1x_1y_1\varphi_{0,1,1}[\xi] - \gamma^2\hbar y_1\varphi_{0,2,0}[\xi] + 2\gamma\hbar a_1y_1\varphi_{0,2,0}[\xi] - \hbar a_1^2y_1\varphi_{0,2,0}[\xi] - \hbar y_1^2\varphi_{1,0,0}[\xi] - y_1\varphi_{1,0,1}[\xi] + T_1y_1\varphi_{1,0,1}[\xi] - \hbar x_1y_1^2\varphi_{1,0,1}[\xi] + \gamma\hbar y_1^2\varphi_{1,1,0}[\xi] - \hbar a_1y_1^2\varphi_{1,1,0}[\xi] - \hbar y_1^3\varphi_{2,0,0}[\xi]\big)\,\epsilon + O[\epsilon]^2\Big]$$

In[∘]:= Last@
$m_{i,b,c\to i}[\text{Œ}_U[\{y_i, a_i, x_i\}_i,\ \xi X_i P0,\ F + \theta_k]\ \text{Œ}_U[\{X_b\}_b,\ 0,\ X_b + \theta_k]\ m_{i\to c}@\text{Œ}_U[\{y_i, a_i, x_i\}_i,\ 0,\ P + \theta_k]]$

Out[∘]=
$$-\frac{y_1}{T_1} + \frac{1}{\hbar T_1}\big(\gamma\hbar^2y_1 - \hbar^2a_1y_1 - \hbar y_1\varphi_{0,0,0}[\xi] - \varphi_{0,0,1}[\xi] + T_1\varphi_{0,0,1}[\xi] - \hbar x_1y_1\varphi_{0,0,1}[\xi] - 2x_1\varphi_{0,0,2}[\xi] + 2T_1x_1\varphi_{0,0,2}[\xi] - \hbar x_1^2y_1\varphi_{0,0,2}[\xi] + \gamma\hbar y_1\varphi_{0,1,0}[\xi] - \hbar a_1y_1\varphi_{0,1,0}[\xi] - a_1\varphi_{0,1,1}[\xi] + a_1T_1\varphi_{0,1,1}[\xi] + \gamma\hbar x_1y_1\varphi_{0,1,1}[\xi] - \hbar a_1x_1y_1\varphi_{0,1,1}[\xi] - \gamma^2\hbar y_1\varphi_{0,2,0}[\xi] + 2\gamma\hbar a_1y_1\varphi_{0,2,0}[\xi] - \hbar a_1^2y_1\varphi_{0,2,0}[\xi] - \hbar y_1^2\varphi_{1,0,0}[\xi] - y_1\varphi_{1,0,1}[\xi] + T_1y_1\varphi_{1,0,1}[\xi] - \hbar x_1y_1^2\varphi_{1,0,1}[\xi] + \gamma\hbar y_1^2\varphi_{1,1,0}[\xi] - \hbar a_1y_1^2\varphi_{1,1,0}[\xi] - \hbar y_1^3\varphi_{2,0,0}[\xi]\big)\,\epsilon + O[\epsilon]^2$$

In[∘]:=
```
rhs = Normal@Last@m_{i,b,c→i}[
    Œ_U[{y_i, a_i, x_i}_i, ξ X_i P0, F + 0_k] Œ_U[{X_b}_b, 0, X_b + 0_k] m_{i→c}@Œ_U[{y_i, a_i, x_i}_i, 0, P + 0_k]]
```

Out[∘]=
$$-\frac{y_1}{T_1}+\frac{1}{\hbar T_1}\in\left(\gamma\hbar^2 y_1-\hbar^2 a_1 y_1-\hbar y_1\varphi_{0,0,0}[\xi]-\varphi_{0,0,1}[\xi]+T_1\varphi_{0,0,1}[\xi]-\hbar x_1 y_1\varphi_{0,0,1}[\xi]-2x_1\varphi_{0,0,2}[\xi]+2T_1x_1\varphi_{0,0,2}[\xi]-\hbar x_1^2y_1\varphi_{0,0,2}[\xi]+\gamma\hbar y_1\varphi_{0,1,0}[\xi]-\hbar a_1y_1\varphi_{0,1,0}[\xi]-a_1\varphi_{0,1,1}[\xi]+a_1T_1\varphi_{0,1,1}[\xi]+\gamma\hbar x_1y_1\varphi_{0,1,1}[\xi]-\hbar a_1x_1y_1\varphi_{0,1,1}[\xi]-\gamma^2\hbar y_1\varphi_{0,2,0}[\xi]+2\gamma\hbar a_1y_1\varphi_{0,2,0}[\xi]-\hbar a_1^2y_1\varphi_{0,2,0}[\xi]-\hbar y_1^2\varphi_{1,0,0}[\xi]-y_1\varphi_{1,0,1}[\xi]+T_1y_1\varphi_{1,0,1}[\xi]-\hbar x_1y_1^2\varphi_{1,0,1}[\xi]+\gamma\hbar y_1^2\varphi_{1,1,0}[\xi]-\hbar a_1y_1^2\varphi_{1,1,0}[\xi]-\hbar y_1^3\varphi_{2,0,0}[\xi]\right)$$

In[∘]:=
```
at0 = (# == 0) & /@ Flatten@CoefficientList[F - 1 /. ξ → 0, {y_i, a_i, x_i}]
```

Out[∘]=
$\{\in\varphi_{0,0,0}[0]==0,\ \in\varphi_{0,0,1}[0]==0,\ \in\varphi_{0,0,2}[0]==0,\ \in\varphi_{0,1,0}[0]==0,\ \in\varphi_{0,1,1}[0]==0,\ \text{True},\ \in\varphi_{0,2,0}[0]==0,\ \text{True},\ \text{True},\ \in\varphi_{1,0,0}[0]==0,\ \in\varphi_{1,0,1}[0]==0,\ \text{True},\ \in\varphi_{1,1,0}[0]==0,\ \text{True},\ \text{True},\ \text{True},\ \text{True},\ \text{True},\ \in\varphi_{2,0,0}[0]==0,\ \text{True},\ \text{True},\ \text{True},\ \text{True},\ \text{True},\ \text{True},\ \text{True},\ \text{True}\}$

In[∘]:=
```
∂_ξ F
```

Out[∘]=
$$\in\left(\varphi_{0,0,0}'[\xi]+x_1\varphi_{0,0,1}'[\xi]+x_1^2\varphi_{0,0,2}'[\xi]+a_1\varphi_{0,1,0}'[\xi]+a_1x_1\varphi_{0,1,1}'[\xi]+a_1^2\varphi_{0,2,0}'[\xi]+y_1\varphi_{1,0,0}'[\xi]+x_1y_1\varphi_{1,0,1}'[\xi]+a_1y_1\varphi_{1,1,0}'[\xi]+y_1^2\varphi_{2,0,0}'[\xi]\right)$$

In[∘]:=
```
P0 X_i F
```

Out[∘]=
$$-\frac{1}{T_1}y_1\left(1+\in\left(\varphi_{0,0,0}[\xi]+x_1\varphi_{0,0,1}[\xi]+x_1^2\varphi_{0,0,2}[\xi]+a_1\varphi_{0,1,0}[\xi]+a_1x_1\varphi_{0,1,1}[\xi]+a_1^2\varphi_{0,2,0}[\xi]+y_1\varphi_{1,0,0}[\xi]+x_1y_1\varphi_{1,0,1}[\xi]+a_1y_1\varphi_{1,1,0}[\xi]+y_1^2\varphi_{2,0,0}[\xi]\right)\right)$$

In[∘]:=
```
CF[P0 X_i F - rhs]
```

Out[∘]=
$$\frac{1}{\hbar T_1}\left(-\gamma\in\hbar^2y_1+\in\hbar^2a_1y_1+\in\varphi_{0,0,1}[\xi]-\in T_1\varphi_{0,0,1}[\xi]+2\in x_1\varphi_{0,0,2}[\xi]-2\in T_1x_1\varphi_{0,0,2}[\xi]-\gamma\in\hbar y_1\varphi_{0,1,0}[\xi]+\in a_1\varphi_{0,1,1}[\xi]-\in a_1T_1\varphi_{0,1,1}[\xi]-\gamma\in\hbar x_1y_1\varphi_{0,1,1}[\xi]+\gamma^2\in\hbar y_1\varphi_{0,2,0}[\xi]-2\gamma\in\hbar a_1y_1\varphi_{0,2,0}[\xi]+\in y_1\varphi_{1,0,1}[\xi]-\in T_1y_1\varphi_{1,0,1}[\xi]-\gamma\in\hbar y_1^2\varphi_{1,1,0}[\xi]\right)$$

In[∘]:=
```
Collect[(∂_ξ F) + P0 X_i F - rhs, {y_i, a_i, x_i}]
```

Out[∘]=
$$-\frac{\in\varphi_{0,0,1}[\xi]}{\hbar}+\frac{\in\varphi_{0,0,1}[\xi]}{\hbar T_1}+\in\varphi_{0,0,0}'[\xi]+x_1\left(-\frac{2\in\varphi_{0,0,2}[\xi]}{\hbar}+\frac{2\in\varphi_{0,0,2}[\xi]}{\hbar T_1}+\in\varphi_{0,0,1}'[\xi]\right)+\in x_1^2\varphi_{0,0,2}'[\xi]+a_1\left(-\frac{\in\varphi_{0,1,1}[\xi]}{\hbar}+\frac{\in\varphi_{0,1,1}[\xi]}{\hbar T_1}+\in\varphi_{0,1,0}'[\xi]+\in x_1\varphi_{0,1,1}'[\xi]\right)+\in a_1^2\varphi_{0,2,0}'[\xi]+y_1\left(-\frac{\gamma\in\hbar}{T_1}-\frac{\gamma\in\varphi_{0,1,0}[\xi]}{T_1}+\frac{\gamma^2\in\varphi_{0,2,0}[\xi]}{T_1}-\frac{\in\varphi_{1,0,1}[\xi]}{\hbar}+\frac{\in\varphi_{1,0,1}[\xi]}{\hbar T_1}+\in\varphi_{1,0,0}'[\xi]+x_1\left(-\frac{\gamma\in\varphi_{0,1,1}[\xi]}{T_1}+\in\varphi_{1,0,1}'[\xi]\right)+a_1\left(\frac{\in\hbar}{T_1}-\frac{2\gamma\in\varphi_{0,2,0}[\xi]}{T_1}+\in\varphi_{1,1,0}'[\xi]\right)\right)+y_1^2\left(-\frac{\gamma\in\varphi_{1,1,0}[\xi]}{T_1}+\in\varphi_{2,0,0}'[\xi]\right)$$

In[∘]:= **atξ = (# == 0) & /@ Flatten@CoefficientList[(∂_ξF) + P0 X_i F - rhs, {y_i, a_i, x_i}]**

Out[∘]= $\left\{-\frac{\epsilon\,\varphi_{0,0,1}[\xi]}{\hbar}+\frac{\epsilon\,\varphi_{0,0,1}[\xi]}{\hbar\,T_1}+\epsilon\,\varphi_{0,0,0}'[\xi]==0,\right.$
$-\frac{2\,\epsilon\,\varphi_{0,0,2}[\xi]}{\hbar}+\frac{2\,\epsilon\,\varphi_{0,0,2}[\xi]}{\hbar\,T_1}+\epsilon\,\varphi_{0,0,1}'[\xi]==0,\ \epsilon\,\varphi_{0,0,2}'[\xi]==0,$
$-\frac{\epsilon\,\varphi_{0,1,1}[\xi]}{\hbar}+\frac{\epsilon\,\varphi_{0,1,1}[\xi]}{\hbar\,T_1}+\epsilon\,\varphi_{0,1,0}'[\xi]==0,\ \epsilon\,\varphi_{0,1,1}'[\xi]==0,\ \text{True},\ \epsilon\,\varphi_{0,2,0}'[\xi]==0,\ \text{True},$
$\text{True},\ -\frac{\gamma\,\epsilon\,\hbar}{T_1}-\frac{\gamma\,\epsilon\,\varphi_{0,1,0}[\xi]}{T_1}+\frac{\gamma^2\,\epsilon\,\varphi_{0,2,0}[\xi]}{T_1}-\frac{\epsilon\,\varphi_{1,0,1}[\xi]}{\hbar}+\frac{\epsilon\,\varphi_{1,0,1}[\xi]}{\hbar\,T_1}+\epsilon\,\varphi_{1,0,0}'[\xi]==0,$
$-\frac{\gamma\,\epsilon\,\varphi_{0,1,1}[\xi]}{T_1}+\epsilon\,\varphi_{1,0,1}'[\xi]==0,\ \text{True},\ \frac{\epsilon\,\hbar}{T_1}-\frac{2\,\gamma\,\epsilon\,\varphi_{0,2,0}[\xi]}{T_1}+\epsilon\,\varphi_{1,1,0}'[\xi]==0,\ \text{True},\ \text{True},\ \text{True},$
$\left.\text{True},\ \text{True},\ -\frac{\gamma\,\epsilon\,\varphi_{1,1,0}[\xi]}{T_1}+\epsilon\,\varphi_{2,0,0}'[\xi]==0,\ \text{True},\ \text{True},\ \text{True},\ \text{True},\ \text{True},\ \text{True},\ \text{True},\ \text{True}\right\}$

In[∘]:= **DSolve[And @@ (at0 ⋃ atξ), φs, ξ][[1]]**

Out[∘]= $\{\varphi_{0,0,0}[\xi]\to 0,\ \varphi_{0,0,1}[\xi]\to 0,\ \varphi_{0,0,2}[\xi]\to 0,\ \varphi_{0,1,0}[\xi]\to 0,\ \varphi_{0,1,1}[\xi]\to 0,$
$\varphi_{0,2,0}[\xi]\to 0,\ \varphi_{1,0,0}[\xi]\to\frac{\gamma\,\xi\,\hbar}{T_1},\ \varphi_{1,0,1}[\xi]\to 0,\ \varphi_{1,1,0}[\xi]\to-\frac{\xi\,\hbar}{T_1},\ \varphi_{2,0,0}[\xi]\to-\frac{\gamma\,\xi^2\,\hbar}{2\,T_1^2}\}$