Pensieve header: The full sl_2 invariant using the Drinfel'd double. Continues 2018-05/ybax.nb, Talks/StonyBrook-1805/ybax.nb, Projects/SL2Portfolio/Logoi.nb.

External Utilities

In[∘]:=
```
HL[ℰ_] := Style[ℰ, Background → Yellow];
```

Program

Program

Internal Utilities

Program

Canonical Form:

Program

In[∘]:=
```
CF[sd_SeriesData] := MapAt[CF, sd, 3];
CF[ℰ_] := ExpandDenominator@
   ExpandNumerator@Together[Expand[ℰ] //. e^x_ e^y_ :→ e^(x+y) /. e^x_ :→ e^CF[x]];
```

Program

The Kronecker δ:

Program

In[∘]:=
```
Kδ /: Kδ_{i_,j_} := If[i === j, 1, 0];
```

Program

Equality, multiplication, and degree-adjustment of perturbed Gaussians; $\mathbb{E}[L, Q, P]$ stands for $e^{L+Q} P$:

Program

In[∘]:=
```
𝔼 /: 𝔼[L1_, Q1_, P1_] ≡ 𝔼[L2_, Q2_, P2_] :=
  CF[L1 == L2] ∧ CF[Q1 == Q2] ∧ CF[Normal[P1 - P2] == 0];
𝔼 /: 𝔼[L1_, Q1_, P1_] 𝔼[L2_, Q2_, P2_] := 𝔼[L1 + L2, Q1 + Q2, P1 * P2];
𝔼[L_, Q_, P_]_{$k_} := 𝔼[L, Q, Series[Normal@P, {ϵ, 0, $k}]];
```

Program

Zip and Bind

Program

Variables and their duals:

Program

In[∘]:=
```
{t*, b*, y*, a*, x*, z*} = {τ, β, η, α, ξ, ζ};
{τ*, β*, η*, α*, ξ*, ζ*} = {t, b, y, a, x, z}; (u_{i_})* := (u*)_i;
```

Program

Finite Zips: (* Perhaps switch Expand to Collect[__, ζ]? *)

Program

```
In[•]:= expand[sd_SeriesData] := MapAt[expand, sd, 3];
expand[ℰ_] := Expand[ℰ];
Zip{}[P_] := P;
Zip{ζ_,ζs___}[P_] := (expand[P // Zip{ζs}] /. f_. ζ^d_. :→ ∂{ζ*,d} f) /. ζ* → 0
```

Program

QZip implements the "Q-level zips" on $\mathbb{E}(L, Q, P) = P e^{L+Q}$. Such zips regard the L variables as scalars.

Program

```
In[•]:= QZip_{ζs_List,simp_}@𝔼[L_, Q_, P_] := Module[{ζ, z, zs, c, ys, ηs, qt, zrule, Q1, Q2},
    zs = Table[ζ*, {ζ, ζs}];
    c = Q /. Alternatives @@ (ζs ⋃ zs) → 0;
    ys = Table[∂_ζ (Q /. Alternatives @@ zs → 0), {ζ, ζs}];
    ηs = Table[∂_z (Q /. Alternatives @@ ζs → 0), {z, zs}];
    qt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ} Q, {ζ, ζs}, {z, zs}];
    zrule = Thread[zs → qt.(zs + ys)];
    Q2 = (Q1 = c + ηs.zs /. zrule) /. Alternatives @@ zs → 0;
    simp /@ 𝔼[L, Q2, Det[qt] e^(-Q2) Zip_ζs[e^Q1 (P /. zrule)]]];
QZip_{ζs_List} := QZip_{ζs,CF};
```

Program

Upper to lower and lower to Upper:

Program

```
In[•]:= U2l = {B_i_^p_. → e^(-p ħ γ b_i), B^p_. → e^(-p ħ γ b), T_i_^p_. → e^(p ħ t_i), T^p_. → e^(p ħ t), 𝒜_i_^p_. → e^(p γ α_i), 𝒜^p_. → e^(p γ α)};
l2U = {e^(c_. b_i_ + d_.) :→ B_i^(-c/(ħ γ)) e^d, e^(c_. b + d_.) :→ B^(-c/(ħ γ)) e^d,
    e^(c_. t_i_ + d_.) :→ T_i^(c/ħ) e^d, e^(c_. t + d_.) :→ T^(c/ħ) e^d,
    e^(c_. α_i_ + d_.) :→ 𝒜_i^(c/γ) e^d, e^(c_. α + d_.) :→ 𝒜^(c/γ) e^d,
    e^ℰ_ :→ e^(Expand@ℰ)};
```

Program

LZip implements the "L-level zips" on $\mathbb{E}(L, Q, P) = P e^{L+Q}$. Such zips regard all of $P e^{Q}$ as a single"P". Here the z's are b and α and the ζ's are β and a.

Program

```
In[•]:= LZip_{ζs_List,simp_}@𝔼[L_, Q_, P_] := Module[{ζ, z, zs, c, ys, ηs, lt, zrule, L1, L2, Q1, Q2},
    zs = Table[ζ*, {ζ, ζs}];
    c = L /. Alternatives @@ (ζs ⋃ zs) → 0;
    ys = Table[∂_ζ (L /. Alternatives @@ zs → 0), {ζ, ζs}];
    ηs = Table[∂_z (L /. Alternatives @@ ζs → 0), {z, zs}];
    lt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ} L, {ζ, ζs}, {z, zs}];
    zrule = Thread[zs → lt.(zs + ys)];
    L2 = (L1 = c + ηs.zs /. zrule) /. Alternatives @@ zs → 0;
    Q2 = (Q1 = Q /. U2l /. zrule) /. Alternatives @@ zs → 0;
    simp /@ 𝔼[L2, Q2, Det[lt] e^(-L2-Q2) Zip_ζs[e^(L1+Q1) (P /. U2l /. zrule)]] //. l2U];
LZip_{ζs_List} := LZip_{ζs,CF};
```

Program

In[∘]:=

```
Bind{}[L_, R_] := L R;
Bind{is__}[L_𝔼, R_𝔼] := Module[{n},
   Times[
      L /. Table[(v : b | B | t | T | a | x | y)i → vn@i, {i, {is}}],
      R /. Table[(v : β | τ | α | 𝒜 | ξ | η)i → vn@i, {i, {is}}]
     ] // LZipFlatten@Table[{βn@i,τn@i,an@i},{i,{is}}] // QZipFlatten@Table[{ξn@i,yn@i},{i,{is}}] ];
Bl_List[L_, R_] := Bindl[L, R]; Bis___[L_, R_] := Bind{is}[L, R];
```

Program

“Define” code

Program

Define[lhs = rhs] defines the lhs to be rhs, except that rhs is computed once and forever yet gets recomputed whenever $k changes. Fancy Mathematica not for the faint of heart. Most readers should ignore.

Program

In[∘]:=

```
SetAttributes[Define, HoldAll];
Define[def_, defs__] := (Define[def]; Define[defs];);
Define[op_is__ = ℰ_] :=
 Module[{SD, ii, jj, kk, isp, nis, nisp, sis}, Block[{i, j, k, l, m, n},
   ReleaseHold[Hold[
      SD[opnisp,$k_Integer, Block[{i, j, k, l, m, n}, opisp,$k = ℰ; opnis,$k]];
      SD[opisp, op{is},$k];
      SD[opsis__, op{sis}];
     ] /. {
      isp → {is} /. {i → i_, j → j_, k → k_},
      nis → {is} /. {i → ii, j → jj, k → kk},
      nisp → {is} /. {i → ii_, j → jj_, k → kk_},
      SD → SetDelayed
     }]
  ]]
```

Program

Booting Up

Program

In[∘]:=

```
$k = 2;
```

Program

In[∘]:=

```
Define[
  am_{i,j→k} = 𝔼[(α_i + α_j) a_k, (e^{-γ α_j} ξ_i + ξ_j) x_k, 1]_{$k},
  bm_{i,j→k} = 𝔼[(β_i + β_j) b_k, (η_i + η_j) y_k, e^{(e^{-ϵ β_i} - 1) η_j y_k}]_{$k},
  R_{i,j} = 𝔼[ℏ a_j b_i, ℏ x_j y_i, e^(Sum_{k=2}^{$k+1} ((1 - e^{γ ϵ ℏ})^k (ℏ y_i x_j)^k) / (k (1 - e^{k γ ϵ ℏ})))]_{$k},
  P_{i,j} = If[$k == 0, 𝔼[β_i α_j / ℏ, η_i ξ_j / ℏ, 1]_0,
    MapAt[(# - ϵ^{$k} Coefficient[(R_{n,m} ~ B_{n,m} ~ ((P_{{n,j},0})_{$k} (P_{{i,m},$k-1})_{$k}))[[3]], ϵ, $k]) &,
     (P_{{i,j},$k-1})_{$k}, 3]],
  aS_i = 𝔼[-α_i a_j, -ξ_i x_i,
      e^{ξ_i x_i} Sum[Expand[(-ℏ γ ϵ)^k / (2^k k!) Nest[Expand[x_i^2 ∂_{{x_i,2}} #] &, e^{-ξ_i e^{ℏ ϵ a_i} x_i}, k]], {k, 0, $k}]]_{$k} ~
    B_{i,j} ~ am_{i,j→i},
  \overline{aS}_i = If[$k == 0, 𝔼[-a_i α_i, -x_i 𝒜_i ξ_i, 1]_0,
    MapAt[(# - ϵ^{$k} Coefficient[((\overline{aS}_{{i},0})_{$k} ~ B_i ~ aS_i ~ B_i ~ (\overline{aS}_{{i},$k-1})_{$k})[[3]], ϵ, $k]) &,
     (\overline{aS}_{{i},$k-1})_{$k}, 3]],
  bS_i = R_{i,n} ~ B_n ~ aS_n ~ B_n ~ P_{i,n},
  \overline{bS}_i = R_{i,n} ~ B_n ~ \overline{aS}_n ~ B_n ~ P_{i,n},
  aΔ_{i→j,k} = (R_{n,j} R_{m,k}) ~ B_{n,m} ~ bm_{n,m→l} ~ B_l ~ P_{l,i},
  bΔ_{i→j,k} = (R_{j,n} R_{k,m}) ~ B_{n,m} ~ am_{n,m→l} ~ B_l ~ P_{i,l},
  dm_{i,j→k} = (𝔼[β_i b_i + α_j a_j, η_i y_i + ξ_j x_j, 1] (aΔ_{i→1,2} ~ B_2 ~ aΔ_{2→2,3}) (bΔ_{j→-1,-2} ~ B_{-2} ~ bΔ_{-2→-2,-3})) ~
    B_3 ~ \overline{aS}_3 ~ B_{-1,3} ~ (P_{-1,3}) ~ B_{-3,1} ~ (P_{-3,1}) ~ B_{2,j,i,-2} ~ (am_{2,j→k} bm_{i,-2→k}),
  dS_i = 𝔼[β_i b_n + α_i a_m, η_i y_n + ξ_i x_m, 1] ~ B_{n,m} ~ (\overline{bS}_n aS_m) ~ B_{n,m} ~ dm_{m,n→i},
  dΔ_{i→j,k} = (bΔ_{i→3,1} aΔ_{i→2,4}) ~ B_{1,2,3,4} ~ (dm_{3,4→k} dm_{1,2→j}),
  \overline{R}_{i,j} = Expand /@ R_{i,j} ~ B_j ~ dS_j,
  CC_i = 𝔼[0, 0, B_i^{1/2} e^{-ℏ ϵ a_i/2}]_{$k},
  \overline{CC}_i = 𝔼[0, 0, B_i^{-1/2} e^{ℏ ϵ a_i/2}]_{$k},
  Kink_i = (R_{1,3} \overline{CC}_2) ~ B_{1,2} ~ dm_{1,2→1} ~ B_{1,3} ~ dm_{1,3→i},
  \overline{Kink}_i = (\overline{R}_{1,3} CC_2) ~ B_{1,2} ~ dm_{1,2→1} ~ B_{1,3} ~ dm_{1,3→i},
  (* t==ϵa-γb and b==-t/γ+ϵa/γ: *)
  b2t_i = 𝔼[α_i a_i - β_i t_i / γ, ξ_i x_i + η_i y_i, e^{ϵ β_i a_i/γ}]_{$k},
  t2b_i = 𝔼[α_i a_j - τ_i γ b_j, ξ_i x_j + η_i y_j, e^{ϵ τ_i a_j}]_{$k}
 ];
```

Testing

In[∘]:=
```
Block[{$k = 1}, {
   am → am_{i,j→k}, bm → bm_{i,j→k}, dm → dm_{i,j→k}, R → R_{i,j}, R̄ → R̄_{i,j}, P → P_{i,j}, aS → aS_i,
   aS̄ → aS̄_i, bS → bS_i, bS̄ → bS̄_i, dS → dS_i, aΔ → aΔ_{i→j,k}, bΔ → bΔ_{i→j,k}, dΔ → dΔ_{i→j,k},
   CC → CC_i, CC̄ → CC̄_i, Kink → Kink_i, Kink̄ → Kink̄_i, b2t → b2t_i, t2b → t2b_i
  }] //
 Column
```

Out[∘]=

$$am \to \mathbb{E}\left[a_k\,(\alpha_i+\alpha_j),\ x_k\left(e^{-\gamma\alpha_j}\xi_i+\xi_j\right),\ 1\right]$$

$$bm \to \mathbb{E}\left[b_k\,(\beta_i+\beta_j),\ y_k\,(\eta_i+\eta_j),\ 1-y_k\,\beta_i\,\eta_j\,\epsilon+O[\epsilon]^2\right]$$

$$\begin{aligned}dm \to \mathbb{E}\Big[&a_k\,\alpha_i+a_k\,\alpha_j+b_k\,\beta_i+b_k\,\beta_j,\ \frac{1}{\hbar\,\mathcal{A}_i\,\mathcal{A}_j}\\&(\hbar\,y_k\,\mathcal{A}_i\,\mathcal{A}_j\,\eta_i+\hbar\,y_k\,\mathcal{A}_j\,\eta_j+\hbar\,x_k\,\mathcal{A}_i\,\xi_i+\mathcal{A}_i\,\mathcal{A}_j\,\eta_j\,\xi_i-B_k\,\mathcal{A}_i\,\mathcal{A}_j\,\eta_j\,\xi_i+\hbar\,x_k\,\mathcal{A}_i\,\mathcal{A}_j\,\xi_j),\\&1+\frac{1}{4\,\hbar\,\mathcal{A}_i\,\mathcal{A}_j}\Big(-4\,\hbar\,y_k\,\mathcal{A}_j\,\beta_i\,\eta_j-4\,\hbar\,x_k\,\mathcal{A}_i\,\beta_j\,\xi_i+4\,\gamma\,\hbar^2\,x_k\,y_k\,\eta_j\,\xi_i+\\&\quad 4\,\hbar\,a_k\,B_k\,\mathcal{A}_i\,\mathcal{A}_j\,\eta_j\,\xi_i+2\,\gamma\,\hbar\,y_k\,\mathcal{A}_j\,\eta_j^2\,\xi_i-6\,\gamma\,\hbar\,B_k\,y_k\,\mathcal{A}_j\,\eta_j^2\,\xi_i+2\,\gamma\,\hbar\,x_k\,\mathcal{A}_i\,\eta_j\,\xi_i^2-\\&\quad 6\,\gamma\,\hbar\,B_k\,x_k\,\mathcal{A}_i\,\eta_j\,\xi_i^2+\gamma\,\mathcal{A}_i\,\mathcal{A}_j\,\eta_j^2\,\xi_i^2-4\,\gamma\,B_k\,\mathcal{A}_i\,\mathcal{A}_j\,\eta_j^2\,\xi_i^2+3\,\gamma\,B_k^2\,\mathcal{A}_i\,\mathcal{A}_j\,\eta_j^2\,\xi_i^2\Big)\,\epsilon+O[\epsilon]^2\Big]\end{aligned}$$

$$R \to \mathbb{E}\left[\hbar\,a_j\,b_i,\ \hbar\,x_j\,y_i,\ 1-\frac{1}{4}\left(\gamma\,\hbar^3\,x_j^2\,y_i^2\right)\epsilon+O[\epsilon]^2\right]$$

$$\overline{R} \to \mathbb{E}\left[-\hbar\,a_j\,b_i,\ -\frac{\hbar\,x_j\,y_i}{B_i},\ 1+\frac{\left(-4\,\hbar^2\,a_j\,B_i\,x_j\,y_i-3\,\gamma\,\hbar^3\,x_j^2\,y_i^2\right)\epsilon}{4\,B_i^2}+O[\epsilon]^2\right]$$

$$P \to \mathbb{E}\left[\frac{\alpha_j\,\beta_i}{\hbar},\ \frac{\eta_i\,\xi_j}{\hbar},\ 1+\frac{\gamma\,\epsilon\,\eta_i^2\,\xi_j^2}{4\,\hbar}\right]$$

$$aS \to \mathbb{E}\left[-a_i\,\alpha_i,\ -x_i\,\mathcal{A}_i\,\xi_i,\ 1+\frac{1}{2}\left(-2\,\hbar\,a_i\,x_i\,\mathcal{A}_i\,\xi_i-\gamma\,\hbar\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2\right)\epsilon+O[\epsilon]^2\right]$$

$$\overline{aS} \to \mathbb{E}\left[-a_i\,\alpha_i,\ -x_i\,\mathcal{A}_i\,\xi_i,\ 1-\frac{1}{2}\,\epsilon\left(-2\,\gamma\,\hbar\,x_i\,\mathcal{A}_i\,\xi_i+2\,\hbar\,a_i\,x_i\,\mathcal{A}_i\,\xi_i+\gamma\,\hbar\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2\right)\right]$$

$$bS \to \mathbb{E}\left[-b_i\,\beta_i,\ -\frac{y_i\,\eta_i}{B_i},\ 1+\frac{\left(-2\,B_i\,y_i\,\beta_i\,\eta_i-\gamma\,\hbar\,y_i^2\,\eta_i^2\right)\epsilon}{2\,B_i^2}+O[\epsilon]^2\right]$$

$$\overline{bS} \to \mathbb{E}\left[-b_i\,\beta_i,\ -\frac{y_i\,\eta_i}{B_i},\ 1+\frac{\left(2\,\gamma\,\hbar\,B_i\,y_i\,\eta_i-2\,B_i\,y_i\,\beta_i\,\eta_i-\gamma\,\hbar\,y_i^2\,\eta_i^2\right)\epsilon}{2\,B_i^2}+O[\epsilon]^2\right]$$

$$\begin{aligned}dS \to \mathbb{E}\Big[&-a_i\,\alpha_i-b_i\,\beta_i,\ \frac{-\hbar\,y_i\,\mathcal{A}_i\,\eta_i-\hbar\,B_i\,x_i\,\mathcal{A}_i\,\xi_i+\mathcal{A}_i\,\eta_i\,\xi_i-B_i\,\mathcal{A}_i\,\eta_i\,\xi_i}{\hbar\,B_i},\\&1+\frac{1}{4\,\hbar\,B_i^2}\Big(4\,\gamma\,\hbar^2\,B_i\,y_i\,\mathcal{A}_i\,\eta_i-4\,\hbar\,B_i\,y_i\,\mathcal{A}_i\,\beta_i\,\eta_i-2\,\gamma\,\hbar^2\,y_i^2\,\mathcal{A}_i^2\,\eta_i^2-4\,\hbar^2\,a_i\,B_i^2\,x_i\,\mathcal{A}_i\,\xi_i-4\,\hbar\,B_i^2\,x_i\,\mathcal{A}_i\,\beta_i\,\xi_i-\\&\quad 4\,\gamma\,\hbar\,B_i\,\mathcal{A}_i\,\eta_i\,\xi_i+4\,\hbar\,a_i\,B_i\,\mathcal{A}_i\,\eta_i\,\xi_i+4\,\gamma\,\hbar\,B_i^2\,\mathcal{A}_i\,\eta_i\,\xi_i-4\,\gamma\,\hbar^2\,B_i\,x_i\,y_i\,\mathcal{A}_i^2\,\eta_i\,\xi_i+\\&\quad 4\,B_i\,\mathcal{A}_i\,\beta_i\,\eta_i\,\xi_i-4\,B_i^2\,\mathcal{A}_i\,\beta_i\,\eta_i\,\xi_i+6\,\gamma\,\hbar\,y_i\,\mathcal{A}_i^2\,\eta_i^2\,\xi_i-2\,\gamma\,\hbar\,B_i\,y_i\,\mathcal{A}_i^2\,\eta_i^2\,\xi_i-2\,\gamma\,\hbar^2\,B_i^2\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2+\\&\quad 6\,\gamma\,\hbar\,B_i\,x_i\,\mathcal{A}_i^2\,\eta_i\,\xi_i^2-2\,\gamma\,\hbar\,B_i^2\,x_i\,\mathcal{A}_i^2\,\eta_i\,\xi_i^2-3\,\gamma\,\mathcal{A}_i^2\,\eta_i^2\,\xi_i^2+4\,\gamma\,B_i\,\mathcal{A}_i^2\,\eta_i^2\,\xi_i^2-\gamma\,B_i^2\,\mathcal{A}_i^2\,\eta_i^2\,\xi_i^2\Big)\,\epsilon+O[\epsilon]^2\Big]\end{aligned}$$

$$a\Delta \to \mathbb{E}\left[a_j\,\alpha_i+a_k\,\alpha_i,\ x_j\,\xi_i+x_k\,\xi_i,\ 1+\frac{1}{2}\left(-2\,\hbar\,a_j\,x_k\,\xi_i+\gamma\,\hbar\,x_j\,x_k\,\xi_i^2\right)\epsilon+O[\epsilon]^2\right]$$

$$b\Delta \to \mathbb{E}\left[b_j\,\beta_i+b_k\,\beta_i,\ B_k\,y_j\,\eta_i+y_k\,\eta_i,\ 1+\frac{1}{2}\,\gamma\,\hbar\,B_k\,y_j\,y_k\,\eta_i^2\,\epsilon+O[\epsilon]^2\right]$$

$$\begin{aligned}d\Delta \to \mathbb{E}\Big[&a_j\,\alpha_i+a_k\,\alpha_i+b_j\,\beta_i+b_k\,\beta_i,\ y_j\,\eta_i+B_j\,y_k\,\eta_i+x_j\,\xi_i+x_k\,\xi_i,\\&1+\frac{1}{2}\left(\gamma\,\hbar\,B_j\,y_j\,y_k\,\eta_i^2-2\,\hbar\,a_j\,x_k\,\xi_i+\gamma\,\hbar\,x_j\,x_k\,\xi_i^2\right)\epsilon+O[\epsilon]^2\Big]\end{aligned}$$

$$CC \to \mathbb{E}\left[0,\ 0,\ \sqrt{B_i}-\frac{1}{2}\left(\hbar\,a_i\,\sqrt{B_i}\right)\epsilon+O[\epsilon]^2\right]$$

$$\overline{CC} \to \mathbb{E}\left[0,\ 0,\ \frac{1}{\sqrt{B_i}}+\frac{\hbar\,a_i\,\epsilon}{2\,\sqrt{B_i}}+O[\epsilon]^2\right]$$

$$Kink \to \mathbb{E}\left[\hbar\,a_i\,b_i,\ \hbar\,x_i\,y_i,\ \frac{1}{\sqrt{B_i}}+\frac{\left(2\,\hbar\,a_i-\gamma\,\hbar^3\,x_i^2\,y_i^2\right)\epsilon}{4\,\sqrt{B_i}}+O[\epsilon]^2\right]$$

$$\overline{Kink} \to \mathbb{E}\left[-\hbar\,a_i\,b_i,\ -\frac{\hbar\,x_i\,y_i}{B_i},\ \sqrt{B_i}+\frac{\left(-2\,\hbar\,a_i\,B_i^2-4\,\hbar^2\,a_i\,B_i\,x_i\,y_i-3\,\gamma\,\hbar^3\,x_i^2\,y_i^2\right)\epsilon}{4\,B_i^{3/2}}+O[\epsilon]^2\right]$$

$$b2t \to \mathbb{E}\left[a_i\,\alpha_i-\frac{t_i\,\beta_i}{\gamma},\ y_i\,\eta_i+x_i\,\xi_i,\ 1+\frac{a_i\,\beta_i\,\epsilon}{\gamma}+O[\epsilon]^2\right]$$

$$t2b \to \mathbb{E}\left[a_j\,\alpha_i-\gamma\,b_j\,\tau_i,\ y_j\,\eta_i+x_j\,\xi_i,\ 1+a_j\,\tau_i\,\epsilon+O[\epsilon]^2\right]$$

Check that on the generators this agrees with our conventions in the handout:

```
In[∘]:= Timing@{{"[a,x]" → ((𝔼[0, 0, a₂ x₁] ~B₁,₂~ am₁,₂→₁)⟦3⟧ - (𝔼[0, 0, a₁ x₂] ~B₁,₂~ am₁,₂→₁)⟦3⟧),
      "[b,y]" → ((𝔼[0, 0, y₂ b₁] ~B₁,₂~ bm₁,₂→₁)⟦3⟧ - (𝔼[0, 0, y₁ b₂] ~B₁,₂~ bm₁,₂→₁)⟦3⟧)} /.
    z_1 → z,
   {"Δ[y]" → Last[𝔼[0, 0, y₁] ~B₁~ bΔ₁→₁,₂],
    "Δ[b]" → Last[𝔼[0, 0, b₁] ~B₁~ bΔ₁→₁,₂],
    "Δ[a]" → Last[𝔼[0, 0, a₁] ~B₁~ aΔ₁→₁,₂],
    "Δ[x]" → Last[𝔼[0, 0, x₁] ~B₁~ aΔ₁→₁,₂]},
   {
     "S(a)" → ((𝔼[0, 0, a₁] ~B₁~ aS₁)⟦3⟧),
     "S(x)" → ((𝔼[0, 0, x₁] ~B₁~ aS₁)⟦3⟧),
     "S(b)" → ((𝔼[0, 0, b₁] ~B₁~ bS₁)⟦3⟧),
     "S(y)" → ((𝔼[0, 0, y₁] ~B₁~ bS₁)⟦3⟧)
   } /. z_1 → z}
```

Out[∘]= $\{0.703125, \{\{[a,x] \to -x\gamma,\ [b,y] \to -y\epsilon + O[\epsilon]^3\},\ \{\Delta[y] \to (B_2 y_1 + y_2) + O[\epsilon]^3,\ \Delta[b] \to (b_1 + b_2) + O[\epsilon]^3,$
$\Delta[a] \to (a_1 + a_2) + O[\epsilon]^3,\ \Delta[x] \to (x_1 + x_2) - \hbar a_1 x_2 \epsilon + \frac{1}{2}\hbar^2 a_1^2 x_2 \epsilon^2 + O[\epsilon]^3\},\ \{S(a) \to -a + O[\epsilon]^3,$
$S(x) \to -x - a x \hbar \epsilon - \frac{1}{2}(a^2 x \hbar^2)\epsilon^2 + O[\epsilon]^3,\ S(b) \to -b + O[\epsilon]^3,\ S(y) \to -\frac{y}{B} + O[\epsilon]^3\}\}\}$

Hopf algebra axioms on both sides separately.

Associativity of am and bm:

```
In[∘]:= Timing@Block[{$k = 3},
   HL /@ {(am₁,₂→₁ ~B₁~ am₁,₃→₁) ≡ (am₂,₃→₂ ~B₂~ am₁,₂→₁), (bm₁,₂→₁ ~B₁~ bm₁,₃→₁) ≡ (bm₂,₃→₂ ~B₂~ bm₁,₂→₁)}
  ]
```

Out[∘]= {0.109375, {True, True}}

R and *P* are inverses:

```
In[∘]:= Timing@Block[{$k = 3}, {Rᵢ,ⱼ, Pᵢ,ₖ, HL[Rᵢ,ⱼ ~Bᵢ~ Pᵢ,ₖ ≡ 𝔼[aⱼ αₖ, xⱼ ξₖ, 1]]}]
```

Out[∘]= $\{0.140625, \{\mathbb{E}[\hbar a_j b_i, \hbar x_j y_i, 1 - \frac{1}{4}(\gamma \hbar^3 x_j^2 y_i^2)\epsilon + (\frac{1}{9}\gamma^2 \hbar^5 x_j^3 y_i^3 + \frac{1}{32}\gamma^2 \hbar^6 x_j^4 y_i^4)\epsilon^2 +$
$\frac{1}{1152}(24\gamma^3 \hbar^5 x_j^2 y_i^2 - 72\gamma^3 \hbar^7 x_j^4 y_i^4 - 32\gamma^3 \hbar^8 x_j^5 y_i^5 - 3\gamma^3 \hbar^9 x_j^6 y_i^6)\epsilon^3 + O[\epsilon]^4],$
$\mathbb{E}[\frac{\alpha_k \beta_i}{\hbar}, \frac{\eta_i \xi_k}{\hbar}, 1 + \frac{\gamma \eta_i^2 \xi_k^2 \epsilon}{4\hbar} + \frac{(36\gamma^2 \hbar^2 \eta_i^2 \xi_k^2 + 40\gamma^2 \hbar \eta_i^3 \xi_k^3 + 9\gamma^2 \eta_i^4 \xi_k^4)\epsilon^2}{288\hbar^2} + \frac{1}{1152\hbar^3}$
$(48\gamma^3 \hbar^4 \eta_i^2 \xi_k^2 + 192\gamma^3 \hbar^3 \eta_i^3 \xi_k^3 + 156\gamma^3 \hbar^2 \eta_i^4 \xi_k^4 + 40\gamma^3 \hbar \eta_i^5 \xi_k^5 + 3\gamma^3 \eta_i^6 \xi_k^6)\epsilon^3 + O[\epsilon]^4], \text{True}\}\}$

as and $\overline{\text{aS}}$ are inverses, bs and $\overline{\text{bS}}$ are inverses:

```
In[∘]:= Timing[HL /@ {a̅S̅₁ ~B₁~ aS₁ ≡ 𝔼[a₁ α₁, x₁ ξ₁, 1], b̅S̅₁ ~B₁~ bS₁ ≡ 𝔼[b₁ β₁, y₁ η₁, 1]}]
```

Out[∘]= {0.375, {True, True}}

(co)-associativity on both sides

In[∘]:=
```
Timing[HL /@
  {(aΔ1→1,2 ~ B2 ~ aΔ2→2,3) ≡ (aΔ1→1,3 ~ B1 ~ aΔ1→1,2), (bΔ1→1,2 ~ B2 ~ bΔ2→2,3) ≡ (bΔ1→1,3 ~ B1 ~ bΔ1→1,2),
   (am1,2→1 ~ B1 ~ am1,3→1) ≡ (am2,3→2 ~ B2 ~ am1,2→1), (bm1,2→1 ~ B1 ~ bm1,3→1) ≡ (bm2,3→2 ~ B2 ~ bm1,2→1)}]
```

Out[∘]= {0.421875, {True, True, True, True}}

Δ is an algebra morphism

In[∘]:=
```
Timing[HL /@ {am1,2→1 ~ B1 ~ aΔ1→1,2 ≡ (aΔ1→1,3 aΔ2→2,4) ~ B1,2,3,4 ~ (am3,4→2 am1,2→1),
   bm1,2→1 ~ B1 ~ bΔ1→1,2 ≡ (bΔ1→1,3 bΔ2→2,4) ~ B1,2,3,4 ~ (bm3,4→2 bm1,2→1)}]
```

Out[∘]= {0.734375, {True, True}}

S is convolution inverse of id

In[∘]:=
```
Timing[HL[# ≡ 𝔼[0, 0, 1]] & /@ {
   (aΔ1→1,2 ~ B1 ~ aS1) ~ B1,2 ~ am1,2→1, (aΔ1→1,2 ~ B2 ~ aS2) ~ B1,2 ~ am1,2→1,
   (bΔ1→1,2 ~ B1 ~ bS1) ~ B1,2 ~ bm1,2→1, (bΔ1→1,2 ~ B2 ~ bS2) ~ B1,2 ~ bm1,2→1}]
```

Out[∘]= {0.640625, {True, True, True, True}}

S is an algebra anti-(co)morphism

In[∘]:=
```
Timing[HL /@ {am1,2→1 ~ B1 ~ aS1 ≡ (aS1 aS2) ~ B1,2 ~ am2,1→1, bm1,2→1 ~ B1 ~ bS1 ≡ (bS1 bS2) ~ B1,2 ~ bm2,1→1,
   aS1 ~ B1 ~ aΔ1→1,2 ≡ aΔ1→2,1 ~ B1,2 ~ (aS1 aS2), bS1 ~ B1 ~ bΔ1→1,2 ≡ bΔ1→2,1 ~ B1,2 ~ (bS1 bS2)}]
```

Out[∘]= {1.03125, {True, True, True, True}}

Pairing axioms

In[∘]:=
```
Timing[HL /@ {(bm1,2→1 𝔼[α3 a3, ξ3 x3, 1]) ~ B1,3 ~ P1,3 ≡
    (𝔼[β1 b1, η1 y1, 1] 𝔼[β2 b2, η2 y2, 1] aΔ3→4,5) ~ B1,4 ~ P1,4 ~ B2,5 ~ P2,5,
   (bΔ1→1,2 𝔼[α3 a3, ξ3 x3, 1] 𝔼[α4 a4, ξ4 x4, 1]) ~ B1,3 ~ P1,3 ~ B2,4 ~ P2,4 ≡
    (𝔼[β1 b1, η1 y1, 1] am3,4→3) ~ B1,3 ~ P1,3 }]
```

Out[∘]= {0.484375, {True, True}}

In[∘]:=
```
Timing[HL /@ {(bS1 𝔼[α2 a2, ξ2 x2, 1]) ~ B1,2 ~ P1,2 ≡ (𝔼[β1 b1, η1 y1, 1] aS2) ~ B1,2 ~ P1,2,
   (b̄S̄1 𝔼[α2 a2, ξ2 x2, 1]) ~ B1,2 ~ P1,2 ≡ (𝔼[β1 b1, η1 y1, 1] āS̄2) ~ B1,2 ~ P1,2}]
```

Out[∘]= {0.328125, {True, True}}

Tests for the double.

Check the double formulas on the generators agree with SL2Portfolio.pdf:

```
In[•]:= Timing@{{
        "[a,y]" → ((𝔼[0, 0, y₂ a₁] ~ B₁,₂ ~ dm₁,₂→₁)⟦3⟧ - (𝔼[0, 0, y₁ a₂] ~ B₁,₂ ~ dm₁,₂→₁)⟦3⟧),
        "[b,x]"  → ((𝔼[0, 0, x₂ b₁] ~ B₁,₂ ~ dm₁,₂→₁)⟦3⟧ - (𝔼[0, 0, x₁ b₂] ~ B₁,₂ ~ dm₁,₂→₁)⟦3⟧),
        "xy-qyx" → ((𝔼[0, 0, x₁ y₂] ~ B₁,₂ ~ dm₁,₂→₁)⟦3⟧ - (1 + ϵ) (𝔼[0, 0, y₁ x₂] ~ B₁,₂ ~ dm₁,₂→₁)⟦3⟧)
       } /. {z_1 → z} // Expand // Factor,
     {
      "Δ(a)" → ((𝔼[0, 0, a₁] ~ B₁ ~ dΔ₁→₁,₂)⟦3⟧),
      "Δ(x)" → ((𝔼[0, 0, x₁] ~ B₁ ~ dΔ₁→₁,₂)⟦3⟧),
      "Δ(b)" → ((𝔼[0, 0, b₁] ~ B₁ ~ dΔ₁→₁,₂)⟦3⟧),
      "Δ(y)" → ((𝔼[0, 0, y₁] ~ B₁ ~ dΔ₁→₁,₂)⟦3⟧)
     } // Simplify,
     {
       "S(a)" → ((𝔼[0, 0, a₁] ~ B₁ ~ dS₁)⟦3⟧),
       "S(x)" → ((𝔼[0, 0, x₁] ~ B₁ ~ dS₁)⟦3⟧),
       "S(b)" → ((𝔼[0, 0, b₁] ~ B₁ ~ dS₁)⟦3⟧),
       "S(y)" → ((𝔼[0, 0, y₁] ~ B₁ ~ dS₁)⟦3⟧)
     } /. {z_1 → z} // Simplify
    }
```

Out[•]= $\{10.8125, \{\{[a,y] \to -y\gamma + O[\epsilon]^3,\ [b,x] \to x\epsilon + O[\epsilon]^3,$
$xy{-}qyx \to \left(-x y + \frac{1 - B + x y \hbar}{\hbar}\right) + (a B - x y + x y \gamma \hbar)\,\epsilon + \frac{1}{2}\left(-a^2 B \hbar + x y \gamma^2 \hbar^2\right)\epsilon^2 + O[\epsilon]^3\},$
$\{\Delta(a) \to (a_1 + a_2) + O[\epsilon]^3,\ \Delta(x) \to (x_1 + x_2) - \hbar\, a_1 x_2\, \epsilon + \frac{1}{2}\hbar^2 a_1^2 x_2\, \epsilon^2 + O[\epsilon]^3,$
$\Delta(b) \to (b_1 + b_2) + O[\epsilon]^3,\ \Delta(y) \to (y_1 + B_1 y_2) + O[\epsilon]^3\},$
$\{S(a) \to -a + O[\epsilon]^3,\ S(x) \to -x - a x \hbar \epsilon - \frac{1}{2}\left(a^2 x \hbar^2\right)\epsilon^2 + O[\epsilon]^3,$
$S(b) \to -b + O[\epsilon]^3,\ S(y) \to -\frac{y}{B} + \frac{y \gamma \hbar \epsilon}{B} - \frac{\left(y \gamma^2 \hbar^2\right)\epsilon^2}{2B} + O[\epsilon]^3\}\}\}$

(co)-associativity

```
In[•]:= Timing[HL /@
      {(dΔ₁→₁,₂ ~ B₂ ~ dΔ₂→₂,₃) ≡ (dΔ₁→₁,₃ ~ B₁ ~ dΔ₁→₁,₂), (dm₁,₂→₁ ~ B₁ ~ dm₁,₃→₁) ≡ (dm₂,₃→₂ ~ B₂ ~ dm₁,₂->₁)}]
Out[•]= {7.23438, {True, True}}
```

Δ is an algebra morphism

```
In[•]:= Timing@HL[dm₁,₂→₁ ~ B₁ ~ dΔ₁→₁,₂ ≡ (dΔ₁→₁,₃ dΔ₂→₂,₄) ~ B₁,₂,₃,₄ ~ (dm₃,₄→₂ dm₁,₂→₁)]
Out[•]= {13.5469, True}
```

S is convolution inverse of id

```
In[•]:= Timing[
      HL[# ≡ 𝔼[0, 0, 1]] & /@ {(dΔ₁→₁,₂ ~ B₁ ~ dS₁) ~ B₁,₂ ~ dm₁,₂→₁, (dΔ₁→₁,₂ ~ B₂ ~ dS₂) ~ B₁,₂ ~ dm₁,₂→₁}]
Out[•]= {11.9688, {True, True}}
```

S is a (co)-algebra anti-morphism

```
In[∘]:= Timing[HL /@
          Expand /@ {dm_{1,2→1} ~ B_1 ~ dS_1 ≡ (dS_1 dS_2) ~ B_{1,2} ~ dm_{2,1→1}, dS_1 ~ B_1 ~ dΔ_{1→1,2} ≡ dΔ_{1→2,1} ~ B_{1,2} ~ (dS_1 dS_2)}]
Out[∘]= {26.125, {True, True}}
```

Quasi-triangular axiom 1:

```
In[∘]:= Timing@HL[R_{1,2} ~ B_1 ~ dΔ_{1->1,3} ≡ (R_{1,4} R_{3,2}) ~ B_{2,4} ~ dm_{2,4→2}]
Out[∘]= {0.6875, True}
```

Quasi-triangular axiom 2:

```
In[∘]:= Timing@HL[((dΔ_{1->1,2} R_{3,4}) ~ B_{1,2,3,4} ~ (dm_{1,3→1} dm_{2,4→2})) ≡ ((dΔ_{1→2,1} R_{3,4}) ~ B_{1,2,3,4} ~ (dm_{3,1→1} dm_{4,2→2}))]
Out[∘]= {11.7344, True}
```

The Drinfel'd element inverse property, $(u_1\,\overline{u}_2)\sim B_{1,2}\sim \mathrm{dm}_{1,2\to 1}\equiv \mathbb{E}[0,0,1]$:

```
In[∘]:= Timing@
          HL[((R_{1,2} ~ B_1 ~ dS_1 ~ B_{1,2} ~ dm_{2,1→i}) (R_{1,2} ~ B_2 ~ dS_2 ~ B_2 ~ dS_2 ~ B_{1,2} ~ dm_{2,1→j})) ~ B_{i,j} ~ dm_{i,j→i} ≡ 𝔼[0, 0, 1]]
Out[∘]= {4.25, True}
```

The ribbon element v satisfies $v^2 = S(u)\,u$. The spinner C=uv^{-1}. It is convenient to compute $z = S(u)\,u^{-1}$ which is something easy.

```
In[∘]:= Timing@Block[{$k = 3},
          (((R_{1,2} ~ B_1 ~ dS_1 ~ B_{1,2} ~ dm_{2,1→i}) ~ B_i ~ dS_i) (R_{1,2} ~ B_2 ~ dS_2 ~ B_2 ~ dS_2 ~ B_{1,2} ~ dm_{2,1→j})) ~ B_{i,j} ~ dm_{i,j→i}]
Out[∘]= {76.0156, 𝔼[0, 0, 1/B_i + ħ a_i ϵ/B_i + ħ^2 a_i^2 ϵ^2/(2 B_i) + ħ^3 a_i^3 ϵ^3/(6 B_i) + O[ϵ]^4]}
```

```
In[∘]:= Timing@Block[{$k = 2}, HL /@ {(CC_i \bar{CC}_j) ~ B_{i,j} ~ dm_{i,j→i} ≡ 𝔼[0, 0, 1], (\bar{CC}_i \bar{CC}_j) ~ B_{i,j} ~ dm_{i,j→i} ≡
          (((R_{1,2} ~ B_1 ~ dS_1 ~ B_{1,2} ~ dm_{2,1→i}) ~ B_i ~ dS_i) (R_{1,2} ~ B_2 ~ dS_2 ~ B_2 ~ dS_2 ~ B_{1,2} ~ dm_{2,1→j})) ~ B_{i,j} ~ dm_{i,j→i}}]
Out[∘]= {5.79688, {True, True}}
```

Reidemeister 2:

```
In[∘]:= Timing[HL[# ≡ 𝔼[0, 0, 1]] & /@
          {(\bar{R}_{1,2} R_{3,4}) ~ B_{1,2,3,4} ~ (dm_{1,3→1} dm_{2,4→2}), (R_{1,2} \bar{R}_{3,4}) ~ B_{1,2,3,4} ~ (dm_{1,3→1} dm_{2,4→2})}]
Out[∘]= {8.26563, {True, True}}
```

Cyclic Reidemeister 2:

```
In[∘]:= Timing@HL[(R_{1,4} \bar{R}_{5,2} \bar{CC}_3) ~ B_{2,4} ~ dm_{2,4→2} ~ B_{1,3} ~ dm_{1,3→1} ~ B_{1,5} ~ dm_{1,5→1} ≡ \bar{CC}_1]
Out[∘]= {6.79688, True}
```

Reidemeister 3:

```
In[∘]:= Timing@HL[((R_{1,2} R_{4,3} R_{5,6}) ~ B_{1,4} ~ dm_{1,4→1} ~ B_{2,5} ~ dm_{2,5→2} ~ B_{3,6} ~ dm_{3,6→3}) ≡
          ((R_{1,6} R_{2,3} R_{4,5}) ~ B_{1,4} ~ dm_{1,4→1} ~ B_{2,5} ~ dm_{2,5→2} ~ B_{3,6} ~ dm_{3,6→3})]
Out[∘]= {5.64063, True}
```

Relations between the four kinks:

In[◦]:=
```
Timing[HL /@ {Kink_i ≡ (R_{3,1} CC_2) ~ B_{1,2} ~ dm_{1,2→1} ~ B_{1,3} ~ dm_{1,3→i},
    OverBar[Kink]_j ≡ (OverBar[R]_{3,1} OverBar[CC]_2) ~ B_{1,2} ~ dm_{1,2→1} ~ B_{1,3} ~ dm_{1,3→j}, (Kink_i OverBar[Kink]_j) ~ B_{i,j} ~ dm_{i,j→1} ≡ 𝔼[0, 0, 1]}]
```

Out[◦]= {10.8594, {True, True, True}}

The Trefoil

In[◦]:=
```
Monitor[Timing@Block[{$k = 1},
    Z = R_{1,5} R_{6,2} R_{3,7} OverBar[CC]_4 OverBar[Kink]_8 OverBar[Kink]_9 OverBar[Kink]_10;
    Do[Z = Z ~ B_{1,r} ~ dm_{1,r→1}, {r, 2, 10}];
    Simplify /@ Z], r]
```

Out[◦]= $\left\{113.813,\ \mathbb{E}\left[0,\ 0,\ \frac{B_1}{1-B_1+B_1^2} - \left(\hbar B_1 \left(-a_1 \left(-1+B_1-B_1^3+B_1^4\right) + \gamma \left(B_1 - 2 B_1^2 - 2 B_1^4 + 2 \hbar x_1 y_1 + B_1^3 \left(3 + 2 \hbar x_1 y_1\right)\right)\right) \epsilon\right) \Big/ \left(1-B_1+B_1^2\right)^3 + O[\epsilon]^2\right]\right\}$