Review for the article *A polynomial time knot polynomial* by D. Bar-Natan and R. van der Veen

The discovery of the Jones polynomial leads to a theory of quantum invariants which brings a huge family of new knot invariants. Theoretically, computing these invariants are very hard in general. Given a knot diagram, the computation of a quantum invariant (such as, the Jones polynomial), requires exponential time with respect to the number of crossings of the input diagram. This makes a sharp contrast with the classical Alexander polynomial which can be computed in polynomial time.

The article under review defines a new knot invariant whose computational complexity is in polynomial time (with respect to certain input of a knot diagram).

The authors introduce a notion of a snarl algebra, an algebraic framework for constructing knot invariants. This is close to the usual operator invariant formalism of tangles. They decompose a knot diagram into fundamental pieces, and the resulting knot invariant is computed by summing up the local contributions of the fundamental pieces. A central issue for the construction of invariant is a construction of snarl algebras. They used q-Weyl algebra, truncated so that $q = 1 + \epsilon$ with $\epsilon^2 = 1$, to construct a snarl algebra and the resulting knot invariant.

Beside the computational complexity, their new invariant and insight are fascinating: First, their construction will post a new point of view in a theory of quantum invariants. Their construction based on q-Weyl algebra suggests there are connection between their invariant and the quantum $\mathfrak{sl}_2$ invariants. Indeed, the author conjecture that their invariants is equal to the second part of Rozansky's expansion of the colored Jones polynomial.

Second, the invariant will be more useful than the previously known invariants. The author observed their new invariant can distinguish many knots and they conjecture that, their new invariant provide an estimate of knot genus, like the Alexander polynomial.

Although this paper is very nice and the reviewer truely recommend publication, its exposition looks sometimes too concise and confusing. Some important notations are used without giving explicit definitions, and this makes a part of the paper hard to track.

In summary, the paper contains stimulating and wonderful ideas and results so the review recommend it for publication, after some revisions.

List of Comments:

1. (General)

   It seems to be helpful to clarify the precise meaning when you say that "an invariant is computable in polynomial time", as you did in Theorem 5 (for example, the statement in Theorem 1)

   To discuss the computational issues, we need what is an input and what kind of complexity you are using – it looks that in this article, the input is snarl diagram and primary you are using the number of X and α of the corresponding snarl algebra as a complexity of input.

   Although it seems this complexity is essentially equivalent to the number of crossings of the diagrams, it still seems to need additional care – when a knot diagram, given in a Morse position as in Section 1.1, has a huge number of $\cup$ or $\cap$, then the size of a matrix for computing the invariant will be large even if the number of crossing is small.

2. (Page 1 line 3) For small knot $\rightarrow$ For knots with small crossing numbers

   A small knot usually means a knot without closed essential surface in its exterior.

3. (Page 1 last line) What is $adj(B)$? (the reviewer understand it is the adjoint matrix of B, but it would be helpful for reader to clarify).

4. (Page 2 Definition 1)
   Are you assuming that the each vertices of a graph G is either one of four-valent ?

5. (Page 3, line -22)
   Strictly speaking, this notation is incorrect: According to the definition $m_k^{i,j}$, once we apply $m_k^{I_1,I_2}$ the element k will become an element of the new label set $L - \{I_1, I_2\} \cup \{k\}$. So the next operation m_k^{k,I_3} is not defined according to the current definition.

   To be precise, a correct definition is that we take temporary labellings $k_2, k_3, \ldots, k_{n-1}$ and then define

   $$m_k^I = m_{k_2}^{I_1,I_2} /\!/ \ m_{k_3}^{k_2,I_3} /\!/ \ \cdots /\!/ \ m_{k_{n-1}}^{k_{n-2},I_{n-1}} /\!/ \ m_k^{k_{n-1},I_n}$$

   It seems the be helpful to extend the definition of $m_k^{i,j}$ to allow the case like $m_i^{i,j}$.

6. (Page 3, Definition 3 equation) It would be helpful to add an explanation that (123) means the sequence $(1, 2, 3)$.

7. (Page 5 line -9)
   if $g = \mathbf{e}^{eQf}$ $\Rightarrow$ if $g = \mathbf{e}^{eQf}$ for some square matrix Q

8. (Page 5 line -6,-4)

   Given square matrices $\to$ Given square matrices W, Q.

9. (Page 5 line -4)

   It seems to be helpful to mention that here you view $Q = (Q_{ij})_{i,j \in S}$ and each Q_{ij} as a formal variable which commutes with x_s and y_s.

10. (Page 6 Lemma 3)

    What is Δ ? (A similar comment applies for Page 9, lemma 5)

11. (Page 6 line -2 – page 7)

    Please clarify the definition of Δ_D and Q_D. (For disjoint union snarls, they are defined by $Z(D) = \Delta_D \mathbf{e}^{eQ_D f}$ ($\Delta_D \in \mathbb{Q}[t^{\pm\frac{1}{2}}]$) – is it correct ?)

    Also, please explain how you take W: the reviewer understand that you are using the label set $\{1, 2, 3\}$ with ordering $1 \prec 2 \prec 3$, and that you are taking W as in Lemma 3. A similar clarification for W (and ordering of the label set) is needed for other cases in page 7, and many of other parts of the paper.

    $(I - WQ)^{-1}$ should be $(I - WQ_D)^{-1}$. (Maybe you write "we put $Q = Q_D$")

12. (Page 7–8 Theorem 4)

    The proof of Theorem 4 is hard to follow since there are various notations which are not defined (at least, in an explicit manner in the main body), or, the same notations are used to represent different objects.

    - (Page 8, line 4) "This proves $\Delta_b = t^{\frac{w}{2}}$ ... " – what is the definition of Δ_b and Q_b ?
    - (Page 8, line 23) The matrix W here seems to be different from the matrix W appeared in page 8, line 16.
    - (Page 8, Section 4 line -11) The equation "$\det(I - WQ_b) = \det_n(I - WQ_b)$" is confusing. The "$b''$ in the left and right hand side seem to represent different object. On the right hand side it seems that b is stitched but on the left hand side b is not stitched.

13. (Page 10, Theorem 5, Proor line 1) Bij $\to$ *By*

14. (Page 11, conjecture 2)

    It seems that the conjecture 2 will follow from the conjecture 3, from Ohtsuki's results on the 2-loop polynomial [T, Ohstuki, On the 2-loop polynomial of knots. Geom. Topol. 11 (2007), 1357–1475]:

    He proved that the degree of the two loop polynomial is bounded above by the genus of the knot as was conjectured by Ronzansky (Theorem 4.6). He also showed that the reduced 2-loop polynomial, a certain reduction of the 2-loop polynomial is $P^{(1)}(t)$ in Ronzansky's expansion of the colored Jones polynomials ($\mathfrak{sl}_2$ weight system reduction of the 2-loop part of the Kontsevich invariant).