Pensieve header: Finding the YB element for NOE-1.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\OneCo-1606"];
<< NOE-1.m
```

Solution to all-but-one equations, as found on 160807:

$$\begin{aligned}
\mathtt{ECAllButOne} = \Big\{ & f_2[x_,y_] :\to y\,f_9[x,y] + y^2 f_{26}[x,y] - g_6[x],\ f_3[x_,z_] :\to x^2 f_{18}[x,z],\\
& f_4[x_,y_] :\to y\,f_{13}[x,y] - g_1[x],\ f_5[x_,y_] :\to y^2 f_{26}[x,y],\ f_8[x_,z_] :\to 0,\\
& f_{10}[x_,z_] :\to 2x\,f_{18}[x,z],\ f_{11}[x_,y_] :\to 2x\,f_{18}[x,y] - 2e^x x\,f_{18}[x,y] - g_4[x],\\
& f_{12}[x_,z_] :\to 0,\ f_{13}[x_,z_] :\to x\,f_{22}[x,z] - g_3[z],\ f_{14}[x_,y_] :\to y\,f_{22}[x,y] - g_2[x],\\
& f_{15}[x_,y_] :\to \frac{1}{\left(-1+e^y\right)x}\Big(-x\,y\,f_{23}[x,y] + e^y x\,y\,f_{23}[x,y] + x\,g_2[x] - e^x x\,g_2[x] -\\
& \quad e^y x\,g_2[x] + e^{x+y} x\,g_2[x] - e^y y\,g_2[y] + e^{x+y} y\,g_2[y] - y\,g_3[y] + e^x y\,g_3[y] - y\,g_4[y] +\\
& \quad e^x y\,g_4[y] + 8y^2 g_5[y] - 8e^x y^2 g_5[y] - 16e^y y^2 g_5[y] + 8e^{2y} y^2 g_5[y] + 16e^{x+y} y^2 g_5[y] -\\
& \quad 8e^{x+2y} y^2 g_5[y] + 4y\,g_7[y] - 4e^x y\,g_7[y] - 4e^y y\,g_7[y] + 4e^{x+y} y\,g_7[y]\Big),\\
& f_{16}[x_,z_] :\to 0,\ f_{17}[x_,y_] :\to 2y\,f_{26}[x,y],\ f_{18}[x_,y_] :\to\\
& \quad -\Big(\big(f_{19}[x,y] + 4g_5[x] - 8e^x g_5[x] + 4e^{2x} g_5[x]\big)\Big/\big(2\left(-1+e^x\right)\big)\Big),\\
& f_{19}[x_,y_] :\to \frac{1}{\left(-1+e^x\right)\left(-1+e^y\right)x^2 y}\Big(-1 + 2e^x - e^{2x} + e^y - 2e^{x+y} + e^{2x+y} +\\
& \quad 2x^2 y\,f_{20}[x,y] - 2e^y x^2 y\,f_{20}[x,y] + x\,y\,g_4[x] - e^x x\,y\,g_4[x] - e^y x\,y\,g_4[x] +\\
& \quad e^{x+y} x\,y\,g_4[x] + y^2 g_4[y] - 2e^x y^2 g_4[y] + e^{2x} y^2 g_4[y] - 4y^3 g_5[y] + 8e^x y^3 g_5[y] -\\
& \quad 4e^{2x} y^3 g_5[y] + 8e^y y^3 g_5[y] - 4e^{2y} y^3 g_5[y] - 16e^{x+y} y^3 g_5[y] + 8e^{2x+y} y^3 g_5[y] +\\
& \quad 8e^{x+2y} y^3 g_5[y] - 4e^{2x+2y} y^3 g_5[y] - e^y y\,h_1[] + 2e^{x+y} y\,h_1[] - e^{2x+y} y\,h_1[]\Big),\\
& f_{21}[x_,z_] :\to 0,\ f_{22}[x_,z_] :\to \frac{-x\,f_{23}[x,z] - 4g_7[z] + 4e^x g_7[z]}{\left(-1+e^x\right)x},\\
& f_{24}[x_,z_] :\to 0,\ f_{25}[x_,z_] :\to 0,\ g_1[y_] :\to y\,g_2[y] + h_1[],\\
& g_2[y_] :\to \frac{1}{2y^2} e^{-y}\Big(-2 + 2e^y - 2y^2 g_3[y] + 8y^3 g_5[y] - 16e^y y^3 g_5[y] +\\
& \quad 8e^{2y} y^3 g_5[y] + 8y^2 g_7[y] - 8e^y y^2 g_7[y] - y\,h_1[] - e^y y\,h_1[]\Big),\\
& f_1[x_,z_] :\to \frac{1}{2}\Big(-\Big(\Big(x\left(-2\left(-1+e^x\right)^2 f_6[x,z] - 2x\,f_{20}[x,z] +\\
& \quad \left(-1+e^x\right)\left(g_4[x] + 4\left(-1+e^x\right)^2 x\,g_5[x]\right)\Big)\Big)\Big/\left(-1+e^x\right)^2\Big) + g_8[z]\Big)\Big\};
\end{aligned}$$

Solution to all equations, as found on 160809:

$$\begin{aligned}
\mathtt{ECAll} = \mathtt{ECAllButOne} \bigcup & \Big\{ f_6[x_,z_] \to \frac{-\frac{x\,f_7[x,z]}{-1+e^x} + g_9[x] + g_{10}[z]}{x},\ g_{10}[y_] \to \frac{1}{2\left(-1+e^y\right)^2 y}\Big(-1 + e^y - y^2 g_4[y] +\\
& \quad 2e^y y^2 g_4[y] - 4y^3 g_5[y] + 8e^y y^3 g_5[y] - 4e^{2y} y^3 g_5[y] + 2e^y y\,g_6[y] - 2e^{2y} y\,g_6[y] -\\
& \quad y\,g_8[y] + e^y y\,g_8[y] - 2y\,g_9[y] + 4e^y y\,g_9[y] - 2e^{2y} y\,g_9[y] - e^y y\,h_1[]\Big)\Big\};
\end{aligned}$$

```
ϵ U[c_j] f_1[b_j, b_k] + ϵ U[c_k] f_2[b_j, b_k] + ϵ U[c_j, c_j] f_3[b_j, b_k] + ϵ U[c_j, c_k] f_4[b_j, b_k] +
  ϵ U[c_k, c_k] f_5[b_j, b_k] + ϵ U[u_j, w_j] f_6[b_j, b_k] + ϵ U[u_j, w_k] f_7[b_j, b_k] +
  ϵ U[u_k, w_j] f_8[b_j, b_k] + ϵ U[u_k, w_k] f_9[b_j, b_k] + ϵ U[c_j, u_j, w_j] f_10[b_j, b_k] +
  ϵ U[c_j, u_j, w_k] f_11[b_j, b_k] + ϵ U[c_j, u_k, w_j] f_12[b_j, b_k] + ϵ U[c_j, u_k, w_k] f_13[b_j, b_k] +
  ϵ U[c_k, u_j, w_j] f_14[b_j, b_k] + ϵ U[c_k, u_j, w_k] f_15[b_j, b_k] + ϵ U[c_k, u_k, w_j] f_16[b_j, b_k] +
  ϵ U[c_k, u_k, w_k] f_17[b_j, b_k] + ϵ U[u_j, u_j, w_j, w_j] f_18[b_j, b_k] + ϵ U[u_j, u_j, w_j, w_k] f_19[b_j, b_k] +
  ϵ U[u_j, u_j, w_k, w_k] f_20[b_j, b_k] + ϵ U[u_j, u_k, w_j, w_j] f_21[b_j, b_k] +
  ϵ U[u_j, u_k, w_j, w_k] f_22[b_j, b_k] + ϵ U[u_j, u_k, w_k, w_k] f_23[b_j, b_k] +
  ϵ U[u_k, u_k, w_j, w_j] f_24[b_j, b_k] + ϵ U[u_k, u_k, w_j, w_k] f_25[b_j, b_k] +
  ϵ U[u_k, u_k, w_k, w_k] f_26[b_j, b_k] /. {U → Times, j → i, k → j}
```

```
ϵ c_i f_1[b_i, b_j] + ϵ c_j f_2[b_i, b_j] + ϵ c_i^2 f_3[b_i, b_j] + ϵ c_i c_j f_4[b_i, b_j] +
 ϵ c_j^2 f_5[b_i, b_j] + ϵ u_i w_i f_6[b_i, b_j] + ϵ u_i w_j f_7[b_i, b_j] + ϵ u_j w_i f_8[b_i, b_j] +
 ϵ u_j w_j f_9[b_i, b_j] + ϵ c_i u_i w_i f_10[b_i, b_j] + ϵ c_i u_i w_j f_11[b_i, b_j] + ϵ c_i u_j w_i f_12[b_i, b_j] +
 ϵ c_i u_j w_j f_13[b_i, b_j] + ϵ c_j u_i w_i f_14[b_i, b_j] + ϵ c_j u_i w_j f_15[b_i, b_j] +
 ϵ c_j u_j w_i f_16[b_i, b_j] + ϵ c_j u_j w_j f_17[b_i, b_j] + ϵ u_i^2 w_i^2 f_18[b_i, b_j] + ϵ u_i^2 w_i w_j f_19[b_i, b_j] +
 ϵ u_i^2 w_j^2 f_20[b_i, b_j] + ϵ u_i u_j w_i^2 f_21[b_i, b_j] + ϵ u_i u_j w_i w_j f_22[b_i, b_j] +
 ϵ u_i u_j w_j^2 f_23[b_i, b_j] + ϵ u_j^2 w_i^2 f_24[b_i, b_j] + ϵ u_j^2 w_i w_j f_25[b_i, b_j] + ϵ u_j^2 w_j^2 f_26[b_i, b_j]
```

```
Rp[i_, j_] := 𝔼[b_i c_j + (e^{b_i} - 1)/b_i u_i w_j]
  (1 + ϵ c_i f_1[b_i, b_j] + ϵ c_j f_2[b_i, b_j] + ϵ c_i^2 f_3[b_i, b_j] + ϵ c_i c_j f_4[b_i, b_j] +
    ϵ c_j^2 f_5[b_i, b_j] + ϵ u_i w_i f_6[b_i, b_j] + ϵ u_i w_j f_7[b_i, b_j] + ϵ u_j w_i f_8[b_i, b_j] +
    ϵ u_j w_j f_9[b_i, b_j] + ϵ c_i u_i w_i f_10[b_i, b_j] + ϵ c_i u_i w_j f_11[b_i, b_j] + ϵ c_i u_j w_i f_12[b_i, b_j] +
    ϵ c_i u_j w_j f_13[b_i, b_j] + ϵ c_j u_i w_i f_14[b_i, b_j] + ϵ c_j u_i w_j f_15[b_i, b_j] +
    ϵ c_j u_j w_i f_16[b_i, b_j] + ϵ c_j u_j w_j f_17[b_i, b_j] + ϵ u_i^2 w_i^2 f_18[b_i, b_j] + ϵ u_i^2 w_i w_j f_19[b_i, b_j] +
    ϵ u_i^2 w_j^2 f_20[b_i, b_j] + ϵ u_i u_j w_i^2 f_21[b_i, b_j] + ϵ u_i u_j w_i w_j f_22[b_i, b_j] +
    ϵ u_i u_j w_j^2 f_23[b_i, b_j] + ϵ u_i^2 w_i^2 f_24[b_i, b_j] + ϵ u_j^2 w_i w_j f_25[b_i, b_j] + ϵ u_j^2 w_j^2 f_26[b_i, b_j])
```

```
Short[t1 = Rp[1, 2] Rp[3, 4] Rp[5, 6] // m[3, 5, x] // m[1, 6, y] // m[2, 4, z]]
```

$$\frac{\mathbb{E}\left[\frac{\ll 1 \gg}{b_x\, b_y}\right]\left(b_x^2\, b_y^2 + \ll 397 \gg + \epsilon\ \ll 5 \gg\right)}{b_x^2\, b_y^2}$$

```
t2 = Rp[1, 2] Rp[3, 4] Rp[5, 6] // m[1, 3, x] // m[2, 5, y] // m[4, 6, z]
```

$$\frac{1}{2\, b_x^2\, b_y^2}\, e^{-2 b_y}\, \mathbb{E}\left[\frac{b_x^2\, b_y\, c_y + \langle\!\langle 10 \rangle\!\rangle + \langle\!\langle 1 \rangle\!\rangle}{b_x\, b_y}\right]$$
$$\left(2\, e^{2 b_y}\, b_x^2\, b_y^2 + \langle\!\langle 662 \rangle\!\rangle + 2\, e^{2 b_y}\, \epsilon\, b_x^2\, b_y^2\, u_z^2\, w_z^2\, f_{26}[b_y, b_z]\right)$$

large output | show less | show more | show all | set size limit...

```
Short[t3 = (t1 ≡ t2)[[1]]]
```

$$\frac{1}{2\, b_x^2\, b_y^2}\, e^{-2 b_y}\, (-4 \ll 6 \gg w_z + 4 \ll 6 \gg w_z + \ll 949 \gg + \ll 1 \gg)$$

```
Short[Errors = CoefficientRules[Expand[t3], {cx, cy, cz, ux, uy, uz, wx, wy, wz}] /.
    {(_ → c_) :> c} /. {bx → x, by → y, bz → z}]
```

$$\left\{-\frac{2\in f_3[x,z]}{x}+\frac{2\,e^x\in f_3[x,z]}{x}+\in f_{10}[x,z]-e^x\in f_{10}[x,z],\right.$$
$$\left.\text{<<56>>},\in f_8[x,z]-e^x\in f_8[x,z]\right\}$$

```
EC = {}; E0 = Errors; gn = hn = 0;
EC = ECAllButOne;
{gn = Cases[EC, g_n_[_] :> n, ∞] // Max, hn = Cases[EC, h_n_[] :> n, ∞] // Max}
```

```
{8, 1}
```

```
EC = ECAll; {gn = Cases[EC, g_n_[_] :> n, ∞] // Max, hn = Cases[EC, h_n_[] :> n, ∞] // Max}
```

```
{10, 1}
```

```
AddRule[ff_, rule_] := (
   Print["As ", e0, ", adding ", rule];
   done = False; EC = EC⋃{rule}
  );
MF[ϕ_, v_] := Module[{t = ϕ, t1}, If[Simplify[t] === 0, 1,
    While[(t1 = Simplify[t /. v → 0]) === 0, t = D[t, v]]; t1
   ]];
done = False; While[!done, done = True;
 E0 = DeleteCases[Simplify[E0 //. EC], 0] // SortBy[LeafCount];
 Print["Length[E0]==", Length[E0],
  "; Length[EC]==", Length[EC], "; {gn,hn}==", {gn, hn}];
 For[k = 1, k ≤ Length@E0, ++k,
  e1 = Factor[e0 = E0[[k]]];
  If[Head[e1] =!= Times, e2 = e1,
   E0[[k]] = e2 = Select[e1, !FreeQ[#, f_[_, _] | g_[_] | h_[]] &] ];
  If[e2 == 1, Print["Panic at ", e0, "! No solutions."]; Break[]];
  If[!FreeQ[e2, f_[_, _]] ∧ (FreeQ[e2, x] ⋁ FreeQ[e2, y] ⋁ FreeQ[e2, z]),
   {ff} = Cases[e2, f_[_, _], {0, ∞}, 1];
   {{sol}} = Solve[e2 == 0, ff];
   rule = ((ff /. {x → x_, y → y_, z → z_}) → (ff /. sol)) /. Rule → RuleDelayed;
   AddRule[ff, rule]; Break[]
  ];
  If[!FreeQ[e2, g_[_]] ∧ (FreeQ[e2, y | z] ⋁ FreeQ[e2, x | z] ⋁ FreeQ[e2, z | y]),
   {gg} = Cases[e2, g_[_], ∞, 1];
   {{sol}} = Solve[e2 == 0, gg];
   rule = ((gg /. {x → x_, y → y_, z → z_}) → (gg /. sol)) /. Rule → RuleDelayed;
   AddRule[gg, rule]; Break[]
  ];
  If[Head[e2] === Plus,
```

```
s = List @@ Collect[e2, f_[_, _], Factor]; s1 = Select[s, FreeQ[f_[_, _]]];
sxy = Cases[s, a_. * f_[x, y]];
sxz = Cases[s, a_. * f_[x, z]]; syz = Cases[s, a_. * f_[y, z]];
Which[
 sxy == {} ⋀ sxz =!= {} ⋀ syz =!= {}, (Print["here"];
  {ff} = Cases[sxz, a_. * fk_[x, z] :> fk[x, z], {1}, 1];
  mf = MF[First@sxz /. f_[x, z] → 1, x]; mf *= MF[First@syz /. f_[y, z] → 1, y];
  s1 = Plus @@ Simplify[s1 / mf];
  sxz = Plus @@ Simplify[sxz / mf]; syz = Plus @@ Simplify[syz / mf];
  Print[{mf, s1, sxz, syz}];
  If[FreeQ[sxz, y] ⋀ FreeQ[syz, x] ⋀
    FreeQ[s1, x | y] ⋀ Simplify[(sxz /. x → y) + syz == 0],
   {{sol}} = Solve[sxz == g++gn[z], ff];
   rule = ((ff /. {x → x_, y → y_, z → z_}) → (ff /. sol)) /. Rule → RuleDelayed;
   AddRule[ff, rule]; Break[]
  ]
 ),
 syz == {} ⋀ sxy =!= {} ⋀ sxz =!= {}, (
  {ff} = Cases[sxy, a_. * fk_[x, y] :> fk[x, y], {1}, 1];
  mf = MF[First@sxy /. f_[x, y] → 1, y]; mf *= MF[First@sxz /. f_[x, z] → 1, z];
  s1 = Plus @@ Simplify[s1 / mf];
  sxy = Plus @@ Simplify[sxy / mf]; sxz = Plus @@ Simplify[sxz / mf];
  If[FreeQ[sxy, z] ⋀ FreeQ[sxz, y] ⋀ FreeQ[s1, y | z] ⋀
    Simplify[(sxz /. z → y) + sxy == 0],
   {{sol}} = Solve[sxy == g++gn[x], ff];
   rule = ((ff /. {x → x_, y → y_, z → z_}) → (ff /. sol)) /. Rule → RuleDelayed;
   AddRule[ff, rule]; Break[]
  ]
 ),
 sxy =!= {} ⋀ sxz =!= {} ⋀ syz =!= {}, (
  kk = Union@Cases[e2, a_. * fk_[x, y] :> k, ∞];
  If[Length[kk] == 1,
   {kk} = kk;
   {{sol}} = Solve[e2 == 0, fkk[x, y]];
   sol = fkk[x, y] /. sol;
   e3 = D[sol, z] // Factor;
   If[FreeQ[e3, f_[_, _]],
    If[Head[e3] === Times,
     e3 = Select[e3, ! FreeQ[#, f_^(0,1)[_, _] | g_[_] | h_[]] &]];
    s = Collect[e3, f_^(0,1)[_, _], Factor];
    s1 = Select[s, FreeQ[f_^(0,1)[_, _]]];
    pxz = Coefficient[s, fkk^(0,1)[x, z]];
    pyz = Coefficient[s, fkk^(0,1)[y, z]];
```

```
      mf = MF[pxz, x]; mf *= MF[pyz, y];
      {s1, pxz, pyz} = Simplify[{s1, pxz, pyz} / mf];
      If[FreeQ[pxz, y] ∧ FreeQ[pyz, x] ∧
        FreeQ[s1, x | y] ∧ Simplify[(pyz /. y → x) + pxz == 0],
       rule = (fkk[x_, z_] → g++gn[z] / pxz + g++gn[x]) /. Rule → RuleDelayed;
       AddRule[fkk[x, z], rule]; Break[]
      ]
     ]
    ]
   ];
   If[FreeQ[e2, f_[_, _]] ∧ ! FreeQ[e2, g_[_]],
    s = List @@ Collect[e2, g_[_], Factor]; s1 = Select[s, FreeQ[g_[_]]];
    sx = Cases[s, a_. * g_[x]]; sy = Cases[s, a_. * g_[y]]; sz = Cases[s, a_. * g_[z]];
    Which[
     FreeQ[e2, x] ∧ sy =!= {} ∧ sz =!= {}, (
      {gg} = Cases[sy, a_. * gk_[y] :> gk[y], {1}, 1];
      mf = MF[First@sy /. g_[y] → 1, y]; mf *= MF[First@sz /. g_[z] → 1, z];
      s1 = Plus @@ Simplify[s1 / mf];
      sy = Plus @@ Simplify[sy / mf]; sz = Plus @@ Simplify[sz / mf];
      If[FreeQ[sx, y] ∧ FreeQ[sz, y] ∧ FreeQ[s1, y | z] ∧
        Simplify[(sz /. z → y) + sy == 0],
       {{sol}} = Solve[sy == h++hn[], gg];
       rule = ((gg /. {x → x_, y → y_, z → z_}) → (gg /. sol)) /. Rule → RuleDelayed;
       AddRule[gg, rule]; Break[]
      ]
     ),
     FreeQ[e2, z] ∧ sy =!= {} ∧ sx =!= {}, (
      {gg} = Cases[sy, a_. * gk_[y] :> gk[y], {1}, 1];
      mf = MF[First@sy /. g_[y] → 1, y]; mf *= MF[First@sx /. g_[x] → 1, x];
      s1 = Plus @@ Simplify[s1 / mf];
      sy = Plus @@ Simplify[sy / mf]; sx = Plus @@ Simplify[sx / mf];
      If[FreeQ[sz, y] ∧ FreeQ[sx, y] ∧ FreeQ[s1, y | x] ∧
        Simplify[(sx /. x → y) + sy == 0],
       {{sol}} = Solve[sy == h++hn[], gg];
       rule = ((gg /. {x → x_, y → y_, z → z_}) → (gg /. sol)) /. Rule → RuleDelayed;
       AddRule[gg, rule]; Break[]
      ]
     )
    ] (* Which *)
   ] (* If *)
  ] (* If *)
 ] (* For *)
```

```
]; (* While *)
E0 = Union[DeleteCases[Simplify[E0 //. EC], 0]] // SortBy[LeafCount];

Length[E0]==0; Length[EC]==24; {gn,hn}=={10, 1}
```

EC

$$\Big\{f_6[x\_,\ z\_] \to \frac{-\frac{x\, f_7[x,z]}{-1+e^x} + g_9[x] + g_{10}[z]}{x},$$

$$g_{10}[y\_] \to \frac{1}{2\,(-1+e^y)^2\,y}\Big(-1 + e^y - y^2\,g_4[y] + 2\,e^y\,y^2\,g_4[y] - 4\,y^3\,g_5[y] + 8\,e^y\,y^3\,g_5[y] - 4\,e^{2\,y}\,y^3\,g_5[y] + 2\,e^y\,y\,g_6[y] - 2\,e^{2\,y}\,y\,g_6[y] - y\,g_8[y] + e^y\,y\,g_8[y] - 2\,y\,g_9[y] + 4\,e^y\,y\,g_9[y] - 2\,e^{2\,y}\,y\,g_9[y] - e^y\,y\,h_1[]\Big),$$

$$f_1[x\_,\ z\_] :\to \frac{1}{2}\Big(-\Big(\Big(x\,\big(-2\,(-1+e^x)^2\,f_6[x,\,z] - 2\,x\,f_{20}[x,\,z] + (-1+e^x)\,\big(g_4[x] + 4\,(-1+e^x)^2\,x\,g_5[x]\big)\big)\Big)\Big/ (-1+e^x)^2\Big) + g_8[z]\Big),$$

$$f_2[x\_,\ y\_] :\to y\,f_9[x,\,y] + y^2\,f_{26}[x,\,y] - g_6[x],\quad f_3[x\_,\ z\_] :\to x^2\,f_{18}[x,\,z],$$

$$f_4[x\_,\ y\_] :\to y\,f_{13}[x,\,y] - g_1[x],$$

$$f_5[x\_,\ y\_] :\to y^2\,f_{26}[x,\,y],$$

$$f_8[x\_,\ z\_] :\to 0,$$

$$f_{10}[x\_,\ z\_] :\to 2\,x\,f_{18}[x,\,z],$$

$$f_{11}[x\_,\ y\_] :\to 2\,x\,f_{18}[x,\,y] - 2\,e^x\,x\,f_{18}[x,\,y] - g_4[x],$$

$$f_{12}[x\_,\ z\_] :\to 0,$$

$$f_{13}[x\_,\ z\_] :\to x\,f_{22}[x,\,z] - g_3[z],$$

$$f_{14}[x\_,\ y\_] :\to y\,f_{22}[x,\,y] - g_2[x],$$

$$f_{15}[x\_,\ y\_] :\to \frac{1}{(-1+e^y)\,x}\Big(-x\,y\,f_{23}[x,\,y] + e^y\,x\,y\,f_{23}[x,\,y] + x\,g_2[x] - e^x\,x\,g_2[x] - e^y\,x\,g_2[x] + e^{x+y}\,x\,g_2[x] - e^y\,y\,g_2[y] + e^{x+y}\,y\,g_2[y] - y\,g_3[y] + e^x\,y\,g_3[y] - y\,g_4[y] + e^x\,y\,g_4[y] + 8\,y^2\,g_5[y] - 8\,e^x\,y^2\,g_5[y] - 16\,e^y\,y^2\,g_5[y] + 8\,e^{2\,y}\,y^2\,g_5[y] + 16\,e^{x+y}\,y^2\,g_5[y] - 8\,e^{x+2\,y}\,y^2\,g_5[y] + 4\,y\,g_7[y] - 4\,e^x\,y\,g_7[y] - 4\,e^y\,y\,g_7[y] + 4\,e^{x+y}\,y\,g_7[y]\Big),$$

$$f_{16}[x\_,\ z\_] :\to 0,\quad f_{17}[x\_,\ y\_] :\to 2\,y\,f_{26}[x,\,y],\quad f_{18}[x\_,\ y\_] :\to -\Big(\big(f_{19}[x,\,y] + 4\,g_5[x] - 8\,e^x\,g_5[x] + 4\,e^{2\,x}\,g_5[x]\big)\Big/(2\,(-1+e^x))\Big),$$

$$f_{19}[x\_,\ y\_] :\to \frac{1}{(-1+e^x)\,(-1+e^y)\,x^2\,y}\Big(-1 + 2\,e^x - e^{2\,x} + e^y - 2\,e^{x+y} + e^{2\,x+y} + 2\,x^2\,y\,f_{20}[x,\,y] - 2\,e^y\,x^2\,y\,f_{20}[x,\,y] + x\,y\,g_4[x] - e^x\,x\,y\,g_4[x] - e^y\,x\,y\,g_4[x] + e^{x+y}\,x\,y\,g_4[x] + y^2\,g_4[y] - 2\,e^x\,y^2\,g_4[y] + e^{2\,x}\,y^2\,g_4[y] - 4\,y^3\,g_5[y] + 8\,e^x\,y^3\,g_5[y] - 4\,e^{2\,x}\,y^3\,g_5[y] + 8\,e^y\,y^3\,g_5[y] - 4\,e^{2\,y}\,y^3\,g_5[y] - 16\,e^{x+y}\,y^3\,g_5[y] + 8\,e^{2\,x+y}\,y^3\,g_5[y] + 8\,e^{x+2\,y}\,y^3\,g_5[y] - 4\,e^{2\,x+2\,y}\,y^3\,g_5[y] - e^y\,y\,h_1[] + 2\,e^{x+y}\,y\,h_1[] - e^{2\,x+y}\,y\,h_1[]\Big),$$

$$f_{21}[x\_,\ z\_] :\to 0,\quad f_{22}[x\_,\ z\_] :\to \frac{-x\,f_{23}[x,\,z] - 4\,g_7[z] + 4\,e^x\,g_7[z]}{(-1+e^x)\,x},$$

$$f_{24}[x\_,\ z\_] :\to 0,$$

$$f_{25}[x\_,\ z\_] :\to 0,$$

$$g_1[y\_] :\to y\,g_2[y] + h_1[],$$

$$g_2[y\_] :\to \frac{1}{2\,y^2}e^{-y}\Big(-2 + 2\,e^y - 2\,y^2\,g_3[y] + 8\,y^3\,g_5[y] - 16\,e^y\,y^3\,g_5[y] + 8\,e^{2\,y}\,y^3\,g_5[y] + 8\,y^2\,g_7[y] - 8\,e^y\,y^2\,g_7[y] - y\,h_1[] - e^y\,y\,h_1[]\Big)\Big\}$$

```
E0
```

```
{}
```

A specific solution of the last remaining equation:

```
E0[[1]] /. {f7[__] → 0, g_[_] → 0, h_[] → 0, f6[x_, y_] :→ 1/(2 x y (e^y - 1))} // Simplify
```

```
0
```

```
eq0 = Collect[E0[[1]]/((-1 + e^x) (-1 + e^y)), f6[__], FullSimplify]
```

$$2\,(-1+e^y)\,x\,y\,f_6[x,y]-2\,(-1+e^y)\,x\,y\,f_6[x,z]+2\,(-1+e^y)\,y^2\,f_6[y,z]+\frac{1}{(-1+e^x)\,(-1+e^y)}\Big(2\,(-1+e^y)^2\,x\,y\,f_7[x,y]-2\,(-1+e^y)^2\,x\,y\,f_7[x,z]+(-1+e^x)\,\Big(1-e^y+y\,\Big(2\,(-1+e^y)\,y\,f_7[y,z]+(y-2\,e^y\,y)\,g_4[y]+g_8[y]+e^y\,\Big(8\,y^2\,(-1+\text{Cosh}[y])\,g_5[y]+2\,(-1+e^y)\,g_6[y]-g_8[y]+h_1[]\Big)\Big)\Big)\Big)$$

```
eq1 = 1/(1/(-1+e^x)^2 2 (-1 + e^y) y) D[eq0, x] // Simplify
```

$$(-1+e^x)^2\,f_6[x,y]-(-1+e^x)^2\,f_6[x,z]-f_7[x,y]+e^x\,f_7[x,y]-e^x\,x\,f_7[x,y]+f_7[x,z]-e^x\,f_7[x,z]+e^x\,x\,f_7[x,z]+x\,f_6^{(1,0)}[x,y]-2\,e^x\,x\,f_6^{(1,0)}[x,y]+e^{2x}\,x\,f_6^{(1,0)}[x,y]-x\,f_6^{(1,0)}[x,z]+2\,e^x\,x\,f_6^{(1,0)}[x,z]-e^{2x}\,x\,f_6^{(1,0)}[x,z]-x\,f_7^{(1,0)}[x,y]+e^x\,x\,f_7^{(1,0)}[x,y]+x\,f_7^{(1,0)}[x,z]-e^x\,x\,f_7^{(1,0)}[x,z]$$

```
(eq1 /. f_[x, y] | f_^(1,0)[x, y] → 0) + (eq1 /. f_[x, z] | f_^(1,0)[x, z] → 0) == eq1 //
 Simplify
```

```
True
```

```
FreeQ[(eq1 /. f_[x, y] | f_^(1,0)[x, y] → 0), y]
```

```
True
```

```
FreeQ[(eq1 /. f_[x, z] | f_^(1,0)[x, z] → 0), z]
```

```
True
```

```
Collect[(-eq1 /. f_[x, y] | f_^(1,0)[x, y] → 0), f6[x, z] | f6^(1,0)[x, z], Simplify]
```

$$(-1+e^x)^2\,f_6[x,z]+(-1-e^x\,(-1+x))\,f_7[x,z]+(-1+e^x)^2\,x\,f_6^{(1,0)}[x,z]+(-1+e^x)\,x\,f_7^{(1,0)}[x,z]$$

```
DSolve[D[a[x] (-1 + e^x)^2 x, x] == (-1 + e^x)^2 a[x], a[x], x]
```

$$\left\{\left\{a[x]\to\frac{C[1]}{(-1+e^x)^2}\right\}\right\}$$

```
eq2 = Collect[1/(-1 + E^x)^2 (-eq1 /. f_[x, y] | f_^(1,0)[x, y] → 0),
   f6[x, z] | f6^(1,0)[x, z], Simplify]
```

$$f_6[x, z] + x\, f_6^{(1,0)}[x, z] + \frac{1}{(-1+e^x)^2}\left((-1 - e^x\,(-1+x))\, f_7[x, z] + (-1+e^x)\, x\, f_7^{(1,0)}[x, z]\right)$$

```
Collect[1/(-1 + E^x)^2 (-eq1 /. f_[x, y] | f_^(1,0)[x, y] → 0),
  f7[x, z] | f7^(1,0)[x, z], Simplify]
```

$$f_6[x, z] - \frac{(1+e^x\,(-1+x))\, f_7[x, z]}{(-1+e^x)^2} + x\, f_6^{(1,0)}[x, z] + \frac{x\, f_7^{(1,0)}[x, z]}{-1+e^x}$$

```
D[x f6[x, z] + (x f7[x, z])/(-1 + E^x), x] - eq2 // Simplify
```

$$0$$

```
Simplify[f6[x, z] /. First@Solve[x f6[x, z] + (x f7[x, z])/(-1 + E^x) == g9[x] + g10[z], f6[x, z]]]
```

$$\frac{-\frac{x\, f_7[x,z]}{-1+e^x} + g_9[x] + g_{10}[z]}{x}$$

```
eq3 = Simplify[eq0 /. f6[x_, z_] → (-(x f7[x,z])/(-1+E^x) + g9[x] + g10[z])/x]
```

$$\frac{1}{-1+e^y}\left(1 - e^y - (-1+2\,e^y)\, y^2\, g_4[y] + 4\,(-1+e^y)^2\, y^3\, g_5[y] - 2\,e^y\, y\, g_6[y] + 2\,e^{2y}\, y\, g_6[y] + y\, g_8[y] - e^y\, y\, g_8[y] + 2\, y\, g_9[y] - 4\,e^y\, y\, g_9[y] + 2\,e^{2y}\, y\, g_9[y] + 2\, y\, g_{10}[y] - 4\,e^y\, y\, g_{10}[y] + 2\,e^{2y}\, y\, g_{10}[y] + e^y\, y\, h_1[]\right)$$

```
First@Solve[eq3 == 0, g10[y]]
```

$$\left\{g_{10}[y] \to \frac{1}{2\,(-1+e^y)^2\, y}\left(-1 + e^y - y^2\, g_4[y] + 2\,e^y\, y^2\, g_4[y] - 4\, y^3\, g_5[y] + 8\,e^y\, y^3\, g_5[y] - 4\,e^{2y}\, y^3\, g_5[y] + 2\,e^y\, y\, g_6[y] - 2\,e^{2y}\, y\, g_6[y] - y\, g_8[y] + e^y\, y\, g_8[y] - 2\, y\, g_9[y] + 4\,e^y\, y\, g_9[y] - 2\,e^{2y}\, y\, g_9[y] - e^y\, y\, h_1[]\right)\right\}$$

```
Simplify[eq0 /. z → y]
```

$$2\,(-1+e^y)\, y^2\, f_6[y, y] + \frac{1}{-1+e^y}\left(1 - e^y + y\left(2\,(-1+e^y)\, y\, f_7[y, y] + (y - 2\,e^y\, y)\, g_4[y] + g_8[y] + e^y\left(8\, y^2\,(-1+\text{Cosh}[y])\, g_5[y] + 2\,(-1+e^y)\, g_6[y] - g_8[y] + h_1[]\right)\right)\right)$$

```
Simplify[eq0 /. x → y]
```

$$\frac{1}{-1+e^y}\left(1 - e^y + 2\,(-1+e^y)^2\, y^2\, f_6[y, y] + 2\,(-1+e^y)\, y^2\, f_7[y, y] + y^2\, g_4[y] - 2\,e^y\, y^2\, g_4[y] - 8\,e^y\, y^3\, g_5[y] + 8\,e^y\, y^3\, \text{Cosh}[y]\, g_5[y] - 2\,e^y\, y\, g_6[y] + 2\,e^{2y}\, y\, g_6[y] + y\, g_8[y] - e^y\, y\, g_8[y] + e^y\, y\, h_1[]\right)$$

```
Simplify[eq0 /. z → x]
```

$$-2\,(-1+e^{y})\,x\,y\,f_6[x,x]+2\,(-1+e^{y})\,x\,y\,f_6[x,y]+2\,(-1+e^{y})\,y^2\,f_6[y,x]+\frac{1}{(-1+e^{x})\,(-1+e^{y})}\Big(-2\,(-1+e^{y})^2\,x\,y\,f_7[x,x]+2\,(-1+e^{y})^2\,x\,y\,f_7[x,y]+(-1+e^{x})\Big(1-e^{y}+y\Big(2\,(-1+e^{y})\,y\,f_7[y,x]+(y-2\,e^{y}\,y)\,g_4[y]+g_8[y]+e^{y}\Big(8\,y^2\,(-1+\text{Cosh}[y])\,g_5[y]+2\,(-1+e^{y})\,g_6[y]-g_8[y]+h_1[]\Big)\Big)\Big)\Big)$$

```
Simplify[E0 /. f6[x_, z_] → (-x f7[x,z]/(-1+e^x) + g9[x] + g10[z])/x /. g10[y_] → 1/(2 (-1+e^y)^2 y)]
```

$$\Big(-1+e^{y}-y^2\,g_4[y]+2\,e^{y}\,y^2\,g_4[y]-4\,y^3\,g_5[y]+8\,e^{y}\,y^3\,g_5[y]-4\,e^{2\,y}\,y^3\,g_5[y]+2\,e^{y}\,y\,g_6[y]-2\,e^{2\,y}\,y\,g_6[y]-y\,g_8[y]+e^{y}\,y\,g_8[y]-2\,y\,g_9[y]+4\,e^{y}\,y\,g_9[y]-2\,e^{2\,y}\,y\,g_9[y]-e^{y}\,y\,h_1[]\Big)\Big]$$

```
{0}
```