

$$\begin{aligned}
 \text{eq0} = & 2 (-1 + e^y) x y f_6[x, y] - 2 (-1 + e^y) x y f_6[x, z] + 2 (-1 + e^y) y^2 f_6[y, z] + \\
 & \frac{1}{(-1 + e^x) (-1 + e^y)} \left(2 (-1 + e^y)^2 x y f_7[x, y] - 2 (-1 + e^y)^2 x y f_7[x, z] + \right. \\
 & \quad \left. (-1 + e^x) \left(1 - e^y + y \left(2 (-1 + e^y) y f_7[y, z] + (y - 2 e^y y) g_4[y] + g_8[y] + \right. \right. \right. \\
 & \quad \quad \left. \left. \left. e^y \left(8 y^2 (-1 + \text{Cosh}[y]) g_5[y] + 2 (-1 + e^y) g_6[y] - g_8[y] + h_1[] \right) \right) \right) \right) \\
 & 2 (-1 + e^y) x y f_6[x, y] - 2 (-1 + e^y) x y f_6[x, z] + 2 (-1 + e^y) y^2 f_6[y, z] + \\
 & \frac{1}{(-1 + e^x) (-1 + e^y)} \left(2 (-1 + e^y)^2 x y f_7[x, y] - 2 (-1 + e^y)^2 x y f_7[x, z] + \right. \\
 & \quad \left. (-1 + e^x) \left(1 - e^y + y \left(2 (-1 + e^y) y f_7[y, z] + (y - 2 e^y y) g_4[y] + g_8[y] + \right. \right. \right. \\
 & \quad \quad \left. \left. \left. e^y \left(8 y^2 (-1 + \text{Cosh}[y]) g_5[y] + 2 (-1 + e^y) g_6[y] - g_8[y] + h_1[] \right) \right) \right) \right)
 \end{aligned}$$

Simplify[eq0 /. z -> 0]

$$\begin{aligned}
 & -2 (-1 + e^y) x y f_6[x, 0] + 2 (-1 + e^y) x y f_6[x, y] + 2 (-1 + e^y) y^2 f_6[y, 0] + \\
 & \frac{1}{(-1 + e^x) (-1 + e^y)} \left(-2 (-1 + e^y)^2 x y f_7[x, 0] + 2 (-1 + e^y)^2 x y f_7[x, y] + \right. \\
 & \quad \left. (-1 + e^x) \left(1 - e^y + y \left(2 (-1 + e^y) y f_7[y, 0] + (y - 2 e^y y) g_4[y] + g_8[y] + \right. \right. \right. \\
 & \quad \quad \left. \left. \left. e^y \left(8 y^2 (-1 + \text{Cosh}[y]) g_5[y] + 2 (-1 + e^y) g_6[y] - g_8[y] + h_1[] \right) \right) \right) \right)
 \end{aligned}$$

{sol0} = Simplify[f6[x, y] /. Solve[(eq0 /. z -> 0) == 0, f6[x, y]]]

$$\left\{ -\frac{1}{2 (-1 + e^y) x y} \left(-2 (-1 + e^y) x y f_6[x, 0] + 2 (-1 + e^y) y^2 f_6[y, 0] + \right. \right. \\
 \frac{1}{(-1 + e^x) (-1 + e^y)} \left(-2 (-1 + e^y)^2 x y f_7[x, 0] + 2 (-1 + e^y)^2 x y f_7[x, y] + \right. \\
 \left. \left. (-1 + e^x) \left(1 - e^y + y \left(2 (-1 + e^y) y f_7[y, 0] + (y - 2 e^y y) g_4[y] + g_8[y] + \right. \right. \right. \right. \\
 \left. \left. \left. e^y \left(8 y^2 (-1 + \text{Cosh}[y]) g_5[y] + 2 (-1 + e^y) g_6[y] - g_8[y] + h_1[] \right) \right) \right) \right) \left. \right\}$$

Simplify[eq0 /. f6[x_, y_] -> sol0]

0

Series[sol0, {y, 0, 0}]

$$\begin{aligned}
 & \frac{1 - h_1[]}{2 x y^2} + \frac{-1 + 2 g_4[0] - 4 g_6[0] + 2 g_8[0]}{4 x y} + \\
 & \frac{1}{24 x} (1 + 24 x f_6[x, 0] - 24 f_7[0, 0] + 12 g_4[0] - 12 g_6[0] - \\
 & \quad 6 g_8[0] + h_1[] + 12 g_4'[0] - 24 g_6'[0] + 12 g_8'[0]) + O[y]^1
 \end{aligned}$$

Series[sol, {y, 0, 1}]

$$\frac{1 - h_1[]}{2 x y^2} + \frac{-1 + 2 g_4[0] - 4 g_6[0] + 2 g_8[0]}{4 x y} +$$

$$\frac{1}{24 x} (1 + 24 x f_6[x, 0] - 24 f_7[0, 0] + 12 g_4[0] - 12 g_6[0] -$$

$$6 g_8[0] + h_1[] + 12 g_4'[0] - 24 g_6'[0] + 12 g_8'[0]) -$$

$$\frac{1}{24 ((-1 + e^x) x)} (-24 f_6[0, 0] + 24 e^x f_6[0, 0] + 12 f_7[0, 0] - 12 e^x f_7[0, 0] - 7 g_4[0] +$$

$$7 e^x g_4[0] - 2 g_6[0] + 2 e^x g_6[0] + g_8[0] - e^x g_8[0] + 12 g_4'[0] - 12 e^x g_4'[0] - 12 g_6'[0] +$$

$$12 e^x g_6'[0] - 6 g_8'[0] + 6 e^x g_8'[0] + 6 g_4''[0] - 6 e^x g_4''[0] - 12 g_6''[0] + 12 e^x g_6''[0] +$$

$$6 g_8''[0] - 6 e^x g_8''[0] + 24 x f_7^{(0,1)}[x, 0] - 24 f_7^{(1,0)}[0, 0] + 24 e^x f_7^{(1,0)}[0, 0]) y + O[y]^2$$

Solve[-1 + 2 g4[0] - 4 g6[0] + 2 g8[0] == 0, g8[0]]

$$\left\{ \left\{ g_8[0] \rightarrow \frac{1}{2} (1 - 2 g_4[0] + 4 g_6[0]) \right\} \right\}$$

sol1 = Simplify[sol /. {h1[] -> 1, g8[0] -> 1/2 (1 - 2 g4[0] + 4 g6[0])}]

$$- \frac{1}{2 (-1 + e^y) x y} \left(-2 (-1 + e^y) x y f_6[x, 0] + 2 (-1 + e^y) y^2 f_6[y, 0] +$$

$$\frac{1}{(-1 + e^x) (-1 + e^y)} \left(-2 (-1 + e^y)^2 x y f_7[x, 0] + 2 (-1 + e^y)^2 x y f_7[x, y] +$$

$$(-1 + e^x) (1 - e^y + y (2 (-1 + e^y) y f_7[y, 0] + (y - 2 e^y y) g_4[y] +$$

$$e^y (1 + 8 y^2 (-1 + \text{Cosh}[y]) g_5[y] + 2 (-1 + e^y) g_6[y] - g_8[y]) + g_8[y]) \right) \right)$$

Normal[Series[sol1, {y, 0, 1}]] /.

$$\left\{ h_1[] \rightarrow 1, g_8[0] \rightarrow \frac{1}{2} (1 - 2 g_4[0] + 4 g_6[0]) \right\} // \text{Simplify}$$

$$- \frac{1}{48 (-1 + e^x) x} \left(-48 (-1 + e^x) x f_6[x, 0] + 48 x y f_7^{(0,1)}[x, 0] +$$

$$(-1 + e^x) (2 + 48 f_7[0, 0] - 36 g_4[0] + 48 g_6[0] - 24 g_4'[0] + 48 g_6'[0] -$$

$$24 g_8'[0] + y (-1 + 48 f_6[0, 0] - 24 f_7[0, 0] + 16 g_4[0] - 24 g_4'[0] + 24 g_6'[0] +$$

$$12 g_8'[0] - 12 g_4''[0] + 24 g_6''[0] - 12 g_8''[0] + 48 f_7^{(1,0)}[0, 0]) \right)$$