Pensieve header: The OneCo project using the (b-ϵc)-scapegoated low algebra.

```
ϵ /: ϵ^2 = 0;
PBWBasis = {c, u, w};
B[U@c, U@w] = -(B[U@w, U@c] = U@w);
B[U@u, U@c] = -(B[U@c, U@u] = U@u);
B[U@w, U@u] = -(B[U@u, U@w] = b U[] - 2 ϵ U@c);
```

```
UU[l___, x_^n_, r___] := UU[l, Sequence @@ Table[x, {n}], r];
UU[l___, 1, r___] := UU[l, r];
UU[] = U[]; UU[l_, r___] := U[l] ** UU[r];
U_i_[ℰ_] := ℰ /. {b → b_i, u_U :> Replace[u, x_ :> x_i, 1]};
```

```
B[x_, x_] = 0;
B[U[(x_)_i_], U[(y_)_i_]] := B[U[x_i], U[y_i]] = U_i[B[U@x, U@y]];
B[U[(x_)_i_], U[(y_)_j_]] /; i =!= j := 0;
B[x_, y_] := x ** y - y ** x;
```

```
x_ ≼ y_ := OrderedQ[{x, y}]; x_ ≺ y_ := ! OrderedQ[{y, x}];
Simp[ℰ_] := Collect[ℰ, _U, Expand];
```

```
Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
0 ** _ = _ ** 0 = 0;
x_ ** U[] := x; U[] ** x_ := x;
(a_ * x_U) ** (b_ * y_U) := If[a b === 0, 0, Simp[a b (x ** y)]];
(a_ * x_U) ** y_ := Simp[a (x ** y)]; x_ ** (a_ * y_U) := Simp[a (x ** y)];
(x_Plus) ** y_ := (# ** y) & /@ x; x_ ** (y_Plus) := (x ** #) & /@ y;
U[x_] ** U[y_] := If[x ≺ y, U[x, y], U[y, x] + B[U@x, U@y]];
U[x_] ** U[y1_, yy__] := If[x ≼ y1, U[x, y1, yy], (U@x ** U@y1) ** U@yy];
U[xx__, xn_] ** U[yy__] := U@xx ** (U@xn ** U@yy);
```

```
U[l___, x_^n_, r___] := U[l, Sequence @@ Table[x, {n}], r];
U[l___, 1, r___] := U[l, r];
```

```
bc_i_ := b_i U[] - ϵ U[c_i];
a_i_,j_ := bc_i ** U@c_j + U@u_i ** U@w_j;
```

```
UExp[ℰ_, n_] := Module[{t = U[], k}, U[] + Sum[t = t ** ℰ / k!, {k, n}]] // Simp
```

```
ToDegree[n_][ℰ_] := Normal[
   Simp[ℰ] /. {ϵ → ħ ϵ, b_i_ :> ħ b_i, b → ħ b, x_U :> ħ^Count[x, u | u_] x} /.
    a_. x_U :> Series[a, {ħ, 0, n}] * x] /. ħ → 1
```

```
LBasis[n_Integer] := LBasis[Range[n]];
LBasis[S_] := DeleteCases[0]@
  Module[{i, j, k, l}, SortBy[(# /. {ϵ → 2, c_ → 2, u_ → 2, w_ → 2, U → Times}) &][
    Union@Flatten[{{U[], ϵ U[]},
       Table[{U@cᵢ, U@uᵢ, U@wᵢ, ϵ U@cᵢ, ϵ U@uᵢ, ϵ U@wᵢ}, {i, S}],
       Table[{U[uᵢ, wⱼ], ϵ U[uᵢ, wⱼ],
         ϵ U @@ Sort@{cᵢ, cⱼ}, ϵ U[cᵢ, uⱼ], ϵ U[cᵢ, wⱼ]}, {i, S}, {j, S}],
       Table[{ϵ U[cᵢ, uⱼ, wₖ], ϵ U @@ Sort@{uᵢ, uⱼ, wₖ}, ϵ U @@ Sort@{uᵢ, wⱼ, wₖ}},
        {i, S}, {j, S}, {k, S}],
       Table[ϵ U @@ Sort@{uᵢ, uⱼ, wₖ, wₗ}, {i, S}, {j, S}, {k, S}, {l, S}] }]
   ]]
```

```
BLBasis[n_Integer] := BLBasis[Range[n]];
BLBasis[S_] := DeleteCases[0]@
  Module[{i, j, k, l}, SortBy[(# /. {ϵ → 2, c_ → 2, u_ → 2, w_ → 2, U → Times}) &][
    Union@Flatten[{{U[], ϵ U[]},
       Table[{U@cᵢ, ϵ U@cᵢ}, {i, S}],
       Table[{U[uᵢ, wⱼ], ϵ U[uᵢ, wⱼ], ϵ U @@ Sort@{cᵢ, cⱼ}}, {i, S}, {j, S}],
       Table[{ϵ U[cᵢ, uⱼ, wₖ]}, {i, S}, {j, S}, {k, S}],
       Table[ϵ U @@ Sort@{uᵢ, uⱼ, wₖ, wₗ}, {i, S}, {j, S}, {k, S}, {l, S}] }]
   ]]
```

```
AutoEa[τ_, i_, j_][x_] := Module[{bas, n, k, f, t, guess, InitCond, DiffCond, sol},
  bas = LBasis@{i, j};
  n = 0; guess = Sum[f₊₊ₙ[t] y, {y, bas}];
  InitCond = Table[0 == Coefficient[(guess /. t → 0) - x, y] /. ϵ → 0, {y, bas}];
  DiffCond =
   Table[0 == Coefficient[D[guess, t] - B[Simp[τ aᵢ,ⱼ], guess], y] /. ϵ → 0, {y, bas}];
  {sol} = DSolve[InitCond ⋃ DiffCond, Table[fₖ[t], {k, n}], t];
  guess /. sol /. t → 1
 ]
```

```
Ea[___][0] = 0;
Ea[x___][a_. * U[]] := a U[];
Ea[x___][y_Plus] := Ea[x] /@ y;
Ea[τ_, i_, j_][U@cᵢ_] := U[cᵢ] - (ℯ^(-τ bᵢ) (-1 + ℯ^(τ bᵢ)) U[uᵢ, wⱼ])/bᵢ +
   (ℯ^(-τ bᵢ) ϵ (-1 + ℯ^(τ bᵢ) - τ bᵢ) U[uᵢ, wⱼ])/bᵢ^2 - (ℯ^(-τ bᵢ) ϵ (-1 + ℯ^(τ bᵢ) - τ bᵢ) U[cᵢ, uᵢ, wⱼ])/bᵢ^2 +
   (ℯ^(-τ bᵢ) ϵ (-1 + ℯ^(τ bᵢ) - τ bᵢ) U[cⱼ, uᵢ, wⱼ])/bᵢ^2 + (ℯ^(-2 τ bᵢ) ϵ (-1 + ℯ^(2 τ bᵢ) - 2 ℯ^(τ bᵢ) τ bᵢ) U[uᵢ, uᵢ, wⱼ, wⱼ])/bᵢ^3;
Ea[τ_, i_, j_][U@cⱼ_] := U[cⱼ] + (ℯ^(-τ bᵢ) (-1 + ℯ^(τ bᵢ)) U[uᵢ, wⱼ])/bᵢ -
   (ℯ^(-τ bᵢ) ϵ (-1 + ℯ^(τ bᵢ) - τ bᵢ) U[uᵢ, wⱼ])/bᵢ^2 + (ℯ^(-τ bᵢ) ϵ (-1 + ℯ^(τ bᵢ) - τ bᵢ) U[cᵢ, uᵢ, wⱼ])/bᵢ^2 -
```

```
    e^(-τ b_i) ϵ (-1 + e^(τ b_i) - τ b_i) U[c_j, u_i, w_j] / b_i^2 - e^(-2 τ b_i) ϵ (-1 + e^(2 τ b_i) - 2 e^(τ b_i) τ b_i) U[u_i, u_i, w_j, w_j] / b_i^3;
Ea[τ_, i_, j_][U@c_k_] /; (k =!= i) ∧ (k =!= j) := U@c_k;
Ea[τ_, i_, j_][U@u_i_] :=
  U[u_i] - ϵ τ U[c_j, u_i] - e^(-τ b_i) ϵ (1 - e^(τ b_i) + e^(τ b_i) τ b_i) U[u_i, u_i, w_j] / b_i^2;
Ea[τ_, i_, j_][U@u_j_] := -(-1 + e^(τ b_i)) b_j U[u_i] / b_i - ϵ (1 - e^(τ b_i) + e^(τ b_i) τ b_i) b_j U[u_i] / b_i^2 +
   e^(τ b_i) U[u_j] + ϵ (1 - e^(τ b_i) + e^(τ b_i) τ b_i) b_j U[c_i, u_i] / b_i^2 - e^(τ b_i) ϵ τ U[c_i, u_j] +
   ϵ (-2 b_i + 2 e^(τ b_i) b_i - b_j + e^(τ b_i) b_j - τ b_i b_j) U[c_j, u_i] / b_i^2 -
   1/b_i^3 e^(-τ b_i) ϵ (-b_i + 2 e^(τ b_i) b_i - e^(2 τ b_i) b_i + 2 e^(τ b_i) b_j - 2 e^(2 τ b_i) b_j + e^(τ b_i) τ b_i b_j + e^(2 τ b_i) τ b_i b_j)
    U[u_i, u_i, w_j] + ϵ (1 - e^(τ b_i) + e^(τ b_i) τ b_i) U[u_i, u_j, w_j] / b_i^2;
Ea[τ_, i_, j_][U@u_k_] /; (k =!= i) ∧ (k =!= j) := U@u_k;
Ea[τ_, i_, j_][U@w_i_] :=
  U[w_i] + e^(-τ b_i) (-1 + e^(τ b_i)) U[w_j] - e^(-τ b_i) ϵ (-1 + e^(τ b_i) + τ b_i) U[c_i, w_j] / b_i + ϵ τ U[c_j, w_i] +
   e^(-τ b_i) ϵ (1 - e^(τ b_i) + e^(τ b_i) τ b_i) U[c_j, w_j] / b_i + e^(-τ b_i) ϵ (1 - e^(τ b_i) + e^(τ b_i) τ b_i) U[u_i, w_i, w_j] / b_i^2 +
   e^(-2 τ b_i) (1 + e^(τ b_i)) ϵ (1 - e^(τ b_i) + e^(τ b_i) τ b_i) U[u_i, w_j, w_j] / b_i^2;
Ea[τ_, i_, j_][U@w_j_] := e^(-τ b_i) U[w_j] + e^(-τ b_i) ϵ τ U[c_i, w_j] -
   e^(-2 τ b_i) ϵ (1 - e^(τ b_i) + e^(τ b_i) τ b_i) U[u_i, w_j, w_j] / b_i^2;
Ea[τ_, i_, j_][U@w_k_] /; (k =!= i) ∧ (k =!= j) := U@w_k;
Ea[x___][a_. * u_U] := Simp[a NonCommutativeMultiply @@ (Ea[x] /@ (U /@ u))];
Ea[i_, j_] := Ea[-1, i, j];
```