OVER THEN UNDER TANGLES

DROR BAR-NATAN, ZSUZSANNA DANCSO, AND ROLAND VAN DER VEEN

Abstract. Brilliant wrong ideas should not be buried and forgotten. Instead, they should be mined for the gold that lies underneath the layer of wrong. In this paper we explain how "over then under tangles" lead to an easy classification of knots, and under the surface, also to some valid mathematics: MORE.

Contents

1. Fheorems and Pfoofs

Discussion 1. In Definition 2 we define "Over then Under" (OU) tangles. Then in Fheorem 3 we pfove[1] that every tangle is an OU tangle and in Theorem 4 we show that if a tangle is OU, its OU diagram is essentially unique. Hence (Corollary 6) the classification of knots is completely frivial: a knot is a long knot which is a tangle with one component which can be brought to a unique OU form!

This discussion continues right after the proof of Corollary 6.

Definition 2. An Over-then-Under (OU) tangle diagram is an oriented tangle diagram each of whose strands can be divided in two, such that in the first part is the "over" strand in every crossing it goes through and in the second part it is the "under" strand in every crossing it goes through, so a journey through each strand looks like an OO. . . OUU. . . U sequence of crossings. An OU tangle is an oriented tangle that can be represented by an OU tangle diagram. Some examples are in Figure 1.

through two crossings, first

Comment: anti-alternating!

Fheorem 3. *Every tangle is an OU tangle.*

Pfoof. As in Figure 2, the pfoof is completely frivial. Assume first that strands 1 and 2 are already in OU form (meaning, all their O crossings come before all their U ones) but strand 3 still needs fixing, because at some point it goes under and then over, as on the left of Figure 2. Simply slide strand 1 forward along and over 3 and slide strand 2 back and under

Date: First edition Not Yet, 2020, this edition Mar. 16, 2020.
2010 *Mathematics Subject Classification.* 57M25.
Key words and phrases. knots, braids, tangles, Drinfel'd double.
This work was partially supported by NSERC grant RGPIN-2018-04350.

[1]Please bear with our spelling.

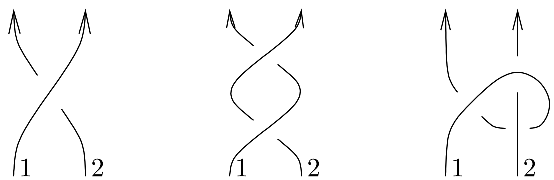
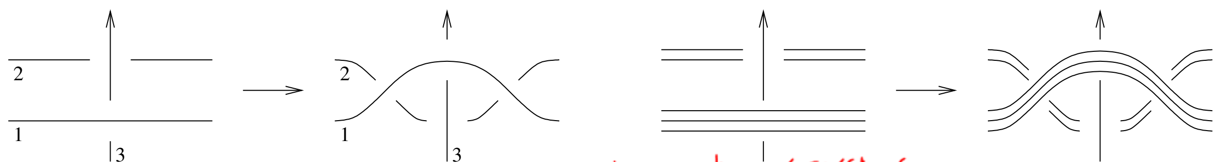


Figure 1. The tangle diagram on the left is OU as strand 1 is all "over" (so it has an empty "U" part) and strand 2 is all "under" (so it has an empty "O" part). The tangle diagram in the middle is not OU: strand 1 is O then U, but strand 2 is U then O. Yet the tangle represented by the middle diagram is OU because it is also represented by the diagram on the right which is OU.



between two crossings

Figure 2. Slide moves and bulk slide moves.

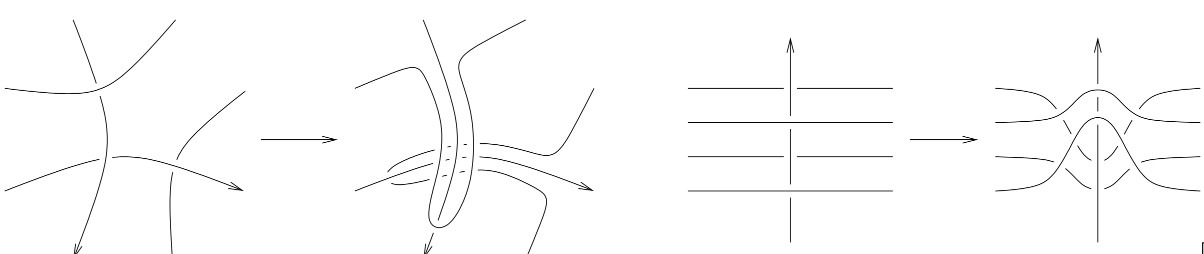
3 as in Figure 2, and the UO interval along 3 is fixed, and nothing is broken on strands 1 and 2 — strand 1 was over and remains over (more precisely, the part of strand 1 that is shown here is the "O" part), and strand 2 is under and remains under.

In fact, it doesn't matter if strands 1 and 2 are already in OU form because as shown in the second part of Figure 2, slide moves can be performed "in bulk". All that the fixing of strand 3 does to strands 1 and 2 is to replace an O by an OOO on strand 1 and a U by a UUU on strand 2, and this does not increase their complexity as UU...UOO...O sequences can be fixed in one go. using bulk slide moves. □

Note that if a tangle diagram is OU then no Reidemeister 3 (R3) moves can be perfomed on it — if one side of an R3 move is OU, the other necessarily isn't. This suggests that perhaps an OU form of a tangle diagram is unique up to Reidemeister 2 (R2) moves. We aim to prove this next.

Theorem 4. *When slide moves (Figure 2) are used to fix a tangle diagram to be OU, the result is independent of the order in which they are used.*

Proof. When UO intervals are apart from each other, their fixing is clearly independent. It remains to see what happens when UO intervals are adjacent, and there are only two distinct cases to consider, as below. Both of these cases are shown here along with their OU fixes, which are clearly independent of the order in which the slide moves are performed:

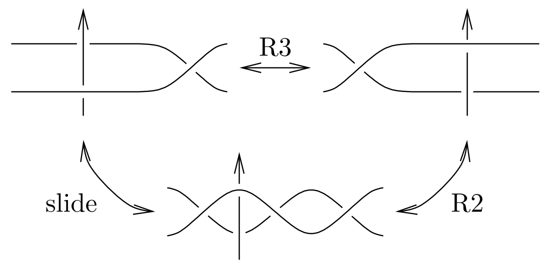


□

what if  ? this is UO⊗UO.

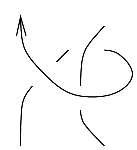
Definition 5. A tangle diagram is called reduced if its crossing number cannot be reduced using only R2 moves.

Corollary 6 (of Fheorem 3 and Theorem 4)**.** *Every tangle has a unique reduced OU diagram. In particulr every long knot has a unique reduced OU diagram, and in particular, the classification of knots is frivial.*



Proof. If two tangle diagrams differ by an R3 move then exactly one of them has a UO interval within the scope of the R3 move, and its elimination via a slide move (which may as well be preformed first) yields the other diagram, up to an R2 move (picture on right). It is also easy to see that R2 moves before a slide become R2 moves after the slide, so the end result of the sliding process of a tangle is unique modulo R2 moves. Finally it is easy to check that within any equivalence class of tangle diagrams modulo R2 moves there is a unique reduced representative. □

Discussion 7. Corollary 6 seems too good to be true, and in fact, it is not true. For while Theorem 4 and its proof are perfectly valid, Fheorem 3 is a Theorem with its T replaced with an F and its pfoof is a spoof with a leaky Halmos. For indeed, while everything we said about slide moves holds true, there is another way a strand may go under and then over — those underpass and overpass may be parts of the very same crossing, as on the right.



It is tempting to dismiss this with "it's only a Reidemeister 1 (R1) issue, and one may anyway slide all kinks to the tail of a strand and count them at the end". Except the same issue can arise in "bulk" UU. . . UOO. . . O situations (as now on the right), where it cannot be easily dismissed. One may attempt to resolve the UUOO situation on the right using single (non-bulk) slide moves. We have no theoretical reason to expect this to work as the lengths of UU. . . U and OO. . . O sequences may build up faster than they are sorted. And indeed, it doesn't work. Figure 3 shows what happens.

2. What hope is left?

3. Theorems and Proofs

Acknowledgement: Oleg Chterental.

References

[Wo] *Wolfram Language & System Documentation Center*, ωεβ/Wolf. See pp. .

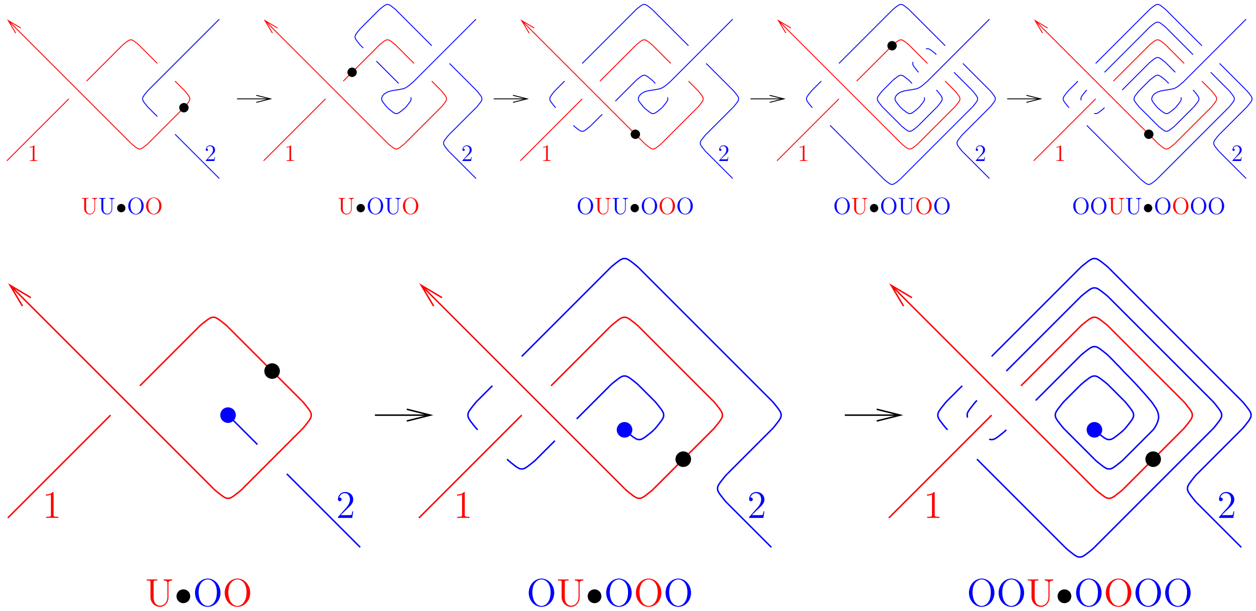


Figure 3. An attempt to fix a non-OU tangle diagram. In each step we use a single slide move to fix the first UO sequence encountered on strand 1 (we mark it with a •), but things get progressively more complicated. The O/U sequences below the diagrams are listed from the perspective of strand 1. (The tangloid/virtual edition is also included).

Department of Mathematics, University of Toronto, Toronto Ontario M5S 2E4, Canada
Email address: `drorbn@math.toronto.edu`
URL: `http://www.math.toronto.edu/drorbn`

School of Mathematics and Statistics, The University of Sydney, Eastern Ave, Camperdown NSW 2006, Australia
Email address: `zsuzsanna.dancso@sydney.edu.au`
URL: `https://sydney.edu.au/science/about/our-people/academic-staff/zsuzsanna.dancso.html`

University of Groningen, Bernoulli Institute, P.O. Box 407, 9700 AK Groningen, The Netherlands
Email address: `roland.mathematics@gmail.com`
URL: `http://www.rolandvdv.nl/`