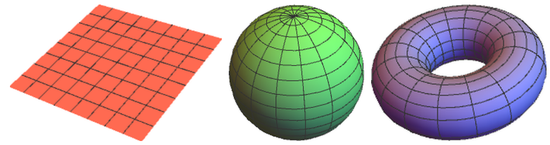
Polynomial Time Knot Polynomials

Summary of Project

Construct, implement, and document poly-time computable polynomial invariants of knots and tangles



Topology is the study of spaces in themselves. What are all the spaces we might be living in? If we were living in 2 dimensions, the answer might have been the plane, or the sphere, or the torus, shown on the right. The list of possibilities is in fact somewhat longer, and topologists understand it quite well. But we live in 4 dimensions (or perhaps more, according to modern physics), and we understand 3- and 4- and higher-dimensional topology much less well. Much is known, but little can be explained in simple terms and on just one page, for even the meaning of the phrase "4-dimensional space" requires some reading.

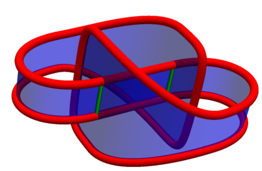
Just one part of topology is easily visible for all: the study of placements of 1-dimensional objects inside 3-dimensional objects, or **Knot Theory**. By luck or by divine justice, the study of knots is both interesting in itself and relevant to the study of higher dimensional topology. My project primarily takes place in knot theory and in closely related subjects in algebra.

Knots are often studied via their *invariants*. Invariants are quantities that can be computed from a picture of a knot and that yet depend only on the intrinsic topological properties of the knot and not on the specifics of how the picture was taken. We know a good number of very useful invariants. Unfortunately, many of them are very difficult to compute and hence can be computed only for knots of relatively small size. Technically speaking, most knot invariants are "exponential time" or worse, whereas we tend to think that truly computable quantities are "polynomial time". It would be extremely desirable, perhaps revolutionary, if we had a few further poly-time computable knot invariants.

The only poly-time invariants we now know are the Alexander polynomial and finite type invariants. The Alexander polynomial can be computed in about n^4 operations (where n is any reasonable measure of the complexity of the knot). Each coefficient of the Alexander polynomial is a numerical invariant, and hence there are infinitely many numerical invariants that can be computed in about n^4 time. Finite type invariants can also be computed in poly-time, but there are only finitely many of them below any n^d time bound. So Alexander stands out as an anomaly or a miracle: an infinite garden within a scarcely planted desert.

This informal picture, of an Alexander miracle in a great desert, is so unusual it must either be proven[1] or refuted. I plan to do the latter by constructing a few new poly-time computable polynomial-valued knot invariants.

I know how to do it (using near-cocommutative 2-dimensional reductions of the Etingof-Kazhdan-Enriquez quantization of classical Yang-Baxter operators and related matters; see "detailed project description"), and I know it will work. I also know it will be hard for the details are daunting (at least for me), though I hope and expect that once the mountains are first climbed, easier paths up will be found. I've been struggling and making progress with these ideas for years now, and I have a unique and uniquely powerful combination of theoretical and computational tools at my disposal. And now is the most critical time within this project: the goal is within reach, yet reaching it would still take a couple years of serious concentration.



I should add that the polynomial invariants I intend to construct extend to tangles and hence stand a fair chance of detecting counterexamples to the "slice equal ribbon" conjecture of 4-dimensional topology (a "ribbon knot", on the right). See my "detailed project description".

[1] Namely, it must be shown that there are no further poly-time invariants.