Loading Pre-Computed Data

In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\HigherRank"];
Once[<< KnotTheory`];
<< Rot.m
```

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
Loading Rot.m from http://drorbn.net/AP/Projects/HigherRank to compute rotation numbers.
```

In[∘]:=
```
CCF[ℰ_] := ExpandDenominator@ExpandNumerator@Together[ℰ];
CCF[ℰ_] := Factor[ℰ];
CF[ℰ_List] := CF /@ ℰ;
CF[ℰ_] := Module[{vs = Cases[ℰ, (x | p | π | g)__, ∞] ⋃ {x, p, ϵ}, ps, c},
   Total[CoefficientRules[Expand[ℰ], vs] /. (ps_ → c_) ⧴ CCF[c] (Times @@ vs^ps)] ];
```

In[∘]:=
```
{
  {r_{0,pxx}[1, i_, j_], r_{0,pxx}[-1, i_, j_]},
  {r_{1,ppx}[1, i_, j_], r_{1,ppx}[-1, i_, j_]},
  {r_{1,rest}[1, i_, j_], r_{1,rest}[-1, i_, j_]},
  γ1[φ_, k_]
 } = CF[Plus[
   {T_1, 1 - T_1 T_2, T_1 (1 - T_1 T_2), T_1 (1 - T_1 T_2)} * Get["px-data.m"],
   -{{0, 0},
     {0, 0},
     {1/2 + T_3 x_{3,i} (p_{3,i+1} - p_{3,j+1}), -1/2 - T_3^-1 x_{3,i} (p_{3,i+1} - p_{3,j+1})},
     φ/2
    },
   {{0, 0},
    {0, 0},
    {α (1 + x_{3,j} (p_{3,j+1} - p_{3,j})) +
        β (1 + x_{3,i} (p_{3,i+1} - p_{3,i}) + (T_3 - 1) x_{3,i} (p_{3,i+1} - p_{3,j+1})) + γ x_{3,i} (p_{3,j+1} - p_{3,j}) +
        δ (x_{3,j} (p_{3,i+1} - p_{3,i}) + (T_3 - 1) x_{3,j} (p_{3,i+1} - p_{3,j+1})) /. {β → 1, γ → -1, δ → 0, α → 0},
     α (1 + x_{3,j} (p_{3,j+1} - p_{3,j})) +
        β (1 + x_{3,i} (p_{3,i+1} - p_{3,i}) + (T_3^-1 - 1) x_{3,i} (p_{3,i+1} - p_{3,j+1})) + γ x_{3,i} (p_{3,j+1} - p_{3,j}) +
        δ (x_{3,j} (p_{3,i+1} - p_{3,i}) + (T_3^-1 - 1) x_{3,j} (p_{3,i+1} - p_{3,j+1})) /. {β → -1, γ → 1, δ → 0, α → 0}},
    0
   }
  ]]
```

Out[∘]=

$$
\begin{aligned}
&\Big\{\Big\{T_1\, p_{3,j}\, x_{1,i}\, x_{2,i} - p_{3,j}\, x_{1,j}\, x_{2,i},\ -\frac{p_{3,j}\, x_{1,i}\, x_{2,i}}{T_1\, T_2} + \frac{p_{3,j}\, x_{1,j}\, x_{2,i}}{T_2}\Big\},\\
&\Big\{(1 - T_1\, T_2)\, p_{1,j}\, p_{2,i}\, x_{3,i} + (-1 + T_1\, T_2)\, p_{1,j}\, p_{2,j}\, x_{3,i},\\
&\quad \frac{(-1 + T_1\, T_2)\, p_{1,j}\, p_{2,i}\, x_{3,i}}{T_1} - \frac{(-1 + T_1\, T_2)\, p_{1,j}\, p_{2,j}\, x_{3,i}}{T_1}\Big\},\\
&\Big\{\frac{1}{2} - T_1\, T_2\, p_{1,j}\, p_{2,j}\, x_{1,i}\, x_{2,i} + \frac{p_{1,j}\, p_{2,i}\, x_{1,j}\, x_{2,i}}{-1 + T_1} + \frac{(-1 + T_1\, T_2)\, p_{1,j}\, p_{2,j}\, x_{1,j}\, x_{2,i}}{-1 + T_1} - \frac{T_1\, p_{1,i}\, p_{2,j}\, x_{1,i}\, x_{2,j}}{-1 + T_1} -\\
&\quad p_{3,i}\, x_{3,i} + T_1\, (-1 + T_1\, T_2)\, p_{1,j}\, p_{3,j}\, x_{1,i}\, x_{3,i} - \frac{p_{1,j}\, p_{3,i}\, x_{1,j}\, x_{3,i}}{-1 + T_1} - \frac{T_1\, (-1 + T_1\, T_2)\, p_{1,j}\, p_{3,j}\, x_{1,j}\, x_{3,i}}{-1 + T_1} +\\
&\quad T_2\, (-1 + T_1\, T_2)\, p_{2,j}\, p_{3,j}\, x_{2,i}\, x_{3,i} + p_{2,j}\, p_{3,i}\, x_{2,j}\, x_{3,i} + \frac{T_1\, p_{1,i}\, p_{3,j}\, x_{1,i}\, x_{3,j}}{-1 + T_1} - T_2\, p_{2,j}\, p_{3,j}\, x_{2,i}\, x_{3,j},\\
&\quad -\frac{1}{2} - \frac{p_{1,j}\, p_{2,i}\, x_{1,i}\, x_{2,i}}{T_1} + \frac{T_1\, (-1 + T_2)\, p_{1,i}\, p_{2,j}\, x_{1,i}\, x_{2,i}}{(-1 + T_1)\, T_2} - \frac{(-T_1 - T_2 + T_1\, T_2)\, p_{1,j}\, p_{2,j}\, x_{1,i}\, x_{2,i}}{T_1\, T_2} -\\
&\quad \frac{p_{1,j}\, p_{2,i}\, x_{1,j}\, x_{2,i}}{-1 + T_1} - p_{1,j}\, p_{2,j}\, x_{1,j}\, x_{2,i} + \frac{T_1\, p_{1,i}\, p_{2,j}\, x_{1,i}\, x_{2,j}}{-1 + T_1} - p_{1,j}\, p_{2,j}\, x_{1,i}\, x_{2,j} + p_{3,i}\, x_{3,i} +\\
&\quad \frac{p_{1,j}\, p_{3,i}\, x_{1,i}\, x_{3,i}}{T_1} - \frac{(-1 + T_1\, T_2)\, p_{1,i}\, p_{3,j}\, x_{1,i}\, x_{3,i}}{(-1 + T_1)\, T_2} + \frac{(-1 + T_1\, T_2)\, p_{1,j}\, p_{3,j}\, x_{1,i}\, x_{3,i}}{T_1\, T_2} +\\
&\quad \frac{p_{1,j}\, p_{3,i}\, x_{1,j}\, x_{3,i}}{-1 + T_1} - \frac{(-1 + T_2)\, p_{2,j}\, p_{3,i}\, x_{2,i}\, x_{3,i}}{T_2} - \frac{(-1 + T_1\, T_2)\, p_{2,i}\, p_{3,j}\, x_{2,i}\, x_{3,i}}{T_1\, T_2} +\\
&\quad \frac{(-1 + 2\, T_2)\, (-1 + T_1\, T_2)\, p_{2,j}\, p_{3,j}\, x_{2,i}\, x_{3,i}}{T_1\, T_2^2} - p_{2,j}\, p_{3,i}\, x_{2,j}\, x_{3,i} + \frac{(-1 + T_1\, T_2)\, p_{2,j}\, p_{3,j}\, x_{2,j}\, x_{3,i}}{T_1\, T_2} -\\
&\quad \frac{T_1\, p_{1,i}\, p_{3,j}\, x_{1,i}\, x_{3,j}}{-1 + T_1} + p_{1,j}\, p_{3,j}\, x_{1,i}\, x_{3,j} + p_{2,j}\, p_{3,j}\, x_{2,i}\, x_{3,j}\Big\},\ -\frac{\varphi}{2} + \varphi\, p_{3,k}\, x_{3,k}\Big\}
\end{aligned}
$$

In[∘]:=

```
q[s_, i_, j_] := Sum[
   x_{ν,i} (p_{ν,i⁺} - p_{ν,i}) + x_{ν,j} (p_{ν,j⁺} - p_{ν,j}) + (T_ν^s - 1) x_{ν,i} (p_{ν,i⁺} - p_{ν,j⁺}),
   {ν, 3}];
ℒ[X_{i_,j_}[s_]] :=
  T_3^s 𝔼[q[s, i, j] + r_{0,pxx}[s, i, j] + ϵ r_{1,ppx}[s, i, j] + ϵ r_{1,rest}[s, i, j] + O[ϵ]^2];
ℒ[C_{k_}[φ_]] := T_3^φ 𝔼[Sum[x_{ν,k} (p_{ν,k⁺} - p_{ν,k}), {ν, 3}] + ϵ γ_1[φ, k] + O[ϵ]^2 ];
ℒ[X_{i,j}[1]]
ℒ[X_{i,j}[-1]]
ℒ[C_i[φ]]
```

Out[*]=

$$T_1 T_2\, \mathbb{E}\Big[\big((-p_{1,i}+p_{1,i^+})\,x_{1,i}+(-1+T_1)\,(p_{1,i^+}-p_{1,j^+})\,x_{1,i}+(-p_{1,j}+p_{1,j^+})\,x_{1,j}+(-p_{2,i}+p_{2,i^+})\,x_{2,i}+(-1+T_2)\,(p_{2,i^+}-p_{2,j^+})\,x_{2,i}+T_1\,p_{3,j}\,x_{1,i}\,x_{2,i}-p_{3,j}\,x_{1,j}\,x_{2,i}+(-p_{2,j}+p_{2,j^+})\,x_{2,j}+(-p_{3,i}+p_{3,i^+})\,x_{3,i}+(-1+T_1 T_2)\,(p_{3,i^+}-p_{3,j^+})\,x_{3,i}+(-p_{3,j}+p_{3,j^+})\,x_{3,j}\big)+\left(\frac{1}{2}-T_1 T_2\,p_{1,j}\,p_{2,j}\,x_{1,i}\,x_{2,i}+\frac{p_{1,j}\,p_{2,i}\,x_{1,j}\,x_{2,i}}{-1+T_1}+\frac{(-1+T_1 T_2)\,p_{1,j}\,p_{2,j}\,x_{1,j}\,x_{2,i}}{-1+T_1}-\frac{T_1\,p_{1,i}\,p_{2,j}\,x_{1,i}\,x_{2,j}}{-1+T_1}+(1-T_1 T_2)\,p_{1,j}\,p_{2,i}\,x_{3,i}+(-1+T_1 T_2)\,p_{1,j}\,p_{2,j}\,x_{3,i}-p_{3,i}\,x_{3,i}+T_1\,(-1+T_1 T_2)\,p_{1,j}\,p_{3,j}\,x_{1,i}\,x_{3,i}-\frac{p_{1,j}\,p_{3,i}\,x_{1,j}\,x_{3,i}}{-1+T_1}-\frac{T_1\,(-1+T_1 T_2)\,p_{1,j}\,p_{3,j}\,x_{1,j}\,x_{3,i}}{-1+T_1}+T_2\,(-1+T_1 T_2)\,p_{2,j}\,p_{3,j}\,x_{2,i}\,x_{3,i}+p_{2,j}\,p_{3,i}\,x_{2,j}\,x_{3,i}+\frac{T_1\,p_{1,i}\,p_{3,j}\,x_{1,i}\,x_{3,j}}{-1+T_1}-T_2\,p_{2,j}\,p_{3,j}\,x_{2,i}\,x_{3,j}\right)\in+O[\in]^2\Big]$$

Out[*]=

$$\frac{1}{T_1 T_2}\,\mathbb{E}\Big[\left((-p_{1,i}+p_{1,i^+})\,x_{1,i}+\left(-1+\frac{1}{T_1}\right)(p_{1,i^+}-p_{1,j^+})\,x_{1,i}+(-p_{1,j}+p_{1,j^+})\,x_{1,j}+(-p_{2,i}+p_{2,i^+})\,x_{2,i}+\left(-1+\frac{1}{T_2}\right)(p_{2,i^+}-p_{2,j^+})\,x_{2,i}-\frac{p_{3,j}\,x_{1,i}\,x_{2,i}}{T_1 T_2}+\frac{p_{3,j}\,x_{1,j}\,x_{2,i}}{T_2}+(-p_{2,j}+p_{2,j^+})\,x_{2,j}+(-p_{3,i}+p_{3,i^+})\,x_{3,i}+\left(-1+\frac{1}{T_1 T_2}\right)(p_{3,i^+}-p_{3,j^+})\,x_{3,i}+(-p_{3,j}+p_{3,j^+})\,x_{3,j}\right)+\left(-\frac{1}{2}-\frac{p_{1,j}\,p_{2,i}\,x_{1,i}\,x_{2,i}}{T_1}+\frac{T_1\,(-1+T_2)\,p_{1,i}\,p_{2,j}\,x_{1,i}\,x_{2,i}}{(-1+T_1)\,T_2}-\frac{(-T_1-T_2+T_1 T_2)\,p_{1,j}\,p_{2,j}\,x_{1,i}\,x_{2,i}}{T_1 T_2}-\frac{p_{1,j}\,p_{2,i}\,x_{1,j}\,x_{2,i}}{-1+T_1}-p_{1,j}\,p_{2,j}\,x_{1,j}\,x_{2,i}+\frac{T_1\,p_{1,i}\,p_{2,j}\,x_{1,i}\,x_{2,j}}{-1+T_1}-p_{1,j}\,p_{2,j}\,x_{1,i}\,x_{2,j}+\frac{(-1+T_1 T_2)\,p_{1,j}\,p_{2,i}\,x_{3,i}}{T_1}-\frac{(-1+T_1 T_2)\,p_{1,j}\,p_{2,j}\,x_{3,i}}{T_1}+p_{3,i}\,x_{3,i}+\frac{p_{1,j}\,p_{3,i}\,x_{1,i}\,x_{3,i}}{T_1}-\frac{(-1+T_1 T_2)\,p_{1,i}\,p_{3,j}\,x_{1,i}\,x_{3,i}}{(-1+T_1)\,T_2}+\frac{(-1+T_1 T_2)\,p_{1,j}\,p_{3,j}\,x_{1,i}\,x_{3,i}}{T_1 T_2}+\frac{p_{1,j}\,p_{3,i}\,x_{1,j}\,x_{3,i}}{-1+T_1}-\frac{(-1+T_2)\,p_{2,j}\,p_{3,i}\,x_{2,i}\,x_{3,i}}{T_2}-\frac{(-1+T_1 T_2)\,p_{2,i}\,p_{3,j}\,x_{2,i}\,x_{3,i}}{T_1 T_2}+\frac{(-1+2 T_2)\,(-1+T_1 T_2)\,p_{2,j}\,p_{3,j}\,x_{2,i}\,x_{3,i}}{T_1 T_2^2}-p_{2,j}\,p_{3,i}\,x_{2,j}\,x_{3,i}+\frac{(-1+T_1 T_2)\,p_{2,j}\,p_{3,j}\,x_{2,j}\,x_{3,i}}{T_1 T_2}-\frac{T_1\,p_{1,i}\,p_{3,j}\,x_{1,i}\,x_{3,j}}{-1+T_1}+p_{1,j}\,p_{3,j}\,x_{1,i}\,x_{3,j}+p_{2,j}\,p_{3,j}\,x_{2,i}\,x_{3,j}\right)\in+O[\in]^2\Big]$$

Out[*]=

$$(T_1 T_2)^{\varphi}\,\mathbb{E}\Big[\big((-p_{1,i}+p_{1,i^+})\,x_{1,i}+(-p_{2,i}+p_{2,i^+})\,x_{2,i}+(-p_{3,i}+p_{3,i^+})\,x_{3,i}\big)+\left(-\frac{\varphi}{2}+\varphi\,p_{3,i}\,x_{3,i}\right)\in+O[\in]^2\Big]$$

In[*]:=

```
{p*, x*, π*, ξ*} = {π, ξ, p, x};  (u_i__)* := (u*)_i;
```

In[∘]:=
```
Zip_{}[ℰ_] := ℰ;
Zip_{ξ_,ξs___}[ℰ_] := (Collect[ℰ // Zip_{ξs}, ξ] /. f_. ξ^d_. :> (D[f, {ξ*, d}])) /. ξ* → 0
```

In[∘]:=
```
px2g[ℰ_] := CF@Module[{ps, xs, Q, α, β},
    ps = Union[Cases[ℰ, p__, ∞]]; xs = Union[Cases[ℰ, x__, ∞]];
    Q = Sum[p0* x0* g_{p0[[2]],x0[[2]],p0[[3]],x0[[3]]}, {p0, ps}, {x0, xs}];
    Expand[Zip_{ps⋃xs}[ℰ e^Q] /. g_{α_,β_,i_,j_} :> If[α === β, g_{α,i,j}, 0]]
  ]
```

In[∘]:= px2g[$p^2_{2,j}\, x_{2,i}\, x_{2,j}$]

Out[∘]=

$$2\, g_{2,j,i}\, g_{2,j,j}$$

In[∘]:= R₁[1, i_, j_] = px2g[r_{1,rest}[1, i, j]]

Out[∘]=

$$\frac{1}{2}+\frac{g_{1,j,j}\, g_{2,i,i}}{-1+T_1}-T_1 T_2\, g_{1,j,i}\, g_{2,j,i}+\frac{(-1+T_1 T_2)\, g_{1,j,j}\, g_{2,j,i}}{-1+T_1}-\frac{T_1\, g_{1,i,i}\, g_{2,j,j}}{-1+T_1}-g_{3,i,i}-\frac{g_{1,j,j}\, g_{3,i,i}}{-1+T_1}+g_{2,j,j}\, g_{3,i,i}+T_1 (-1+T_1 T_2)\, g_{1,j,i}\, g_{3,j,i}-\frac{T_1 (-1+T_1 T_2)\, g_{1,j,j}\, g_{3,j,i}}{-1+T_1}+T_2 (-1+T_1 T_2)\, g_{2,j,i}\, g_{3,j,i}+\frac{T_1\, g_{1,i,i}\, g_{3,j,j}}{-1+T_1}-T_2\, g_{2,j,i}\, g_{3,j,j}$$

In[∘]:= R₁[-1, i_, j_] = px2g[r_{1,rest}[-1, i, j]]

Out[∘]=

$$-\frac{1}{2}-\frac{g_{1,j,i}\, g_{2,i,i}}{T_1}-\frac{g_{1,j,j}\, g_{2,i,i}}{-1+T_1}+\frac{T_1 (-1+T_2)\, g_{1,i,i}\, g_{2,j,i}}{(-1+T_1)\, T_2}-\frac{(-T_1-T_2+T_1 T_2)\, g_{1,j,i}\, g_{2,j,i}}{T_1 T_2}-g_{1,j,j}\, g_{2,j,i}+\frac{T_1\, g_{1,i,i}\, g_{2,j,j}}{-1+T_1}-g_{1,j,i}\, g_{2,j,j}+g_{3,i,i}+\frac{g_{1,j,i}\, g_{3,i,i}}{T_1}+\frac{g_{1,j,j}\, g_{3,i,i}}{-1+T_1}-\frac{(-1+T_2)\, g_{2,j,i}\, g_{3,i,i}}{T_2}-g_{2,j,j}\, g_{3,i,i}-\frac{(-1+T_1 T_2)\, g_{1,i,i}\, g_{3,j,i}}{(-1+T_1)\, T_2}+\frac{(-1+T_1 T_2)\, g_{1,j,i}\, g_{3,j,i}}{T_1 T_2}-\frac{(-1+T_1 T_2)\, g_{2,i,i}\, g_{3,j,i}}{T_1 T_2}+\frac{(-1+2 T_2)(-1+T_1 T_2)\, g_{2,j,i}\, g_{3,j,i}}{T_1 T_2^2}+\frac{(-1+T_1 T_2)\, g_{2,j,j}\, g_{3,j,i}}{T_1 T_2}-\frac{T_1\, g_{1,i,i}\, g_{3,j,j}}{-1+T_1}+g_{1,j,i}\, g_{3,j,j}+g_{2,j,i}\, g_{3,j,j}$$

In[∘]:= px2g[r_{0,pxx}[1, i0, j0] r_{1,ppx}[1, i1, j1]]

Out[∘]=

$$-T_1 (-1+T_1 T_2)\, g_{1,j1,i0}\, g_{2,i1,i0}\, g_{3,j0,i1}+(-1+T_1 T_2)\, g_{1,j1,j0}\, g_{2,i1,i0}\, g_{3,j0,i1}+T_1 (-1+T_1 T_2)\, g_{1,j1,i0}\, g_{2,j1,i0}\, g_{3,j0,i1}+(1-T_1 T_2)\, g_{1,j1,j0}\, g_{2,j1,i0}\, g_{3,j0,i1}$$

In[◦]:=
```
θ[{1, i0_, j0_}, {1, i1_, j1_}] = px2g[r0,pxx[1, i0, j0] r1,ppx[1, i1, j1]]
θ[{1, i0_, j0_}, {-1, i1_, j1_}] = px2g[r0,pxx[1, i0, j0] r1,ppx[-1, i1, j1]]
θ[{-1, i0_, j0_}, {1, i1_, j1_}] = px2g[r0,pxx[-1, i0, j0] r1,ppx[1, i1, j1]]
θ[{-1, i0_, j0_}, {-1, i1_, j1_}] = px2g[r0,pxx[-1, i0, j0] r1,ppx[-1, i1, j1]]
```

Out[◦]=

$$-T_1\,(-1+T_1T_2)\,g_{1,j1,i0}\,g_{2,i1,i0}\,g_{3,j0,i1}+(-1+T_1T_2)\,g_{1,j1,j0}\,g_{2,i1,i0}\,g_{3,j0,i1}+T_1\,(-1+T_1T_2)\,g_{1,j1,i0}\,g_{2,j1,i0}\,g_{3,j0,i1}+(1-T_1T_2)\,g_{1,j1,j0}\,g_{2,j1,i0}\,g_{3,j0,i1}$$

Out[◦]=

$$(-1+T_1T_2)\,g_{1,j1,i0}\,g_{2,i1,i0}\,g_{3,j0,i1}-\frac{(-1+T_1T_2)\,g_{1,j1,j0}\,g_{2,i1,i0}\,g_{3,j0,i1}}{T_1}+(1-T_1T_2)\,g_{1,j1,i0}\,g_{2,j1,i0}\,g_{3,j0,i1}+\frac{(-1+T_1T_2)\,g_{1,j1,j0}\,g_{2,j1,i0}\,g_{3,j0,i1}}{T_1}$$

Out[◦]=

$$\frac{(-1+T_1T_2)\,g_{1,j1,i0}\,g_{2,i1,i0}\,g_{3,j0,i1}}{T_1T_2}-\frac{(-1+T_1T_2)\,g_{1,j1,j0}\,g_{2,i1,i0}\,g_{3,j0,i1}}{T_2}-\frac{(-1+T_1T_2)\,g_{1,j1,i0}\,g_{2,j1,i0}\,g_{3,j0,i1}}{T_1T_2}+\frac{(-1+T_1T_2)\,g_{1,j1,j0}\,g_{2,j1,i0}\,g_{3,j0,i1}}{T_2}$$

Out[◦]=

$$-\frac{(-1+T_1T_2)\,g_{1,j1,i0}\,g_{2,i1,i0}\,g_{3,j0,i1}}{T_1^2T_2}+\frac{(-1+T_1T_2)\,g_{1,j1,j0}\,g_{2,i1,i0}\,g_{3,j0,i1}}{T_1T_2}+\frac{(-1+T_1T_2)\,g_{1,j1,i0}\,g_{2,j1,i0}\,g_{3,j0,i1}}{T_1^2T_2}-\frac{(-1+T_1T_2)\,g_{1,j1,j0}\,g_{2,j1,i0}\,g_{3,j0,i1}}{T_1T_2}$$

In[◦]:=
```
CF[θ[{1, i0, j0}, {1, i1, j1}] + (θ[{-1, i0, j0}, {-1, i1, j1}] /. T_i_ :> T_i^-1)]
```

Out[◦]=

0

In[◦]:=
```
CF[θ[{1, i0, j0}, {-1, i1, j1}] + (θ[{-1, i0, j0}, {1, i1, j1}] /. T_i_ :> T_i^-1)]
```

Out[◦]=

0

In[◦]:=
```
Γ1[φ_, k_] = px2g[γ1[φ, k]]
```

Out[◦]=

$$-\frac{\varphi}{2}+\varphi\,g_{3,k,k}$$

The Programs

In[∘]:=
```
T3 = T1 T2;
θ[K_] := θ[K] = Module[{Cs, φ, n, A, s, i, j, k, Δ, G, ν, α, β, gEval, Y, yEval, c, z},
   {Cs, φ} = Rot[K]; n = Length[Cs];
   A = IdentityMatrix[2 n + 1];
   Cases[Cs, {s_, i_, j_} :> (A[[{i, j}, {i + 1, j + 1}]] += ( -T^s  T^s - 1
                                                              0     -1    ))];
   Δ = T^((-Total[φ] - Total[Cs[[All, 1]]])/2) Det[A];
   G = Inverse[A]; gEval[ℰ_] := CCF[ℰ /. g_{ν_, α_, β_} :> (G[[α, β]] /. T → T_ν)];
   z = gEval[Sum[Sum[θ[Cs[[k1]], Cs[[k2]]], {k2, 1, n}], {k1, 1, n}]];
   z += gEval[Sum[R1 @@ Cs[[k]], {k, 1, n}]];
   z += gEval[Sum[Γ1[φ[[k]], k], {k, 1, 2 n}]];
   {Δ, (Δ /. T → T1) (Δ /. T → T2) (Δ /. T → T3) z} // CCF
  ];
```

In[∘]:=
```
θ_{T1_, T2_}[K_] := θ_{T1, T2}[K] = Module[{Cs, φ, n, A, s, i, j, k, Δ, G, gEval, Y, yEval, c, z = 0},
   {Cs, φ} = Rot[K]; n = Length[Cs];
   temp0 = PrintTemporary["At work, n=", n];
   A = IdentityMatrix[2 n + 1];
   Cases[Cs, {s_, i_, j_} :> (A[[{i, j}, {i + 1, j + 1}]] += ( -T^s  T^s - 1
                                                              0     -1    ))];
   Δ[0] := Δ[0] = T^((-Total[φ] - Total[Cs[[All, 1]]])/2) Det[A];
   G[0] := G[0] = Inverse[A];
   {Δ[1], G[1]} = If[NumberQ@T1,
     {Det[A /. T → T1], Inverse[A /. T → T1]}, {Δ[0], G[0]} /. T → T1 ];
   temp = PrintTemporary@"Done with {Δ[1],G[1]}.";
   {Δ[2], G[2]} = If[NumberQ@T2,
     {Det[A /. T → T2], Inverse[A /. T → T2]}, {Δ[0], G[0]} /. T → T2 ];
   NotebookDelete[temp]; temp = PrintTemporary@"Done with {Δ[2],G[2]}.";
   {Δ[3], G[3]} = If[NumberQ[T1 T2],
     {Det[A /. T → T1 T2], Inverse[A /. T → T1 T2]}, {Δ[0], G[0]} /. T → T1 T2 ];
   NotebookDelete[temp]; temp = PrintTemporary@"Done with {Δ[3],G[3]}.";
   gEval[ℰ_] := CCF[ℰ //. {T1 → T1, T2 → T2, g_{ν_, α_, β_} :> G[ν][[α, β]]}];
   Do[z += gEval[θ[Cs[[k1]], Cs[[k2]]]], {k1, n}, {k2, n}];
   Do[z += gEval[R1 @@ Cs[[k]]], {k, n}];
   Do[z += gEval[Γ1[φ[[k]], k]], {k, 2 n}];
   NotebookDelete[temp0]; NotebookDelete[temp];
   {{Δ[1], Δ[2], Δ[3]}, Δ[1] Δ[2] Δ[3] z} // CCF
  ];
```

In[•]:=
```
TestSymmetries[K_] := Module[{θ0, θ1},
  {θ0, θ1} = {θ[K][[2]], θ[Mirror@K][[2]]};
  Simplify@And[
    θ0 == (θ0 /. {T1 → T2, T2 → T1}),
    θ0 == -θ1,
    θ0 == (θ0 /. T_i_ :→ T_i^-1),
    θ0 == (θ0 /. T2 → T1^-1 T2^-1)
   ]
 ]
```

In[•]:=
```
hex = Table[{Cos[α], Sin[α]} / Cos[2 π / 12] / 2, {α, 2 π / 12, 2 π, 2 π / 6}];
PolyPlot__[0] = Graphics[{}];
PolyPlot_{T1_,T2_}[p_] := PolyPlot_{Hexagon,T1,T2}[p]
PolyPlot_{shape_,T1_,T2_}[p_] := Module[{crs, m1, m2, maxc, minc, s},
  crs = CoefficientRules[T1^(m1=-Exponent[p,T1,Min]) T2^(m2=-Exponent[p,T2,Min]) p, {T1, T2}];
  maxc = Max@Abs[Last /@ crs];
  minc = Min@Select[Abs[Last /@ crs], # > 0 &];
  If[minc == maxc,
   s[_] = 0,
   s[c_] := s[c] =
     N[Interpolation[{{Log@minc, 1}, {Log@maxc, 0}}, InterpolationOrder → 1][Log@c]]
  ];
  Graphics[crs /. ({x1_, x2_} → c_) :→ {
      If[c == 0, White, Lighter[If[c > 0, Red, Blue], 0.88 s[Abs@c]]],
      Switch[shape,
       Disk, Disk[{{1, -1/2}, {0, √3/2}}.{x1 + m1, x2 + m2}, 0.5],
       Hexagon, Polygon[({{1, -1/2}, {0, √3/2}}.{x1 + m1, x2 + m2} + #) & /@ hex]
      ]
     }]
 ]
```

Sporadic Testing

In[∘]:=
```
K = Knot[3, 1]; Timing[Expand[θ[K]]]
TestSymmetries[K]
```

KnotTheory: Loading precomputed data in PD4Knots`.

Out[∘]=

$$\left\{0.,\ \left\{-1+\frac{1}{T}+T,\ -\frac{1}{T_1^2}-T_1^2-\frac{1}{T_2^2}-\frac{1}{T_1^2 T_2^2}+\frac{1}{T_1 T_2^2}+\frac{1}{T_1^2 T_2}+\frac{T_1}{T_2}+\frac{T_2}{T_1}+T_1^2 T_2-T_2^2+T_1 T_2^2-T_1^2 T_2^2\right\}\right\}$$

Out[∘]=

True

In[∘]:=
```
K = Knot[8, 19]; Timing[Expand[θ[K]]]
TestSymmetries[K]
```

Out[∘]=

$$\Big\{0.015625,$$

$$\Big\{1+\frac{1}{T^3}-\frac{1}{T^2}-T^2+T^3,\ \frac{3}{T_1^6}-\frac{3}{T_1^4}+\frac{4}{T_1^3}-\frac{1}{T_1^2}-T_1^2+4T_1^3-3T_1^4+3T_1^6+\frac{3}{T_2^6}+\frac{3}{T_1^6T_2^6}-\frac{3}{T_1^5T_2^6}+\frac{3}{T_1^3T_2^6}-\frac{3}{T_1T_2^6}-$$

$$\frac{3}{T_1^6T_2^5}+\frac{3}{T_1^4T_2^5}-\frac{3}{T_1^3T_2^5}-\frac{3}{T_1^2T_2^5}+\frac{3}{T_1T_2^5}-\frac{3T_1}{T_2^5}-\frac{3}{T_2^4}+\frac{3}{T_1^5T_2^4}-\frac{3}{T_1^4T_2^4}+\frac{3}{T_1^2T_2^4}+\frac{3T_1}{T_2^4}+\frac{4}{T_2^3}+\frac{3}{T_1^6T_2^3}-\frac{3}{T_1^5T_2^3}+$$

$$\frac{4}{T_1^3T_2^3}-\frac{2}{T_1^2T_2^3}-\frac{2}{T_1T_2^3}-\frac{3T_1^2}{T_2^3}+\frac{3T_1^3}{T_2^3}-\frac{1}{T_2^2}-\frac{3}{T_1^5T_2^2}+\frac{3}{T_1^4T_2^2}-\frac{2}{T_1^3T_2^2}-\frac{1}{T_1^2T_2^2}+\frac{1}{T_1T_2^2}-\frac{2T_1}{T_2^2}+\frac{3T_1^2}{T_2^2}-$$

$$\frac{3T_1^3}{T_2^2}-\frac{3}{T_1^6T_2}+\frac{3}{T_1^5T_2}-\frac{2}{T_1^3T_2}+\frac{1}{T_1^2T_2}+\frac{T_1}{T_2}-\frac{2T_1^2}{T_2}+\frac{3T_1^4}{T_2}-\frac{3T_1^5}{T_2}-\frac{3T_2}{T_1^5}+\frac{3T_2}{T_1^4}-\frac{2T_2}{T_1^2}+\frac{T_2}{T_1}+T_1^2T_2-$$

$$2T_1^3T_2+3T_1^5T_2-3T_1^6T_2-T_2^2-\frac{3T_2^2}{T_1^3}+\frac{3T_2^2}{T_1^2}-\frac{2T_2^2}{T_1}+T_1T_2^2-T_1^2T_2^2-2T_1^3T_2^2+3T_1^4T_2^2-3T_1^5T_2^2+4T_2^3+$$

$$\frac{3T_2^3}{T_1^3}-\frac{3T_2^3}{T_1^2}-2T_1T_2^3-2T_1^2T_2^3+4T_1^3T_2^3-3T_1^5T_2^3+3T_1^6T_2^3-3T_2^4+\frac{3T_2^4}{T_1}+3T_1^2T_2^4-3T_1^4T_2^4+3T_1^5T_2^4-$$

$$\frac{3T_2^5}{T_1}+3T_1T_2^5-3T_1^2T_2^5-3T_1^3T_2^5+3T_1^4T_2^5-3T_1^6T_2^5+3T_2^6-3T_1T_2^6+3T_1^3T_2^6-3T_1^5T_2^6+3T_1^6T_2^6\Big\}\Big\}$$

Out[∘]=

True

In[∘]:= Timing[Expand@θ_{T_1,T_2}[Knot[3, 1]]]

Out[∘]=

$$\Big\{0.,\ \Big\{\Big\{-1+\frac{1}{T_1}+T_1,\ -1+\frac{1}{T_2}+T_2,\ -1+\frac{1}{T_1T_2}+T_1T_2\Big\},$$

$$-\frac{1}{T_1^2}-T_1^2-\frac{1}{T_2^2}-\frac{1}{T_1^2T_2^2}+\frac{1}{T_1T_2^2}+\frac{1}{T_1^2T_2}+\frac{T_1}{T_2}+\frac{T_2}{T_1}+T_1^2T_2-T_2^2+T_1T_2^2-T_1^2T_2^2\Big\}\Big\}$$

In[∘]:=
```
K = Knot[4, 1]; Timing[θ[K]]
TestSymmetries[K]
```

Out[∘]=

$$\left\{0.,\ \left\{-\frac{1-3\,T+T^2}{T},\ 0\right\}\right\}$$

Out[∘]=

True

In[∘]:=
```
K = Knot["K11n34"]; Timing[θ[K]]
TestSymmetries[K]
```

KnotTheory: Loading precomputed data in DTCode4KnotsTo11`.

KnotTheory: The GaussCode to PD conversion was written by Siddarth Sankaran at the University of Toronto in the summer of 2005.

Out[∘]=

$$\Big\{0.,\ \Big\{1,\ -\frac{1}{T_1^6\,T_2^6}\,\big(T_1^2-2\,T_1^3+T_1^4-2\,T_1\,T_2+2\,T_1^2\,T_2+2\,T_1^5\,T_2-2\,T_1^6\,T_2+T_2^2+2\,T_1\,T_2^2-2\,T_1^2\,T_2^2-2\,T_1^4\,T_2^2-2\,T_1^6\,T_2^2+2\,T_1^7\,T_2^2+T_1^8\,T_2^2-2\,T_2^3+T_1^4\,T_2^3+T_1^5\,T_2^3-2\,T_1^9\,T_2^3+T_2^4-2\,T_1^2\,T_2^4+T_1^3\,T_2^4+2\,T_1^4\,T_2^4+2\,T_1^6\,T_2^4+T_1^7\,T_2^4-2\,T_1^8\,T_2^4+T_1^{10}\,T_2^4+2\,T_1\,T_2^5+T_1^3\,T_2^5-4\,T_1^5\,T_2^5-4\,T_1^6\,T_2^5+T_1^8\,T_2^5+2\,T_1^{10}\,T_2^5-2\,T_1\,T_2^6-2\,T_1^2\,T_2^6+2\,T_1^4\,T_2^6-4\,T_1^5\,T_2^6+12\,T_1^6\,T_2^6-4\,T_1^7\,T_2^6+2\,T_1^8\,T_2^6-2\,T_1^{10}\,T_2^6-2\,T_1^{11}\,T_2^6+2\,T_1^2\,T_2^7+T_1^4\,T_2^7-4\,T_1^6\,T_2^7-4\,T_1^7\,T_2^7+T_1^9\,T_2^7+2\,T_1^{11}\,T_2^7+T_1^2\,T_2^8-2\,T_1^4\,T_2^8+T_1^5\,T_2^8+2\,T_1^6\,T_2^8+2\,T_1^8\,T_2^8+T_1^9\,T_2^8-2\,T_1^{10}\,T_2^8+T_1^{12}\,T_2^8-2\,T_1^3\,T_2^9+T_1^7\,T_2^9+T_1^8\,T_2^9-2\,T_1^{12}\,T_2^9+T_1^4\,T_2^{10}+2\,T_1^5\,T_2^{10}-2\,T_1^6\,T_2^{10}-2\,T_1^8\,T_2^{10}-2\,T_1^{10}\,T_2^{10}+2\,T_1^{11}\,T_2^{10}+T_1^{12}\,T_2^{10}-2\,T_1^6\,T_2^{11}+2\,T_1^7\,T_2^{11}+2\,T_1^{10}\,T_2^{11}-2\,T_1^{11}\,T_2^{11}+T_1^8\,T_2^{12}-2\,T_1^9\,T_2^{12}+T_1^{10}\,T_2^{12}\big)\Big\}\Big\}$$

Out[∘]=

True

In[∘]:=
```
K = Knot["K11n42"]; Timing[θ[K]]
TestSymmetries[K]
```

Out[∘]=

$$\Big\{0.03125,\ \Big\{1,\ \frac{1}{T_1^3\,T_2^3}\,\big(T_1+T_1^2+T_2-2\,T_1\,T_2-2\,T_1^2\,T_2-2\,T_1^3\,T_2+T_1^4\,T_2+T_2^2-2\,T_1\,T_2^2+2\,T_1^2\,T_2^2+2\,T_1^3\,T_2^2-2\,T_1^4\,T_2^2+T_1^5\,T_2^2-2\,T_1\,T_2^3+2\,T_1^2\,T_2^3+2\,T_1^4\,T_2^3-2\,T_1^5\,T_2^3+T_1\,T_2^4-2\,T_1^2\,T_2^4+2\,T_1^3\,T_2^4+2\,T_1^4\,T_2^4-2\,T_1^5\,T_2^4+T_1^6\,T_2^4+T_1^2\,T_2^5-2\,T_1^3\,T_2^5-2\,T_1^4\,T_2^5-2\,T_1^5\,T_2^5+T_1^6\,T_2^5+T_1^4\,T_2^6+T_1^5\,T_2^6\big)\Big\}\Big\}$$

Out[∘]=

True

In[∘]:=
```
PD[GST48] = PD[X[1, 15, 2, 14], X[29, 2, 30, 3], X[40, 4, 41, 3],
   X[4, 44, 5, 43], X[5, 26, 6, 27], X[95, 7, 96, 6], X[7, 1, 8, 96], X[8, 14, 9, 13],
   X[28, 9, 29, 10], X[41, 11, 42, 10], X[11, 43, 12, 42], X[12, 27, 13, 28],
   X[15, 31, 16, 30], X[61, 16, 62, 17], X[72, 17, 73, 18], X[83, 18, 84, 19],
   X[34, 20, 35, 19], X[20, 89, 21, 90], X[92, 21, 93, 22], X[22, 79, 23, 80],
   X[23, 68, 24, 69], X[24, 57, 25, 58], X[56, 25, 57, 26], X[31, 63, 32, 62],
   X[32, 74, 33, 73], X[33, 85, 34, 84], X[35, 50, 36, 51], X[81, 37, 82, 36],
   X[70, 38, 71, 37], X[59, 39, 60, 38], X[54, 39, 55, 40], X[55, 45, 56, 44],
   X[45, 59, 46, 58], X[46, 70, 47, 69], X[47, 81, 48, 80], X[91, 49, 92, 48],
   X[49, 91, 50, 90], X[82, 52, 83, 51], X[71, 53, 72, 52], X[60, 54, 61, 53],
   X[74, 63, 75, 64], X[85, 64, 86, 65], X[65, 76, 66, 77], X[66, 87, 67, 88],
   X[94, 67, 95, 68], X[86, 75, 87, 76], X[77, 88, 78, 89], X[93, 78, 94, 79]];
```

In[∘]:=
```
K = GST48; AbsoluteTiming[Short@θ[K]]
TestSymmetries[K]
```

Out[∘]=

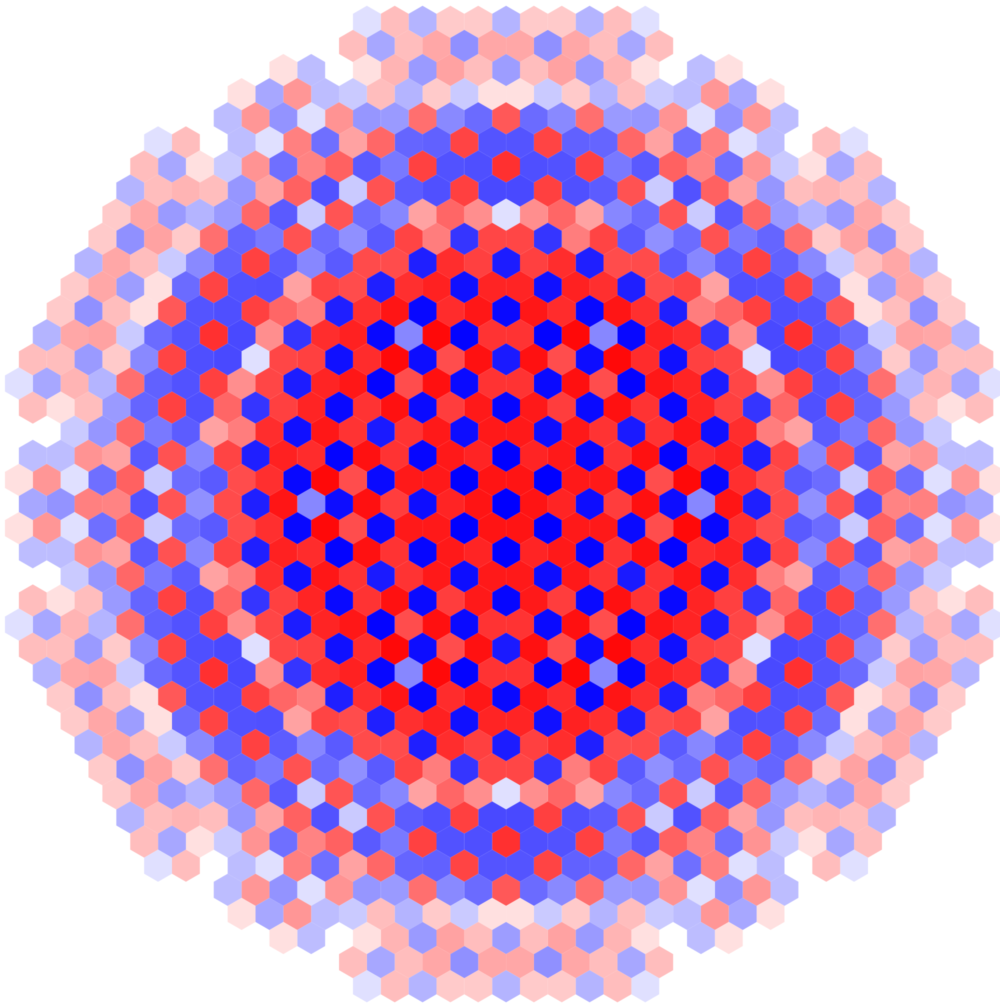
$$\left\{13.5155, \left\{-\frac{\left(-1+2\,T-T^2-T^3+2\,T^4-T^5+T^8\right)\,\left(-1+\ll 7\gg+T^8\right)}{T^8}, \frac{\ll 1764\gg+T_1^{\ll 2\gg}\ \ll 1\gg}{T_1^{20}\,T_2^{20}}\right\}\right\}$$

Out[∘]=

True

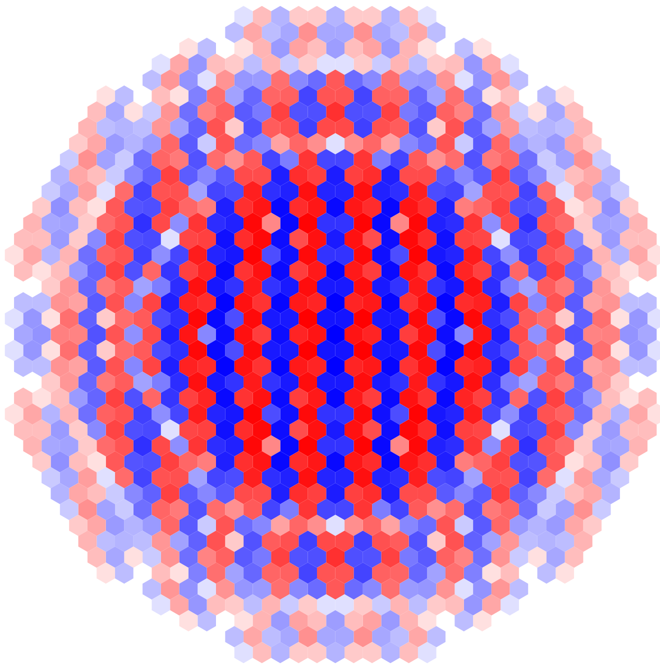
In[•]:= $\mathbf{PolyPlot}_{T_1,T_2}[-\Theta[\mathbf{GST48}][[2]]]$

Out[•]=



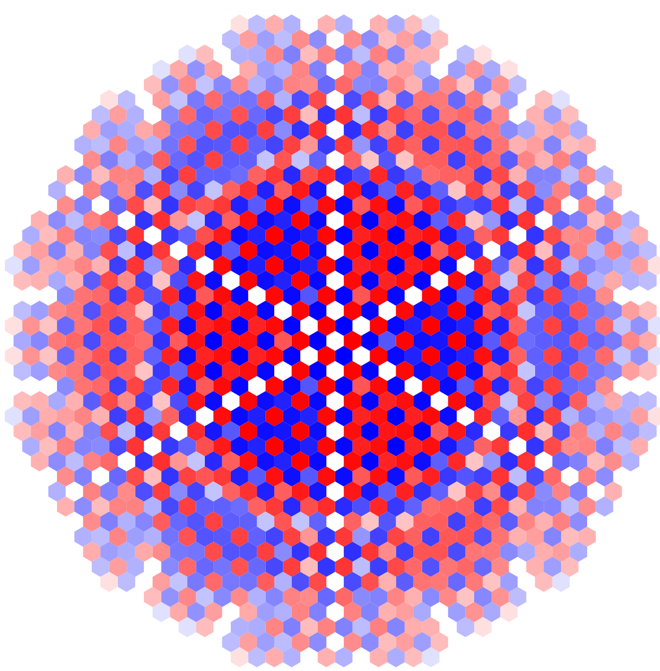
In[ ]:= $\mathbf{PolyPlot}_{T_1,T_2}[-\theta[\mathbf{GST48}][[2]]\ /.\ \{T_1 \to T_1,\ T_2 \to -T_2\}]$

Out[ ]=



In[ ]:= $\mathbf{PolyPlot}_{T_1,T_2}\left[\theta[\mathbf{GST48}][[2]]\ \left(T_1 + T_2 - T_3 - T_1^{-1} - T_2^{-1} + T_3^{-1}\right)\right]$

Out[ ]=



In[ ]:= $\mathbf{AbsoluteTiming}[\theta_{T_1,T_2}[\mathbf{GST48}];]$

Out[ ]=

{60.1527, Null}

In[∘]:= `AbsoluteTiming[`$\theta_{22/7,34/21}$`[GST48]]`

Out[∘]=

$$\left\{0.440852, \left\{\left\{-\frac{1\,422\,357\,287\,561\,349\,859\,889}{10\,190\,414\,377\,180\,576}, -\frac{486\,885\,265\,100\,293\,177\,259\,569}{15\,915\,006\,754\,796\,041\,036\,704}, -\frac{6\,215\,902\,990\,719\,340\,337\,664\,427\,997\,383\,765\,280\,900\,656\,009}{162\,180\,513\,646\,999\,558\,542\,864\,476\,199\,651\,861\,504}\right\},\right.\right.$$
$$21\,304\,335\,657\,502\,800\,961\,104\,521\,150\,882\,906\,491\,585\,928\,445\,141\,602\,977\,673\,772\,524\,921\,333\,287\,756\,546\,740\,936\,248\,585\,046\,107\,073\,499 \,/$$
$$\left.\left. 4\,728\,039\,585\,290\,312\,086\,302\,386\,002\,441\,018\,010\,474\,726\,543\,601\,178\,518\,697\,564\,845\,087\,356\,778\,405\,506\,577\,334\,272\right\}\right\}$$

Systematic Testing

In[∘]:= `DuplicateFreeQ[θ /@ AllKnots[{3, 10}]]`

Out[∘]=

True

In[∘]:= `Total[TestSymmetries /@ AllKnots[{3, 10}]]`

Out[∘]=

249 True

In[∘]:= `DuplicateFreeQ[θ /@ AllKnots[{3, 12}]]`

KnotTheory: Loading precomputed data in KnotTheory/12A.dts.

KnotTheory: Loading precomputed data in KnotTheory/12N.dts.

Out[∘]=

False

In[∘]:= `tab11 = Table[K → θ@K, {K, AllKnots[{3, 11}] }]`

Out[∘]=

$$\left\{\text{Knot}[3,1]\to\left\{\frac{1-T+T^2}{T}, -\frac{1-T_1+T_1^2-T_2-T_1^3 T_2+T_2^2+T_1^4 T_2^2-T_1 T_2^3-T_1^4 T_2^3+T_1^2 T_2^4-T_1^3 T_2^4+T_1^4 T_2^4}{T_1^2 T_2^2}\right\}, \text{Knot}[4,1]\to\left\{-\frac{1-3T+T^2}{T}, 0\right\},\right.$$
$$\text{Knot}[5,1]\to\left\{\frac{1-T+T^2-T^3+T^4}{T^2}, -\frac{\langle\!\langle 53\rangle\!\rangle+2T_1^8T_2^8}{T_1^4T_2^4}\right\}, \langle\!\langle 795\rangle\!\rangle, \text{Knot}[11,\text{NonAlternating},183]\to\left\{\frac{\langle\!\langle 1\rangle\!\rangle}{T^3}, \langle\!\langle 1\rangle\!\rangle\right\},$$
$$\text{Knot}[11,\text{NonAlternating},184]\to\left\{\frac{(1-T+T^2)(2-7T+11T^2-7T^3+2T^4)}{T^3}, \frac{9-41T_1+92T_1^2-115T_1^3+\langle\!\langle 166\rangle\!\rangle+92T_1^{10}T_2^{12}-41T_1^{11}T_2^{12}+9T_1^{12}T_2^{12}}{T_1^6T_2^6}\right\},$$
$$\text{Knot}[11,\text{NonAlternating},185]\to$$
$$\left\{-\frac{(1-3T+T^2)(1-T+T^2)(2-3T+2T^2)}{T^3}, -\frac{1}{T_1^6T_2^6}\left(17-93T_1+202T_1^2-261T_1^3+202T_1^4-93T_1^5+17T_1^6-93T_2+416T_1T_2-593T_1^2T_2+321T_1^3T_2+\right.\right.$$
$$\left.\left.\left.\langle\!\langle 153\rangle\!\rangle+416T_1^{11}T_2^{11}-93T_1^{12}T_2^{11}+17T_1^6T_2^{12}-93T_1^7T_2^{12}+202T_1^8T_2^{12}-261T_1^9T_2^{12}+202T_1^{10}T_2^{12}-93T_1^{11}T_2^{12}+17T_1^{12}T_2^{12}\right)\right\}\right\}$$

Full expression not available (original memory size: 33.2 MB)

In[•]:=
```
Gather[tab11, Last[#1] === Last[#2] &]
```

Out[•]=

$$\left\{\left\{\text{Knot}[3,1]\to\left\{\frac{1-T+T^2}{T},-\frac{1-T_1+T_1^2-T_2-T_1^3T_2+T_2^2+T_1^4T_2^2-T_1T_2^3-T_1^4T_2^3+T_1^2T_2^4-T_1^3T_2^4+T_1^4T_2^4}{T_1^2T_2^2}\right\}\right\},\left\{\text{Knot}[4,1]\to\left\{-\frac{1-3T+T^2}{T},0\right\}\right\},\right.$$

$$\left\{\text{Knot}[5,1]\to\left\{\frac{1-T+T^2-T^3+T^4}{T^2},-\frac{\cdots 53\cdots+2T_1^8T_2^8}{T_1^4T_2^4}\right\}\right\},\ \cdots 792\cdots,\ \left\{\text{Knot}[11,\text{NonAlternating},183]\to\left\{\frac{\cdots 1\cdots}{T^3},\cdots 1\cdots\right\}\right\},$$

$$\left\{\text{Knot}[11,\text{NonAlternating},184]\to\left\{\frac{(1-T+T^2)(2-7T+11T^2-7T^3+2T^4)}{T^3},\frac{9-41T_1+\cdots 169\cdots+92T_1^{10}T_2^{12}-41T_1^{11}T_2^{12}+9T_1^{12}T_2^{12}}{T_1^6T_2^6}\right\}\right\},$$

$$\{\text{Knot}[11,\text{NonAlternating},185]\to$$

$$\left\{-\frac{(1-3T+T^2)(1-T+T^2)(2-3T+2T^2)}{T^3},-\frac{1}{T_1^6T_2^6}\left(17-93T_1+202T_1^2-261T_1^3+202T_1^4-93T_1^5+17T_1^6-93T_2+416T_1T_2-593T_1^2T_2+321T_1^3T_2+\right.\right.$$

$$\left.\left.\left.\cdots 153\cdots+416T_1^{11}T_2^{11}-93T_1^{12}T_2^{11}+17T_1^6T_2^{12}-93T_1^7T_2^{12}+202T_1^8T_2^{12}-261T_1^9T_2^{12}+202T_1^{10}T_2^{12}-93T_1^{11}T_2^{12}+17T_1^{12}T_2^{12}\right)\right\}\right\}\right\}$$

Full expression not available (original memory size: 33.2 MB)

In[•]:=
```
Select[Gather[tab11, Last[#1] === Last[#2] &], Length[#] > 1 &]
```

Out[•]=

$$\{\{\text{Knot}[11,\text{Alternating},44]\to$$

$$\left\{\frac{(1-T+T^2)^2(1-3T+5T^2-3T^3+T^4)}{T^4},-\frac{1}{T_1^6T_2^6}2\left(1-T_1+T_1^2\right)\left(1-T_2+T_2^2\right)\left(1-T_1T_2+T_1^2T_2^2\right)\right.$$

$$\left(T_1-2T_1^2+T_1^3+T_2-5T_1T_2+5T_1^2T_2+5T_1^3T_2-5T_1^4T_2+T_1^5T_2-2T_2^2+5T_1T_2^2+5T_1^2T_2^2-26T_1^3T_2^2+\right.$$
$$5T_1^4T_2^2+5T_1^5T_2^2-2T_1^6T_2^2+T_2^3+5T_1T_2^3-26T_1^2T_2^3+32T_1^3T_2^3+32T_1^4T_2^3-26T_1^5T_2^3+5T_1^6T_2^3+T_1^7T_2^3-$$
$$5T_1T_2^4+5T_1^2T_2^4+32T_1^3T_2^4-96T_1^4T_2^4+32T_1^5T_2^4+5T_1^6T_2^4-5T_1^7T_2^4+T_1T_2^5+5T_1^2T_2^5-26T_1^3T_2^5+$$
$$32T_1^4T_2^5+32T_1^5T_2^5-26T_1^6T_2^5+5T_1^7T_2^5+T_1^8T_2^5-2T_1^2T_2^6+5T_1^3T_2^6+5T_1^4T_2^6-26T_1^5T_2^6+5T_1^6T_2^6+$$
$$\left.\left.5T_1^7T_2^6-2T_1^8T_2^6+T_1^3T_2^7-5T_1^4T_2^7+5T_1^5T_2^7+5T_1^6T_2^7-5T_1^7T_2^7+T_1^8T_2^7+T_1^5T_2^8-2T_1^6T_2^8+T_1^7T_2^8\right)\right\},$$

$$\text{Knot}[11,\text{Alternating},47]\to\left\{\frac{(1-T+T^2)^2(1-3T+5T^2-3T^3+T^4)}{T^4},\right.$$

$$-\frac{1}{T_1^6T_2^6}2\left(1-T_1+T_1^2\right)\left(1-T_2+T_2^2\right)\left(1-T_1T_2+T_1^2T_2^2\right)$$

$$\left(T_1-2T_1^2+T_1^3+T_2-5T_1T_2+5T_1^2T_2+5T_1^3T_2-5T_1^4T_2+T_1^5T_2-2T_2^2+5T_1T_2^2+5T_1^2T_2^2-26T_1^3T_2^2+\right.$$
$$5T_1^4T_2^2+5T_1^5T_2^2-2T_1^6T_2^2+T_2^3+5T_1T_2^3-26T_1^2T_2^3+32T_1^3T_2^3+32T_1^4T_2^3-26T_1^5T_2^3+5T_1^6T_2^3+T_1^7T_2^3-$$
$$5T_1T_2^4+5T_1^2T_2^4+32T_1^3T_2^4-96T_1^4T_2^4+32T_1^5T_2^4+5T_1^6T_2^4-5T_1^7T_2^4+T_1T_2^5+5T_1^2T_2^5-26T_1^3T_2^5+$$
$$32T_1^4T_2^5+32T_1^5T_2^5-26T_1^6T_2^5+5T_1^7T_2^5+T_1^8T_2^5-2T_1^2T_2^6+5T_1^3T_2^6+5T_1^4T_2^6-26T_1^5T_2^6+5T_1^6T_2^6+$$
$$\left.\left.5T_1^7T_2^6-2T_1^8T_2^6+T_1^3T_2^7-5T_1^4T_2^7+5T_1^5T_2^7+5T_1^6T_2^7-5T_1^7T_2^7+T_1^8T_2^7+T_1^5T_2^8-2T_1^6T_2^8+T_1^7T_2^8\right)\right\}\},$$

$$\left\{\text{Knot}[11,\text{Alternating},57]\to\left\{-\frac{(1-T+T^2)^2(1-3T+3T^2-3T^3+T^4)}{T^4},\right.\right.$$

$$\frac{1}{T_1^8T_2^8}\left(1-T_1+T_1^2\right)\left(1-T_2+T_2^2\right)\left(1-T_1T_2+T_1^2T_2^2\right)$$

$$\left(1-4T_1+7T_1^2-9T_1^3+7T_1^4-4T_1^5+T_1^6-4T_2+12T_1T_2-12T_1^2T_2+8T_1^3T_2+8T_1^4T_2-12T_1^5T_2+\right.$$
$$12T_1^6T_2-4T_1^7T_2+7T_2^2-12T_1T_2^2-8T_1^2T_2^2+25T_1^3T_2^2-52T_1^4T_2^2+25T_1^5T_2^2-8T_1^6T_2^2-12T_1^7T_2^2+$$
$$7T_1^8T_2^2-9T_2^3+8T_1T_2^3+25T_1^2T_2^3-32T_1^3T_2^3+37T_1^4T_2^3+37T_1^5T_2^3-32T_1^6T_2^3+25T_1^7T_2^3+8T_1^8T_2^3-$$
$$9T_1^9T_2^3+7T_2^4+8T_1T_2^4-52T_1^2T_2^4+37T_1^3T_2^4-6T_1^4T_2^4-68T_1^5T_2^4-6T_1^6T_2^4+37T_1^7T_2^4-52T_1^8T_2^4+$$
$$8T_1^9T_2^4+7T_1^{10}T_2^4-4T_2^5-12T_1T_2^5+25T_1^2T_2^5+37T_1^3T_2^5-68T_1^4T_2^5+66T_1^5T_2^5+66T_1^6T_2^5-68T_1^7T_2^5+$$
$$37T_1^8T_2^5+25T_1^9T_2^5-12T_1^{10}T_2^5-4T_1^{11}T_2^5+T_2^6+12T_1T_2^6-8T_1^2T_2^6-32T_1^3T_2^6-6T_1^4T_2^6+66T_1^5T_2^6-$$
$$156T_1^6T_2^6+66T_1^7T_2^6-6T_1^8T_2^6-32T_1^9T_2^6-8T_1^{10}T_2^6+12T_1^{11}T_2^6+T_1^{12}T_2^6-4T_1T_2^7-12T_1^2T_2^7+$$

$$25\,T_1^3\,T_2^7+37\,T_1^4\,T_2^7-68\,T_1^5\,T_2^7+66\,T_1^6\,T_2^7+66\,T_1^7\,T_2^7-68\,T_1^8\,T_2^7+37\,T_1^9\,T_2^7+25\,T_1^{10}\,T_2^7-12\,T_1^{11}\,T_2^7-$$
$$4\,T_1^{12}\,T_2^7+7\,T_1^2\,T_2^8+8\,T_1^3\,T_2^8-52\,T_1^4\,T_2^8+37\,T_1^5\,T_2^8-6\,T_1^6\,T_2^8-68\,T_1^7\,T_2^8-6\,T_1^8\,T_2^8+37\,T_1^9\,T_2^8-52\,T_1^{10}\,T_2^8+$$
$$8\,T_1^{11}\,T_2^8+7\,T_1^{12}\,T_2^8-9\,T_1^3\,T_2^9+8\,T_1^4\,T_2^9+25\,T_1^5\,T_2^9-32\,T_1^6\,T_2^9+37\,T_1^7\,T_2^9+37\,T_1^8\,T_2^9-32\,T_1^9\,T_2^9+$$
$$25\,T_1^{10}\,T_2^9+8\,T_1^{11}\,T_2^9-9\,T_1^{12}\,T_2^9+7\,T_1^4\,T_2^{10}-12\,T_1^5\,T_2^{10}-8\,T_1^6\,T_2^{10}+25\,T_1^7\,T_2^{10}-52\,T_1^8\,T_2^{10}+25\,T_1^9\,T_2^{10}-$$
$$8\,T_1^{10}\,T_2^{10}-12\,T_1^{11}\,T_2^{10}+7\,T_1^{12}\,T_2^{10}-4\,T_1^5\,T_2^{11}+12\,T_1^6\,T_2^{11}-12\,T_1^7\,T_2^{11}+8\,T_1^8\,T_2^{11}+8\,T_1^9\,T_2^{11}-12\,T_1^{10}\,T_2^{11}+$$
$$12\,T_1^{11}\,T_2^{11}-4\,T_1^{12}\,T_2^{11}+T_1^6\,T_2^{12}-4\,T_1^7\,T_2^{12}+7\,T_1^8\,T_2^{12}-9\,T_1^9\,T_2^{12}+7\,T_1^{10}\,T_2^{12}-4\,T_1^{11}\,T_2^{12}+T_1^{12}\,T_2^{12}\big)\Big\},$$

$$\text{Knot}[11,\text{Alternating},231]\to\Big\{-\frac{\left(1-T+T^2\right)^2\,\left(1-3\,T+3\,T^2-3\,T^3+T^4\right)}{T^4},$$

$$\frac{1}{T_1^8\,T_2^8}\,\left(1-T_1+T_1^2\right)\,\left(1-T_2+T_2^2\right)\,\left(1-T_1\,T_2+T_1^2\,T_2^2\right)$$
$$\big(1-4\,T_1+7\,T_1^2-9\,T_1^3+7\,T_1^4-4\,T_1^5+T_1^6-4\,T_2+12\,T_1\,T_2-12\,T_1^2\,T_2+8\,T_1^3\,T_2+8\,T_1^4\,T_2-12\,T_1^5\,T_2+$$
$$12\,T_1^6\,T_2-4\,T_1^7\,T_2+7\,T_2^2-12\,T_1\,T_2^2-8\,T_1^2\,T_2^2+25\,T_1^3\,T_2^2-52\,T_1^4\,T_2^2+25\,T_1^5\,T_2^2-8\,T_1^6\,T_2^2-12\,T_1^7\,T_2^2+$$
$$7\,T_1^8\,T_2^2-9\,T_2^3+8\,T_1\,T_2^3+25\,T_1^2\,T_2^3-32\,T_1^3\,T_2^3+37\,T_1^4\,T_2^3+37\,T_1^5\,T_2^3-32\,T_1^6\,T_2^3+25\,T_1^7\,T_2^3+8\,T_1^8\,T_2^3-$$
$$9\,T_1^9\,T_2^3+7\,T_2^4+8\,T_1\,T_2^4-52\,T_1^2\,T_2^4+37\,T_1^3\,T_2^4-6\,T_1^4\,T_2^4-68\,T_1^5\,T_2^4-6\,T_1^6\,T_2^4+37\,T_1^7\,T_2^4-52\,T_1^8\,T_2^4+$$
$$8\,T_1^9\,T_2^4+7\,T_1^{10}\,T_2^4-4\,T_2^5-12\,T_1\,T_2^5+25\,T_1^2\,T_2^5+37\,T_1^3\,T_2^5-68\,T_1^4\,T_2^5+66\,T_1^5\,T_2^5+66\,T_1^6\,T_2^5-68\,T_1^7\,T_2^5+$$
$$37\,T_1^8\,T_2^5+25\,T_1^9\,T_2^5-12\,T_1^{10}\,T_2^5-4\,T_1^{11}\,T_2^5+T_2^6+12\,T_1\,T_2^6-8\,T_1^2\,T_2^6-32\,T_1^3\,T_2^6-6\,T_1^4\,T_2^6+66\,T_1^5\,T_2^6-$$
$$156\,T_1^6\,T_2^6+66\,T_1^7\,T_2^6-6\,T_1^8\,T_2^6-32\,T_1^9\,T_2^6-8\,T_1^{10}\,T_2^6+12\,T_1^{11}\,T_2^6+T_1^{12}\,T_2^6-4\,T_1\,T_2^7-12\,T_1^2\,T_2^7+$$
$$25\,T_1^3\,T_2^7+37\,T_1^4\,T_2^7-68\,T_1^5\,T_2^7+66\,T_1^6\,T_2^7+66\,T_1^7\,T_2^7-68\,T_1^8\,T_2^7+37\,T_1^9\,T_2^7+25\,T_1^{10}\,T_2^7-12\,T_1^{11}\,T_2^7-$$
$$4\,T_1^{12}\,T_2^7+7\,T_1^2\,T_2^8+8\,T_1^3\,T_2^8-52\,T_1^4\,T_2^8+37\,T_1^5\,T_2^8-6\,T_1^6\,T_2^8-68\,T_1^7\,T_2^8-6\,T_1^8\,T_2^8+37\,T_1^9\,T_2^8-52\,T_1^{10}\,T_2^8+$$
$$8\,T_1^{11}\,T_2^8+7\,T_1^{12}\,T_2^8-9\,T_1^3\,T_2^9+8\,T_1^4\,T_2^9+25\,T_1^5\,T_2^9-32\,T_1^6\,T_2^9+37\,T_1^7\,T_2^9+37\,T_1^8\,T_2^9-32\,T_1^9\,T_2^9+$$
$$25\,T_1^{10}\,T_2^9+8\,T_1^{11}\,T_2^9-9\,T_1^{12}\,T_2^9+7\,T_1^4\,T_2^{10}-12\,T_1^5\,T_2^{10}-8\,T_1^6\,T_2^{10}+25\,T_1^7\,T_2^{10}-52\,T_1^8\,T_2^{10}+25\,T_1^9\,T_2^{10}-$$
$$8\,T_1^{10}\,T_2^{10}-12\,T_1^{11}\,T_2^{10}+7\,T_1^{12}\,T_2^{10}-4\,T_1^5\,T_2^{11}+12\,T_1^6\,T_2^{11}-12\,T_1^7\,T_2^{11}+8\,T_1^8\,T_2^{11}+8\,T_1^9\,T_2^{11}-12\,T_1^{10}\,T_2^{11}+$$
$$12\,T_1^{11}\,T_2^{11}-4\,T_1^{12}\,T_2^{11}+T_1^6\,T_2^{12}-4\,T_1^7\,T_2^{12}+7\,T_1^8\,T_2^{12}-9\,T_1^9\,T_2^{12}+7\,T_1^{10}\,T_2^{12}-4\,T_1^{11}\,T_2^{12}+T_1^{12}\,T_2^{12}\big)\Big\}\Big\},$$

$$\Big\{\text{Knot}[11,\text{NonAlternating},73]\to\Big\{\frac{\left(1-T+T^2\right)^2}{T^2},$$
$$-\frac{2\,\left(1-T_1+T_1^2\right)\,\left(1-T_2+T_2^2\right)\,\left(1-T_1\,T_2+T_1^2\,T_2^2\right)\,\left(1+T_1+T_2-6\,T_1\,T_2+T_1^2\,T_2+T_1\,T_2^2+T_1^2\,T_2^2\right)}{T_1^3\,T_2^3}\Big\},$$

$$\text{Knot}[11,\text{NonAlternating},74]\to$$
$$\Big\{\frac{\left(1-T+T^2\right)^2}{T^2},$$
$$-\frac{2\,\left(1-T_1+T_1^2\right)\,\left(1-T_2+T_2^2\right)\,\left(1-T_1\,T_2+T_1^2\,T_2^2\right)\,\left(1+T_1+T_2-6\,T_1\,T_2+T_1^2\,T_2+T_1\,T_2^2+T_1^2\,T_2^2\right)}{T_1^3\,T_2^3}\Big\}\Big\}\Big\}$$

In[◦]:=
```
tab12 = Table[K → θ@K, {K, AllKnots[{3, 12}] }]
```

KnotTheory: Loading precomputed data in KnotTheory/12A.dts.

KnotTheory: Loading precomputed data in KnotTheory/12N.dts.

Out[∘]=

$$\left\{\text{Knot}[3,1]\to\left\{\frac{1-T+T^2}{T},\ -\frac{1-T_1+T_1^2-T_2-T_1^3T_2+T_2^2+T_1^4T_2^2-T_1T_2^3-T_1^4T_2^3+T_1^2T_2^4-T_1^3T_2^4+T_1^4T_2^4}{T_1^2T_2^2}\right\},\ \text{Knot}[4,1]\to\left\{-\frac{1-3T+T^2}{T},\ 0\right\},\right.$$
$$\text{Knot}[5,1]\to\left\{\frac{1-T+T^2-T^3+T^4}{T^2},\ -\frac{\langle\langle 53\rangle\rangle+2T_1^8T_2^8}{T_1^4T_2^4}\right\},\ \langle\langle 2971\rangle\rangle,\ \text{Knot}[12,\text{NonAlternating},886]\to\left\{\frac{\langle\langle 1\rangle\rangle}{T^3},\ \langle\langle 1\rangle\rangle\right\},$$
$$\text{Knot}[12,\text{NonAlternating},887]\to\left\{\frac{1-6T+16T^2-25T^3+29T^4-25T^5+16T^6-6T^7+T^8}{T^4},\ \frac{2-12T_1+\langle\langle 327\rangle\rangle+2T_1^{16}T_2^{16}}{T_1^8T_2^8}\right\},$$
$$\text{Knot}[12,\text{NonAlternating},888]\to\left\{\frac{(1-T+T^2)^2\,(1+T-2T^2+T^3-2T^4+T^5+T^6)}{T^5},\right.$$
$$\frac{1}{T_1^{10}T_2^{10}}\left(1-T_1+T_1^2\right)\left(1-T_2+T_2^2\right)\left(\langle\langle 1\rangle\rangle\right)\left(5-10T_1^2+20T_1^3-25T_1^4+20T_1^5-10T_1^6+5T_1^8-10T_2^2+11T_1^2T_2^2-39T_1^3T_2^2+\right.$$
$$\left.\left.\left.\langle\langle 208\rangle\rangle+11T_1^{14}T_2^{14}-10T_1^{16}T_2^{14}+5T_1^8T_2^{16}-10T_1^{10}T_2^{16}+20T_1^{11}T_2^{16}-25T_1^{12}T_2^{16}+20T_1^{13}T_2^{16}-10T_1^{14}T_2^{16}+5T_1^{16}T_2^{16}\right)\right\}\right\}$$

Full expression not available (original memory size: 150.5 MB)

In[∘]:=
```
dup12 = Map[First, Select[Gather[tab12, Last[#1] === Last[#2] &], Length[#] > 1 &], {2}]
```

Out[∘]=
```
{{Knot[10, 106], Knot[12, NonAlternating, 369]},
 {Knot[11, Alternating, 44], Knot[11, Alternating, 47]},
 {Knot[11, Alternating, 57], Knot[11, Alternating, 231]},
 {Knot[11, NonAlternating, 73], Knot[11, NonAlternating, 74]},
 {Knot[12, Alternating, 30], Knot[12, Alternating, 33]},
 {Knot[12, Alternating, 122], Knot[12, Alternating, 182]},
 {Knot[12, Alternating, 164], Knot[12, Alternating, 166]},
 {Knot[12, Alternating, 167], Knot[12, Alternating, 692]},
 {Knot[12, Alternating, 273], Knot[12, Alternating, 890]},
 {Knot[12, Alternating, 341], Knot[12, Alternating, 627]},
 {Knot[12, Alternating, 427], Knot[12, Alternating, 435], Knot[12, Alternating, 990]},
 {Knot[12, Alternating, 458], Knot[12, Alternating, 887]},
 {Knot[12, Alternating, 510], Knot[12, Alternating, 821]},
 {Knot[12, NonAlternating, 56], Knot[12, NonAlternating, 57]},
 {Knot[12, NonAlternating, 60], Knot[12, NonAlternating, 61]},
 {Knot[12, NonAlternating, 62], Knot[12, NonAlternating, 66]},
 {Knot[12, NonAlternating, 144], Knot[12, NonAlternating, 507]},
 {Knot[12, NonAlternating, 313], Knot[12, NonAlternating, 430]}}
```

In[∘]:=
```
Length /@ dup12
```

Out[∘]=
```
{2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 2, 2, 2, 2, 2, 2, 2}
```

In[∘]:=
```
Total[(Length /@ dup12) - 1]
```

Out[∘]=
```
19
```

In[∘]:=
```
Length /@
 Select[Gather[tab12 /. {T₁ → 22 / 7, T₂ → 13 / 21}, Last[#1] === Last[#2] &], Length[#] > 1 &]
```

Out[∘]=
```
{2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 2, 2, 2, 2, 2, 2, 2}
```

In[∘]:=
```
Put[tab12 /. {T₁ → T1, T₂ → T2}, "Data12.m"]
```

Ribbon Knots

In[◦]:=
```
Table[K → θ[K],
 {K, {Knot[6, 1], Knot[8, 8], Knot[8, 9], Knot[8, 20], Knot[9, 27], Knot[9, 41],
   Knot[9, 46], Knot[10, 3], Knot[10, 22], Knot[10, 35], Knot[10, 42], Knot[10, 48],
   Knot[10, 75], Knot[10, 87], Knot[10, 99], Knot[10, 123], Knot[10, 129],
   Knot[10, 137], Knot[10, 140], Knot[10, 153], Knot[10, 155]}}]
```

Out[◦]=

$$\Big\{\text{Knot}[6,1]\to$$
$$\Big\{-\frac{(-2+T)\ (-1+2\,T)}{T},\ \frac{1}{T_1^2\,T_2^2}\big(1-3\,T_1+T_1^2-3\,T_2+6\,T_1\,T_2+6\,T_1^2\,T_2-3\,T_1^3\,T_2+T_2^2+6\,T_1\,T_2^2-24\,T_1^2\,T_2^2+$$
$$6\,T_1^3\,T_2^2+T_1^4\,T_2^2-3\,T_1\,T_2^3+6\,T_1^2\,T_2^3+6\,T_1^3\,T_2^3-3\,T_1^4\,T_2^3+T_1^2\,T_2^4-3\,T_1^3\,T_2^4+T_1^4\,T_2^4\big)\Big\},$$

$$\text{Knot}[8,8]\to\Big\{\frac{\left(2-2\,T+T^2\right)\ \left(1-2\,T+2\,T^2\right)}{T^2},\ \frac{1}{T_1^4\,T_2^4}\ \big(1-3\,T_1+5\,T_1^2-3\,T_1^3+T_1^4-3\,T_2+6\,T_1\,T_2-$$
$$6\,T_1^2\,T_2-6\,T_1^3\,T_2+6\,T_1^4\,T_2-3\,T_1^5\,T_2+5\,T_2^2-6\,T_1\,T_2^2+9\,T_1^2\,T_2^2+5\,T_1^3\,T_2^2+9\,T_1^4\,T_2^2-6\,T_1^5\,T_2^2+5\,T_1^6\,T_2^2-$$
$$3\,T_2^3-6\,T_1\,T_2^3+5\,T_1^2\,T_2^3-18\,T_1^3\,T_2^3-18\,T_1^4\,T_2^3+5\,T_1^5\,T_2^3-6\,T_1^6\,T_2^3-3\,T_1^7\,T_2^3+T_2^4+6\,T_1\,T_2^4+9\,T_1^2\,T_2^4-$$
$$18\,T_1^3\,T_2^4+60\,T_1^4\,T_2^4-18\,T_1^5\,T_2^4+9\,T_1^6\,T_2^4+6\,T_1^7\,T_2^4+T_1^8\,T_2^4-3\,T_1\,T_2^5-6\,T_1^2\,T_2^5+5\,T_1^3\,T_2^5-18\,T_1^4\,T_2^5-$$
$$18\,T_1^5\,T_2^5+5\,T_1^6\,T_2^5-6\,T_1^7\,T_2^5-3\,T_1^8\,T_2^5+5\,T_1^2\,T_2^6-6\,T_1^3\,T_2^6+9\,T_1^4\,T_2^6+5\,T_1^5\,T_2^6+9\,T_1^6\,T_2^6-6\,T_1^7\,T_2^6+5\,T_1^8\,T_2^6-$$
$$3\,T_1^3\,T_2^7+6\,T_1^4\,T_2^7-6\,T_1^5\,T_2^7-6\,T_1^6\,T_2^7+6\,T_1^7\,T_2^7-3\,T_1^8\,T_2^7+T_1^4\,T_2^8-3\,T_1^5\,T_2^8+5\,T_1^6\,T_2^8-3\,T_1^7\,T_2^8+T_1^8\,T_2^8\big)\Big\},$$

$$\text{Knot}[8,9]\to\Big\{-\frac{\left(-1+T-2\,T^2+T^3\right)\ \left(-1+2\,T-T^2+T^3\right)}{T^3},\ 0\Big\},$$

$$\text{Knot}[8,20]\to$$
$$\Big\{\frac{\left(1-T+T^2\right)^2}{T^2},\ -\frac{1}{T_1^2\,T_2^2}2\ \big(3-4\,T_1+3\,T_1^2-4\,T_2+T_1\,T_2+T_1^2\,T_2-4\,T_1^3\,T_2+3\,T_2^2+T_1\,T_2^2+$$
$$T_1^3\,T_2^2+3\,T_1^4\,T_2^2-4\,T_1\,T_2^3+T_1^2\,T_2^3+T_1^3\,T_2^3-4\,T_1^4\,T_2^3+3\,T_1^2\,T_2^4-4\,T_1^3\,T_2^4+3\,T_1^4\,T_2^4\big)\Big\},$$

$$\text{Knot}[9,27]\to\Big\{-\frac{\left(-1+2\,T-3\,T^2+T^3\right)\ \left(-1+3\,T-2\,T^2+T^3\right)}{T^3},$$
$$-\frac{1}{T_1^4\,T_2^4}\ \big(1-T_1+T_1^2-T_1^3+T_1^4-T_2-8\,T_1\,T_2+4\,T_1^2\,T_2+4\,T_1^3\,T_2-8\,T_1^4\,T_2-T_1^5\,T_2+T_2^2+4\,T_1\,T_2^2+$$
$$49\,T_1^2\,T_2^2-67\,T_1^3\,T_2^2+49\,T_1^4\,T_2^2+4\,T_1^5\,T_2^2+T_1^6\,T_2^2-T_2^3+4\,T_1\,T_2^3-67\,T_1^2\,T_2^3+20\,T_1^3\,T_2^3+$$
$$20\,T_1^4\,T_2^3-67\,T_1^5\,T_2^3+4\,T_1^6\,T_2^3-T_1^7\,T_2^3+T_2^4-8\,T_1\,T_2^4+49\,T_1^2\,T_2^4+20\,T_1^3\,T_2^4-12\,T_1^4\,T_2^4+$$
$$20\,T_1^5\,T_2^4+49\,T_1^6\,T_2^4-8\,T_1^7\,T_2^4+T_1^8\,T_2^4-T_1\,T_2^5+4\,T_1^2\,T_2^5-67\,T_1^3\,T_2^5+20\,T_1^4\,T_2^5+20\,T_1^5\,T_2^5-$$
$$67\,T_1^6\,T_2^5+4\,T_1^7\,T_2^5-T_1^8\,T_2^5+T_1^2\,T_2^6+4\,T_1^3\,T_2^6+49\,T_1^4\,T_2^6-67\,T_1^5\,T_2^6+49\,T_1^6\,T_2^6+4\,T_1^7\,T_2^6+T_1^8\,T_2^6-$$
$$T_1^3\,T_2^7-8\,T_1^4\,T_2^7+4\,T_1^5\,T_2^7+4\,T_1^6\,T_2^7-8\,T_1^7\,T_2^7-T_1^8\,T_2^7+T_1^4\,T_2^8-T_1^5\,T_2^8+T_1^6\,T_2^8-T_1^7\,T_2^8+T_1^8\,T_2^8\big)\Big\},$$

$$\text{Knot}[9,41]\to\Big\{\frac{\left(3-3\,T+T^2\right)\ \left(1-3\,T+3\,T^2\right)}{T^2},\ -\frac{1}{T_1^4\,T_2^4}\ \big(3-15\,T_1+27\,T_1^2-15\,T_1^3+3\,T_1^4-15\,T_2+$$
$$58\,T_1\,T_2-56\,T_1^2\,T_2-56\,T_1^3\,T_2+58\,T_1^4\,T_2-15\,T_1^5\,T_2+27\,T_2^2-56\,T_1\,T_2^2-81\,T_1^2\,T_2^2+333\,T_1^3\,T_2^2-81\,T_1^4\,T_2^2-$$
$$56\,T_1^5\,T_2^2+27\,T_1^6\,T_2^2-15\,T_2^3-56\,T_1\,T_2^3+333\,T_1^2\,T_2^3-396\,T_1^3\,T_2^3-396\,T_1^4\,T_2^3+333\,T_1^5\,T_2^3-56\,T_1^6\,T_2^3-$$
$$15\,T_1^7\,T_2^3+3\,T_2^4+58\,T_1\,T_2^4-81\,T_1^2\,T_2^4-396\,T_1^3\,T_2^4+1188\,T_1^4\,T_2^4-396\,T_1^5\,T_2^4-81\,T_1^6\,T_2^4+58\,T_1^7\,T_2^4+$$
$$3\,T_1^8\,T_2^4-15\,T_1\,T_2^5-56\,T_1^2\,T_2^5+333\,T_1^3\,T_2^5-396\,T_1^4\,T_2^5-396\,T_1^5\,T_2^5+333\,T_1^6\,T_2^5-56\,T_1^7\,T_2^5-15\,T_1^8\,T_2^5+$$

$$27\,T_1^2\,T_2^6-56\,T_1^3\,T_2^6-81\,T_1^4\,T_2^6+333\,T_1^5\,T_2^6-81\,T_1^6\,T_2^6-56\,T_1^7\,T_2^6+27\,T_1^8\,T_2^6-15\,T_1^3\,T_2^7+58\,T_1^4\,T_2^7-$$
$$56\,T_1^5\,T_2^7-56\,T_1^6\,T_2^7+58\,T_1^7\,T_2^7-15\,T_1^8\,T_2^7+3\,T_1^4\,T_2^8-15\,T_1^5\,T_2^8+27\,T_1^6\,T_2^8-15\,T_1^7\,T_2^8+3\,T_1^8\,T_2^8\Big)\Big\},$$

$$\text{Knot}[9,\,46]\to\Big\{-\frac{(-2+T)\,(-1+2\,T)}{T},\;\frac{1}{T_1^2\,T_2^2}3\,\Big(1-3\,T_1+T_1^2-3\,T_2+6\,T_1\,T_2+6\,T_1^2\,T_2-3\,T_1^3\,T_2+$$
$$T_2^2+6\,T_1\,T_2^2-24\,T_1^2\,T_2^2+6\,T_1^3\,T_2^2+T_1^4\,T_2^2-3\,T_1\,T_2^3+6\,T_1^2\,T_2^3+6\,T_1^3\,T_2^3-3\,T_1^4\,T_2^3+T_1^2\,T_2^4-3\,T_1^3\,T_2^4+T_1^4\,T_2^4\Big)\Big\},$$

$$\text{Knot}[10,\,3]\to\Big\{-\frac{(-3+2\,T)\,(-2+3\,T)}{T},\;\frac{1}{T_1^2\,T_2^2}\Big(45-101\,T_1+45\,T_1^2-101\,T_2+126\,T_1\,T_2+$$
$$126\,T_1^2\,T_2-101\,T_1^3\,T_2+45\,T_2^2+126\,T_1\,T_2^2-420\,T_1^2\,T_2^2+126\,T_1^3\,T_2^2+45\,T_1^4\,T_2^2-$$
$$101\,T_1\,T_2^3+126\,T_1^2\,T_2^3+126\,T_1^3\,T_2^3-101\,T_1^4\,T_2^3+45\,T_1^2\,T_2^4-101\,T_1^3\,T_2^4+45\,T_1^4\,T_2^4\Big)\Big\},$$

$$\text{Knot}[10,\,22]\to\Big\{-\frac{\left(-2+2\,T-2\,T^2+T^3\right)\,\left(-1+2\,T-2\,T^2+2\,T^3\right)}{T^3},$$
$$-\frac{1}{T_1^6\,T_2^6}\Big(1-3\,T_1+5\,T_1^2-7\,T_1^3+5\,T_1^4-3\,T_1^5+T_1^6-3\,T_2+6\,T_1\,T_2-6\,T_1^2\,T_2+6\,T_1^3\,T_2+6\,T_1^4\,T_2-6\,T_1^5\,T_2+$$
$$6\,T_1^6\,T_2-3\,T_1^7\,T_2+5\,T_2^2-6\,T_1\,T_2^2+3\,T_1^2\,T_2^2-9\,T_1^3\,T_2^2-5\,T_1^4\,T_2^2-9\,T_1^5\,T_2^2+3\,T_1^6\,T_2^2-6\,T_1^7\,T_2^2+$$
$$5\,T_1^8\,T_2^2-7\,T_2^3+6\,T_1\,T_2^3-9\,T_1^2\,T_2^3+30\,T_1^3\,T_2^3+4\,T_1^4\,T_2^3+4\,T_1^5\,T_2^3+30\,T_1^6\,T_2^3-9\,T_1^7\,T_2^3+6\,T_1^8\,T_2^3-$$
$$7\,T_1^9\,T_2^3+5\,T_2^4+6\,T_1\,T_2^4-5\,T_1^2\,T_2^4+4\,T_1^3\,T_2^4-89\,T_1^4\,T_2^4+63\,T_1^5\,T_2^4-89\,T_1^6\,T_2^4+4\,T_1^7\,T_2^4-5\,T_1^8\,T_2^4+$$
$$6\,T_1^9\,T_2^4+5\,T_1^{10}\,T_2^4-3\,T_2^5-6\,T_1\,T_2^5-9\,T_1^2\,T_2^5+4\,T_1^3\,T_2^5+63\,T_1^4\,T_2^5+22\,T_1^5\,T_2^5+22\,T_1^6\,T_2^5+63\,T_1^7\,T_2^5+$$
$$4\,T_1^8\,T_2^5-9\,T_1^9\,T_2^5-6\,T_1^{10}\,T_2^5-3\,T_1^{11}\,T_2^5+T_2^6+6\,T_1\,T_2^6+3\,T_1^2\,T_2^6+30\,T_1^3\,T_2^6-89\,T_1^4\,T_2^6+22\,T_1^5\,T_2^6-$$
$$108\,T_1^6\,T_2^6+22\,T_1^7\,T_2^6-89\,T_1^8\,T_2^6+30\,T_1^9\,T_2^6+3\,T_1^{10}\,T_2^6+6\,T_1^{11}\,T_2^6+T_1^{12}\,T_2^6-3\,T_1\,T_2^7-6\,T_1^2\,T_2^7-$$
$$9\,T_1^3\,T_2^7+4\,T_1^4\,T_2^7+63\,T_1^5\,T_2^7+22\,T_1^6\,T_2^7+22\,T_1^7\,T_2^7+63\,T_1^8\,T_2^7+4\,T_1^9\,T_2^7-9\,T_1^{10}\,T_2^7-6\,T_1^{11}\,T_2^7-$$
$$3\,T_1^{12}\,T_2^7+5\,T_1^2\,T_2^8+6\,T_1^3\,T_2^8-5\,T_1^4\,T_2^8+4\,T_1^5\,T_2^8-89\,T_1^6\,T_2^8+63\,T_1^7\,T_2^8-89\,T_1^8\,T_2^8+4\,T_1^9\,T_2^8-$$
$$5\,T_1^{10}\,T_2^8+6\,T_1^{11}\,T_2^8+5\,T_1^{12}\,T_2^8-7\,T_1^3\,T_2^9+6\,T_1^4\,T_2^9-9\,T_1^5\,T_2^9+30\,T_1^6\,T_2^9+4\,T_1^7\,T_2^9+4\,T_1^8\,T_2^9+30\,T_1^9\,T_2^9-$$
$$9\,T_1^{10}\,T_2^9+6\,T_1^{11}\,T_2^9-7\,T_1^{12}\,T_2^9+5\,T_1^4\,T_2^{10}-6\,T_1^5\,T_2^{10}+3\,T_1^6\,T_2^{10}-9\,T_1^7\,T_2^{10}-5\,T_1^8\,T_2^{10}-9\,T_1^9\,T_2^{10}+$$
$$3\,T_1^{10}\,T_2^{10}-6\,T_1^{11}\,T_2^{10}+5\,T_1^{12}\,T_2^{10}-3\,T_1^5\,T_2^{11}+6\,T_1^6\,T_2^{11}-6\,T_1^7\,T_2^{11}+6\,T_1^8\,T_2^{11}+6\,T_1^9\,T_2^{11}-6\,T_1^{10}\,T_2^{11}+$$
$$6\,T_1^{11}\,T_2^{11}-3\,T_1^{12}\,T_2^{11}+T_1^6\,T_2^{12}-3\,T_1^7\,T_2^{12}+5\,T_1^8\,T_2^{12}-7\,T_1^9\,T_2^{12}+5\,T_1^{10}\,T_2^{12}-3\,T_1^{11}\,T_2^{12}+T_1^{12}\,T_2^{12}\Big)\Big\},$$

$$\text{Knot}[10,\,35]\to\Big\{\frac{\left(2-4\,T+T^2\right)\,\left(1-4\,T+2\,T^2\right)}{T^2},\;\frac{1}{T_1^4\,T_2^4}\Big(1-7\,T_1+13\,T_1^2-7\,T_1^3+T_1^4-7\,T_2+$$
$$42\,T_1\,T_2-42\,T_1^2\,T_2-42\,T_1^3\,T_2+42\,T_1^4\,T_2-7\,T_1^5\,T_2+13\,T_2^2-42\,T_1\,T_2^2-148\,T_1^2\,T_2^2+426\,T_1^3\,T_2^2-$$
$$148\,T_1^4\,T_2^2-42\,T_1^5\,T_2^2+13\,T_1^6\,T_2^2-7\,T_2^3-42\,T_1\,T_2^3+426\,T_1^2\,T_2^3-468\,T_1^3\,T_2^3-468\,T_1^4\,T_2^3+426\,T_1^5\,T_2^3-$$
$$42\,T_1^6\,T_2^3-7\,T_1^7\,T_2^3+T_2^4+42\,T_1\,T_2^4-148\,T_1^2\,T_2^4-468\,T_1^3\,T_2^4+1392\,T_1^4\,T_2^4-468\,T_1^5\,T_2^4-148\,T_1^6\,T_2^4+$$
$$42\,T_1^7\,T_2^4+T_1^8\,T_2^4-7\,T_1\,T_2^5-42\,T_1^2\,T_2^5+426\,T_1^3\,T_2^5-468\,T_1^4\,T_2^5-468\,T_1^5\,T_2^5+426\,T_1^6\,T_2^5-42\,T_1^7\,T_2^5-$$
$$7\,T_1^8\,T_2^5+13\,T_1^2\,T_2^6-42\,T_1^3\,T_2^6-148\,T_1^4\,T_2^6+426\,T_1^5\,T_2^6-148\,T_1^6\,T_2^6-42\,T_1^7\,T_2^6+13\,T_1^8\,T_2^6-7\,T_1^3\,T_2^7+$$
$$42\,T_1^4\,T_2^7-42\,T_1^5\,T_2^7-42\,T_1^6\,T_2^7+42\,T_1^7\,T_2^7-7\,T_1^8\,T_2^7+T_1^4\,T_2^8-7\,T_1^5\,T_2^8+13\,T_1^6\,T_2^8-7\,T_1^7\,T_2^8+T_1^8\,T_2^8\Big)\Big\},$$

$$\text{Knot}[10,\,42]\to\Big\{-\frac{\left(-1+3\,T-4\,T^2+T^3\right)\,\left(-1+4\,T-3\,T^2+T^3\right)}{T^3},$$
$$-\frac{1}{T_1^4\,T_2^4}\Big(6-24\,T_1+38\,T_1^2-24\,T_1^3+6\,T_1^4-24\,T_2+72\,T_1\,T_2-54\,T_1^2\,T_2-54\,T_1^3\,T_2+72\,T_1^4\,T_2-$$
$$24\,T_1^5\,T_2+38\,T_2^2-54\,T_1\,T_2^2-109\,T_1^2\,T_2^2+279\,T_1^3\,T_2^2-109\,T_1^4\,T_2^2-54\,T_1^5\,T_2^2+38\,T_1^6\,T_2^2-24\,T_2^3-$$
$$54\,T_1\,T_2^3+279\,T_1^2\,T_2^3-222\,T_1^3\,T_2^3-222\,T_1^4\,T_2^3+279\,T_1^5\,T_2^3-54\,T_1^6\,T_2^3-24\,T_1^7\,T_2^3+6\,T_2^4+72\,T_1\,T_2^4-$$
$$109\,T_1^2\,T_2^4-222\,T_1^3\,T_2^4+552\,T_1^4\,T_2^4-222\,T_1^5\,T_2^4-109\,T_1^6\,T_2^4+72\,T_1^7\,T_2^4+6\,T_1^8\,T_2^4-24\,T_1\,T_2^5-$$
$$54\,T_1^2\,T_2^5+279\,T_1^3\,T_2^5-222\,T_1^4\,T_2^5-222\,T_1^5\,T_2^5+279\,T_1^6\,T_2^5-54\,T_1^7\,T_2^5-24\,T_1^8\,T_2^5+38\,T_1^2\,T_2^6-$$

$$54 T_1^3 T_2^6 - 109 T_1^4 T_2^6 + 279 T_1^5 T_2^6 - 109 T_1^6 T_2^6 - 54 T_1^7 T_2^6 + 38 T_1^8 T_2^6 - 24 T_1^3 T_2^7 + 72 T_1^4 T_2^7 -$$
$$54 T_1^5 T_2^7 - 54 T_1^6 T_2^7 + 72 T_1^7 T_2^7 - 24 T_1^8 T_2^7 + 6 T_1^4 T_2^8 - 24 T_1^5 T_2^8 + 38 T_1^6 T_2^8 - 24 T_1^7 T_2^8 + 6 T_1^8 T_2^8\Big)\Big\},$$

$$\text{Knot}[10, 48] \to \Big\{\frac{\left(1 - T + 2 T^2 - 2 T^3 + T^4\right)\left(1 - 2 T + 2 T^2 - T^3 + T^4\right)}{T^4},$$

$$-\frac{1}{T_1^6 T_2^6}\Big(1 - 2 T_1 + 2 T_1^2 - T_1^3 + 2 T_1^4 - 2 T_1^5 + T_1^6 - 2 T_2 + 2 T_1 T_2 + T_1^2 T_2 - 3 T_1^3 T_2 - 3 T_1^4 T_2 + T_1^5 T_2 + 2 T_1^6 T_2 -$$
$$2 T_1^7 T_2 + 2 T_2^2 + T_1 T_2^2 - 9 T_1^2 T_2^2 + 9 T_1^3 T_2^2 + T_1^4 T_2^2 + 9 T_1^5 T_2^2 - 9 T_1^6 T_2^2 + T_1^7 T_2^2 + 2 T_1^8 T_2^2 - T_2^3 - 3 T_1 T_2^3 +$$
$$9 T_1^2 T_2^3 + 2 T_1^3 T_2^3 - 10 T_1^4 T_2^3 - 10 T_1^5 T_2^3 + 2 T_1^6 T_2^3 + 9 T_1^7 T_2^3 - 3 T_1^8 T_2^3 - T_1^9 T_2^3 + 2 T_2^4 - 3 T_1 T_2^4 + T_1^2 T_2^4 -$$
$$10 T_1^3 T_2^4 - 5 T_1^4 T_2^4 + 29 T_1^5 T_2^4 - 5 T_1^6 T_2^4 - 10 T_1^7 T_2^4 + T_1^8 T_2^4 - 3 T_1^9 T_2^4 + 2 T_1^{10} T_2^4 - 2 T_2^5 + T_1 T_2^5 + 9 T_1^2 T_2^5 -$$
$$10 T_1^3 T_2^5 + 29 T_1^4 T_2^5 - 22 T_1^5 T_2^5 - 22 T_1^6 T_2^5 + 29 T_1^7 T_2^5 - 10 T_1^8 T_2^5 + 9 T_1^9 T_2^5 + T_1^{10} T_2^5 - 2 T_1^{11} T_2^5 + T_2^6 +$$
$$2 T_1 T_2^6 - 9 T_1^2 T_2^6 + 2 T_1^3 T_2^6 - 5 T_1^4 T_2^6 - 22 T_1^5 T_2^6 + 48 T_1^6 T_2^6 - 22 T_1^7 T_2^6 - 5 T_1^8 T_2^6 + 2 T_1^9 T_2^6 - 9 T_1^{10} T_2^6 +$$
$$2 T_1^{11} T_2^6 + T_1^{12} T_2^6 - 2 T_1 T_2^7 + T_1^2 T_2^7 + 9 T_1^3 T_2^7 - 10 T_1^4 T_2^7 + 29 T_1^5 T_2^7 - 22 T_1^6 T_2^7 - 22 T_1^7 T_2^7 + 29 T_1^8 T_2^7 -$$
$$10 T_1^9 T_2^7 + 9 T_1^{10} T_2^7 + T_1^{11} T_2^7 - 2 T_1^{12} T_2^7 + 2 T_1^2 T_2^8 - 3 T_1^3 T_2^8 + T_1^4 T_2^8 - 10 T_1^5 T_2^8 - 5 T_1^6 T_2^8 + 29 T_1^7 T_2^8 -$$
$$5 T_1^8 T_2^8 - 10 T_1^9 T_2^8 + T_1^{10} T_2^8 - 3 T_1^{11} T_2^8 + 2 T_1^{12} T_2^8 - T_1^3 T_2^9 - 3 T_1^4 T_2^9 + 9 T_1^5 T_2^9 + 2 T_1^6 T_2^9 - 10 T_1^7 T_2^9 -$$
$$10 T_1^8 T_2^9 + 2 T_1^9 T_2^9 + 9 T_1^{10} T_2^9 - 3 T_1^{11} T_2^9 - T_1^{12} T_2^9 + 2 T_1^4 T_2^{10} + T_1^5 T_2^{10} - 9 T_1^6 T_2^{10} + 9 T_1^7 T_2^{10} + T_1^8 T_2^{10} +$$
$$9 T_1^9 T_2^{10} - 9 T_1^{10} T_2^{10} + T_1^{11} T_2^{10} + 2 T_1^{12} T_2^{10} - 2 T_1^5 T_2^{11} + 2 T_1^6 T_2^{11} + T_1^7 T_2^{11} - 3 T_1^8 T_2^{11} - 3 T_1^9 T_2^{11} + T_1^{10} T_2^{11} +$$
$$2 T_1^{11} T_2^{11} - 2 T_1^{12} T_2^{11} + T_1^6 T_2^{12} - 2 T_1^7 T_2^{12} + 2 T_1^8 T_2^{12} - T_1^9 T_2^{12} + 2 T_1^{10} T_2^{12} - 2 T_1^{11} T_2^{12} + T_1^{12} T_2^{12}\Big)\Big\},$$

$$\text{Knot}[10, 75] \to \Big\{-\frac{\left(-1 + 3 T - 4 T^2 + T^3\right)\left(-1 + 4 T - 3 T^2 + T^3\right)}{T^3},$$

$$\frac{1}{T_1^4 T_2^4}\Big(2 - 8 T_1 + 16 T_1^2 - 8 T_1^3 + 2 T_1^4 - 8 T_2 + 18 T_1 T_2 - 32 T_1^2 T_2 - 32 T_1^3 T_2 + 18 T_1^4 T_2 - 8 T_1^5 T_2 + 16 T_2^2 -$$
$$32 T_1 T_2^2 + 75 T_1^2 T_2^2 + 75 T_1^3 T_2^2 + 75 T_1^4 T_2^2 - 32 T_1^5 T_2^2 + 16 T_1^6 T_2^2 - 8 T_2^3 - 32 T_1 T_2^3 + 75 T_1^2 T_2^3 - 256 T_1^3 T_2^3 -$$
$$256 T_1^4 T_2^3 + 75 T_1^5 T_2^3 - 32 T_1^6 T_2^3 - 8 T_1^7 T_2^3 + 2 T_2^4 + 18 T_1 T_2^4 + 75 T_1^2 T_2^4 - 256 T_1^3 T_2^4 + 900 T_1^4 T_2^4 - 256 T_1^5 T_2^4 +$$
$$75 T_1^6 T_2^4 + 18 T_1^7 T_2^4 + 2 T_1^8 T_2^4 - 8 T_1 T_2^5 - 32 T_1^2 T_2^5 + 75 T_1^3 T_2^5 - 256 T_1^4 T_2^5 - 256 T_1^5 T_2^5 + 75 T_1^6 T_2^5 -$$
$$32 T_1^7 T_2^5 - 8 T_1^8 T_2^5 + 16 T_1^2 T_2^6 - 32 T_1^3 T_2^6 + 75 T_1^4 T_2^6 + 75 T_1^5 T_2^6 + 75 T_1^6 T_2^6 - 32 T_1^7 T_2^6 + 16 T_1^8 T_2^6 - 8 T_1^3 T_2^7 +$$
$$18 T_1^4 T_2^7 - 32 T_1^5 T_2^7 - 32 T_1^6 T_2^7 + 18 T_1^7 T_2^7 - 8 T_1^8 T_2^7 + 2 T_1^4 T_2^8 - 8 T_1^5 T_2^8 + 16 T_1^6 T_2^8 - 8 T_1^7 T_2^8 + 2 T_1^8 T_2^8\Big)\Big\},$$

$$\text{Knot}[10, 87] \to \Big\{-\frac{(-2 + T)(-1 + 2 T)\left(1 - T + T^2\right)^2}{T^3},$$

$$-\frac{1}{T_1^6 T_2^6}\Big(1 - 4 T_1 + 8 T_1^2 - 11 T_1^3 + 8 T_1^4 - 4 T_1^5 + T_1^6 - 4 T_2 + 12 T_1 T_2 - 16 T_1^2 T_2 + 12 T_1^3 T_2 + 12 T_1^4 T_2 - 16 T_1^5 T_2 +$$
$$12 T_1^6 T_2 - 4 T_1^7 T_2 + 8 T_2^2 - 16 T_1 T_2^2 + 12 T_1^2 T_2^2 - 13 T_1^3 T_2^2 - 18 T_1^4 T_2^2 - 13 T_1^5 T_2^2 + 12 T_1^6 T_2^2 - 16 T_1^7 T_2^2 +$$
$$8 T_1^8 T_2^2 - 11 T_2^3 + 12 T_1 T_2^3 - 13 T_1^2 T_2^3 + 62 T_1^3 T_2^3 + 9 T_1^4 T_2^3 + 9 T_1^5 T_2^3 + 62 T_1^6 T_2^3 - 13 T_1^7 T_2^3 + 12 T_1^8 T_2^3 -$$
$$11 T_1^9 T_2^3 + 8 T_2^4 + 12 T_1 T_2^4 - 18 T_1^2 T_2^4 + 9 T_1^3 T_2^4 - 296 T_1^4 T_2^4 + 290 T_1^5 T_2^4 - 296 T_1^6 T_2^4 + 9 T_1^7 T_2^4 - 18 T_1^8 T_2^4 +$$
$$12 T_1^9 T_2^4 + 8 T_1^{10} T_2^4 - 4 T_2^5 - 16 T_1 T_2^5 - 13 T_1^2 T_2^5 + 9 T_1^3 T_2^5 + 290 T_1^4 T_2^5 - 32 T_1^5 T_2^5 - 32 T_1^6 T_2^5 + 290 T_1^7 T_2^5 +$$
$$9 T_1^8 T_2^5 - 13 T_1^9 T_2^5 - 16 T_1^{10} T_2^5 - 4 T_1^{11} T_2^5 + T_2^6 + 12 T_1 T_2^6 + 12 T_1^2 T_2^6 + 62 T_1^3 T_2^6 - 296 T_1^4 T_2^6 - 32 T_1^5 T_2^6 -$$
$$72 T_1^6 T_2^6 - 32 T_1^7 T_2^6 - 296 T_1^8 T_2^6 + 62 T_1^9 T_2^6 + 12 T_1^{10} T_2^6 + 12 T_1^{11} T_2^6 + T_1^{12} T_2^6 - 4 T_1 T_2^7 - 16 T_1^2 T_2^7 -$$
$$13 T_1^3 T_2^7 + 9 T_1^4 T_2^7 + 290 T_1^5 T_2^7 - 32 T_1^6 T_2^7 - 32 T_1^7 T_2^7 + 290 T_1^8 T_2^7 + 9 T_1^9 T_2^7 - 13 T_1^{10} T_2^7 - 16 T_1^{11} T_2^7 -$$
$$4 T_1^{12} T_2^7 + 8 T_1^2 T_2^8 + 12 T_1^3 T_2^8 - 18 T_1^4 T_2^8 + 9 T_1^5 T_2^8 - 296 T_1^6 T_2^8 + 290 T_1^7 T_2^8 - 296 T_1^8 T_2^8 + 9 T_1^9 T_2^8 -$$
$$18 T_1^{10} T_2^8 + 12 T_1^{11} T_2^8 + 8 T_1^{12} T_2^8 - 11 T_1^3 T_2^9 + 12 T_1^4 T_2^9 - 13 T_1^5 T_2^9 + 62 T_1^6 T_2^9 + 9 T_1^7 T_2^9 + 9 T_1^8 T_2^9 + 62 T_1^9 T_2^9 -$$
$$13 T_1^{10} T_2^9 + 12 T_1^{11} T_2^9 - 11 T_1^{12} T_2^9 + 8 T_1^4 T_2^{10} - 16 T_1^5 T_2^{10} + 12 T_1^6 T_2^{10} - 13 T_1^7 T_2^{10} - 18 T_1^8 T_2^{10} - 13 T_1^9 T_2^{10} +$$
$$12 T_1^{10} T_2^{10} - 16 T_1^{11} T_2^{10} + 8 T_1^{12} T_2^{10} - 4 T_1^5 T_2^{11} + 12 T_1^6 T_2^{11} - 16 T_1^7 T_2^{11} + 12 T_1^8 T_2^{11} + 12 T_1^9 T_2^{11} - 16 T_1^{10} T_2^{11} +$$
$$12 T_1^{11} T_2^{11} - 4 T_1^{12} T_2^{11} + T_1^6 T_2^{12} - 4 T_1^7 T_2^{12} + 8 T_1^8 T_2^{12} - 11 T_1^9 T_2^{12} + 8 T_1^{10} T_2^{12} - 4 T_1^{11} T_2^{12} + T_1^{12} T_2^{12}\Big)\Big\},$$

$$\text{Knot}[10, 99] \to \left\{\frac{\left(1 - T + T^2\right)^4}{T^4}, 0\right\}, \text{Knot}[10, 123] \to$$
$$\left\{\frac{\left(1 - 3\,T + 3\,T^2 - 3\,T^3 + T^4\right)^2}{T^4},\right.$$
$$\left. 0\right\},$$

$$\text{Knot}[10, 129] \to \left\{\frac{\left(2 - 2\,T + T^2\right)\,\left(1 - 2\,T + 2\,T^2\right)}{T^2},\right.$$
$$\frac{1}{T_1^4\,T_2^4}\,\left(1 - 2\,T_1 + 3\,T_1^2 - 2\,T_1^3 + T_1^4 - 2\,T_2 + 4\,T_1\,T_2 - 2\,T_1^2\,T_2 - 2\,T_1^3\,T_2 + 4\,T_1^4\,T_2 - 2\,T_1^5\,T_2 + 3\,T_2^2 - 2\,T_1\,T_2^2 -\right.$$
$$31\,T_1^2\,T_2^2 + 43\,T_1^3\,T_2^2 - 31\,T_1^4\,T_2^2 - 2\,T_1^5\,T_2^2 + 3\,T_1^6\,T_2^2 - 2\,T_2^3 - 2\,T_1\,T_2^3 + 43\,T_1^2\,T_2^3 - 14\,T_1^3\,T_2^3 - 14\,T_1^4\,T_2^3 +$$
$$43\,T_1^5\,T_2^3 - 2\,T_1^6\,T_2^3 - 2\,T_1^7\,T_2^3 + T_2^4 + 4\,T_1\,T_2^4 - 31\,T_1^2\,T_2^4 - 14\,T_1^3\,T_2^4 + 12\,T_1^4\,T_2^4 - 14\,T_1^5\,T_2^4 - 31\,T_1^6\,T_2^4 + 4\,T_1^7\,T_2^4 +$$
$$T_1^8\,T_2^4 - 2\,T_1\,T_2^5 - 2\,T_1^2\,T_2^5 + 43\,T_1^3\,T_2^5 - 14\,T_1^4\,T_2^5 - 14\,T_1^5\,T_2^5 + 43\,T_1^6\,T_2^5 - 2\,T_1^7\,T_2^5 - 2\,T_1^8\,T_2^5 + 3\,T_1^2\,T_2^6 -$$
$$2\,T_1^3\,T_2^6 - 31\,T_1^4\,T_2^6 + 43\,T_1^5\,T_2^6 - 31\,T_1^6\,T_2^6 - 2\,T_1^7\,T_2^6 + 3\,T_1^8\,T_2^6 - 2\,T_1^3\,T_2^7 + 4\,T_1^4\,T_2^7 - 2\,T_1^5\,T_2^7 - 2\,T_1^6\,T_2^7 +$$
$$\left.\left. 4\,T_1^7\,T_2^7 - 2\,T_1^8\,T_2^7 + T_1^4\,T_2^8 - 2\,T_1^5\,T_2^8 + 3\,T_1^6\,T_2^8 - 2\,T_1^7\,T_2^8 + T_1^8\,T_2^8\right)\right\}, \text{Knot}[10, 137] \to \left\{\frac{\left(1 - 3\,T + T^2\right)^2}{T^2},\right.$$
$$\left. -\frac{2\,\left(1 - 3\,T_1 + T_1^2\right)\,\left(1 - 3\,T_2 + T_2^2\right)\,\left(1 - 3\,T_1\,T_2 + T_1^2\,T_2^2\right)\,\left(1 + T_1 + T_2 - 6\,T_1\,T_2 + T_1^2\,T_2 + T_1\,T_2^2 + T_1^2\,T_2^2\right)}{T_1^3\,T_2^3}\right\},$$

$$\text{Knot}[10, 140] \to \left\{\frac{\left(1 - T + T^2\right)^2}{T^2}, -\frac{1}{T_1^2\,T_2^2} 4\,\left(3 - 4\,T_1 + 3\,T_1^2 - 4\,T_2 + T_1\,T_2 + T_1^2\,T_2 - 4\,T_1^3\,T_2 + 3\,T_2^2 +\right.\right.$$
$$\left.\left. T_1\,T_2^2 + T_1^3\,T_2^2 + 3\,T_1^4\,T_2^2 - 4\,T_1\,T_2^3 + T_1^2\,T_2^3 + T_1^3\,T_2^3 - 4\,T_1^4\,T_2^3 + 3\,T_1^2\,T_2^4 - 4\,T_1^3\,T_2^4 + 3\,T_1^4\,T_2^4\right)\right\},$$

$$\text{Knot}[10, 153] \to \left\{\frac{\left(1 - T + T^3\right)\,\left(1 - T^2 + T^3\right)}{T^3},\right.$$
$$-\frac{1}{T_1^6\,T_2^6}\,\left(1 - T_1 - T_1^2 + 3\,T_1^3 - T_1^4 - T_1^5 + T_1^6 - T_2 - 2\,T_1\,T_2 + 5\,T_1^2\,T_2 - 4\,T_1^3\,T_2 - 4\,T_1^4\,T_2 + 5\,T_1^5\,T_2 - 2\,T_1^6\,T_2 - T_1^7\,T_2 -\right.$$
$$T_2^2 + 5\,T_1\,T_2^2 - 2\,T_1^2\,T_2^2 - 4\,T_1^3\,T_2^2 + 10\,T_1^4\,T_2^2 - 4\,T_1^5\,T_2^2 - 2\,T_1^6\,T_2^2 + 5\,T_1^7\,T_2^2 - T_1^8\,T_2^2 + 3\,T_2^3 - 4\,T_1\,T_2^3 -$$
$$4\,T_1^2\,T_2^3 + 10\,T_1^3\,T_2^3 - 6\,T_1^4\,T_2^3 - 6\,T_1^5\,T_2^3 + 10\,T_1^6\,T_2^3 - 4\,T_1^7\,T_2^3 - 4\,T_1^8\,T_2^3 + 3\,T_1^9\,T_2^3 - T_2^4 - 4\,T_1\,T_2^4 + 10\,T_1^2\,T_2^4 -$$
$$6\,T_1^3\,T_2^4 - 10\,T_1^4\,T_2^4 + 18\,T_1^5\,T_2^4 - 10\,T_1^6\,T_2^4 - 6\,T_1^7\,T_2^4 + 10\,T_1^8\,T_2^4 - 4\,T_1^9\,T_2^4 - T_1^{10}\,T_2^4 - T_2^5 + 5\,T_1\,T_2^5 -$$
$$4\,T_1^2\,T_2^5 - 6\,T_1^3\,T_2^5 + 18\,T_1^4\,T_2^5 - 10\,T_1^5\,T_2^5 - 10\,T_1^6\,T_2^5 + 18\,T_1^7\,T_2^5 - 6\,T_1^8\,T_2^5 - 4\,T_1^9\,T_2^5 + 5\,T_1^{10}\,T_2^5 - T_1^{11}\,T_2^5 +$$
$$T_2^6 - 2\,T_1\,T_2^6 - 2\,T_1^2\,T_2^6 + 10\,T_1^3\,T_2^6 - 10\,T_1^4\,T_2^6 - 10\,T_1^5\,T_2^6 + 24\,T_1^6\,T_2^6 - 10\,T_1^7\,T_2^6 - 10\,T_1^8\,T_2^6 + 10\,T_1^9\,T_2^6 -$$
$$2\,T_1^{10}\,T_2^6 - 2\,T_1^{11}\,T_2^6 + T_1^{12}\,T_2^6 - T_1\,T_2^7 + 5\,T_1^2\,T_2^7 - 4\,T_1^3\,T_2^7 - 6\,T_1^4\,T_2^7 + 18\,T_1^5\,T_2^7 - 10\,T_1^6\,T_2^7 - 10\,T_1^7\,T_2^7 +$$
$$18\,T_1^8\,T_2^7 - 6\,T_1^9\,T_2^7 - 4\,T_1^{10}\,T_2^7 + 5\,T_1^{11}\,T_2^7 - T_1^{12}\,T_2^7 - T_1^2\,T_2^8 - 4\,T_1^3\,T_2^8 + 10\,T_1^4\,T_2^8 - 6\,T_1^5\,T_2^8 - 10\,T_1^6\,T_2^8 +$$
$$18\,T_1^7\,T_2^8 - 10\,T_1^8\,T_2^8 - 6\,T_1^9\,T_2^8 + 10\,T_1^{10}\,T_2^8 - 4\,T_1^{11}\,T_2^8 - T_1^{12}\,T_2^8 + 3\,T_1^3\,T_2^9 - 4\,T_1^4\,T_2^9 - 4\,T_1^5\,T_2^9 + 10\,T_1^6\,T_2^9 -$$
$$6\,T_1^7\,T_2^9 - 6\,T_1^8\,T_2^9 + 10\,T_1^9\,T_2^9 - 4\,T_1^{10}\,T_2^9 - 4\,T_1^{11}\,T_2^9 + 3\,T_1^{12}\,T_2^9 - T_1^4\,T_2^{10} + 5\,T_1^5\,T_2^{10} - 2\,T_1^6\,T_2^{10} - 4\,T_1^7\,T_2^{10} +$$
$$10\,T_1^8\,T_2^{10} - 4\,T_1^9\,T_2^{10} - 2\,T_1^{10}\,T_2^{10} + 5\,T_1^{11}\,T_2^{10} - T_1^{12}\,T_2^{10} - T_1^5\,T_2^{11} - 2\,T_1^6\,T_2^{11} + 5\,T_1^7\,T_2^{11} - 4\,T_1^8\,T_2^{11} - 4\,T_1^9\,T_2^{11} +$$
$$\left.\left. 5\,T_1^{10}\,T_2^{11} - 2\,T_1^{11}\,T_2^{11} - T_1^{12}\,T_2^{11} + T_1^6\,T_2^{12} - T_1^7\,T_2^{12} - T_1^8\,T_2^{12} + 3\,T_1^9\,T_2^{12} - T_1^{10}\,T_2^{12} - T_1^{11}\,T_2^{12} + T_1^{12}\,T_2^{12}\right)\right\},$$

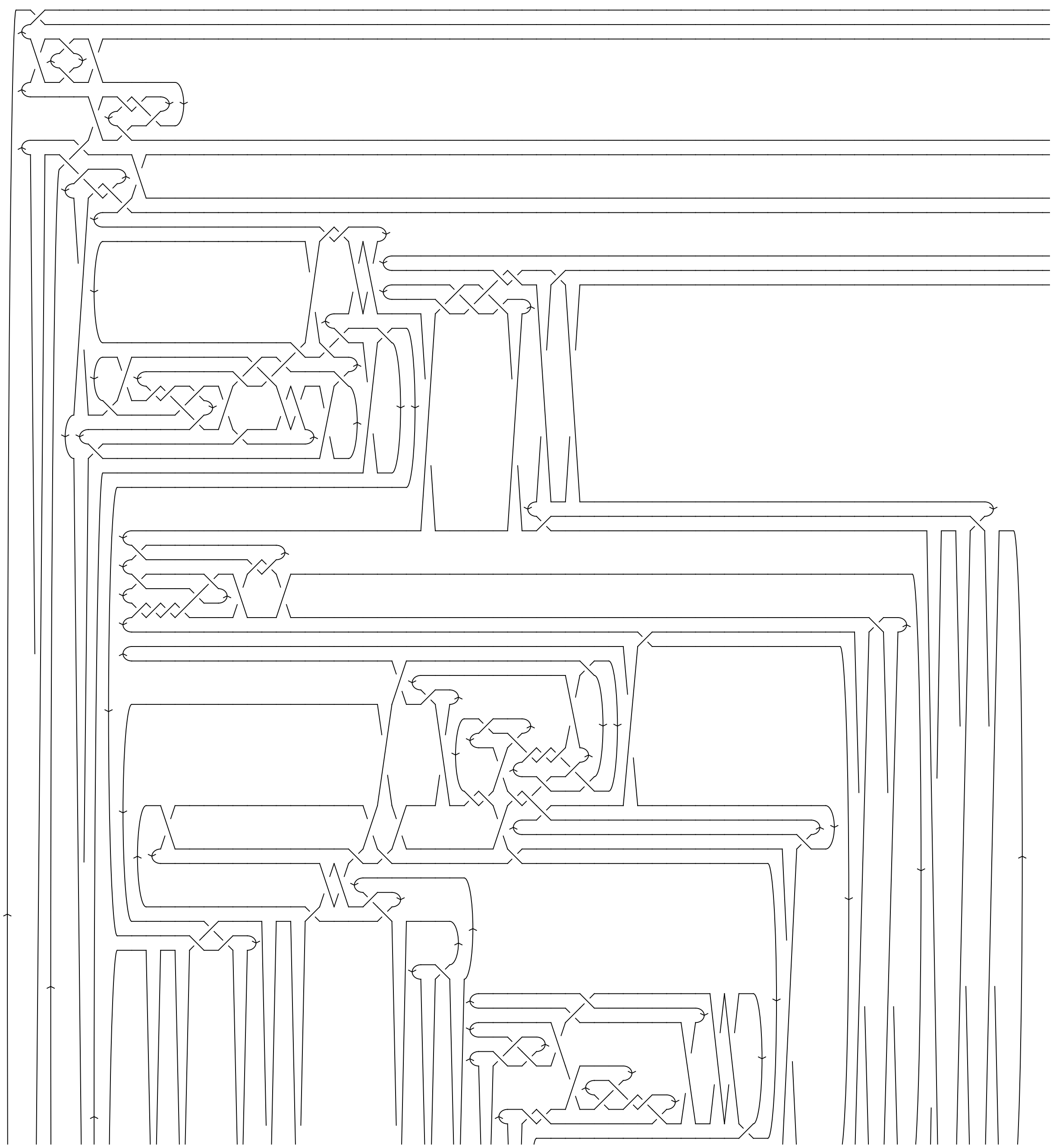
$$\text{Knot}[10, 155] \to \left\{-\frac{\left(-1 + T - 2\,T^2 + T^3\right)\,\left(-1 + 2\,T - T^2 + T^3\right)}{T^3},\right.$$
$$-\frac{1}{T_1^4\,T_2^4}\,2\,\left(1 - 4\,T_1 + 5\,T_1^2 - 4\,T_1^3 + T_1^4 - 4\,T_2 + 11\,T_1\,T_2 - 3\,T_1^2\,T_2 - 3\,T_1^3\,T_2 + 11\,T_1^4\,T_2 - 4\,T_1^5\,T_2 + 5\,T_2^2 -\right.$$
$$3\,T_1\,T_2^2 - 26\,T_1^2\,T_2^2 + 24\,T_1^3\,T_2^2 - 26\,T_1^4\,T_2^2 - 3\,T_1^5\,T_2^2 + 5\,T_1^6\,T_2^2 - 4\,T_2^3 - 3\,T_1\,T_2^3 + 24\,T_1^2\,T_2^3 + 4\,T_1^3\,T_2^3 +$$

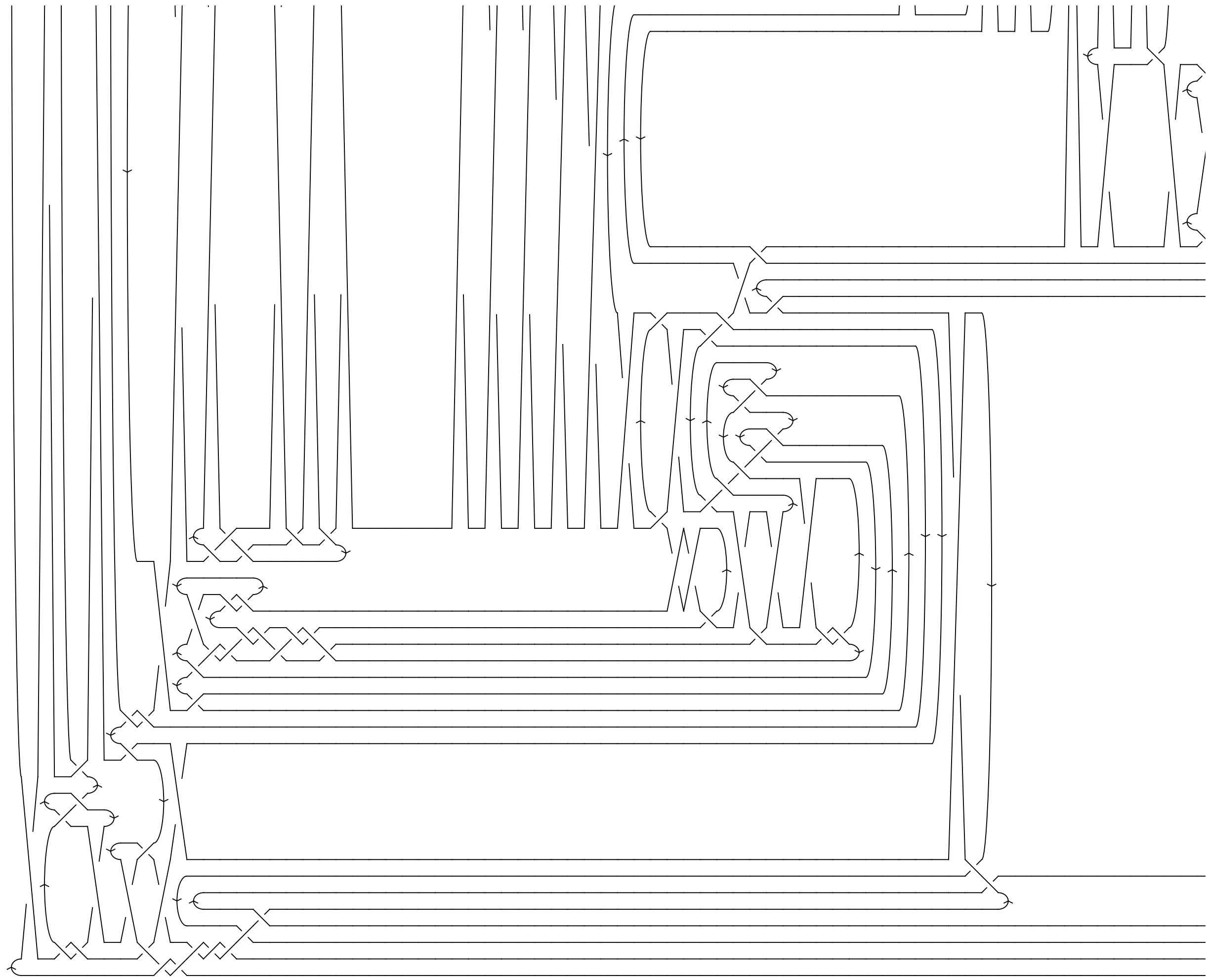
$$4\, T_1^4 T_2^3 + 24\, T_1^5 T_2^3 - 3\, T_1^6 T_2^3 - 4\, T_1^7 T_2^3 + T_2^4 + 11\, T_1 T_2^4 - 26\, T_1^2 T_2^4 + 4\, T_1^3 T_2^4 - 30\, T_1^4 T_2^4 + 4\, T_1^5 T_2^4 -$$
$$26\, T_1^6 T_2^4 + 11\, T_1^7 T_2^4 + T_1^8 T_2^4 - 4\, T_1 T_2^5 - 3\, T_1^2 T_2^5 + 24\, T_1^3 T_2^5 + 4\, T_1^4 T_2^5 + 4\, T_1^5 T_2^5 + 24\, T_1^6 T_2^5 - 3\, T_1^7 T_2^5 -$$
$$4\, T_1^8 T_2^5 + 5\, T_1^2 T_2^6 - 3\, T_1^3 T_2^6 - 26\, T_1^4 T_2^6 + 24\, T_1^5 T_2^6 - 26\, T_1^6 T_2^6 - 3\, T_1^7 T_2^6 + 5\, T_1^8 T_2^6 - 4\, T_1^3 T_2^7 +$$
$$11\, T_1^4 T_2^7 - 3\, T_1^5 T_2^7 - 3\, T_1^6 T_2^7 + 11\, T_1^7 T_2^7 - 4\, T_1^8 T_2^7 + T_1^4 T_2^8 - 4\, T_1^5 T_2^8 + 5\, T_1^6 T_2^8 - 4\, T_1^7 T_2^8 + T_1^8 T_2^8) \}\}$$

In[∘]:=
```
DunfieldKnots = ReadList["../../People/Dunfield/nmd_random_knots"] /. k_Integer :> k + 1;
DK[n_] := DunfieldKnots⟦n - 2⟧
```

In[∘]:=
```
DrawMorseLink[DK[250]]
```

Out[∘]=





In[•]:= Crossings[DK[576]]

Out[•]=

576

In[•]:= AbsoluteTiming[θ[DK[3]]]

Out[•]=

$$\left\{0.0061176, \left\{\frac{1 - T + T^2}{T}, \frac{1 - T_1 + T_1^2 - T_2 - T_1^3 T_2 + T_2^2 + T_1^4 T_2^2 - T_1 T_2^3 - T_1^4 T_2^3 + T_1^2 T_2^4 - T_1^3 T_2^4 + T_1^4 T_2^4}{T_1^2 T_2^2}\right\}\right\}$$

In[•]:= AbsoluteTiming[θ[DK[30]];]

Out[•]=

{2.91933, Null}

In[•]:= AbsoluteTiming[θ[DK[60]];]

Out[•]=

{27.4555, Null}

```
In[•]:= AbsoluteTiming[θ[DK[90]];]
Out[•]=
{227.389, Null}

In[•]:= AbsoluteTiming[θ120 = θ[DK[120]];]
Out[•]=
{0.0003743, Null}

In[•]:= Put[θ120, "Theta4DK120.m"]

In[•]:= AbsoluteTiming[θ[DK[150]];]
Out[•]=
{2357.39, Null}
```

(during the previous computation I biked home, so the AbsoluteTiming is too much)

```
In[•]:= AbsoluteTiming[θ[DK[180]];]
Out[•]=
{5391.24, Null}

In[•]:= AbsoluteTiming[θ[DK[210]];]
Out[•]=
{9613.68, Null}

In[•]:= AbsoluteTiming[θ[DK[240]];]
Out[•]=
{22 462.4, Null}

In[•]:= AbsoluteTiming[θ[DK[270]];]
```

Mathematica crashed while trying the above computation.

```
In[•]:= AbsoluteTiming[θ[DK[300]];]

In[•]:= Do[Echo /@ AbsoluteTiming[n → θ_{22/7,34/21}[DK[n]]], {n, 100, 500, 100}]
```

» 1.34291

» $100 \to \Big\{\Big\{\frac{35\,388\,936\,522\,490\,931\,938\,908\,923\,343\,364\,558\,590\,414\,632\,463\,375\,508\,742\,089}{264\,554\,736\,545\,069\,605\,885\,631\,471\,128\,764\,401\,339\,301\,535\,744},$
$\frac{525\,106\,180\,586\,933\,014\,293\,865\,927\,609\,379\,271\,742\,972\,076\,277\,257\,025\,413\,914\,338\,499}{37\,324\,734\,431\,368\,634\,257\,516\,595\,221\,111\,791\,096\,751\,570\,183\,668\,795\,296\,664\,772\,608},$
50 463 574 955 913 231 815 385 186 261 134 862 814 456 979 779 055 953 806 229 018 368 595 827 102 502 222 063 -
117 299 430 053 887 387 799 738 329 099 644 807 147 011 110 057 363 /
78 995 482 272 843 339 527 758 555 299 340 636 345 228 530 305 737 655 210 586 135 944 082 585 735 874 960 483 -
459 404 024 632 393 880 311 552 802 816},
1 528 310 677 820 715 321 034 523 399 570 065 191 062 105 455 458 377 892 190 819 455 810 946 769 247 237 972 364 -
715 885 979 420 470 551 351 869 219 633 193 553 826 417 257 308 347 635 722 740 692 821 508 135 020 135 287 456 -
366 121 034 149 470 683 332 248 166 617 909 950 793 807 487 984 811 798 893 565 093 125 255 348 183 610 375 623 -
605 724 602 385 /
2 021 852 735 124 190 443 601 930 854 750 557 817 293 581 868 158 419 077 273 632 660 683 611 032 469 645 660 -
194 599 371 946 978 382 921 589 937 447 245 226 083 493 038 643 718 762 601 638 877 904 972 518 050 153 046 -
861 446 451 966 589 617 326 519 650 102 159 919 340 894 781 095 877 211 742 319 673 344}

» 6.88529

» 200 →

{{-(72 941 025 249 230 622 091 769 886 034 332 903 937 878 867 275 035 495 850 289 152 467 601 139 729 946 680 -
691 983 449 444 238 470 173 260 899 434 879 455 547 646 677 /
79 780 391 006 864 379 747 986 053 920 193 038 680 545 693 079 622 955 011 027 668 359 182 291 645 896 903 -
218 461 275 510 571 008),
13 469 039 288 358 770 844 889 186 746 410 419 403 949 987 382 833 567 787 469 752 570 946 087 488 964 056 464 -
083 956 449 441 872 952 430 656 158 262 269 810 083 547 830 189 003 289 443 154 125 /
4 240 161 130 043 882 037 823 084 995 205 726 632 691 185 572 237 933 032 456 552 833 243 815 216 744 170 971 -
881 548 991 957 331 738 797 061 590 095 303 559 046 326 968 215 750 967 296,
3 058 236 953 956 402 226 943 593 388 603 713 021 071 954 699 338 326 371 450 792 000 285 430 803 814 324 110 -
911 806 690 348 020 780 088 584 382 124 603 092 971 693 299 841 778 094 187 288 377 810 035 496 408 283 188 -
130 224 093 352 681 965 580 164 395 682 496 054 504 489 551 954 332 992 465 733 972 977 594 735 369 459 115 -
633 590 163 189 798 671 672 600 349 071 866 872 120 468 309 375 /
7 389 876 778 587 670 278 409 931 856 936 212 530 694 800 372 408 625 530 583 166 986 417 139 021 654 981 203 -
589 910 511 227 601 136 991 125 732 955 086 827 137 765 975 954 473 403 792 833 419 463 344 119 138 486 741 -
874 061 457 114 480 552 952 530 491 222 541 669 872 799 328 574 041 719 777 250 405 019 238 495 420 416},
-(35 533 798 751 418 160 350 916 090 870 874 408 685 758 076 531 957 553 028 308 354 367 936 952 715 320 377 112 -
933 900 291 194 748 021 391 980 122 119 460 697 184 063 729 775 201 344 517 723 397 729 781 282 842 088 707 -
536 733 758 752 195 455 093 509 038 015 678 684 681 626 418 035 519 803 604 439 397 416 661 432 511 206 560 -
127 326 980 562 590 565 142 398 059 299 186 452 157 584 572 312 347 570 546 167 881 173 768 455 447 102 478 -
378 052 565 824 989 035 759 718 349 901 555 797 046 487 367 735 873 953 550 250 292 996 462 075 359 706 165 -
962 265 760 112 833 307 407 741 496 584 457 563 023 053 844 158 922 142 850 482 681 009 343 615 561 563 933 -
345 073 931 843 736 416 605 341 872 288 994 025 512 080 297 221 469 946 108 375 450 764 191 881 092 403 125 /
452 396 514 172 443 948 090 596 720 075 743 969 379 888 907 838 827 526 625 786 124 662 888 374 624 411 285 -
068 305 310 109 452 395 752 503 075 302 027 422 247 590 129 306 367 202 635 464 223 536 884 780 523 952 041 -
663 218 284 564 278 956 217 013 122 499 393 566 958 337 419 775 741 184 128 728 079 197 011 880 897 341 842 -
707 105 674 675 400 895 701 799 815 201 160 823 272 081 093 378 813 312 065 293 550 562 986 036 284 189 802 -
691 770 038 253 906 432 703 028 717 411 518 620 619 761 249 698 953 156 705 113 477 290 132 297 634 963 280 -
748 846 254 218 110 952 588 168 734 038 774 458 591 745 251 980 483 140 272 217 590 873 158 475 317 248)}

» 81.2757

```
» 300 →
  {{54 300 428 014 802 247 763 147 703 343 836 297 447 025 108 824 684 772 425 762 525 822 095 039 545 899 375 981 -
      953 473 178 602 586 048 430 534 584 880 163 873 723 541 762 115 735 883 067 341 959 560 581 371 283 178 656 -
      972 648 408 925 263 946 669 /
    6 741 838 682 197 306 940 008 962 116 848 220 280 436 936 971 437 572 995 472 014 771 688 913 708 639 211 514 -
      814 195 885 491 758 038 709 972 366 558 512 006 372 340 250 849 089 814 593 530 683 936 627 298 651 512 766 -
      464,
   1 084 128 382 249 743 436 824 663 986 171 685 150 273 646 351 713 912 937 150 171 700 202 730 323 922 010 700 -
      294 161 035 743 289 238 368 194 879 507 950 682 627 574 784 328 439 797 605 967 434 628 113 238 619 877 448 -
      933 104 349 915 804 145 167 106 117 098 828 582 214 168 974 179 /
    458 816 114 715 914 322 691 410 371 538 510 819 835 906 604 695 828 488 701 592 446 861 566 683 983 329 916 -
      364 046 021 667 534 630 113 436 786 891 827 119 466 479 256 930 424 597 743 983 452 685 367 746 981 696 618 -
      500 346 273 956 034 473 567 578 882 100 679 941 606 973 898 752,
   - (158 777 874 852 495 582 515 909 215 389 994 852 546 352 653 931 705 508 650 307 891 657 053 561 609 520 779 -
        186 320 897 348 004 451 340 565 961 074 347 535 242 136 402 407 084 832 097 701 971 876 894 887 835 991 169 -
        195 699 017 190 487 685 513 574 819 025 748 109 103 168 978 452 501 811 090 422 603 306 747 210 926 095 970 -
        770 670 185 035 477 605 544 327 410 988 587 473 792 754 126 636 018 339 393 952 001 669 899 686 164 600 864 -
        484 927 816 109 847 962 066 717 003 302 534 438 301 515 100 500 581 439 281 502 338 168 771 925 334 310 271 -
        437 341 818 561 /
      8 446 673 524 619 204 540 662 248 188 364 579 654 962 149 362 100 111 349 567 813 607 145 180 164 671 139 -
        617 365 814 596 293 558 611 877 467 632 393 708 787 160 491 479 639 500 826 381 376 300 773 027 876 197 -
        170 955 833 764 004 216 082 452 919 975 997 020 526 350 495 894 405 720 336 559 612 735 646 735 734 155 -
        554 395 961 189 410 159 575 680 771 895 729 613 390 941 354 707 084 783 892 152 666 711 430 746 078 787 -
        591 302 278 416 571 017 951 710 864 634 193 356 469 295 526 911 091 658 361 659 195 392)},
  - (8 598 040 329 900 132 178 849 810 392 065 575 015 656 948 332 717 228 018 818 196 986 408 406 885 151 173 114 -
      729 742 371 657 327 870 129 553 797 167 264 600 601 461 737 612 762 883 778 056 461 125 303 156 682 177 822 -
      387 597 941 597 676 133 555 775 929 651 554 558 568 826 851 193 016 325 730 344 539 614 484 324 504 069 552 -
      066 916 711 741 608 633 404 825 059 528 743 681 819 722 488 192 923 953 808 035 534 926 597 091 591 375 719 -
      708 970 825 214 204 090 352 696 010 613 508 905 010 815 827 512 539 197 920 217 378 414 243 201 536 885 840 -
      502 206 086 625 497 141 347 632 796 621 737 879 816 174 936 164 213 142 719 738 496 243 651 312 127 236 569 -
      658 634 354 493 624 215 745 814 083 607 554 164 979 886 252 364 586 458 746 627 111 732 001 798 098 411 377 -
      469 694 277 623 092 049 862 323 332 740 569 060 876 937 876 842 372 536 879 611 798 159 751 897 313 972 600 -
      034 650 421 307 830 475 711 279 585 030 859 630 220 299 694 039 396 624 252 597 049 438 680 591 984 366 793 -
      943 898 999 135 971 052 027 731 897 551 216 899 484 829 106 288 612 686 258 405 660 999 234 795 832 964 806 -
      965 873 /
    812 678 875 896 339 039 067 670 285 655 670 982 811 813 572 505 939 516 526 515 895 464 038 253 655 962 698 -
      534 470 158 273 453 471 969 412 225 862 875 287 695 193 281 294 047 490 514 554 993 940 744 345 155 227 044 -
      324 133 303 532 650 014 900 112 879 006 801 216 964 275 606 874 592 888 221 306 209 845 338 126 393 770 242 -
      035 421 093 568 450 115 306 673 002 230 112 041 735 438 509 121 588 709 168 122 196 789 163 088 049 032 402 -
      421 831 930 165 869 146 969 680 446 412 107 255 774 061 135 012 374 209 095 972 722 550 879 186 845 119 609 -
      905 916 389 810 258 595 619 812 363 193 227 320 810 658 099 006 534 020 402 904 912 489 467 165 903 133 321 -
      063 930 316 828 893 776 965 178 816 926 996 966 709 051 510 488 188 756 691 086 660 277 067 356 140 651 827 -
      003 820 730 966 021 344 695 355 788 718 822 044 920 980 326 904 411 394 046 648 037 199 883 563 233 621 627 -
      831 014 801 221 912 882 289 230 772 061 896 822 000 174 973 211 954 770 033 843 643 470 936 514 389 292 275 -
      081 706 404 767 643 586 504 062 664 704)}
```

Out[∘]=

$Aborted

The following crashes at n=700:

In[∘]:= `Do[Echo /@ AbsoluteTiming[n → θ_{22/7,34/21}[DK[n]]], {n, 500, 1000, 100}]`

Knot Genus

In[ ]:=
```
Import["KnotGenusFromKnotInfo.csv"][[2 ;;]] /.
  {K_String, pd_String, g_Integer} :> (Genus[Knot[K]] = g);
```

In[ ]:=
```
ρ[K_] := ρ[K] = Module[{R, Cs, φ, n, A, s, i, j, k, Δ, G, ρ1},
   R[s_, i_, j_] := s (g_{j i} (g_{j+, j} + g_{j, j+} - g_{i j}) - g_{i i} (g_{j, j+} - 1) - 1 / 2);
   {Cs, φ} = Rot[K]; n = Length[Cs];
   A = IdentityMatrix[2 n + 1];
   Cases[Cs, {s_, i_, j_} :> (A[[{i, j}, {i + 1, j + 1}]] += ({{-T^s, T^s - 1}, {0, -1}}))];
   Δ = T^((-Total[φ] - Total[Cs[[All, 1]]]) / 2) Det[A];
   G = Inverse[A];
   ρ1 = Sum[R @@ Cs[[k]], {k, 1, n}] - Sum[φ[[k]] (g_{k k} - 1 / 2), {k, 1, 2 n}];
   Factor@{Δ, Δ^2 ρ1 /. α_+ :> α + 1 /. g_{α_, β_} :> G[[α, β]]}];
```

In[∘]:=
```
Table[
  {K, Genus[K], Exponent[θ[K]〚1〛, T], Exponent[ρ[K]〚2〛, T] / 2, Exponent[θ[K]〚2〛, T1] / 2,
   Exponent[θ[K]〚2〛 /. {T1 → T, T2 → T^2}, T] / 6}, {K, AllKnots[{3, 8}]}] // MatrixForm
```

Out[∘]//MatrixForm=

Knot[3, 1]	1	1	1	1	1
Knot[4, 1]	1	1	−∞	−∞	−∞
Knot[5, 1]	2	2	2	2	2
Knot[5, 2]	1	1	1	1	1
Knot[6, 1]	1	1	1	1	1
Knot[6, 2]	2	2	2	2	2
Knot[6, 3]	2	2	−∞	−∞	−∞
Knot[7, 1]	3	3	3	3	3
Knot[7, 2]	1	1	1	1	1
Knot[7, 3]	2	2	2	2	2
Knot[7, 4]	1	1	1	1	1
Knot[7, 5]	2	2	2	2	2
Knot[7, 6]	2	2	2	2	2
Knot[7, 7]	2	2	1	1	1
Knot[8, 1]	1	1	1	1	1
Knot[8, 2]	3	3	3	3	3
Knot[8, 3]	1	1	−∞	−∞	−∞
Knot[8, 4]	2	2	2	2	2
Knot[8, 5]	3	3	3	3	3
Knot[8, 6]	2	2	2	2	2
Knot[8, 7]	3	3	3	3	3
Knot[8, 8]	2	2	2	2	2
Knot[8, 9]	3	3	−∞	−∞	−∞
Knot[8, 10]	3	3	3	3	3
Knot[8, 11]	2	2	2	2	2
Knot[8, 12]	2	2	−∞	−∞	−∞
Knot[8, 13]	2	2	2	2	2
Knot[8, 14]	2	2	2	2	2
Knot[8, 15]	2	2	2	2	2
Knot[8, 16]	3	3	3	3	3
Knot[8, 17]	3	3	−∞	−∞	−∞
Knot[8, 18]	3	3	−∞	−∞	−∞
Knot[8, 19]	3	3	3	3	3
Knot[8, 20]	2	2	1	1	1
Knot[8, 21]	2	2	2	2	2

In[∘]:=
```
Select[AllKnots[{3, 12}], Genus[#] < Exponent[θ[#]〚1〛, T] &]
```

KnotTheory: Loading precomputed data in PD4Knots`.

KnotTheory: Loading precomputed data in DTCode4KnotsTo11`.

KnotTheory: The GaussCode to PD conversion was written by Siddarth Sankaran at the University of Toronto in the summer of 2005.

KnotTheory: Loading precomputed data in KnotTheory/12A.dts.

General: Further output of KnotTheory::loading will be suppressed during this calculation.

Out[∘]=

{}

```
In[ ]:= Select[AllKnots[{3, 11}], Genus[#] > Exponent[θ[#][[1]], T] &]
Out[ ]=
{Knot[11, NonAlternating, 34], Knot[11, NonAlternating, 42],
 Knot[11, NonAlternating, 45], Knot[11, NonAlternating, 67], Knot[11, NonAlternating, 73],
 Knot[11, NonAlternating, 97], Knot[11, NonAlternating, 152]}

In[ ]:= Select[AllKnots[{3, 12}], Genus[#] < Ceiling[Exponent[θ[#][[2]], T1] / 2] &]
Out[ ]=
{}

In[ ]:= Union[(2 Genus[#] - Exponent[θ[#][[2]], T1]) & /@ AllKnots[{3, 12}]]
Out[ ]=
{0, 1, 2, 3, ∞}

In[ ]:= Select[AllKnots[{3, 10}], Genus[#] > Ceiling[Exponent[θ[#][[2]], T1] / 2] &]
Out[ ]=
{Knot[4, 1], Knot[6, 3], Knot[7, 7], Knot[8, 3], Knot[8, 9], Knot[8, 12],
 Knot[8, 17], Knot[8, 18], Knot[8, 20], Knot[9, 24], Knot[9, 27], Knot[9, 30],
 Knot[9, 33], Knot[9, 34], Knot[10, 17], Knot[10, 33], Knot[10, 37], Knot[10, 42],
 Knot[10, 43], Knot[10, 45], Knot[10, 48], Knot[10, 60], Knot[10, 71],
 Knot[10, 75], Knot[10, 79], Knot[10, 81], Knot[10, 88], Knot[10, 91], Knot[10, 96],
 Knot[10, 99], Knot[10, 104], Knot[10, 109], Knot[10, 115], Knot[10, 118],
 Knot[10, 123], Knot[10, 140], Knot[10, 141], Knot[10, 155], Knot[10, 158]}

In[ ]:= Select[AllKnots[{3, 12}], Genus[#] < Ceiling[Exponent[ρ[#][[2]], T] / 2] &]
Out[ ]=
{}

In[ ]:= Select[AllKnots[{3, 10}], 2 Genus[#] > Exponent[ρ[#][[2]], T] &]
Out[ ]=
{Knot[4, 1], Knot[6, 3], Knot[7, 7], Knot[8, 3], Knot[8, 9], Knot[8, 12], Knot[8, 17],
 Knot[8, 18], Knot[8, 20], Knot[9, 24], Knot[9, 27], Knot[9, 30], Knot[9, 33], Knot[9, 34],
 Knot[9, 44], Knot[10, 17], Knot[10, 31], Knot[10, 33], Knot[10, 37], Knot[10, 42],
 Knot[10, 43], Knot[10, 45], Knot[10, 48], Knot[10, 60], Knot[10, 71], Knot[10, 75],
 Knot[10, 79], Knot[10, 81], Knot[10, 88], Knot[10, 91], Knot[10, 96], Knot[10, 99],
 Knot[10, 104], Knot[10, 107], Knot[10, 109], Knot[10, 115], Knot[10, 118], Knot[10, 123],
 Knot[10, 132], Knot[10, 137], Knot[10, 140], Knot[10, 141], Knot[10, 155], Knot[10, 158]}

In[ ]:= Select[AllKnots[{3, 10}], Exponent[ρ[#][[2]], T] > Exponent[θ[#][[2]], T1] &]
Out[ ]=
{}

In[ ]:= Select[AllKnots[{3, 10}], Exponent[ρ[#][[2]], T] < Exponent[θ[#][[2]], T1] &]
Out[ ]=
{Knot[9, 24], Knot[10, 31], Knot[10, 107], Knot[10, 158]}
```

```
In[•]:= Factor@Alexander[Knot[10, 107]][T]
Out[•]=
```

$$-\frac{1-8\,T+22\,T^2-31\,T^3+22\,T^4-8\,T^5+T^6}{T^3}$$

```
In[•]:= Select[AllKnots[{3, 12}], Genus[#] < Exponent[θ[#][[2]] /. {T₁ → T, T₂ → T²}, T] / 6 &]
Out[•]=
{}

In[•]:= Select[AllKnots[{3, 10}], Genus[#] > Exponent[θ[#][[2]] /. {T₁ → T, T₂ → T²}, T] / 6 &]
Out[•]=
{Knot[4, 1], Knot[6, 3], Knot[7, 7], Knot[8, 3], Knot[8, 9], Knot[8, 12], Knot[8, 17],
 Knot[8, 18], Knot[8, 20], Knot[9, 24], Knot[9, 27], Knot[9, 30], Knot[9, 33], Knot[9, 34],
 Knot[9, 44], Knot[10, 17], Knot[10, 31], Knot[10, 33], Knot[10, 37], Knot[10, 42],
 Knot[10, 43], Knot[10, 45], Knot[10, 48], Knot[10, 60], Knot[10, 71], Knot[10, 75],
 Knot[10, 79], Knot[10, 81], Knot[10, 88], Knot[10, 91], Knot[10, 96], Knot[10, 99],
 Knot[10, 104], Knot[10, 107], Knot[10, 109], Knot[10, 115], Knot[10, 118], Knot[10, 123],
 Knot[10, 132], Knot[10, 137], Knot[10, 140], Knot[10, 141], Knot[10, 155], Knot[10, 158]}

In[•]:= sel = Select[AllKnots[{3, 12}], Exponent[ρ[#][[2]], T] / 2 > Exponent[θ[#][[2]], T₁] / 2 &]
Out[•]=
{}

In[•]:= Select[AllKnots[{3, 11}], Ceiling[Exponent[ρ[#][[2]], T] / 2] < Exponent[θ[#][[2]], T₁] / 2 &]
Out[•]=
{Knot[10, 107], Knot[11, Alternating, 209], Knot[11, Alternating, 228],
 Knot[11, NonAlternating, 34], Knot[11, NonAlternating, 45], Knot[11, NonAlternating, 152]}
```

In[∘]:=
```
Table[{K, Genus[K], Exponent[θ[K][[1]], T], Exponent[ρ[K][[2]], T] / 2,
   Exponent[θ[K][[2]], T1] / 2 , Exponent[θ[K][[2]] /. {T1 → T, T2 → T^2}, T] / 6},
  {K, AllKnots[{3, 7}] ~Join~ {Knot[9, 24], Knot[9, 30], Knot[10, 107],
     Knot["K11n34"], Knot["K11n42"], GST48}}] // MatrixForm
```

Out[∘]//MatrixForm=

Knot[3, 1]	1	1	1	1	1
Knot[4, 1]	1	1	$-\infty$	$-\infty$	$-\infty$
Knot[5, 1]	2	2	2	2	2
Knot[5, 2]	1	1	1	1	1
Knot[6, 1]	1	1	1	1	1
Knot[6, 2]	2	2	2	2	2
Knot[6, 3]	2	2	$-\infty$	$-\infty$	$-\infty$
Knot[7, 1]	3	3	3	3	3
Knot[7, 2]	1	1	1	1	1
Knot[7, 3]	2	2	2	2	2
Knot[7, 4]	1	1	1	1	1
Knot[7, 5]	2	2	2	2	2
Knot[7, 6]	2	2	2	2	2
Knot[7, 7]	2	2	1	1	1
Knot[9, 24]	3	3	$\frac{3}{2}$	2	$\frac{11}{6}$
Knot[9, 30]	3	3	2	2	$\frac{11}{6}$
Knot[10, 107]	3	3	2	$\frac{5}{2}$	$\frac{7}{3}$
Knot[11, NonAlternating, 34]	3	0	$\frac{3}{2}$	3	$\frac{8}{3}$
Knot[11, NonAlternating, 42]	2	0	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{4}{3}$
GST48	Genus[GST48]	8	8	10	$\frac{55}{6}$

In[∘]:=
```
Union@Table[Simplify[ρ[K][[2]] == (-θ[K][[2]] /. {T1 → T, T2 → 1})], {K, AllKnots[{3, 11}]}]
```

Out[∘]=

{True}

In[∘]:=
```
K = Knot[3, 1]; Factor@{(ρ[K][[2]] /. T → T^3), (-θ[K][[2]] /. {T1 → T, T2 → T^2})}
```

Out[∘]=

$$\left\{\frac{(-1+T)^2\,(1+T^2)\,(1+T+T^2)^2\,(1-T^2+T^4)}{T^6},\ \frac{(-1+T)^2\,(1+T^4)\,(1+T+T^2+T^3+T^4+T^5+T^6)}{T^6}\right\}$$

In[ ]:= $\text{PolyPlot}_{T_1,T_2}[\theta[\text{Knot}[10, 107]]⟦2⟧]$

Out[ ]=