

Loading Pre-Computed Data

```
In[*]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\HigherRank"];
Once[<< KnotTheory`];
<< Rot.m
```

Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.

Read more at <http://katlas.org/wiki/KnotTheory>.

Loading Rot.m from <http://drorbn.net/AP/Projects/HigherRank> to compute rotation numbers.

```
In[*]:= CCF[ $\mathcal{E}$ _] := ExpandDenominator@ExpandNumerator@Together[ $\mathcal{E}$ ];
CCF[ $\mathcal{E}$ _] := Factor[ $\mathcal{E}$ ];
CF[ $\mathcal{E}$ _List] := CF /@  $\mathcal{E}$ ;
CF[ $\mathcal{E}$ _] := Module[{vs = Cases[ $\mathcal{E}$ , (x | p |  $\pi$  | g)_,  $\infty$ ] U {x, p,  $\epsilon$ }, ps, c},
  Total[CoefficientRules[Expand[ $\mathcal{E}$ ], vs] /. (ps_ -> c_) := CCF[c] (Times @@ vsps) ]];
```

Must get rid of the "+1"s below:

```

In[ ]:= {
  {r0,pxx[1, i_, j_], r0,pxx[-1, i_, j_]},
  {r1,ppx[1, i_, j_], r1,ppx[-1, i_, j_]},
  {r1,rest[1, i_, j_], r1,rest[-1, i_, j_]},
   $\gamma_1[\varphi, k]$ 
} = CF[Plus[
  {T1, 1 - T1 T2, T1 (1 - T1 T2), T1 (1 - T1 T2)} * Get["px-data.m"],
  -{0, 0},
  {0, 0},
  {1/2 + T3 x3,i (p3,i+1 - p3,j+1), -1/2 - T3^-1 x3,i (p3,i+1 - p3,j+1)},
   $\varphi/2$ 
}
]]

```

Out[]=

$$\left\{ \left\{ T_1 p_{3,j} x_{1,i} x_{2,i} - p_{3,j} x_{1,j} x_{2,i}, -\frac{p_{3,j} x_{1,i} x_{2,i}}{T_1 T_2} + \frac{p_{3,j} x_{1,j} x_{2,i}}{T_2} \right\}, \right.$$

$$\left. \left\{ (1 - T_1 T_2) p_{1,j} p_{2,i} x_{3,i} + (-1 + T_1 T_2) p_{1,j} p_{2,j} x_{3,i}, \right. \right.$$

$$\left. \frac{(-1 + T_1 T_2) p_{1,j} p_{2,i} x_{3,i}}{T_1} - \frac{(-1 + T_1 T_2) p_{1,j} p_{2,j} x_{3,i}}{T_1} \right\},$$

$$\left\{ -\frac{1}{2} - T_1 T_2 p_{1,j} p_{2,j} x_{1,i} x_{2,i} + \frac{p_{1,j} p_{2,i} x_{1,j} x_{2,i}}{-1 + T_1} + \frac{(-1 + T_1 T_2) p_{1,j} p_{2,j} x_{1,j} x_{2,i}}{-1 + T_1} - \right.$$

$$\frac{T_1 p_{1,i} p_{2,j} x_{1,i} x_{2,j}}{-1 + T_1} - T_3 p_{3,1+i} x_{3,i} - p_{3,j} x_{3,i} + T_3 p_{3,1+j} x_{3,i} + T_1 (-1 + T_1 T_2) p_{1,j} p_{3,j} x_{1,i} x_{3,i} -$$

$$\frac{p_{1,j} p_{3,i} x_{1,j} x_{3,i}}{-1 + T_1} - \frac{T_1 (-1 + T_1 T_2) p_{1,j} p_{3,j} x_{1,j} x_{3,i}}{-1 + T_1} + T_2 (-1 + T_1 T_2) p_{2,j} p_{3,j} x_{2,i} x_{3,i} +$$

$$p_{2,j} p_{3,i} x_{2,j} x_{3,i} + \frac{T_1 p_{1,i} p_{3,j} x_{1,i} x_{3,j}}{-1 + T_1} - T_2 p_{2,j} p_{3,j} x_{2,i} x_{3,j}, \frac{1}{2} - \frac{p_{1,j} p_{2,i} x_{1,i} x_{2,i}}{T_1} +$$

$$\frac{T_1 (-1 + T_2) p_{1,i} p_{2,j} x_{1,i} x_{2,i}}{(-1 + T_1) T_2} - \frac{(-T_1 - T_2 + T_1 T_2) p_{1,j} p_{2,j} x_{1,i} x_{2,i}}{T_1 T_2} - \frac{p_{1,j} p_{2,i} x_{1,j} x_{2,i}}{-1 + T_1} -$$

$$p_{1,j} p_{2,j} x_{1,j} x_{2,i} + \frac{T_1 p_{1,i} p_{2,j} x_{1,i} x_{2,j}}{-1 + T_1} - p_{1,j} p_{2,j} x_{1,i} x_{2,j} + \frac{p_{3,1+i} x_{3,i}}{T_3} + p_{3,j} x_{3,i} - \frac{p_{3,1+j} x_{3,i}}{T_3} +$$

$$\frac{p_{1,j} p_{3,i} x_{1,i} x_{3,i}}{T_1} - \frac{(-1 + T_1 T_2) p_{1,i} p_{3,j} x_{1,i} x_{3,i}}{(-1 + T_1) T_2} + \frac{(-1 + T_1 T_2) p_{1,j} p_{3,j} x_{1,i} x_{3,i}}{T_1 T_2} +$$

$$\frac{p_{1,j} p_{3,i} x_{1,j} x_{3,i}}{-1 + T_1} - \frac{(-1 + T_2) p_{2,j} p_{3,i} x_{2,i} x_{3,i}}{T_2} - \frac{(-1 + T_1 T_2) p_{2,i} p_{3,j} x_{2,i} x_{3,i}}{T_1 T_2} +$$

$$\frac{(-1 + 2 T_2) (-1 + T_1 T_2) p_{2,j} p_{3,j} x_{2,i} x_{3,i}}{T_1 T_2^2} - p_{2,j} p_{3,i} x_{2,j} x_{3,i} + \frac{(-1 + T_1 T_2) p_{2,j} p_{3,j} x_{2,j} x_{3,i}}{T_1 T_2} -$$

$$\left. \frac{T_1 p_{1,i} p_{3,j} x_{1,i} x_{3,j}}{-1 + T_1} + p_{1,j} p_{3,j} x_{1,i} x_{3,j} + p_{2,j} p_{3,j} x_{2,i} x_{3,j} \right\}, -\frac{\varphi}{2} + \varphi p_{3,k} x_{3,k} \left. \right\}$$

In[*]:= **r_{1,rest}[1, 4, 5]**

Out[*]=

$$\begin{aligned}
 & -\frac{1}{2} - T_1 T_2 p_{1,5} p_{2,5} x_{1,4} x_{2,4} + \frac{p_{1,5} p_{2,4} x_{1,5} x_{2,4}}{-1 + T_1} + \frac{(-1 + T_1 T_2) p_{1,5} p_{2,5} x_{1,5} x_{2,4}}{-1 + T_1} - \\
 & \frac{T_1 p_{1,4} p_{2,5} x_{1,4} x_{2,5}}{-1 + T_1} - p_{3,5} x_{3,4} - T_1 T_2 p_{3,5} x_{3,4} + T_1 T_2 p_{3,6} x_{3,4} + \\
 & T_1 (-1 + T_1 T_2) p_{1,5} p_{3,5} x_{1,4} x_{3,4} - \frac{p_{1,5} p_{3,4} x_{1,5} x_{3,4}}{-1 + T_1} - \frac{T_1 (-1 + T_1 T_2) p_{1,5} p_{3,5} x_{1,5} x_{3,4}}{-1 + T_1} + \\
 & T_2 (-1 + T_1 T_2) p_{2,5} p_{3,5} x_{2,4} x_{3,4} + p_{2,5} p_{3,4} x_{2,5} x_{3,4} + \frac{T_1 p_{1,4} p_{3,5} x_{1,4} x_{3,5}}{-1 + T_1} - T_2 p_{2,5} p_{3,5} x_{2,4} x_{3,5}
 \end{aligned}$$

In[*]:= **{p*, x*, pi*, xi*} = {pi, xi, p, x}; (u_{i_})* := (u*)_i;**

In[*]:= **Zip_{i}[\mathcal{E}_] := \mathcal{E};**

Zip_{\{\xi, \xi s_{---}\}}[\mathcal{E}_] := (Collect[\mathcal{E} // Zip_{\{\xi s\}}, \mathcal{S}^d] /. f_ . \mathcal{S}^d_ . \Rightarrow (D[f, \{\xi s^*, d\}])) /. \mathcal{S}^* \rightarrow \theta

In[*]:= **px2g[\mathcal{E}_] := CF@Module[{ps, xs, Q, \alpha, \beta},
 ps = Union[Cases[\mathcal{E}, p_, \infty]]; xs = Union[Cases[\mathcal{E}, x_, \infty]];
 Q = Sum[p0* x0* g_{p0[[2]], x0[[2]], p0[[3]], x0[[3]], {p0, ps}, {x0, xs}];
 Expand[Zip_{ps \cup xs}[\mathcal{E} e^Q] /. g_{\alpha, \beta, i, j} \Rightarrow If[\alpha == \beta, g_{\alpha, i, j}, \theta]]
]**

In[*]:= **px2g[p_{2,j}^2 x_{2,i} x_{2,j}]**

Out[*]=

$$2 g_{2,j,i} g_{2,j,j}$$

In[*]:= **R₁[1, i_, j_] = px2g[r_{1,rest}[1, i, j]]**

Out[*]=

$$\begin{aligned}
 & -\frac{1}{2} + \frac{g_{1,j,j} g_{2,i,i}}{-1 + T_1} - T_1 T_2 g_{1,j,i} g_{2,j,i} + \frac{(-1 + T_1 T_2) g_{1,j,j} g_{2,j,i}}{-1 + T_1} - \frac{T_1 g_{1,i,i} g_{2,j,j}}{-1 + T_1} - \frac{g_{1,j,j} g_{3,i,i}}{-1 + T_1} + \\
 & g_{2,j,j} g_{3,i,i} - T_3 g_{3,1+i,i} - g_{3,j,i} + T_1 (-1 + T_1 T_2) g_{1,j,i} g_{3,j,i} - \frac{T_1 (-1 + T_1 T_2) g_{1,j,j} g_{3,j,i}}{-1 + T_1} + \\
 & T_2 (-1 + T_1 T_2) g_{2,j,i} g_{3,j,i} + \frac{T_1 g_{1,i,i} g_{3,j,j}}{-1 + T_1} - T_2 g_{2,j,i} g_{3,j,j} + T_3 g_{3,1+j,i}
 \end{aligned}$$

In[*]:= $R_1[-1, i_-, j_-] = px2g[r_{1,rest}[-1, i, j]]$

Out[*]=

$$\frac{1}{2} \frac{g_{1,j,i} g_{2,i,i} - g_{1,j,j} g_{2,i,i}}{T_1 - 1 + T_1} + \frac{T_1 (-1 + T_2) g_{1,i,i} g_{2,j,i}}{(-1 + T_1) T_2} - \frac{(-T_1 - T_2 + T_1 T_2) g_{1,j,i} g_{2,j,i}}{T_1 T_2} -$$

$$g_{1,j,j} g_{2,j,i} + \frac{T_1 g_{1,i,i} g_{2,j,j}}{-1 + T_1} - g_{1,j,i} g_{2,j,j} + \frac{g_{1,j,i} g_{3,i,i}}{T_1} + \frac{g_{1,j,j} g_{3,i,i}}{-1 + T_1} -$$

$$\frac{(-1 + T_2) g_{2,j,i} g_{3,i,i}}{T_2} - g_{2,j,j} g_{3,i,i} + \frac{g_{3,1+i,i}}{T_3} + g_{3,j,i} - \frac{(-1 + T_1 T_2) g_{1,i,i} g_{3,j,i}}{(-1 + T_1) T_2} +$$

$$\frac{(-1 + T_1 T_2) g_{1,j,i} g_{3,j,i}}{T_1 T_2} - \frac{(-1 + T_1 T_2) g_{2,i,i} g_{3,j,i}}{T_1 T_2} + \frac{(-1 + 2 T_2) (-1 + T_1 T_2) g_{2,j,i} g_{3,j,i}}{T_1 T_2^2} +$$

$$\frac{(-1 + T_1 T_2) g_{2,j,j} g_{3,j,i}}{T_1 T_2} - \frac{T_1 g_{1,i,i} g_{3,j,j}}{-1 + T_1} + g_{1,j,i} g_{3,j,j} + g_{2,j,i} g_{3,j,j} - \frac{g_{3,1+j,i}}{T_3}$$

In[*]:= $px2g[r_{\theta,pxx}[1, i_0, j_0] r_{1,ppx}[1, i_1, j_1]]$

Out[*]=

$$-T_1 (-1 + T_1 T_2) g_{1,j_1,i_0} g_{2,i_1,i_0} g_{3,j_0,i_1} + (-1 + T_1 T_2) g_{1,j_1,j_0} g_{2,i_1,i_0} g_{3,j_0,i_1} +$$

$$T_1 (-1 + T_1 T_2) g_{1,j_1,i_0} g_{2,j_1,i_0} g_{3,j_0,i_1} + (1 - T_1 T_2) g_{1,j_1,j_0} g_{2,j_1,i_0} g_{3,j_0,i_1}$$

In[*]:= $\theta[\{1, i_0, j_0\}, \{1, i_1, j_1\}] = px2g[r_{\theta,pxx}[1, i_0, j_0] r_{1,ppx}[1, i_1, j_1]]$
 $\theta[\{1, i_0, j_0\}, \{-1, i_1, j_1\}] = px2g[r_{\theta,pxx}[1, i_0, j_0] r_{1,ppx}[-1, i_1, j_1]]$
 $\theta[\{-1, i_0, j_0\}, \{1, i_1, j_1\}] = px2g[r_{\theta,pxx}[-1, i_0, j_0] r_{1,ppx}[1, i_1, j_1]]$
 $\theta[\{-1, i_0, j_0\}, \{-1, i_1, j_1\}] = px2g[r_{\theta,pxx}[-1, i_0, j_0] r_{1,ppx}[-1, i_1, j_1]]$

Out[*]=

$$-T_1 (-1 + T_1 T_2) g_{1,j_1,i_0} g_{2,i_1,i_0} g_{3,j_0,i_1} + (-1 + T_1 T_2) g_{1,j_1,j_0} g_{2,i_1,i_0} g_{3,j_0,i_1} +$$

$$T_1 (-1 + T_1 T_2) g_{1,j_1,i_0} g_{2,j_1,i_0} g_{3,j_0,i_1} + (1 - T_1 T_2) g_{1,j_1,j_0} g_{2,j_1,i_0} g_{3,j_0,i_1}$$

Out[*]=

$$(-1 + T_1 T_2) g_{1,j_1,i_0} g_{2,i_1,i_0} g_{3,j_0,i_1} - \frac{(-1 + T_1 T_2) g_{1,j_1,j_0} g_{2,i_1,i_0} g_{3,j_0,i_1}}{T_1} +$$

$$(1 - T_1 T_2) g_{1,j_1,i_0} g_{2,j_1,i_0} g_{3,j_0,i_1} + \frac{(-1 + T_1 T_2) g_{1,j_1,j_0} g_{2,j_1,i_0} g_{3,j_0,i_1}}{T_1}$$

Out[*]=

$$\frac{(-1 + T_1 T_2) g_{1,j_1,i_0} g_{2,i_1,i_0} g_{3,j_0,i_1}}{T_1 T_2} - \frac{(-1 + T_1 T_2) g_{1,j_1,j_0} g_{2,i_1,i_0} g_{3,j_0,i_1}}{T_2} -$$

$$\frac{(-1 + T_1 T_2) g_{1,j_1,i_0} g_{2,j_1,i_0} g_{3,j_0,i_1}}{T_1 T_2} + \frac{(-1 + T_1 T_2) g_{1,j_1,j_0} g_{2,j_1,i_0} g_{3,j_0,i_1}}{T_2}$$

Out[*]=

$$- \frac{(-1 + T_1 T_2) g_{1,j_1,i_0} g_{2,i_1,i_0} g_{3,j_0,i_1}}{T_1^2 T_2} + \frac{(-1 + T_1 T_2) g_{1,j_1,j_0} g_{2,i_1,i_0} g_{3,j_0,i_1}}{T_1 T_2} +$$

$$\frac{(-1 + T_1 T_2) g_{1,j_1,i_0} g_{2,j_1,i_0} g_{3,j_0,i_1}}{T_1^2 T_2} - \frac{(-1 + T_1 T_2) g_{1,j_1,j_0} g_{2,j_1,i_0} g_{3,j_0,i_1}}{T_1 T_2}$$

```
In[*]:= CF[ $\theta$ [{1, i0, j0}], {1, i1, j1}] + ( $\theta$ [-1, i0, j0], {-1, i1, j1}] /. Ti-  $\Rightarrow$  Ti--1)]
Out[*]=
0
```

```
In[*]:= CF[ $\theta$ [{1, i0, j0}], {-1, i1, j1}] + ( $\theta$ [-1, i0, j0], {1, i1, j1}] /. Ti-  $\Rightarrow$  Ti--1)]
Out[*]=
0
```

```
In[*]:=  $\Gamma_1[\varphi, k] = \text{px2g}[\gamma_1[\varphi, k]]$ 
```

```
Out[*]=
 $-\frac{\varphi}{2} + \varphi \mathfrak{g}_{3,k,k}$ 
```

The Programs

```
In[*]:= T3 = T1 T2;
 $\theta$ [K_] := Module[{Cs,  $\varphi$ , n, A, s, i, j, k,  $\Delta$ , G,  $\nu$ ,  $\alpha$ ,  $\beta$ , gEval, Y, yEval, c, z},
  {Cs,  $\varphi$ } = Rot[K]; n = Length[Cs];
  A = IdentityMatrix[2 n + 1];
  Cases[Cs, {s_, i_, j_}  $\Rightarrow$  (A[[{i, j}, {i + 1, j + 1}]] += ( $\begin{matrix} -T^s & T^s & -1 \\ \mathbf{0} & & -1 \end{matrix}$ ))];
   $\Delta = T^{(-\text{Total}[\varphi] - \text{Total}[\text{Cs}[\text{All}, 1]])/2} \text{Det}[A]$ ;
  G = Inverse[A]; gEval[ $\mathcal{E}$ ] := CCF[ $\mathcal{E}$  /. g $\nu$ ,  $\alpha$ ,  $\beta$ ]  $\Rightarrow$  (G[[ $\alpha$ ,  $\beta$ ]] /. T  $\rightarrow$  T $\nu$ );
  z = gEval[ $\sum_{k1=1}^n \sum_{k2=1}^n \theta$ [Cs[[k1]], Cs[[k2]]]];
  z += gEval[ $\sum_{k=1}^n R_1$  @@ Cs[[k]]];
  z += gEval[ $\sum_{k=1}^{2^n} \Gamma_1$ [ $\varphi$ [[k]], k]];
  { $\Delta$ , ( $\Delta$  /. T  $\rightarrow$  T1) ( $\Delta$  /. T  $\rightarrow$  T2) ( $\Delta$  /. T  $\rightarrow$  T3) z} // CCF
];
```

```

In[*]:=  $\theta_{T_1, T_2}[K_] := \text{Module}[\{\text{Cs}, \varphi, n, A, s, i, j, k, \Delta, G, \text{gEval}, Y, \text{yEval}, c, z = \mathbf{0}\},$ 
   $\{\text{Cs}, \varphi\} = \text{Rot}[K]; n = \text{Length}[\text{Cs}];$ 
  temp0 = PrintTemporary["At work, n=", n];
  A = IdentityMatrix[2 n + 1];
  Cases[Cs, {s_, i_, j_}  $\Rightarrow$   $\left( A[[\{i, j\}, \{i+1, j+1\}]] += \begin{pmatrix} -T^s & T^s & -1 \\ \mathbf{0} & & -1 \end{pmatrix} \right)$ ];
   $\Delta[0] := \Delta[0] = T^{(-\text{Total}[\varphi] - \text{Total}[\text{Cs}[[\text{All}, 1]])]/2} \text{Det}[A];$ 
  G[0] := G[0] = Inverse[A];
   $\{\Delta[1], G[1]\} = \text{If}[\text{NumberQ}[T_1],$ 
     $\{\text{Det}[A /. T \rightarrow T_1], \text{Inverse}[A /. T \rightarrow T_1]\}, \{\Delta[0], G[0]\} /. T \rightarrow T_1];$ 
  temp = PrintTemporary@"Done with  $\{\Delta[1], G[1]\}$ .";
   $\{\Delta[2], G[2]\} = \text{If}[\text{NumberQ}[T_2],$ 
     $\{\text{Det}[A /. T \rightarrow T_2], \text{Inverse}[A /. T \rightarrow T_2]\}, \{\Delta[0], G[0]\} /. T \rightarrow T_2];$ 
  NotebookDelete[temp]; temp = PrintTemporary@"Done with  $\{\Delta[2], G[2]\}$ .";
   $\{\Delta[3], G[3]\} = \text{If}[\text{NumberQ}[T_1 T_2],$ 
     $\{\text{Det}[A /. T \rightarrow T_1 T_2], \text{Inverse}[A /. T \rightarrow T_1 T_2]\}, \{\Delta[0], G[0]\} /. T \rightarrow T_1 T_2];$ 
  NotebookDelete[temp]; temp = PrintTemporary@"Done with  $\{\Delta[3], G[3]\}$ .";
  gEval[ $\mathcal{E}_-$ ] := CCF[ $\mathcal{E}$  //  $\{T_1 \rightarrow T_1, T_2 \rightarrow T_2, g_{v, \alpha, \beta} \Rightarrow G[v][[\alpha, \beta]]\}$ ];
  Do[z += gEval[ $\theta$ [Cs[[k1], Cs[[k2]]], {k1, n}, {k2, n}];
  Do[z += gEval[R1@@Cs[[k]], {k, n}];
  Do[z += gEval[T1[ $\varphi$ [[k], k]], {k, 2 n}];
  NotebookDelete[temp0]; NotebookDelete[temp];
   $\{\{\Delta[1], \Delta[2], \Delta[3]\}, \Delta[1] \Delta[2] \Delta[3] z\} // \text{CCF}$ 
];

```

```

In[*]:= TestSymmetries[K_] := Module[{ $\theta_0, \theta_1$ },
   $\{\theta_0, \theta_1\} = \{\theta[K][[2]], \theta[\text{Mirror}[K][[2]]\};$ 
  Simplify@And[
     $\theta_0 = (\theta_0 /. \{T_1 \rightarrow T_2, T_2 \rightarrow T_1\}),$ 
     $\theta_0 = -\theta_1,$ 
     $\theta_0 = (\theta_0 /. T_{i-} \Rightarrow T_i^{-1}),$ 
     $\theta_0 = (\theta_0 /. T_2 \rightarrow T_1^{-1} T_2^{-1})$ 
  ]
];

```

```

In[ ]:= PolyPlotT1, T2[p_] := Module[{crs, m1, m2, mc},
  crs = CoefficientRules[T1m1 == -Exponent[p, T1, Min] T2m2 == -Exponent[p, T2, Min] p, {T1, T2}];
  mc = Max@Abs[Last /@ crs];
  Graphics[crs /. ({x1_, x2_} -> c_) -> {
    Which[
      c == 0, White,
      c > 0, Lighter[Red, 1 - c / mc],
      c < 0, Lighter[Blue, 1 + c / mc]
    ],
    Disk[ $\left(\begin{matrix} 1 & -1/2 \\ 0 & \sqrt{3}/2 \end{matrix}\right) \cdot \{x1 + m1, x2 + m2\}, 0.5]$ 
  ]
]

```

Sporadic Testing

```

In[ ]:= K = Knot[3, 1]; Timing[Expand[Theta[K]]]
TestSymmetries[K]

```

 KnotTheory: Loading precomputed data in PD4Knots`.

Out[]:=

$$\left\{ 0., \left\{ -1 + \frac{1}{T_1} + T_1, -\frac{1}{T_1^2} - T_1^2 - \frac{1}{T_2^2} - \frac{1}{T_1^2 T_2^2} + \frac{1}{T_1 T_2^2} + \frac{1}{T_1^2 T_2} + \frac{T_1}{T_2} + \frac{T_2}{T_1} + T_1^2 T_2 - T_2^2 + T_1 T_2^2 - T_1^2 T_2^2 \right\} \right\}$$

Out[]:=

True

```
In[*]:= K = Knot[8, 19]; Timing[Expand[Theta[K]]]
TestSymmetries[K]
```

Out[*]=

$$\left\{ 0.015625, \left\{ 1 + \frac{1}{T^3} - \frac{1}{T^2} - T^2 + T^3, \frac{3}{T_1^6} - \frac{3}{T_1^4} + \frac{4}{T_1^3} - \frac{1}{T_1^2} - T_1^2 + 4 T_1^3 - 3 T_1^4 + 3 T_1^6 + \frac{3}{T_2^6} + \frac{3}{T_1^6 T_2^6} - \frac{3}{T_1^5 T_2^6} + \frac{3}{T_1^3 T_2^6} - \frac{3}{T_1 T_2^6} - \frac{3}{T_1^6 T_2^5} + \frac{3}{T_1^4 T_2^5} - \frac{3}{T_1^3 T_2^5} - \frac{3}{T_1^2 T_2^5} + \frac{3}{T_1 T_2^5} - \frac{3 T_1}{T_2^5} - \frac{3}{T_2^4} + \frac{3}{T_1^5 T_2^4} - \frac{3}{T_1^4 T_2^4} + \frac{3}{T_1^2 T_2^4} + \frac{3 T_1}{T_2^4} + \frac{4}{T_2^3} + \frac{3}{T_1^6 T_2^3} - \frac{3}{T_1^5 T_2^3} + \frac{4}{T_1^3 T_2^3} - \frac{2}{T_1 T_2^3} - \frac{2}{T_1^2 T_2^3} - \frac{3 T_1^2}{T_2^3} + \frac{3 T_1^3}{T_2^3} - \frac{1}{T_2^2} - \frac{3}{T_1^5 T_2^2} + \frac{3}{T_1^4 T_2^2} - \frac{2}{T_1^3 T_2^2} - \frac{1}{T_1^2 T_2^2} + \frac{1}{T_1 T_2^2} - \frac{2 T_1}{T_2^2} + \frac{3 T_1^2}{T_2^2} - \frac{3 T_1^3}{T_2^2} - \frac{3}{T_1^6 T_2} + \frac{3}{T_1^5 T_2} - \frac{2}{T_1^3 T_2} + \frac{1}{T_1^2 T_2} + \frac{T_1}{T_2} - \frac{2 T_1^2}{T_2} + \frac{3 T_1^4}{T_2} - \frac{3 T_1^5}{T_2} - \frac{3 T_2}{T_1^5} + \frac{3 T_2}{T_1^4} - \frac{2 T_2}{T_1^2} + \frac{T_2}{T_1} + T_1^2 T_2 - 2 T_1^3 T_2 + 3 T_1^5 T_2 - 3 T_1^6 T_2 - T_2^2 - \frac{3 T_2^2}{T_1^3} + \frac{3 T_2^2}{T_1^2} - \frac{2 T_2^2}{T_1} + T_1 T_2^2 - T_1^2 T_2^2 - 2 T_1^3 T_2^2 + 3 T_1^4 T_2^2 - 3 T_1^5 T_2^2 + 4 T_2^3 + \frac{3 T_2^3}{T_1^3} - \frac{3 T_2^3}{T_1^2} - 2 T_1 T_2^3 - 2 T_1^2 T_2^3 + 4 T_1^3 T_2^3 - 3 T_1^5 T_2^3 + 3 T_1^6 T_2^3 - 3 T_2^4 + \frac{3 T_2^4}{T_1} + 3 T_1^2 T_2^4 - 3 T_1^4 T_2^4 + 3 T_1^5 T_2^4 - \frac{3 T_2^5}{T_1} + 3 T_1 T_2^5 - 3 T_1^2 T_2^5 - 3 T_1^3 T_2^5 + 3 T_1^4 T_2^5 - 3 T_1^6 T_2^5 + 3 T_2^6 - 3 T_1 T_2^6 + 3 T_1^3 T_2^6 - 3 T_1^5 T_2^6 + 3 T_1^6 T_2^6 \right\} \right\}$$

Out[*]=

True

```
In[*]:= Timing[Expand@Theta_{T1, T2}[Knot[3, 1]]]
```

Out[*]=

$$\left\{ 0., \left\{ \left\{ -1 + \frac{1}{T_1} + T_1, -1 + \frac{1}{T_2} + T_2, -1 + \frac{1}{T_1 T_2} + T_1 T_2 \right\}, -\frac{1}{T_1^2} - T_1^2 - \frac{1}{T_2^2} - \frac{1}{T_1^2 T_2^2} + \frac{1}{T_1 T_2^2} + \frac{1}{T_1^2 T_2} + \frac{T_1}{T_2} + \frac{T_2}{T_1} + T_1^2 T_2 - T_2^2 + T_1 T_2^2 - T_1^2 T_2^2 \right\} \right\}$$

```
In[*]:= K = Knot[4, 1]; Timing[Theta[K]]
TestSymmetries[K]
```

Out[*]=

$$\left\{ 0., \left\{ -\frac{1 - 3 T + T^2}{T}, \emptyset \right\} \right\}$$

Out[*]=

True


```
In[*]:= K = Knot["K11n34"]; Timing[Θ[K]]
TestSymmetries[K]
```

Out[*]=

$$\left\{ \emptyset., \left\{ 1, -\frac{1}{T_1^6 T_2^6} \left(T_1^2 - 2 T_1^3 + T_1^4 - 2 T_1 T_2 + 2 T_1^2 T_2 + 2 T_1^5 T_2 - 2 T_1^6 T_2 + T_2^2 + 2 T_1 T_2^2 - 2 T_1^2 T_2^2 - 2 T_1^4 T_2^2 - 2 T_1^6 T_2^2 + 2 T_1^7 T_2^2 + T_1^8 T_2^2 - 2 T_2^3 + T_1^4 T_2^3 + T_1^5 T_2^3 - 2 T_1^9 T_2^3 + T_2^4 - 2 T_1^2 T_2^4 + T_1^3 T_2^4 + 2 T_1^4 T_2^4 + 2 T_1^6 T_2^4 + T_1^7 T_2^4 - 2 T_1^8 T_2^4 + T_1^{10} T_2^4 + 2 T_1 T_2^5 + T_1^3 T_2^5 - 4 T_1^5 T_2^5 - 4 T_1^6 T_2^5 + T_1^8 T_2^5 + 2 T_1^{10} T_2^5 - 2 T_1 T_2^6 - 2 T_1^2 T_2^6 + 2 T_1^4 T_2^6 - 4 T_1^5 T_2^6 + 12 T_1^6 T_2^6 - 4 T_1^7 T_2^6 + 2 T_1^8 T_2^6 - 2 T_1^{10} T_2^6 - 2 T_1^{11} T_2^6 + 2 T_1^2 T_2^7 + T_1^4 T_2^7 - 4 T_1^6 T_2^7 - 4 T_1^7 T_2^7 + T_1^9 T_2^7 + 2 T_1^{11} T_2^7 + T_1^2 T_2^8 - 2 T_1^4 T_2^8 + T_1^5 T_2^8 + 2 T_1^6 T_2^8 + 2 T_1^8 T_2^8 + T_1^9 T_2^8 - 2 T_1^{10} T_2^8 + T_1^{12} T_2^8 - 2 T_1^3 T_2^9 + T_1^7 T_2^9 + T_1^8 T_2^9 - 2 T_1^{12} T_2^9 + T_1^4 T_2^{10} + 2 T_1^5 T_2^{10} - 2 T_1^6 T_2^{10} - 2 T_1^8 T_2^{10} - 2 T_1^{10} T_2^{10} + 2 T_1^{11} T_2^{10} + T_1^{12} T_2^{10} - 2 T_1^6 T_2^{11} + 2 T_1^7 T_2^{11} + 2 T_1^{10} T_2^{11} - 2 T_1^{11} T_2^{11} + T_1^8 T_2^{12} - 2 T_1^9 T_2^{12} + T_1^{10} T_2^{12} \right) \right\} \right\}$$

Out[*]=

True

```
In[*]:= K = Knot["K11n42"]; Timing[Θ[K]]
TestSymmetries[K]
```

Out[*]=

$$\left\{ \emptyset.015625, \left\{ 1, \frac{1}{T_1^3 T_2^3} \left(T_1 + T_1^2 + T_2 - 2 T_1 T_2 - 2 T_1^2 T_2 - 2 T_1^3 T_2 + T_1^4 T_2 + T_2^2 - 2 T_1 T_2^2 + 2 T_1^2 T_2^2 + 2 T_1^3 T_2^2 - 2 T_1^4 T_2^2 + 2 T_1^5 T_2^2 - 2 T_1 T_2^3 + 2 T_1^2 T_2^3 + 2 T_1^4 T_2^3 - 2 T_1^5 T_2^3 + T_1 T_2^4 - 2 T_1^2 T_2^4 + 2 T_1^3 T_2^4 + 2 T_1^4 T_2^4 - 2 T_1^5 T_2^4 + T_1^6 T_2^4 + T_1^2 T_2^5 - 2 T_1^3 T_2^5 - 2 T_1^4 T_2^5 - 2 T_1^5 T_2^5 + T_1^6 T_2^5 + T_1^4 T_2^6 + T_1^5 T_2^6 \right) \right\} \right\}$$

Out[*]=

True

```
In[*]:= PD[GST48] = PD[X[1, 15, 2, 14], X[29, 2, 30, 3], X[40, 4, 41, 3],
X[4, 44, 5, 43], X[5, 26, 6, 27], X[95, 7, 96, 6], X[7, 1, 8, 96], X[8, 14, 9, 13],
X[28, 9, 29, 10], X[41, 11, 42, 10], X[11, 43, 12, 42], X[12, 27, 13, 28],
X[15, 31, 16, 30], X[61, 16, 62, 17], X[72, 17, 73, 18], X[83, 18, 84, 19],
X[34, 20, 35, 19], X[20, 89, 21, 90], X[92, 21, 93, 22], X[22, 79, 23, 80],
X[23, 68, 24, 69], X[24, 57, 25, 58], X[56, 25, 57, 26], X[31, 63, 32, 62],
X[32, 74, 33, 73], X[33, 85, 34, 84], X[35, 50, 36, 51], X[81, 37, 82, 36],
X[70, 38, 71, 37], X[59, 39, 60, 38], X[54, 39, 55, 40], X[55, 45, 56, 44],
X[45, 59, 46, 58], X[46, 70, 47, 69], X[47, 81, 48, 80], X[91, 49, 92, 48],
X[49, 91, 50, 90], X[82, 52, 83, 51], X[71, 53, 72, 52], X[60, 54, 61, 53],
X[74, 63, 75, 64], X[85, 64, 86, 65], X[65, 76, 66, 77], X[66, 87, 67, 88],
X[94, 67, 95, 68], X[86, 75, 87, 76], X[77, 88, 78, 89], X[93, 78, 94, 79]];
```

```
In[ ]:= K = GST48; AbsoluteTiming[Short@θ[K]]
TestSymmetries[K]
```

Out[]=

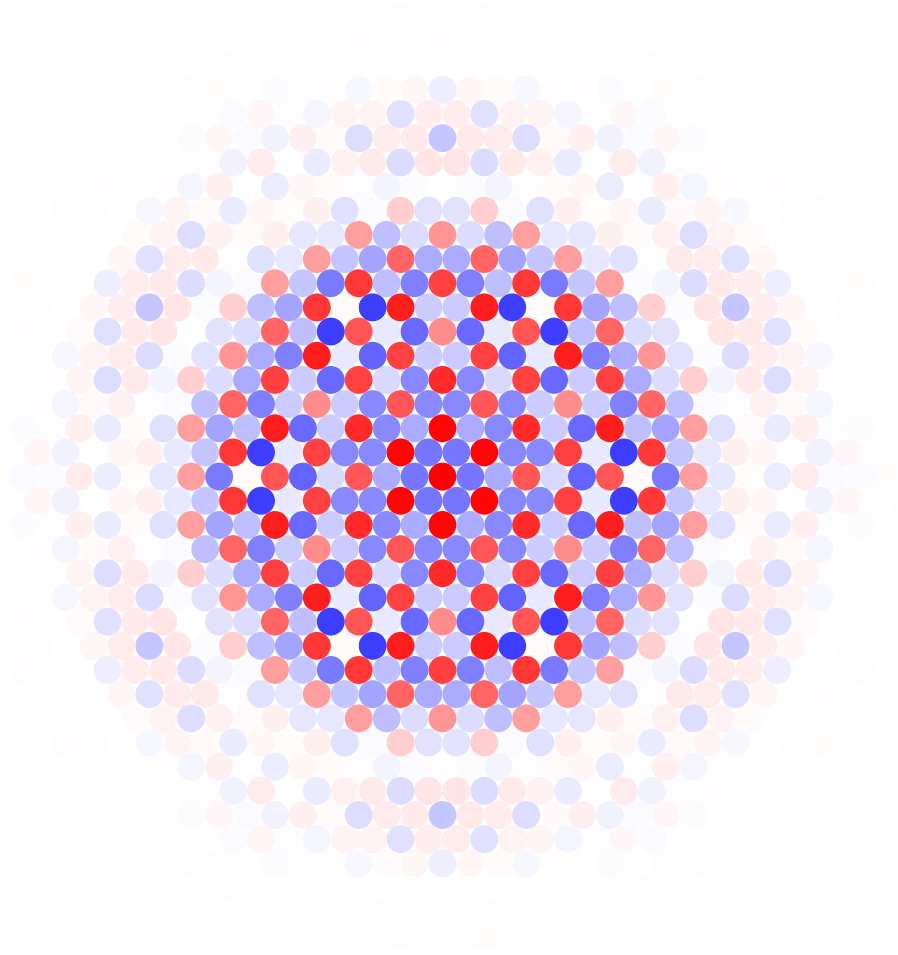
$$\left\{ 11.9122, \left\{ -\frac{(-1 + 2T - T^2 - T^3 + 2T^4 - T^5 + T^8)(-1 + \langle\langle 7 \rangle\rangle + T^8)}{T^8}, \frac{\langle\langle 1764 \rangle\rangle + T_1^{35} T_2^{40}}{T_1^{20} T_2^{20}} \right\} \right\}$$

Out[]=

True

```
In[ ]:= PolyPlot_{T_1, T_2}[θ[GST48][[2]]]
```

Out[]=



```
In[ ]:= AbsoluteTiming[θ_{T_1, T_2}[GST48];]
```

Out[]=

{60.1527, Null}

In[*]:= AbsoluteTiming[$\theta_{22/7,34/21}$ [GST48]]

Out[*]=

$$\left\{ 0.36564, \left\{ \left\{ -\frac{1\,422\,357\,287\,561\,349\,859\,889}{10\,190\,414\,377\,180\,576}, -\frac{486\,885\,265\,100\,293\,177\,259\,569}{15\,915\,006\,754\,796\,041\,036\,704} \right\}, -\frac{6\,215\,902\,990\,719\,340\,337\,664\,427\,997\,383\,765\,280\,900\,656\,009}{162\,180\,513\,646\,999\,558\,542\,864\,476\,199\,651\,861\,504} \right\}, \right. \\ \left. 21\,304\,335\,657\,502\,800\,961\,104\,521\,150\,882\,906\,491\,585\,928\,445\,141\,602\,977\,673\,772\,524\,921\,333\,287\,756 \ ; \right. \\ \left. 546\,740\,936\,248\,585\,046\,107\,073\,499 \ / \right. \\ \left. 4\,728\,039\,585\,290\,312\,086\,302\,386\,002\,441\,018\,010\,474\,726\,543\,601\,178\,518\,697\,564\,845\,087\,356\,778\,405 \ ; \right. \\ \left. 506\,577\,334\,272 \right\} \left. \right\}$$

Systematic Testing

In[*]:= DuplicateFreeQ[θ /@ AllKnots [{3, 10}]]

Out[*]=

True

In[*]:= Total[TestSymmetries /@ AllKnots [{3, 10}]]

Out[*]=

249 True

In[*]:= DuplicateFreeQ[θ /@ AllKnots [{3, 12}]]

 KnotTheory: Loading precomputed data in KnotTheory/12A.dts.

 KnotTheory: Loading precomputed data in KnotTheory/12N.dts.


Out[*]=

False

In[*]:= tab11 = Table[K \rightarrow θ @K, {K, AllKnots [{3, 11}]}]

Out[*]=

$$\left\{ \text{Knot}[3, 1] \rightarrow \left\{ \frac{1-T+T^2}{T}, -\frac{1-T_1+T_1^2-T_2-T_1^3-T_2+T_1^4-T_1^3-T_2-T_1^4+T_1^3+T_1^4-T_1^3-T_2+T_1^4}{T_1^2 T_2^2} \right\}, \text{Knot}[4, 1] \rightarrow \left\{ -\frac{1-3T+T^2}{T}, \emptyset \right\}, \right. \\ \left. \text{Knot}[5, 1] \rightarrow \left\{ \frac{1-T+T^2-T^3+T^4}{T^2}, -\frac{\dots 53 \dots + 2 T_1^3 T_2^2}{T_1^4 T_2^2} \right\}, \dots 795 \dots, \text{Knot}[11, \text{NonAlternating}, 183] \rightarrow \left\{ \frac{\dots 1 \dots}{T^3}, \dots 1 \dots \right\}, \right. \\ \left. \text{Knot}[11, \text{NonAlternating}, 184] \rightarrow \left\{ \frac{(1-T+T^2)(2-7T+11T^2-7T^3+2T^4)}{T^3}, \frac{9-41T_1+92T_1^2-115T_1^3+\dots 166 \dots + 92T_1^9 T_2^{12}-41T_1^{11} T_2^{12}-9T_1^{12} T_2^{12}}{T_1^9 T_2^9} \right\}, \right. \\ \left. \text{Knot}[11, \text{NonAlternating}, 185] \rightarrow \left\{ -\frac{(1-3T+T^2)(1-T+T^2)(2-3T+2T^2)}{T^3}, \right. \right. \\ \left. \left. -\frac{1}{T_1^9 T_2^9} \left(17-93T_1+202T_1^2-261T_1^3+202T_1^4-93T_1^5+17T_1^6-93T_2+416T_1T_2-593T_1^2T_2+321T_1^3T_2+\dots 153 \dots + \right. \right. \right. \\ \left. \left. \left. 416T_1^{11}T_2^{11}-93T_1^{12}T_2^{11}+17T_1^6T_2^{12}-93T_1^7T_2^{12}+202T_1^8T_2^{12}-261T_1^9T_2^{12}+202T_1^{10}T_2^{12}-93T_1^{11}T_2^{12}+17T_1^{12}T_2^{12} \right) \right\} \right\}$$

Full expression not available (original memory size: 33.2 MB) 

In[*]:= Gather[tab11, Last[#1] === Last[#2] &]

Out[*]=

$$\left\{ \left\{ \text{Knot}[3, 1] \rightarrow \left\{ \frac{1-T+T^2}{T}, -\frac{1-T_1+T_1^2-T_2-T_1^3 T_2+T_2^2+T_1^4 T_2^2-T_1 T_2^3-T_1^2 T_2^3+T_1^3 T_2^3-T_1^4 T_2^3+T_1^4 T_2^4}{T_1^2 T_2^2} \right\} \right\}, \left\{ \text{Knot}[4, 1] \rightarrow \left\{ -\frac{1-3T+T^2}{T}, \emptyset \right\} \right\}, \right.$$

$$\left. \left\{ \text{Knot}[5, 1] \rightarrow \left\{ \frac{1-T+T^2-T^3+T^4}{T^2}, -\frac{\dots 53 \dots +2 T_1^8 T_2^8}{T_1^4 T_2^4} \right\} \right\}, \dots 792 \dots, \left\{ \text{Knot}[11, \text{NonAlternating}, 183] \rightarrow \left\{ -\frac{\dots 1 \dots}{T^3}, \dots 1 \dots \right\} \right\}, \right.$$

$$\left\{ \text{Knot}[11, \text{NonAlternating}, 184] \rightarrow \left\{ \frac{(1-T+T^2)(2-7T+11T^2-7T^3+2T^4)}{T^3}, \frac{9-41 T_1+\dots 169 \dots +92 T_1^{10} T_2^{12}-41 T_1^{11} T_2^{12}+9 T_1^{12} T_2^{12}}{T_1^5 T_2^5} \right\} \right\},$$

$$\left\{ \text{Knot}[11, \text{NonAlternating}, 185] \rightarrow \left\{ -\frac{(1-3T+T^2)(1-T+T^2)(2-3T+2T^2)}{T^3}, \right.$$

$$\left. -\frac{1}{T_1^5 T_2^5} (17-93 T_1+202 T_1^2-261 T_1^3+202 T_1^4-93 T_1^5+17 T_1^6-93 T_2+416 T_1 T_2-593 T_1^2 T_2+321 T_1^3 T_2+\dots 153 \dots + \right.$$

$$\left. 416 T_1^{11} T_2^{11}-93 T_1^{12} T_2^{11}+17 T_1^6 T_1^{12} T_2^2-93 T_1^7 T_1^{12} T_2^2+202 T_1^8 T_1^{12} T_2^2-261 T_1^9 T_1^{12} T_2^2+202 T_1^{10} T_1^{12} T_2^2-93 T_1^{11} T_1^{12} T_2^2+17 T_1^{12} T_1^{12} T_2^2) \right\} \right\}$$

Full expression not available (original memory size: 33.2 MB)

In[*]:= Select[Gather[tab11, Last[#1] === Last[#2] &], Length[#] > 1 &]

Out[*]=

$$\left\{ \left\{ \text{Knot}[11, \text{Alternating}, 44] \rightarrow \right.$$

$$\left. \left\{ \frac{(1-T+T^2)^2(1-3T+5T^2-3T^3+T^4)}{T^4}, -\frac{1}{T_1^6 T_2^6} 2(1-T_1+T_1^2)(1-T_2+T_2^2)(1-T_1 T_2+T_1^2 T_2^2) \right. \right.$$

$$\left. \left. \left(T_1-2 T_1^2+T_1^3+T_2-5 T_1 T_2+5 T_1^2 T_2+5 T_1^3 T_2-5 T_1^4 T_2+T_1^5 T_2-2 T_2^2+5 T_1 T_2^2+5 T_1^2 T_2^2-26 T_1^3 T_2^2+ \right. \right.$$

$$\left. \left. 5 T_1^4 T_2^2+5 T_1^5 T_2^2-2 T_1^6 T_2^2+T_2^3+5 T_1 T_2^3-26 T_1^2 T_2^3+32 T_1^3 T_2^3+32 T_1^4 T_2^3-26 T_1^5 T_2^3+5 T_1^6 T_2^3+T_1^7 T_2^3- \right. \right.$$

$$\left. \left. 5 T_1 T_2^4+5 T_1^2 T_2^4+32 T_1^3 T_2^4-96 T_1^4 T_2^4+32 T_1^5 T_2^4+5 T_1^6 T_2^4-5 T_1^7 T_2^4+T_1 T_2^5+5 T_1^2 T_2^5-26 T_1^3 T_2^5+ \right. \right.$$

$$\left. \left. 32 T_1^4 T_2^5+32 T_1^5 T_2^5-26 T_1^6 T_2^5+5 T_1^7 T_2^5+T_1^8 T_2^5-2 T_1 T_2^6+5 T_1^3 T_2^6+5 T_1^4 T_2^6-26 T_1^5 T_2^6+5 T_1^6 T_2^6+ \right. \right.$$

$$\left. \left. 5 T_1^7 T_2^6-2 T_1^8 T_2^6+T_1^3 T_2^7-5 T_1^4 T_2^7+5 T_1^5 T_2^7+5 T_1^6 T_2^7-5 T_1^7 T_2^7+T_1^8 T_2^7+T_1^5 T_2^8-2 T_1^6 T_2^8+T_1^7 T_2^8) \right\} \right\},$$

$$\text{Knot}[11, \text{Alternating}, 47] \rightarrow \left\{ \frac{(1-T+T^2)^2(1-3T+5T^2-3T^3+T^4)}{T^4}, \right.$$

$$\left. -\frac{1}{T_1^6 T_2^6} 2(1-T_1+T_1^2)(1-T_2+T_2^2)(1-T_1 T_2+T_1^2 T_2^2) \right.$$

$$\left. \left. \left(T_1-2 T_1^2+T_1^3+T_2-5 T_1 T_2+5 T_1^2 T_2+5 T_1^3 T_2-5 T_1^4 T_2+T_1^5 T_2-2 T_2^2+5 T_1 T_2^2+5 T_1^2 T_2^2-26 T_1^3 T_2^2+ \right. \right.$$

$$\left. \left. 5 T_1^4 T_2^2+5 T_1^5 T_2^2-2 T_1^6 T_2^2+T_2^3+5 T_1 T_2^3-26 T_1^2 T_2^3+32 T_1^3 T_2^3+32 T_1^4 T_2^3-26 T_1^5 T_2^3+5 T_1^6 T_2^3+T_1^7 T_2^3- \right. \right.$$

$$\left. \left. 5 T_1 T_2^4+5 T_1^2 T_2^4+32 T_1^3 T_2^4-96 T_1^4 T_2^4+32 T_1^5 T_2^4+5 T_1^6 T_2^4-5 T_1^7 T_2^4+T_1 T_2^5+5 T_1^2 T_2^5-26 T_1^3 T_2^5+ \right. \right.$$

$$\left. \left. 32 T_1^4 T_2^5+32 T_1^5 T_2^5-26 T_1^6 T_2^5+5 T_1^7 T_2^5+T_1^8 T_2^5-2 T_1 T_2^6+5 T_1^3 T_2^6+5 T_1^4 T_2^6-26 T_1^5 T_2^6+5 T_1^6 T_2^6+ \right. \right.$$

$$\left. \left. 5 T_1^7 T_2^6-2 T_1^8 T_2^6+T_1^3 T_2^7-5 T_1^4 T_2^7+5 T_1^5 T_2^7+5 T_1^6 T_2^7-5 T_1^7 T_2^7+T_1^8 T_2^7+T_1^5 T_2^8-2 T_1^6 T_2^8+T_1^7 T_2^8) \right\} \right\},$$

$$\left\{ \text{Knot}[11, \text{Alternating}, 57] \rightarrow \left\{ -\frac{(1-T+T^2)^2(1-3T+3T^2-3T^3+T^4)}{T^4}, \right.$$

$$\left. \frac{1}{T_1^8 T_2^8} (1-T_1+T_1^2)(1-T_2+T_2^2)(1-T_1 T_2+T_1^2 T_2^2) \right.$$

$$\left. \left. \left(1-4 T_1+7 T_1^2-9 T_1^3+7 T_1^4-4 T_1^5+T_1^6-4 T_2+12 T_1 T_2-12 T_1^2 T_2+8 T_1^3 T_2+8 T_1^4 T_2-12 T_1^5 T_2+ \right. \right.$$

$$\left. \left. 12 T_1^6 T_2-4 T_1^7 T_2+7 T_2^2-12 T_1 T_2^2-8 T_1^2 T_2^2+25 T_1^3 T_2^2-52 T_1^4 T_2^2+25 T_1^5 T_2^2-8 T_1^6 T_2^2-12 T_1^7 T_2^2+ \right. \right.$$

$$\left. \left. 7 T_1^8 T_2^2-9 T_2^3+8 T_1 T_2^3+25 T_1^2 T_2^3-32 T_1^3 T_2^3+37 T_1^4 T_2^3+37 T_1^5 T_2^3-32 T_1^6 T_2^3+25 T_1^7 T_2^3+8 T_1^8 T_2^3- \right. \right.$$

$$\left. \left. 9 T_1^9 T_2^3+7 T_2^4+8 T_1 T_2^4-52 T_1^2 T_2^4+37 T_1^3 T_2^4-6 T_1^4 T_2^4-68 T_1^5 T_2^4-6 T_1^6 T_2^4+37 T_1^7 T_2^4-52 T_1^8 T_2^4+ \right. \right.$$

$$\left. \left. 8 T_1^9 T_2^4+7 T_1^{10} T_2^4-4 T_2^5-12 T_1 T_2^5+25 T_1^2 T_2^5+37 T_1^3 T_2^5-68 T_1^4 T_2^5+66 T_1^5 T_2^5+66 T_1^6 T_2^5-68 T_1^7 T_2^5+ \right. \right.$$

$$\left. \left. 37 T_1^8 T_2^5+25 T_1^9 T_2^5-12 T_1^{10} T_2^5-4 T_1^{11} T_2^5+T_2^6+12 T_1 T_2^6-8 T_1^2 T_2^6-32 T_1^3 T_2^6-6 T_1^4 T_2^6+66 T_1^5 T_2^6- \right. \right.$$

$$\begin{aligned}
 & 156 T_1^6 T_2^6 + 66 T_1^7 T_2^6 - 6 T_1^8 T_2^6 - 32 T_1^9 T_2^6 - 8 T_1^{10} T_2^6 + 12 T_1^{11} T_2^6 + T_1^{12} T_2^6 - 4 T_1 T_2^7 - 12 T_1^2 T_2^7 + \\
 & 25 T_1^3 T_2^7 + 37 T_1^4 T_2^7 - 68 T_1^5 T_2^7 + 66 T_1^6 T_2^7 + 66 T_1^7 T_2^7 - 68 T_1^8 T_2^7 + 37 T_1^9 T_2^7 + 25 T_1^{10} T_2^7 - 12 T_1^{11} T_2^7 - \\
 & 4 T_1^{12} T_2^7 + 7 T_1^2 T_2^8 + 8 T_1^3 T_2^8 - 52 T_1^4 T_2^8 + 37 T_1^5 T_2^8 - 6 T_1^6 T_2^8 - 68 T_1^7 T_2^8 - 6 T_1^8 T_2^8 + 37 T_1^9 T_2^8 - 52 T_1^{10} T_2^8 + \\
 & 8 T_1^{11} T_2^8 + 7 T_1^{12} T_2^8 - 9 T_1^3 T_2^9 + 8 T_1^4 T_2^9 + 25 T_1^5 T_2^9 - 32 T_1^6 T_2^9 + 37 T_1^7 T_2^9 + 37 T_1^8 T_2^9 - 32 T_1^9 T_2^9 + \\
 & 25 T_1^{10} T_2^9 + 8 T_1^{11} T_2^9 - 9 T_1^{12} T_2^9 + 7 T_1^4 T_2^{10} - 12 T_1^5 T_2^{10} - 8 T_1^6 T_2^{10} + 25 T_1^7 T_2^{10} - 52 T_1^8 T_2^{10} + 25 T_1^9 T_2^{10} - \\
 & 8 T_1^{10} T_2^{10} - 12 T_1^{11} T_2^{10} + 7 T_1^{12} T_2^{10} - 4 T_1^5 T_2^{11} + 12 T_1^6 T_2^{11} - 12 T_1^7 T_2^{11} + 8 T_1^8 T_2^{11} + 8 T_1^9 T_2^{11} - 12 T_1^{10} T_2^{11} + \\
 & 12 T_1^{11} T_2^{11} - 4 T_1^{12} T_2^{11} + T_1^6 T_2^{12} - 4 T_1^7 T_2^{12} + 7 T_1^8 T_2^{12} - 9 T_1^9 T_2^{12} + 7 T_1^{10} T_2^{12} - 4 T_1^{11} T_2^{12} + T_1^{12} T_2^{12} \} ,
 \end{aligned}$$

$$\text{Knot}[11, \text{Alternating}, 231] \rightarrow \left\{ -\frac{(1 - T + T^2)^2 (1 - 3T + 3T^2 - 3T^3 + T^4)}{T^4}, \right.$$

$$\left. \frac{1}{T_1^8 T_2^8} (1 - T_1 + T_1^2) (1 - T_2 + T_2^2) (1 - T_1 T_2 + T_1^2 T_2^2) \right.$$

$$\begin{aligned}
 & (1 - 4 T_1 + 7 T_1^2 - 9 T_1^3 + 7 T_1^4 - 4 T_1^5 + T_1^6 - 4 T_2 + 12 T_1 T_2 - 12 T_1^2 T_2 + 8 T_1^3 T_2 + 8 T_1^4 T_2 - 12 T_1^5 T_2 + \\
 & 12 T_1^6 T_2 - 4 T_1^7 T_2 + 7 T_2^2 - 12 T_1 T_2^2 - 8 T_1^2 T_2^2 + 25 T_1^3 T_2^2 - 52 T_1^4 T_2^2 + 25 T_1^5 T_2^2 - 8 T_1^6 T_2^2 - 12 T_1^7 T_2^2 + \\
 & 7 T_1^8 T_2^2 - 9 T_2^3 + 8 T_1 T_2^3 + 25 T_1^2 T_2^3 - 32 T_1^3 T_2^3 + 37 T_1^4 T_2^3 + 37 T_1^5 T_2^3 - 32 T_1^6 T_2^3 + 25 T_1^7 T_2^3 + 8 T_1^8 T_2^3 - \\
 & 9 T_1^9 T_2^3 + 7 T_2^4 + 8 T_1 T_2^4 - 52 T_1^2 T_2^4 + 37 T_1^3 T_2^4 - 6 T_1^4 T_2^4 - 68 T_1^5 T_2^4 - 6 T_1^6 T_2^4 + 37 T_1^7 T_2^4 - 52 T_1^8 T_2^4 + \\
 & 8 T_1^9 T_2^4 + 7 T_1^{10} T_2^4 - 4 T_2^5 - 12 T_1 T_2^5 + 25 T_1^2 T_2^5 + 37 T_1^3 T_2^5 - 68 T_1^4 T_2^5 + 66 T_1^5 T_2^5 + 66 T_1^6 T_2^5 - 68 T_1^7 T_2^5 + \\
 & 37 T_1^8 T_2^5 + 25 T_1^9 T_2^5 - 12 T_1^{10} T_2^5 - 4 T_1^{11} T_2^5 + T_2^6 + 12 T_1 T_2^6 - 8 T_1^2 T_2^6 - 32 T_1^3 T_2^6 - 6 T_1^4 T_2^6 + 66 T_1^5 T_2^6 - \\
 & 156 T_1^6 T_2^6 + 66 T_1^7 T_2^6 - 6 T_1^8 T_2^6 - 32 T_1^9 T_2^6 - 8 T_1^{10} T_2^6 + 12 T_1^{11} T_2^6 + T_1^{12} T_2^6 - 4 T_1 T_2^7 - 12 T_1^2 T_2^7 + \\
 & 25 T_1^3 T_2^7 + 37 T_1^4 T_2^7 - 68 T_1^5 T_2^7 + 66 T_1^6 T_2^7 + 66 T_1^7 T_2^7 - 68 T_1^8 T_2^7 + 37 T_1^9 T_2^7 + 25 T_1^{10} T_2^7 - 12 T_1^{11} T_2^7 - \\
 & 4 T_1^{12} T_2^7 + 7 T_1^2 T_2^8 + 8 T_1^3 T_2^8 - 52 T_1^4 T_2^8 + 37 T_1^5 T_2^8 - 6 T_1^6 T_2^8 - 68 T_1^7 T_2^8 - 6 T_1^8 T_2^8 + 37 T_1^9 T_2^8 - 52 T_1^{10} T_2^8 + \\
 & 8 T_1^{11} T_2^8 + 7 T_1^{12} T_2^8 - 9 T_1^3 T_2^9 + 8 T_1^4 T_2^9 + 25 T_1^5 T_2^9 - 32 T_1^6 T_2^9 + 37 T_1^7 T_2^9 + 37 T_1^8 T_2^9 - 32 T_1^9 T_2^9 + \\
 & 25 T_1^{10} T_2^9 + 8 T_1^{11} T_2^9 - 9 T_1^{12} T_2^9 + 7 T_1^4 T_2^{10} - 12 T_1^5 T_2^{10} - 8 T_1^6 T_2^{10} + 25 T_1^7 T_2^{10} - 52 T_1^8 T_2^{10} + 25 T_1^9 T_2^{10} - \\
 & 8 T_1^{10} T_2^{10} - 12 T_1^{11} T_2^{10} + 7 T_1^{12} T_2^{10} - 4 T_1^5 T_2^{11} + 12 T_1^6 T_2^{11} - 12 T_1^7 T_2^{11} + 8 T_1^8 T_2^{11} + 8 T_1^9 T_2^{11} - 12 T_1^{10} T_2^{11} + \\
 & 12 T_1^{11} T_2^{11} - 4 T_1^{12} T_2^{11} + T_1^6 T_2^{12} - 4 T_1^7 T_2^{12} + 7 T_1^8 T_2^{12} - 9 T_1^9 T_2^{12} + 7 T_1^{10} T_2^{12} - 4 T_1^{11} T_2^{12} + T_1^{12} T_2^{12} \} \} ,
 \end{aligned}$$

$$\left\{ \text{Knot}[11, \text{NonAlternating}, 73] \rightarrow \left\{ \frac{(1 - T + T^2)^2}{T^2}, \right. \right.$$

$$\left. \left. - \frac{2 (1 - T_1 + T_1^2) (1 - T_2 + T_2^2) (1 - T_1 T_2 + T_1^2 T_2^2) (1 + T_1 + T_2 - 6 T_1 T_2 + T_1^2 T_2 + T_1 T_2^2 + T_1^2 T_2^2)}{T_1^3 T_2^3} \right\} \right\} ,$$

$$\text{Knot}[11, \text{NonAlternating}, 74] \rightarrow$$

$$\left\{ \frac{(1 - T + T^2)^2}{T^2}, \right.$$

$$\left. \left. - \frac{2 (1 - T_1 + T_1^2) (1 - T_2 + T_2^2) (1 - T_1 T_2 + T_1^2 T_2^2) (1 + T_1 + T_2 - 6 T_1 T_2 + T_1^2 T_2 + T_1 T_2^2 + T_1^2 T_2^2)}{T_1^3 T_2^3} \right\} \right\} \}$$

In[*]:= `tab12 = Table[K -> e@K, {K, AllKnots[{3, 12]}]}`

 KnotTheory: Loading precomputed data in KnotTheory/12A.dts.

 KnotTheory: Loading precomputed data in KnotTheory/12N.dts.

Out[]=

$$\left\{ \text{Knot}[3, 1] \rightarrow \left\{ \frac{1-T+T^2}{T}, -\frac{1-T_1+T_1^2-T_2-T_1^3 T_2+T_1^4 T_2^2-T_1 T_2^3-T_1^4 T_2^3+T_1^5 T_2^3-T_1^4 T_2^4+T_1^4 T_2^4}{T_1^2 T_2^2} \right\}, \text{Knot}[4, 1] \rightarrow \left\{ -\frac{1-3T+T^2}{T}, 0 \right\}, \right.$$

$$\text{Knot}[5, 1] \rightarrow \left\{ \frac{1-T+T^2-T^3+T^4}{T^2}, -\frac{\dots 53 \dots + 2 T_1^8 T_2^8}{T_1^4 T_2^4} \right\}, \dots 2971 \dots, \text{Knot}[12, \text{NonAlternating}, 886] \rightarrow \left\{ \frac{\dots 1 \dots}{T^3}, \dots 1 \dots \right\},$$

$$\text{Knot}[12, \text{NonAlternating}, 887] \rightarrow \left\{ \frac{1-6T+16T^2-25T^3+29T^4-25T^5+16T^6-6T^7+T^8}{T^4}, \frac{2-12T_1+\dots 327 \dots + 2 T_1^{16} T_2^{16}}{T_1^8 T_2^8} \right\},$$

$$\text{Knot}[12, \text{NonAlternating}, 888] \rightarrow \left\{ \frac{(1-T+T^2)^2 (1+T-2T^2+T^3-2T^4+T^5+T^6)}{T^5}, \frac{1}{T_1^{10} T_2^{10}} \right.$$

$$\left. \left((1-T_1+T_1^2) (1-T_2+T_2^2) (\dots 1 \dots) (5-10T_1^2+20T_1^3-25T_1^4+20T_1^5-10T_1^6+5T_1^8-10T_2^2+11T_1^2 T_2^2-39T_1^3 T_2^2 + \dots 208 \dots + 11T_1^{14} T_2^{14} - 10T_1^{16} T_2^{14} + 5T_1^8 T_2^{16} - 10T_1^{10} T_2^{16} + 20T_1^{11} T_2^{16} - 25T_1^{12} T_2^{16} + 20T_1^{13} T_2^{16} - 10T_1^{14} T_2^{16} + 5T_1^{16} T_2^{16}) \right\} \right\}$$

Full expression not available (original memory size: 150.5 MB) ⚙️

In[]:= **dup12 = Map[First, Select[Gather[tab12, Last[#1] === Last[#2] &], Length[#] > 1 &], {2}]**

Out[]=

- ```
{ {Knot[10, 106], Knot[12, NonAlternating, 369] },
 {Knot[11, Alternating, 44], Knot[11, Alternating, 47] },
 {Knot[11, Alternating, 57], Knot[11, Alternating, 231] },
 {Knot[11, NonAlternating, 73], Knot[11, NonAlternating, 74] },
 {Knot[12, Alternating, 30], Knot[12, Alternating, 33] },
 {Knot[12, Alternating, 122], Knot[12, Alternating, 182] },
 {Knot[12, Alternating, 164], Knot[12, Alternating, 166] },
 {Knot[12, Alternating, 167], Knot[12, Alternating, 692] },
 {Knot[12, Alternating, 273], Knot[12, Alternating, 890] },
 {Knot[12, Alternating, 341], Knot[12, Alternating, 627] },
 {Knot[12, Alternating, 427], Knot[12, Alternating, 435], Knot[12, Alternating, 990] },
 {Knot[12, Alternating, 458], Knot[12, Alternating, 887] },
 {Knot[12, Alternating, 510], Knot[12, Alternating, 821] },
 {Knot[12, NonAlternating, 56], Knot[12, NonAlternating, 57] },
 {Knot[12, NonAlternating, 60], Knot[12, NonAlternating, 61] },
 {Knot[12, NonAlternating, 62], Knot[12, NonAlternating, 66] },
 {Knot[12, NonAlternating, 144], Knot[12, NonAlternating, 507] },
 {Knot[12, NonAlternating, 313], Knot[12, NonAlternating, 430] } }
```

In[ ]:= **Length /@ dup12**

Out[ ]=

- ```
{2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 2, 2, 2, 2, 2, 2, 2}
```

In[]:= **Total[(Length /@ dup12) - 1]**

Out[]=

19

In[]:= **Length /@ Select[Gather[tab12 /. {T1 -> 22 / 7, T2 -> 13 / 21}, Last[#1] === Last[#2] &], Length[#] > 1 &]**

Out[]=

- ```
{2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 2, 2, 2, 2, 2, 2, 2}
```

In[ ]:= **Put[tab12 /. {T1 -> T1, T2 -> T2}, "Data12.m"]**

## Ribbon Knots

In[\*]:= Table[K →  $\Theta$ [K],

{K, {Knot[6, 1], Knot[8, 8], Knot[8, 9], Knot[8, 20], Knot[9, 27], Knot[9, 41],  
 Knot[9, 46], Knot[10, 3], Knot[10, 22], Knot[10, 35], Knot[10, 42], Knot[10, 48],  
 Knot[10, 75], Knot[10, 87], Knot[10, 99], Knot[10, 123], Knot[10, 129],  
 Knot[10, 137], Knot[10, 140], Knot[10, 153], Knot[10, 155]}}

Out[\*]=

$$\left\{ \text{Knot}[6, 1] \rightarrow \left\{ -\frac{(-2 + T)(-1 + 2T)}{T}, \frac{1}{T_1^2 T_2^2} \left( 1 - 3T_1 + T_1^2 - 3T_2 + 6T_1 T_2 + 6T_1^2 T_2 - 3T_1^3 T_2 + T_2^2 + 6T_1 T_2^2 - 24T_1^2 T_2^2 + 6T_1^3 T_2^2 + T_1^4 T_2^2 - 3T_1 T_2^3 + 6T_1^2 T_2^3 + 6T_1^3 T_2^3 - 3T_1^4 T_2^3 + T_1^2 T_2^4 - 3T_1^3 T_2^4 + T_1^4 T_2^4 \right) \right\}, \right.$$

$$\text{Knot}[8, 8] \rightarrow \left\{ \frac{(2 - 2T + T^2)(1 - 2T + 2T^2)}{T^2}, \frac{1}{T_1^4 T_2^2} \left( 1 - 3T_1 + 5T_1^2 - 3T_1^3 + T_1^4 - 3T_2 + 6T_1 T_2 - 6T_1^2 T_2 - 6T_1^3 T_2 + 6T_1^4 T_2 - 3T_1^5 T_2 + 5T_2^2 - 6T_1 T_2^2 + 9T_1^2 T_2^2 + 5T_1^3 T_2^2 + 9T_1^4 T_2^2 - 6T_1^5 T_2^2 + 5T_1^6 T_2^2 - 3T_2^3 - 6T_1 T_2^3 + 5T_1^2 T_2^3 - 18T_1^3 T_2^3 - 18T_1^4 T_2^3 + 5T_1^5 T_2^3 - 6T_1^6 T_2^3 - 3T_1^7 T_2^3 + T_2^4 + 6T_1 T_2^4 + 9T_1^2 T_2^4 - 18T_1^3 T_2^4 + 60T_1^4 T_2^4 - 18T_1^5 T_2^4 + 9T_1^6 T_2^4 + 6T_1^7 T_2^4 + T_2^5 + 8T_1 T_2^5 - 3T_1^2 T_2^5 - 6T_1^3 T_2^5 + 5T_1^4 T_2^5 - 18T_1^5 T_2^5 - 18T_1^6 T_2^5 + 5T_1^7 T_2^5 - 6T_1^8 T_2^5 + 5T_1^9 T_2^5 - 6T_1^{10} T_2^5 + 9T_1^{11} T_2^5 - 6T_1^{12} T_2^5 + 5T_1^{13} T_2^5 - 3T_1^{14} T_2^5 + 6T_1^{15} T_2^5 - 6T_1^{16} T_2^5 + 6T_1^{17} T_2^5 - 3T_1^{18} T_2^5 + T_2^6 + 8T_1 T_2^6 - 3T_1^2 T_2^6 + 5T_1^3 T_2^6 - 6T_1^4 T_2^6 + 9T_1^5 T_2^6 - 6T_1^6 T_2^6 + 5T_1^7 T_2^6 - 6T_1^8 T_2^6 + 5T_1^9 T_2^6 - 3T_1^{10} T_2^6 + T_2^7 + 8T_1 T_2^7 - 3T_1^2 T_2^7 + 5T_1^3 T_2^7 - 6T_1^4 T_2^7 + 9T_1^5 T_2^7 - 6T_1^6 T_2^7 + 5T_1^7 T_2^7 - 6T_1^8 T_2^7 + 5T_1^9 T_2^7 - 3T_1^{10} T_2^7 + T_2^8 + 8T_1 T_2^8 - 3T_1^2 T_2^8 + 5T_1^3 T_2^8 - 6T_1^4 T_2^8 + 9T_1^5 T_2^8 - 6T_1^6 T_2^8 + 5T_1^7 T_2^8 - 6T_1^8 T_2^8 + 5T_1^9 T_2^8 - 3T_1^{10} T_2^8 + T_2^9 + 8T_1 T_2^9 - 3T_1^2 T_2^9 + 5T_1^3 T_2^9 - 6T_1^4 T_2^9 + 9T_1^5 T_2^9 - 6T_1^6 T_2^9 + 5T_1^7 T_2^9 - 6T_1^8 T_2^9 + 5T_1^9 T_2^9 - 3T_1^{10} T_2^9 + T_2^{10} \right) \right\},$$

$$\text{Knot}[8, 9] \rightarrow \left\{ -\frac{(-1 + T - 2T^2 + T^3)(-1 + 2T - T^2 + T^3)}{T^3}, \emptyset \right\},$$

$$\text{Knot}[8, 20] \rightarrow \left\{ \frac{(1 - T + T^2)^2}{T^2}, -\frac{1}{T_1^2 T_2^2} 2 \left( 3 - 4T_1 + 3T_1^2 - 4T_2 + T_1 T_2 + T_1^2 T_2 - 4T_1^3 T_2 + 3T_2^2 + T_1 T_2^2 + T_1^3 T_2^2 + 3T_1^4 T_2^2 - 4T_1 T_2^3 + T_1^2 T_2^3 + T_1^3 T_2^3 - 4T_1^4 T_2^3 + 3T_1^2 T_2^4 - 4T_1^3 T_2^4 + 3T_1^4 T_2^4 \right) \right\},$$

$$\text{Knot}[9, 27] \rightarrow \left\{ -\frac{(-1 + 2T - 3T^2 + T^3)(-1 + 3T - 2T^2 + T^3)}{T^3}, \right.$$

$$-\frac{1}{T_1^4 T_2^4} \left( 1 - T_1 + T_1^2 - T_1^3 + T_1^4 - T_2 - 8T_1 T_2 + 4T_1^2 T_2 + 4T_1^3 T_2 - 8T_1^4 T_2 - T_1^5 T_2 + T_2^2 + 4T_1 T_2^2 + 49T_1^2 T_2^2 - 67T_1^3 T_2^2 + 49T_1^4 T_2^2 + 4T_1^5 T_2^2 + T_1^6 T_2^2 - T_2^3 + 4T_1 T_2^3 - 67T_1^2 T_2^3 + 20T_1^3 T_2^3 + 20T_1^4 T_2^3 - 67T_1^5 T_2^3 + 4T_1^6 T_2^3 - T_1^7 T_2^3 + T_2^4 - 8T_1 T_2^4 + 49T_1^2 T_2^4 + 20T_1^3 T_2^4 - 12T_1^4 T_2^4 + 20T_1^5 T_2^4 + 49T_1^6 T_2^4 - 8T_1^7 T_2^4 + T_1^8 T_2^4 - T_1 T_2^5 + 4T_1^2 T_2^5 - 67T_1^3 T_2^5 + 20T_1^4 T_2^5 + 20T_1^5 T_2^5 - 67T_1^6 T_2^5 + 4T_1^7 T_2^5 - T_1^8 T_2^5 + T_1^2 T_2^6 + 4T_1^3 T_2^6 + 49T_1^4 T_2^6 - 67T_1^5 T_2^6 + 49T_1^6 T_2^6 + 4T_1^7 T_2^6 + T_1^8 T_2^6 - T_1^3 T_2^7 - 8T_1^4 T_2^7 + 4T_1^5 T_2^7 + 4T_1^6 T_2^7 - 8T_1^7 T_2^7 - T_1^8 T_2^7 + T_1^4 T_2^8 - T_1^5 T_2^8 + T_1^6 T_2^8 - T_1^7 T_2^8 + T_1^8 T_2^8 \right) \left. \right\}, \text{Knot}[9, 41] \rightarrow$$

$$\left\{ \frac{(3 - 3T + T^2)(1 - 3T + 3T^2)}{T^2}, -\frac{1}{T_1^4 T_2^4} \left( 3 - 15T_1 + 27T_1^2 - 15T_1^3 + 3T_1^4 - 15T_2 + 58T_1 T_2 - 56T_1^2 T_2 - 56T_1^3 T_2 + 58T_1^4 T_2 - 15T_1^5 T_2 + 27T_2^2 - 56T_1 T_2^2 - 81T_1^2 T_2^2 + 333T_1^3 T_2^2 - 81T_1^4 T_2^2 - 56T_1^5 T_2^2 + 27T_1^6 T_2^2 - 15T_2^3 - 56T_1 T_2^3 + 333T_1^2 T_2^3 - 396T_1^3 T_2^3 - 396T_1^4 T_2^3 + 333T_1^5 T_2^3 - 56T_1^6 T_2^3 - 15T_1^7 T_2^3 + 3T_2^4 + 58T_1 T_2^4 - 81T_1^2 T_2^4 - 396T_1^3 T_2^4 + 1188T_1^4 T_2^4 - 396T_1^5 T_2^4 - 81T_1^6 T_2^4 + 58T_1^7 T_2^4 + 3T_1^8 T_2^4 - 15T_1 T_2^5 - 56T_1^2 T_2^5 + 333T_1^3 T_2^5 - 396T_1^4 T_2^5 - 396T_1^5 T_2^5 + 333T_1^6 T_2^5 - 56T_1^7 T_2^5 - 15T_1^8 T_2^5 + \right.$$

$$27 T_1^2 T_2^6 - 56 T_1^3 T_2^6 - 81 T_1^4 T_2^6 + 333 T_1^5 T_2^6 - 81 T_1^6 T_2^6 - 56 T_1^7 T_2^6 + 27 T_1^8 T_2^6 - 15 T_1^3 T_2^7 + 58 T_1^4 T_2^7 - 56 T_1^5 T_2^7 - 56 T_1^6 T_2^7 + 58 T_1^7 T_2^7 - 15 T_1^8 T_2^7 + 3 T_1^4 T_2^8 - 15 T_1^5 T_2^8 + 27 T_1^6 T_2^8 - 15 T_1^7 T_2^8 + 3 T_1^8 T_2^8 \},$$

$$\text{Knot}[9, 46] \rightarrow \left\{ -\frac{(-2+T)(-1+2T)}{T}, \frac{1}{T_1^2 T_2^2} 3 \left( 1 - 3 T_1 + T_1^2 - 3 T_2 + 6 T_1 T_2 + 6 T_1^2 T_2 - 3 T_1^3 T_2 + T_2^2 + 6 T_1 T_2^2 - 24 T_1^2 T_2^2 + 6 T_1^3 T_2^2 + T_1^4 T_2^2 - 3 T_1 T_2^3 + 6 T_1^2 T_2^3 + 6 T_1^3 T_2^3 - 3 T_1^4 T_2^3 + T_1^2 T_2^4 - 3 T_1^3 T_2^4 + T_1^4 T_2^4 \right) \right\},$$

$$\text{Knot}[10, 3] \rightarrow \left\{ -\frac{(-3+2T)(-2+3T)}{T}, \frac{1}{T_1^2 T_2^2} \left( 45 - 101 T_1 + 45 T_1^2 - 101 T_2 + 126 T_1 T_2 + 126 T_1^2 T_2 - 101 T_1^3 T_2 + 45 T_2^2 + 126 T_1 T_2^2 - 420 T_1^2 T_2^2 + 126 T_1^3 T_2^2 + 45 T_1^4 T_2^2 - 101 T_1 T_2^3 + 126 T_1^2 T_2^3 + 126 T_1^3 T_2^3 - 101 T_1^4 T_2^3 + 45 T_1^2 T_2^4 - 101 T_1^3 T_2^4 + 45 T_1^4 T_2^4 \right) \right\},$$

$$\text{Knot}[10, 22] \rightarrow \left\{ -\frac{(-2+2T-2T^2+T^3)(-1+2T-2T^2+2T^3)}{T^3}, \right.$$

$$-\frac{1}{T_1^6 T_2^6} \left( 1 - 3 T_1 + 5 T_1^2 - 7 T_1^3 + 5 T_1^4 - 3 T_1^5 + T_1^6 - 3 T_2 + 6 T_1 T_2 - 6 T_1^2 T_2 + 6 T_1^3 T_2 + 6 T_1^4 T_2 - 6 T_1^5 T_2 + 6 T_1^6 T_2 - 3 T_1^7 T_2 + 5 T_1^8 T_2 - 6 T_1 T_2^2 + 3 T_1^2 T_2^2 - 9 T_1^3 T_2^2 - 5 T_1^4 T_2^2 - 9 T_1^5 T_2^2 + 3 T_1^6 T_2^2 - 6 T_1^7 T_2^2 + 5 T_1^8 T_2^2 - 7 T_2^3 + 6 T_1 T_2^3 - 9 T_1^2 T_2^3 + 30 T_1^3 T_2^3 + 4 T_1^4 T_2^3 + 4 T_1^5 T_2^3 + 30 T_1^6 T_2^3 - 9 T_1^7 T_2^3 + 6 T_1^8 T_2^3 - 7 T_1^9 T_2^3 + 5 T_2^4 + 6 T_1 T_2^4 - 5 T_1^2 T_2^4 + 4 T_1^3 T_2^4 - 89 T_1^4 T_2^4 + 63 T_1^5 T_2^4 - 89 T_1^6 T_2^4 + 4 T_1^7 T_2^4 - 5 T_1^8 T_2^4 + 6 T_1^9 T_2^4 + 5 T_1^{10} T_2^4 - 3 T_2^5 - 6 T_1 T_2^5 - 9 T_1^2 T_2^5 + 4 T_1^3 T_2^5 + 63 T_1^4 T_2^5 + 22 T_1^5 T_2^5 + 22 T_1^6 T_2^5 + 63 T_1^7 T_2^5 + 4 T_1^8 T_2^5 - 9 T_1^9 T_2^5 - 6 T_1^{10} T_2^5 - 3 T_1^{11} T_2^5 + T_2^6 + 6 T_1 T_2^6 + 3 T_1^2 T_2^6 + 30 T_1^3 T_2^6 - 89 T_1^4 T_2^6 + 22 T_1^5 T_2^6 - 108 T_1^6 T_2^6 + 22 T_1^7 T_2^6 - 89 T_1^8 T_2^6 + 30 T_1^9 T_2^6 + 3 T_1^{10} T_2^6 + 6 T_1^{11} T_2^6 + T_1^{12} T_2^6 - 3 T_1 T_2^7 - 6 T_1^2 T_2^7 - 9 T_1^3 T_2^7 + 4 T_1^4 T_2^7 + 63 T_1^5 T_2^7 + 22 T_1^6 T_2^7 + 22 T_1^7 T_2^7 + 63 T_1^8 T_2^7 + 4 T_1^9 T_2^7 - 9 T_1^{10} T_2^7 - 6 T_1^{11} T_2^7 - 3 T_1^{12} T_2^7 + 5 T_2^8 + 6 T_1 T_2^8 + 6 T_1^2 T_2^8 - 89 T_1^3 T_2^8 + 63 T_1^4 T_2^8 - 89 T_1^5 T_2^8 + 4 T_1^6 T_2^8 - 5 T_1^{10} T_2^8 + 6 T_1^{11} T_2^8 + 5 T_1^{12} T_2^8 - 7 T_2^9 + 6 T_1 T_2^9 + 4 T_1^2 T_2^9 - 9 T_1^3 T_2^9 + 30 T_1^4 T_2^9 + 4 T_1^5 T_2^9 + 4 T_1^6 T_2^9 + 30 T_1^7 T_2^9 - 9 T_1^8 T_2^9 + 6 T_1^9 T_2^9 + 6 T_1^{10} T_2^9 - 7 T_2^{10} + 5 T_1 T_2^{10} - 6 T_1^2 T_2^{10} + 3 T_1^3 T_2^{10} - 9 T_1^4 T_2^{10} - 5 T_1^5 T_2^{10} - 9 T_1^6 T_2^{10} + 3 T_1^7 T_2^{10} - 6 T_1^8 T_2^{10} + 5 T_1^9 T_2^{10} - 3 T_1^{10} T_2^{10} - 6 T_1^{11} T_2^{10} + 5 T_1^{12} T_2^{10} - 3 T_2^{11} + 6 T_1 T_2^{11} + 6 T_1^2 T_2^{11} - 6 T_1^3 T_2^{11} + 6 T_1^4 T_2^{11} - 6 T_1^5 T_2^{11} + 6 T_1^6 T_2^{11} - 6 T_1^7 T_2^{11} - 3 T_1^8 T_2^{11} - 3 T_1^9 T_2^{11} + T_2^{12} + T_1 T_2^{12} + 6 T_1^2 T_2^{12} - 3 T_1^3 T_2^{12} - 3 T_1^4 T_2^{12} + 5 T_1^5 T_2^{12} - 7 T_1^6 T_2^{12} + 5 T_1^7 T_2^{12} - 3 T_1^8 T_2^{12} - 3 T_1^9 T_2^{12} + T_1^{10} T_2^{12} - 3 T_1^{11} T_2^{12} + T_1^{12} T_2^{12} \right) \},$$

$$\text{Knot}[10, 35] \rightarrow \left\{ \frac{(2-4T+T^2)(1-4T+2T^2)}{T^2}, \frac{1}{T_1^4 T_2^4} \left( 1 - 7 T_1 + 13 T_1^2 - 7 T_1^3 + T_1^4 - 7 T_2 + 42 T_1 T_2 - 42 T_1^2 T_2 - 42 T_1^3 T_2 + 42 T_1^4 T_2 - 7 T_1^5 T_2 + 13 T_2^2 - 42 T_1 T_2^2 - 148 T_1^2 T_2^2 + 426 T_1^3 T_2^2 - 148 T_1^4 T_2^2 - 42 T_1^5 T_2^2 + 13 T_1^6 T_2^2 - 7 T_2^3 - 42 T_1 T_2^3 + 426 T_1^2 T_2^3 - 468 T_1^3 T_2^3 - 468 T_1^4 T_2^3 + 426 T_1^5 T_2^3 - 42 T_1^6 T_2^3 - 7 T_1^7 T_2^3 + T_2^4 + 42 T_1 T_2^4 - 148 T_1^2 T_2^4 - 468 T_1^3 T_2^4 + 1392 T_1^4 T_2^4 - 468 T_1^5 T_2^4 - 148 T_1^6 T_2^4 + 42 T_1^7 T_2^4 + T_1^8 T_2^4 - 7 T_1 T_2^5 - 42 T_1^2 T_2^5 + 426 T_1^3 T_2^5 - 468 T_1^4 T_2^5 - 468 T_1^5 T_2^5 + 426 T_1^6 T_2^5 - 42 T_1^7 T_2^5 - 7 T_1^8 T_2^5 + 13 T_1^9 T_2^5 - 42 T_1^3 T_2^6 - 148 T_1^4 T_2^6 + 426 T_1^5 T_2^6 - 148 T_1^6 T_2^6 - 42 T_1^7 T_2^6 + 13 T_1^8 T_2^6 - 7 T_1^3 T_2^7 + 42 T_1^4 T_2^7 - 42 T_1^5 T_2^7 + 42 T_1^6 T_2^7 - 7 T_1^8 T_2^7 + T_1^4 T_2^8 - 7 T_1^5 T_2^8 + 13 T_1^6 T_2^8 - 7 T_1^7 T_2^8 + T_1^8 T_2^8 \right) \},$$

$$\text{Knot}[10, 42] \rightarrow \left\{ -\frac{(-1+3T-4T^2+T^3)(-1+4T-3T^2+T^3)}{T^3}, \right.$$

$$-\frac{1}{T_1^4 T_2^4} \left( 6 - 24 T_1 + 38 T_1^2 - 24 T_1^3 + 6 T_1^4 - 24 T_2 + 72 T_1 T_2 - 54 T_1^2 T_2 - 54 T_1^3 T_2 + 72 T_1^4 T_2 - 24 T_1^5 T_2 + 38 T_2^2 - 54 T_1 T_2^2 - 109 T_1^2 T_2^2 + 279 T_1^3 T_2^2 - 109 T_1^4 T_2^2 - 54 T_1^5 T_2^2 + 38 T_1^6 T_2^2 - 24 T_2^3 - 54 T_1 T_2^3 + 279 T_1^2 T_2^3 - 222 T_1^3 T_2^3 - 222 T_1^4 T_2^3 + 279 T_1^5 T_2^3 - 54 T_1^6 T_2^3 - 24 T_1^7 T_2^3 + 6 T_2^4 + 72 T_1 T_2^4 - 109 T_1^2 T_2^4 - 222 T_1^3 T_2^4 + 552 T_1^4 T_2^4 - 222 T_1^5 T_2^4 - 109 T_1^6 T_2^4 + 72 T_1^7 T_2^4 + 6 T_1^8 T_2^4 - 24 T_1 T_2^5 - 54 T_1^2 T_2^5 + 279 T_1^3 T_2^5 - 222 T_1^4 T_2^5 - 222 T_1^5 T_2^5 + 279 T_1^6 T_2^5 - 54 T_1^7 T_2^5 - 24 T_1^8 T_2^5 + 38 T_1^9 T_2^5 - 24 T_1^2 T_2^6 - 109 T_1^3 T_2^6 + 279 T_1^4 T_2^6 - 109 T_1^5 T_2^6 - 54 T_1^6 T_2^6 + 38 T_1^7 T_2^6 - 24 T_1^8 T_2^6 + 38 T_1^9 T_2^6 - 24 T_1 T_2^7 - 109 T_1^2 T_2^7 + 279 T_1^3 T_2^7 - 109 T_1^4 T_2^7 - 54 T_1^5 T_2^7 + 38 T_1^6 T_2^7 - 24 T_1^7 T_2^7 + 6 T_2^8 + 72 T_1 T_2^8 - 109 T_1^2 T_2^8 - 222 T_1^3 T_2^8 + 552 T_1^4 T_2^8 - 222 T_1^5 T_2^8 - 109 T_1^6 T_2^8 + 72 T_1^7 T_2^8 + 6 T_1^8 T_2^8 - 24 T_1 T_2^9 - 54 T_1^2 T_2^9 + 279 T_1^3 T_2^9 - 222 T_1^4 T_2^9 - 222 T_1^5 T_2^9 + 279 T_1^6 T_2^9 - 54 T_1^7 T_2^9 - 24 T_1^8 T_2^9 + 38 T_1^9 T_2^9 - 24 T_1 T_2^{10} - 109 T_1^2 T_2^{10} + 279 T_1^3 T_2^{10} - 109 T_1^4 T_2^{10} - 54 T_1^5 T_2^{10} + 38 T_1^6 T_2^{10} - 24 T_1^7 T_2^{10} + 6 T_2^{11} + 72 T_1 T_2^{11} - 109 T_1^2 T_2^{11} - 222 T_1^3 T_2^{11} + 552 T_1^4 T_2^{11} - 222 T_1^5 T_2^{11} - 109 T_1^6 T_2^{11} + 72 T_1^7 T_2^{11} + 6 T_1^8 T_2^{11} - 24 T_1 T_2^{12} - 54 T_1^2 T_2^{12} + 279 T_1^3 T_2^{12} - 222 T_1^4 T_2^{12} - 222 T_1^5 T_2^{12} + 279 T_1^6 T_2^{12} - 54 T_1^7 T_2^{12} - 24 T_1^8 T_2^{12} + 38 T_1^9 T_2^{12} - 24 T_1 T_2^{13} - 109 T_1^2 T_2^{13} + 279 T_1^3 T_2^{13} - 109 T_1^4 T_2^{13} - 54 T_1^5 T_2^{13} + 38 T_1^6 T_2^{13} - 24 T_1^7 T_2^{13} + 6 T_2^{14} + 72 T_1 T_2^{14} - 109 T_1^2 T_2^{14} - 222 T_1^3 T_2^{14} + 552 T_1^4 T_2^{14} - 222 T_1^5 T_2^{14} - 109 T_1^6 T_2^{14} + 72 T_1^7 T_2^{14} + 6 T_1^8 T_2^{14} - 24 T_1 T_2^{15} - 54 T_1^2 T_2^{15} + 279 T_1^3 T_2^{15} - 222 T_1^4 T_2^{15} - 222 T_1^5 T_2^{15} + 279 T_1^6 T_2^{15} - 54 T_1^7 T_2^{15} - 24 T_1^8 T_2^{15} + 38 T_1^9 T_2^{15} - 24 T_1 T_2^{16} - 109 T_1^2 T_2^{16} + 279 T_1^3 T_2^{16} - 109 T_1^4 T_2^{16} - 54 T_1^5 T_2^{16} + 38 T_1^6 T_2^{16} - 24 T_1^7 T_2^{16} + 6 T_2^{17} + 72 T_1 T_2^{17} - 109 T_1^2 T_2^{17} - 222 T_1^3 T_2^{17} + 552 T_1^4 T_2^{17} - 222 T_1^5 T_2^{17} - 109 T_1^6 T_2^{17} + 72 T_1^7 T_2^{17} + 6 T_1^8 T_2^{17} - 24 T_1 T_2^{18} - 54 T_1^2 T_2^{18} + 279 T_1^3 T_2^{18} - 222 T_1^4 T_2^{18} - 222 T_1^5 T_2^{18} + 279 T_1^6 T_2^{18} - 54 T_1^7 T_2^{18} - 24 T_1^8 T_2^{18} + 38 T_1^9 T_2^{18} - 24 T_1 T_2^{19} - 109 T_1^2 T_2^{19} + 279 T_1^3 T_2^{19} - 109 T_1^4 T_2^{19} - 54 T_1^5 T_2^{19} + 38 T_1^6 T_2^{19} - 24 T_1^7 T_2^{19} + 6 T_2^{20} + 72 T_1 T_2^{20} - 109 T_1^2 T_2^{20} - 222 T_1^3 T_2^{20} + 552 T_1^4 T_2^{20} - 222 T_1^5 T_2^{20} - 109 T_1^6 T_2^{20} + 72 T_1^7 T_2^{20} + 6 T_1^8 T_2^{20} - 24 T_1 T_2^{21} - 54 T_1^2 T_2^{21} + 279 T_1^3 T_2^{21} - 222 T_1^4 T_2^{21} - 222 T_1^5 T_2^{21} + 279 T_1^6 T_2^{21} - 54 T_1^7 T_2^{21} - 24 T_1^8 T_2^{21} + 38 T_1^9 T_2^{21} - 24 T_1 T_2^{22} - 109 T_1^2 T_2^{22} + 279 T_1^3 T_2^{22} - 109 T_1^4 T_2^{22} - 54 T_1^5 T_2^{22} + 38 T_1^6 T_2^{22} - 24 T_1^7 T_2^{22} + 6 T_2^{23} + 72 T_1 T_2^{23} - 109 T_1^2 T_2^{23} - 222 T_1^3 T_2^{23} + 552 T_1^4 T_2^{23} - 222 T_1^5 T_2^{23} - 109 T_1^6 T_2^{23} + 72 T_1^7 T_2^{23} + 6 T_1^8 T_2^{23} - 24 T_1 T_2^{24} - 54 T_1^2 T_2^{24} + 279 T_1^3 T_2^{24} - 222 T_1^4 T_2^{24} - 222 T_1^5 T_2^{24} + 279 T_1^6 T_2^{24} - 54 T_1^7 T_2^{24} - 24 T_1^8 T_2^{24} + 38 T_1^9 T_2^{24} - 24 T_1 T_2^{25} - 109 T_1^2 T_2^{25} + 279 T_1^3 T_2^{25} - 109 T_1^4 T_2^{25} - 54 T_1^5 T_2^{25} + 38 T_1^6 T_2^{25} - 24 T_1^7 T_2^{25} + 6 T_2^{26} + 72 T_1 T_2^{26} - 109 T_1^2 T_2^{26} - 222 T_1^3 T_2^{26} + 552 T_1^4 T_2^{26} - 222 T_1^5 T_2^{26} - 109 T_1^6 T_2^{26} + 72 T_1^7 T_2^{26} + 6 T_1^8 T_2^{26} - 24 T_1 T_2^{27} - 54 T_1^2 T_2^{27} + 279 T_1^3 T_2^{27} - 222 T_1^4 T_2^{27} - 222 T_1^5 T_2^{27} + 279 T_1^6 T_2^{27} - 54 T_1^7 T_2^{27} - 24 T_1^8 T_2^{27} + 38 T_1^9 T_2^{27} - 24 T_1 T_2^{28} - 109 T_1^2 T_2^{28} + 279 T_1^3 T_2^{28} - 109 T_1^4 T_2^{28} - 54 T_1^5 T_2^{28} + 38 T_1^6 T_2^{28} - 24 T_1^7 T_2^{28} + 6 T_2^{29} + 72 T_1 T_2^{29} - 109 T_1^2 T_2^{29} - 222 T_1^3 T_2^{29} + 552 T_1^4 T_2^{29} - 222 T_1^5 T_2^{29} - 109 T_1^6 T_2^{29} + 72 T_1^7 T_2^{29} + 6 T_1^8 T_2^{29} - 24 T_1 T_2^{30} - 54 T_1^2 T_2^{30} + 279 T_1^3 T_2^{30} - 222 T_1^4 T_2^{30} - 222 T_1^5 T_2^{30} + 279 T_1^6 T_2^{30} - 54 T_1^7 T_2^{30} - 24 T_1^8 T_2^{30} + 38 T_1^9 T_2^{30} - 24 T_1 T_2^{31} - 109 T_1^2 T_2^{31} + 279 T_1^3 T_2^{31} - 109 T_1^4 T_2^{31} - 54 T_1^5 T_2^{31} + 38 T_1^6 T_2^{31} - 24 T_1^7 T_2^{31} + 6 T_2^{32} + 72 T_1 T_2^{32} - 109 T_1^2 T_2^{32} - 222 T_1^3 T_2^{32} + 552 T_1^4 T_2^{32} - 222 T_1^5 T_2^{32} - 109 T_1^6 T_2^{32} + 72 T_1^7 T_2^{32} + 6 T_1^8 T_2^{32} - 24 T_1 T_2^{33} - 54 T_1^2 T_2^{33} + 279 T_1^3 T_2^{33} - 222 T_1^4 T_2^{33} - 222 T_1^5 T_2^{33} + 279 T_1^6 T_2^{33} - 54 T_1^7 T_2^{33} - 24 T_1^8 T_2^{33} + 38 T_1^9 T_2^{33} - 24 T_1 T_2^{34} - 109 T_1^2 T_2^{34} + 279 T_1^3 T_2^{34} - 109 T_1^4 T_2^{34} - 54 T_1^5 T_2^{34} + 38 T_1^6 T_2^{34} - 24 T_1^7 T_2^{34} + 6 T_2^{35} + 72 T_1 T_2^{35} - 109 T_1^2 T_2^{35} - 222 T_1^3 T_2^{35} + 552 T_1^4 T_2^{35} - 222 T_1^5 T_2^{35} - 109 T_1^6 T_2^{35} + 72 T_1^7 T_2^{35} + 6 T_1^8 T_2^{35} - 24 T_1 T_2^{36} - 54 T_1^2 T_2^{36} + 279 T_1^3 T_2^{36} - 222 T_1^4 T_2^{36} - 222 T_1^5 T_2^{36} + 279 T_1^6 T_2^{36} - 54 T_1^7 T_2^{36} - 24 T_1^8 T_2^{36} + 38 T_1^9 T_2^{36} - 24 T_1 T_2^{37} - 109 T_1^2 T_2^{37} + 279 T_1^3 T_2^{37} - 109 T_1^4 T_2^{37} - 54 T_1^5 T_2^{37} + 38 T_1^6 T_2^{37} - 24 T_1^7 T_2^{37} + 6 T_2^{38} + 72 T_1 T_2^{38} - 109 T_1^2 T_2^{38} - 222 T_1^3 T_2^{38} + 552 T_1^4 T_2^{38} - 222 T_1^5 T_2^{38} - 109 T_1^6 T_2^{38} + 72 T_1^7 T_2^{38} + 6 T_1^8 T_2^{38} - 24 T_1 T_2^{39} - 54 T_1^2 T_2^{39} + 279 T_1^3 T_2^{39} - 222 T_1^4 T_2^{39} - 222 T_1^5 T_2^{39} + 279 T_1^6 T_2^{39} - 54 T_1^7 T_2^{39} - 24 T_1^8 T_2^{39} + 38 T_1^9 T_2^{39} - 24 T_1 T_2^{40} - 109 T_1^2 T_2^{40} + 279 T_1^3 T_2^{40} - 109 T_1^4 T_2^{40} - 54 T_1^5 T_2^{40} + 38 T_1^6 T_2^{40} - 24 T_1^7 T_2^{40} + 6 T_2^{41} + 72 T_1 T_2^{41} - 109 T_1^2 T_2^{41} - 222 T_1^3 T_2^{41} + 552 T_1^4 T_2^{41} - 222 T_1^5 T_2^{41} - 109 T_1^6 T_2^{41} + 72 T_1^7 T_2^{41} + 6 T_1^8 T_2^{41} - 24 T_1 T_2^{42} - 54 T_1^2 T_2^{42} + 279 T_1^3 T_2^{42} - 222 T_1^4 T_2^{42} - 222 T_1^5 T_2^{42} + 279 T_1^6 T_2^{42} - 54 T_1^7 T_2^{42} - 24 T_1^8 T_2^{42} + 38 T_1^9 T_2^{42} - 24 T_1 T_2^{43} - 109 T_1^2 T_2^{43} + 279 T_1^3 T_2^{43} - 109 T_1^4 T_2^{43} - 54 T_1^5 T_2^{43} + 38 T_1^6 T_2^{43} - 24 T_1^7 T_2^{43} + 6 T_2^{44} + 72 T_1 T_2^{44} - 109 T_1^2 T_2^{44} - 222 T_1^3 T_2^{44} + 552 T_1^4 T_2^{44} - 222 T_1^5 T_2^{44} - 109 T_1^6 T_2^{44} + 72 T_1^7 T_2^{44} + 6 T_1^8 T_2^{44} - 24 T_1 T_2^{45} - 54 T_1^2 T_2^{45} + 279 T_1^3 T_2^{45} - 222 T_1^4 T_2^{45} - 222 T_1^5 T_2^{45} + 279 T_1^6 T_2^{45} - 54 T_1^7 T_2^{45} - 24 T_1^8 T_2^{45} + 38 T_1^9 T_2^{45} - 24 T_1 T_2^{46} - 109 T_1^2 T_2^{46} + 279 T_1^3 T_2^{46} - 109 T_1^4 T_2^{46} - 54 T_1^5 T_2^{46} + 38 T_1^6 T_2^{46} - 24 T_1^7 T_2^{46} + 6 T_2^{47} + 72 T_1 T_2^{47} - 109 T_1^2 T_2^{47} - 222 T_1^3 T_2^{47} + 552 T_1^4 T_2^{47} - 222 T_1^5 T_2^{47} - 109 T_1^6 T_2^{47} + 72 T_1^7 T_2^{47} + 6 T_1^8 T_2^{47} - 24 T_1 T_2^{48} - 54 T_1^2 T_2^{48} + 279 T_1^3 T_2^{48} - 222 T_1^4 T_2^{48} - 222 T_1^5 T_2^{48} + 279 T_1^6 T_2^{48} - 54 T_1^7 T_2^{48} - 24 T_1^8 T_2^{48} + 38 T_1^9 T_2^{48} - 24 T_1 T_2^{49} - 109 T_1^2 T_2^{49} + 279 T_1^3 T_2^{49} - 109 T_1^4 T_2^{49} - 54 T_1^5 T_2^{49} + 38 T_1^6 T_2^{49} - 24 T_1^7 T_2^{49} + 6 T_2^{50} + 72 T_1 T_2^{50} - 109 T_1^2 T_2^{50} - 222 T_1^3 T_2^{50} + 552 T_1^4 T_2^{50} - 222 T_1^5 T_2^{50} - 109 T_1^6 T_2^{50} + 72 T_1^7 T_2^{50} + 6 T_1^8 T_2^{50} - 24 T_1 T_2^{51} - 54 T_1^2 T_2^{51} + 279 T_1^3 T_2^{51} - 222 T_1^4 T_2^{51} - 222 T_1^5 T_2^{51} + 279 T_1^6 T_2^{51} - 54 T_1^7 T_2^{51} - 24 T_1^8 T_2^{51} + 38 T_1^9 T_2^{51} - 24 T_1 T_2^{52} - 109 T_1^2 T_2^{52} + 279 T_1^3 T_2^{52} - 109 T_1^4 T_2^{52} - 54 T_1^5 T_2^{52} + 38 T_1^6 T_2^{52} - 24 T_1^7 T_2^{52} + 6 T_2^{53} + 72 T_1 T_2^{53} - 109 T_1^2 T_2^{53} - 222 T_1^3 T_2^{53} + 552 T_1^4 T_2^{53} - 222 T_1^5 T_2^{53} - 109 T_1^6 T_2^{53} + 72 T_1^7 T_2^{53} + 6 T_1^8 T_2^{53} - 24 T_1 T_2^{54} - 54 T_1^2 T_2^{54} + 279 T_1^3 T_2^{54} - 222 T_1^4 T_2^{54} - 222 T_1^5 T_2^{54} + 279 T_1^6 T_2^{54} - 54 T_1^7 T_2^{54} - 24 T_1^8 T_2^{54} + 38 T_1^9 T_2^{54} - 24 T_1 T_2^{55} - 109 T_1^2 T_2^{55} + 279 T_1^3 T_2^{55} - 109 T_1^4 T_2^{55} - 54 T_1^5 T_2^{55} + 38 T_1^6 T_2^{55} - 24 T_1^7 T_2^{55} + 6 T_2^{56} + 72 T_1 T_2^{56} - 109 T_1^2 T_2^{56} - 222 T_1^3 T_2^{56} + 552 T_1^4 T_2^{56} - 222 T_1^5 T_2^{56} - 109 T_1^6 T_2^{56} + 72 T_1^7 T_2^{56} + 6 T_1^8 T_2^{56} - 24 T_1 T_2^{57} - 54 T_1^2 T_2^{57} + 279 T_1^3 T_2^{57} - 222 T_1^4 T_2^{57} - 222 T_1^5 T_2^{57} + 279 T_1^6 T_2^{57} - 54 T_1^7 T_2^{57} - 24 T_1^8 T_2^{57} + 38 T_1^9 T_2^{57} - 24 T_1 T_2^{58} - 109 T_1^2 T_2^{58} + 279 T_1^3 T_2^{58} - 109 T_1^4 T_2^{58} - 54 T_1^5 T_2^{58} + 38 T_1^6 T_2^{58} - 24 T_1^7 T_2^{58} + 6 T_2^{59} + 72 T_1 T_2^{59} - 109 T_1^2 T_2^{59} - 222 T_1^3 T_2^{59} + 552 T_1^4 T_2^{59} - 222 T_1^5 T_2^{59} - 109 T_1^6 T_2^{59} + 72 T_1^7 T_2^{59} + 6 T_1^8 T_2^{59} - 24 T_1 T_2^{60} - 54 T_1^2 T_2^{60} + 279 T_1^3 T_2^{60} - 222 T_1^4 T_2^{60} - 222 T_1^5 T_2^{60} + 279 T_1^6 T_2^{60} - 54 T_1^7 T_2^{60} - 24 T_1^8 T_2^{60} + 38 T_1^9 T_2^{60} - 24 T_1 T_2^{61} - 109 T_1^2 T_2^{61} + 279 T_1^3 T_2^{61} - 109 T_1^4 T_2^{61} - 54 T_1^5 T_2^{61} + 38 T_1^6 T_2^{61} - 24 T_1^7 T_2^{61} + 6 T_2^{62} + 72 T_1 T_2^{62} - 109 T_1^2 T_2^{62} - 222 T_1^3 T_2^{62} + 552 T_1^4 T_2^{62} - 222 T_1^5 T_2^{62} - 109 T_1^6 T_2^{62} + 72 T_1^7 T_2^{62} + 6 T_1^8 T_2^{62} - 24 T_1 T_2^{63} - 54 T_1^2 T_2^{63} + 279 T_1^3 T_2^{63} - 222 T_1^4 T_2^{63} - 222 T_1^5 T_2^{63} + 279 T_1^6 T_2^{63} - 54 T_1^7 T_2^{63} - 24 T_1^8 T_2^{63} + 38 T_1^9 T_2^{63} - 24 T_1 T_2^{64} - 109 T_1^2 T_2^{64} + 279 T_1^3 T_2^{64} - 109 T_1^4 T_2^{64} - 54 T_1^5 T_2^{64} + 38 T_1^6 T_2^{64} - 24 T_1^7 T_2^{64} + 6 T_2^{65} + 72 T_1 T_2^{65} - 109 T_1^2 T_2^{65} - 222 T_1^3 T_2^{65} + 552 T_1^4 T_2^{65} - 222 T_1^5 T_2^{65} - 109 T_1^6 T_2^{65} + 72 T_1^7 T_2^{65} + 6 T_$$



$$\begin{aligned}
 & 54 T_1^3 T_2^6 - 109 T_1^4 T_2^6 + 279 T_1^5 T_2^6 - 109 T_1^6 T_2^6 - 54 T_1^7 T_2^6 + 38 T_1^8 T_2^6 - 24 T_1^3 T_2^7 + 72 T_1^4 T_2^7 - \\
 & 54 T_1^5 T_2^7 - 54 T_1^6 T_2^7 + 72 T_1^7 T_2^7 - 24 T_1^8 T_2^7 + 6 T_1^4 T_2^8 - 24 T_1^5 T_2^8 + 38 T_1^6 T_2^8 - 24 T_1^7 T_2^8 + 6 T_1^8 T_2^8 \} , \\
 \text{Knot [10, 48]} \rightarrow & \left\{ \frac{(1 - T + 2 T^2 - 2 T^3 + T^4) (1 - 2 T + 2 T^2 - T^3 + T^4)}{T^4}, \right. \\
 & - \frac{1}{T_1^6 T_2^6} (1 - 2 T_1 + 2 T_1^2 - T_1^3 + 2 T_1^4 - 2 T_1^5 + T_1^6 - 2 T_2 + 2 T_1 T_2 + T_1^2 T_2 - 3 T_1^3 T_2 - 3 T_1^4 T_2 + T_1^5 T_2 + 2 T_1^6 T_2 - \\
 & 2 T_1^7 T_2 + 2 T_1^2 T_2^2 + T_1 T_2^2 - 9 T_1^2 T_2^2 + 9 T_1^3 T_2^2 + T_1^4 T_2^2 + 9 T_1^5 T_2^2 - 9 T_1^6 T_2^2 + T_1^7 T_2^2 + 2 T_1^8 T_2^2 - T_2^3 - 3 T_1 T_2^3 + \\
 & 9 T_1^2 T_2^3 + 2 T_1^3 T_2^3 - 10 T_1^4 T_2^3 - 10 T_1^5 T_2^3 + 2 T_1^6 T_2^3 + 9 T_1^7 T_2^3 - 3 T_1^8 T_2^3 - T_1^9 T_2^3 + 2 T_2^4 - 3 T_1 T_2^4 + T_1^2 T_2^4 - \\
 & 10 T_1^3 T_2^4 - 5 T_1^4 T_2^4 + 29 T_1^5 T_2^4 - 5 T_1^6 T_2^4 - 10 T_1^7 T_2^4 + T_1^8 T_2^4 - 3 T_1^9 T_2^4 + 2 T_1^{10} T_2^4 - 2 T_2^5 + T_1 T_2^5 + 9 T_1^2 T_2^5 - \\
 & 10 T_1^3 T_2^5 + 29 T_1^4 T_2^5 - 22 T_1^5 T_2^5 - 22 T_1^6 T_2^5 + 29 T_1^7 T_2^5 - 10 T_1^8 T_2^5 + 9 T_1^9 T_2^5 + T_1^{10} T_2^5 - 2 T_1^{11} T_2^5 + T_2^6 + \\
 & 2 T_1 T_2^6 - 9 T_1^2 T_2^6 + 2 T_1^3 T_2^6 - 5 T_1^4 T_2^6 - 22 T_1^5 T_2^6 + 48 T_1^6 T_2^6 - 22 T_1^7 T_2^6 - 5 T_1^8 T_2^6 + 2 T_1^9 T_2^6 - 9 T_1^{10} T_2^6 + \\
 & 2 T_1^{11} T_2^6 + T_2^7 - 2 T_1 T_2^7 + T_1^2 T_2^7 + 9 T_1^3 T_2^7 - 10 T_1^4 T_2^7 + 29 T_1^5 T_2^7 - 22 T_1^6 T_2^7 - 22 T_1^7 T_2^7 + 29 T_1^8 T_2^7 - \\
 & 10 T_1^9 T_2^7 + 9 T_1^{10} T_2^7 + T_1^{11} T_2^7 - 2 T_1^{12} T_2^7 + 2 T_1^2 T_2^8 - 3 T_1^3 T_2^8 + T_1^4 T_2^8 - 10 T_1^5 T_2^8 - 5 T_1^6 T_2^8 + 29 T_1^7 T_2^8 - \\
 & 5 T_1^8 T_2^8 - 10 T_1^9 T_2^8 + T_1^{10} T_2^8 - 3 T_1^{11} T_2^8 + 2 T_1^{12} T_2^8 - T_2^9 - 3 T_1 T_2^9 + 9 T_1^2 T_2^9 + 2 T_1^6 T_2^9 - 10 T_1^7 T_2^9 - \\
 & 10 T_1^8 T_2^9 + 2 T_1^9 T_2^9 + 9 T_1^{10} T_2^9 - 3 T_1^{11} T_2^9 - T_1^{12} T_2^9 + 2 T_1^4 T_2^{10} + T_1^5 T_2^{10} - 9 T_1^6 T_2^{10} + 9 T_1^7 T_2^{10} + T_1^8 T_2^{10} + \\
 & 9 T_1^9 T_2^{10} - 9 T_1^{10} T_2^{10} + T_1^{11} T_2^{10} + 2 T_1^{12} T_2^{10} - 2 T_1^5 T_2^{11} + 2 T_1^6 T_2^{11} + T_1^7 T_2^{11} - 3 T_1^8 T_2^{11} - 3 T_1^9 T_2^{11} + T_1^{10} T_2^{11} + \\
 & 2 T_1^{11} T_2^{11} - 2 T_1^{12} T_2^{11} + T_1^6 T_2^{12} - 2 T_1^7 T_2^{12} + 2 T_1^8 T_2^{12} - T_1^9 T_2^{12} + 2 T_1^{10} T_2^{12} - 2 T_1^{11} T_2^{12} + T_1^{12} T_2^{12}) \} , \\
 \text{Knot [10, 75]} \rightarrow & \left\{ - \frac{(-1 + 3 T - 4 T^2 + T^3) (-1 + 4 T - 3 T^2 + T^3)}{T^3}, \right.
 \end{aligned}$$

$$\begin{aligned}
 & - \frac{1}{T_1^4 T_2^4} (2 - 8 T_1 + 16 T_1^2 - 8 T_1^3 + 2 T_1^4 - 8 T_2 + 18 T_1 T_2 - 32 T_1^2 T_2 - 32 T_1^3 T_2 + 18 T_1^4 T_2 - \\
 & 8 T_1^5 T_2 + 16 T_2^2 - 32 T_1 T_2^2 + 75 T_1^2 T_2^2 + 75 T_1^3 T_2^2 + 75 T_1^4 T_2^2 - 32 T_1^5 T_2^2 + 16 T_1^6 T_2^2 - 8 T_2^3 - \\
 & 32 T_1 T_2^3 + 75 T_1^2 T_2^3 - 256 T_1^3 T_2^3 - 256 T_1^4 T_2^3 + 75 T_1^5 T_2^3 - 32 T_1^6 T_2^3 - 8 T_1^7 T_2^3 + 2 T_2^4 + 18 T_1 T_2^4 + \\
 & 75 T_1^2 T_2^4 - 256 T_1^3 T_2^4 + 900 T_1^4 T_2^4 - 256 T_1^5 T_2^4 + 75 T_1^6 T_2^4 + 18 T_1^7 T_2^4 + 2 T_1^8 T_2^4 - 8 T_1 T_2^5 - \\
 & 32 T_1^2 T_2^5 + 75 T_1^3 T_2^5 - 256 T_1^4 T_2^5 - 256 T_1^5 T_2^5 + 75 T_1^6 T_2^5 - 32 T_1^7 T_2^5 - 8 T_1^8 T_2^5 + 16 T_1^2 T_2^6 - \\
 & 32 T_1^3 T_2^6 + 75 T_1^4 T_2^6 + 75 T_1^5 T_2^6 + 75 T_1^6 T_2^6 - 32 T_1^7 T_2^6 + 16 T_1^8 T_2^6 - 8 T_1^3 T_2^7 + 18 T_1^4 T_2^7 - \\
 & 32 T_1^5 T_2^7 - 32 T_1^6 T_2^7 + 18 T_1^7 T_2^7 - 8 T_1^8 T_2^7 + 2 T_1^4 T_2^8 - 8 T_1^5 T_2^8 + 16 T_1^6 T_2^8 - 8 T_1^7 T_2^8 + 2 T_1^8 T_2^8) \} ,
 \end{aligned}$$

$$\begin{aligned}
 \text{Knot [10, 87]} \rightarrow & \left\{ - \frac{(-2 + T) (-1 + 2 T) (1 - T + T^2)^2}{T^3}, \right. \\
 & - \frac{1}{T_1^6 T_2^6} (1 - 4 T_1 + 8 T_1^2 - 11 T_1^3 + 8 T_1^4 - 4 T_1^5 + T_1^6 - 4 T_2 + 12 T_1 T_2 - 16 T_1^2 T_2 + 12 T_1^3 T_2 + 12 T_1^4 T_2 - \\
 & 16 T_1^5 T_2 + 12 T_1^6 T_2 - 4 T_1^7 T_2 + 8 T_2^2 - 16 T_1 T_2^2 + 12 T_1^2 T_2^2 - 13 T_1^3 T_2^2 - 18 T_1^4 T_2^2 - 13 T_1^5 T_2^2 + \\
 & 12 T_1^6 T_2^2 - 16 T_1^7 T_2^2 + 8 T_1^8 T_2^2 - 11 T_2^3 + 12 T_1 T_2^3 - 13 T_1^2 T_2^3 + 62 T_1^3 T_2^3 + 9 T_1^4 T_2^3 + 9 T_1^5 T_2^3 + \\
 & 62 T_1^6 T_2^3 - 13 T_1^7 T_2^3 + 12 T_1^8 T_2^3 - 11 T_1^9 T_2^3 + 8 T_2^4 + 12 T_1 T_2^4 - 18 T_1^2 T_2^4 + 9 T_1^3 T_2^4 - 296 T_1^4 T_2^4 + \\
 & 290 T_1^5 T_2^4 - 296 T_1^6 T_2^4 + 9 T_1^7 T_2^4 - 18 T_1^8 T_2^4 + 12 T_1^9 T_2^4 + 8 T_1^{10} T_2^4 - 4 T_2^5 - 16 T_1 T_2^5 - 13 T_1^2 T_2^5 + \\
 & 9 T_1^3 T_2^5 + 290 T_1^4 T_2^5 - 32 T_1^5 T_2^5 - 32 T_1^6 T_2^5 + 290 T_1^7 T_2^5 + 9 T_1^8 T_2^5 - 13 T_1^9 T_2^5 - 16 T_1^{10} T_2^5 - 4 T_1^{11} T_2^5 + \\
 & T_2^6 + 12 T_1 T_2^6 + 12 T_1^2 T_2^6 + 62 T_1^3 T_2^6 - 296 T_1^4 T_2^6 - 32 T_1^5 T_2^6 - 72 T_1^6 T_2^6 - 32 T_1^7 T_2^6 - 296 T_1^8 T_2^6 + \\
 & 62 T_1^9 T_2^6 + 12 T_1^{10} T_2^6 + 12 T_1^{11} T_2^6 + T_2^7 - 4 T_1 T_2^7 - 16 T_1^2 T_2^7 - 13 T_1^3 T_2^7 + 9 T_1^4 T_2^7 + 290 T_1^5 T_2^7 - \\
 & 32 T_1^6 T_2^7 - 32 T_1^7 T_2^7 + 290 T_1^8 T_2^7 + 9 T_1^9 T_2^7 - 13 T_1^{10} T_2^7 - 16 T_1^{11} T_2^7 - 4 T_1^{12} T_2^7 + 8 T_1^2 T_2^8 + 12 T_1^3 T_2^8 - \\
 & 18 T_1^4 T_2^8 + 9 T_1^5 T_2^8 - 296 T_1^6 T_2^8 + 290 T_1^7 T_2^8 - 296 T_1^8 T_2^8 + 9 T_1^9 T_2^8 - 18 T_1^{10} T_2^8 + 12 T_1^{11} T_2^8 + 8 T_1^{12} T_2^8 - \\
 & 11 T_1^3 T_2^9 + 12 T_1^4 T_2^9 - 13 T_1^5 T_2^9 + 62 T_1^6 T_2^9 + 9 T_1^7 T_2^9 + 9 T_1^8 T_2^9 + 62 T_1^9 T_2^9 - 13 T_1^{10} T_2^9 + 12 T_1^{11} T_2^9 - \\
 & 11 T_1^{12} T_2^9 + 8 T_1^4 T_2^{10} - 16 T_1^5 T_2^{10} + 12 T_1^6 T_2^{10} - 13 T_1^7 T_2^{10} - 18 T_1^8 T_2^{10} - 13 T_1^9 T_2^{10} + 12 T_1^{10} T_2^{10} - \\
 & 16 T_1^{11} T_2^{10} + 8 T_1^{12} T_2^{10} - 4 T_1^5 T_2^{11} + 12 T_1^6 T_2^{11} - 16 T_1^7 T_2^{11} + 12 T_1^8 T_2^{11} + 12 T_1^9 T_2^{11} - 16 T_1^{10} T_2^{11} +
 \end{aligned}$$

$$\begin{aligned}
& \left. 12 T_1^{11} T_2^{11} - 4 T_1^{12} T_2^{11} + T_1^6 T_2^{12} - 4 T_1^7 T_2^{12} + 8 T_1^8 T_2^{12} - 11 T_1^9 T_2^{12} + 8 T_1^{10} T_2^{12} - 4 T_1^{11} T_2^{12} + T_1^{12} T_2^{12} \right\}, \\
\text{Knot}[10, 99] & \rightarrow \left\{ \frac{(1 - T + T^2)^4}{T^4}, \emptyset \right\}, \text{Knot}[10, 123] \rightarrow \\
& \left\{ \frac{(1 - 3T + 3T^2 - 3T^3 + T^4)^2}{T^4}, \right. \\
& \left. \emptyset \right\}, \\
\text{Knot}[10, 129] & \rightarrow \left\{ \frac{(2 - 2T + T^2)(1 - 2T + 2T^2)}{T^2}, \right. \\
& \frac{1}{T_1^4 T_2^4} \left( 1 - 2T_1 + 3T_1^2 - 2T_1^3 + T_1^4 - 2T_2 + 4T_1 T_2 - 2T_1^2 T_2 - 2T_1^3 T_2 + 4T_1^4 T_2 - 2T_1^5 T_2 + 3T_2^2 - 2T_1 T_2^2 - \right. \\
& 31T_1^2 T_2^2 + 43T_1^3 T_2^2 - 31T_1^4 T_2^2 - 2T_1^5 T_2^2 + 3T_1^6 T_2^2 - 2T_2^3 - 2T_1 T_2^3 + 43T_1^2 T_2^3 - 14T_1^3 T_2^3 - 14T_1^4 T_2^3 + \\
& 43T_1^5 T_2^3 - 2T_1^6 T_2^3 - 2T_1^7 T_2^3 + T_2^4 + 4T_1 T_2^4 - 31T_1^2 T_2^4 - 14T_1^3 T_2^4 + 12T_1^4 T_2^4 - 14T_1^5 T_2^4 - 31T_1^6 T_2^4 + 4T_1^7 T_2^4 + \\
& T_1^8 T_2^4 - 2T_1 T_2^5 - 2T_1^2 T_2^5 + 43T_1^3 T_2^5 - 14T_1^4 T_2^5 - 14T_1^5 T_2^5 + 43T_1^6 T_2^5 - 2T_1^7 T_2^5 - 2T_1^8 T_2^5 + 3T_1^2 T_2^6 - \\
& 2T_1^3 T_2^6 - 31T_1^4 T_2^6 + 43T_1^5 T_2^6 - 31T_1^6 T_2^6 - 2T_1^7 T_2^6 + 3T_1^8 T_2^6 - 2T_1^3 T_2^7 + 4T_1^4 T_2^7 - 2T_1^5 T_2^7 - 2T_1^6 T_2^7 + \\
& \left. 4T_1^7 T_2^7 - 2T_1^8 T_2^7 + T_1^4 T_2^8 - 2T_1^5 T_2^8 + 3T_1^6 T_2^8 - 2T_1^7 T_2^8 + T_1^8 T_2^8 \right) \left. \right\}, \text{Knot}[10, 137] \rightarrow \left\{ \frac{(1 - 3T + T^2)^2}{T^2}, \right. \\
& \left. - \frac{2(1 - 3T_1 + T_1^2)(1 - 3T_2 + T_2^2)(1 - 3T_1 T_2 + T_1^2 T_2^2)(1 + T_1 + T_2 - 6T_1 T_2 + T_1^2 T_2 + T_1 T_2^2 + T_1^2 T_2^2)}{T_1^3 T_2^3} \right\}, \\
\text{Knot}[10, 140] & \rightarrow \left\{ \frac{(1 - T + T^2)^2}{T^2}, \right. \\
& - \frac{1}{T_1^2 T_2^2} 4(3 - 4T_1 + 3T_1^2 - 4T_2 + T_1 T_2 + T_1^2 T_2 - 4T_1^3 T_2 + 3T_2^2 + T_1 T_2^2 + T_1^3 T_2^2 + \\
& \left. 3T_1^4 T_2^2 - 4T_1 T_2^3 + T_1^2 T_2^3 + T_1^3 T_2^3 - 4T_1^4 T_2^3 + 3T_1^2 T_2^4 - 4T_1^3 T_2^4 + 3T_1^4 T_2^4) \right\}, \\
\text{Knot}[10, 153] & \rightarrow \left\{ \frac{(1 - T + T^3)(1 - T^2 + T^3)}{T^3}, \right. \\
& - \frac{1}{T_1^6 T_2^6} \left( 1 - T_1 - T_1^2 + 3T_1^3 - T_1^4 - T_1^5 + T_1^6 - T_2 - 2T_1 T_2 + 5T_1^2 T_2 - 4T_1^3 T_2 - 4T_1^4 T_2 + 5T_1^5 T_2 - 2T_1^6 T_2 - \right. \\
& T_1^7 T_2 - T_2^2 + 5T_1 T_2^2 - 2T_1^2 T_2^2 - 4T_1^3 T_2^2 + 10T_1^4 T_2^2 - 4T_1^5 T_2^2 - 2T_1^6 T_2^2 + 5T_1^7 T_2^2 - T_1^8 T_2^2 + 3T_2^3 - 4T_1 T_2^3 - \\
& 4T_1^2 T_2^3 + 10T_1^3 T_2^3 - 6T_1^4 T_2^3 - 6T_1^5 T_2^3 + 10T_1^6 T_2^3 - 4T_1^7 T_2^3 - 4T_1^8 T_2^3 + 3T_1^9 T_2^3 - T_2^4 - 4T_1 T_2^4 + 10T_1^2 T_2^4 - \\
& 6T_1^3 T_2^4 - 10T_1^4 T_2^4 + 18T_1^5 T_2^4 - 10T_1^6 T_2^4 - 6T_1^7 T_2^4 + 10T_1^8 T_2^4 - 4T_1^9 T_2^4 - T_1^{10} T_2^4 - T_2^5 + 5T_1 T_2^5 - \\
& 4T_1^2 T_2^5 - 6T_1^3 T_2^5 + 18T_1^4 T_2^5 - 10T_1^5 T_2^5 - 10T_1^6 T_2^5 + 18T_1^7 T_2^5 - 6T_1^8 T_2^5 - 4T_1^9 T_2^5 + 5T_1^{10} T_2^5 - T_1^{11} T_2^5 + \\
& T_2^6 - 2T_1 T_2^6 - 2T_1^2 T_2^6 + 10T_1^3 T_2^6 - 10T_1^4 T_2^6 - 10T_1^5 T_2^6 + 24T_1^6 T_2^6 - 10T_1^7 T_2^6 - 10T_1^8 T_2^6 + 10T_1^9 T_2^6 - \\
& 2T_1^{10} T_2^6 - 2T_1^{11} T_2^6 + T_1^{12} T_2^6 - T_1 T_2^7 + 5T_1^2 T_2^7 - 4T_1^3 T_2^7 - 6T_1^4 T_2^7 + 18T_1^5 T_2^7 - 10T_1^6 T_2^7 - 10T_1^7 T_2^7 + \\
& 18T_1^8 T_2^7 - 6T_1^9 T_2^7 - 4T_1^{10} T_2^7 + 5T_1^{11} T_2^7 - T_1^{12} T_2^7 - T_1^2 T_2^8 - 4T_1^3 T_2^8 + 10T_1^4 T_2^8 - 6T_1^5 T_2^8 - 10T_1^6 T_2^8 + \\
& 18T_1^7 T_2^8 - 10T_1^8 T_2^8 - 6T_1^9 T_2^8 + 10T_1^{10} T_2^8 - 4T_1^{11} T_2^8 - T_1^{12} T_2^8 + 3T_1^3 T_2^9 - 4T_1^4 T_2^9 - 4T_1^5 T_2^9 + 10T_1^6 T_2^9 - \\
& 6T_1^7 T_2^9 - 6T_1^8 T_2^9 + 10T_1^9 T_2^9 - 4T_1^{10} T_2^9 - 4T_1^{11} T_2^9 + 3T_1^{12} T_2^9 - T_1^4 T_2^{10} + 5T_1^5 T_2^{10} - 2T_1^6 T_2^{10} - 4T_1^7 T_2^{10} + \\
& 10T_1^8 T_2^{10} - 4T_1^9 T_2^{10} - 2T_1^{10} T_2^{10} + 5T_1^{11} T_2^{10} - T_1^{12} T_2^{10} - T_1^5 T_2^{11} - 2T_1^6 T_2^{11} + 5T_1^7 T_2^{11} - 4T_1^8 T_2^{11} - 4T_1^9 T_2^{11} + \\
& \left. 5T_1^{10} T_2^{11} - 2T_1^{11} T_2^{11} - T_1^{12} T_2^{11} + T_1^6 T_2^{12} - T_1^7 T_2^{12} - T_1^8 T_2^{12} + 3T_1^9 T_2^{12} - T_1^{10} T_2^{12} - T_1^{11} T_2^{12} + T_1^{12} T_2^{12} \right) \left. \right\}, \\
\text{Knot}[10, 155] & \rightarrow \left\{ - \frac{(-1 + T - 2T^2 + T^3)(-1 + 2T - T^2 + T^3)}{T^3}, \right.
\end{aligned}$$

$$\begin{aligned}
 & - \frac{1}{T_1^4 T_2^4} 2 \left( 1 - 4 T_1 + 5 T_1^2 - 4 T_1^3 + T_1^4 - 4 T_2 + 11 T_1 T_2 - 3 T_1^2 T_2 - 3 T_1^3 T_2 + 11 T_1^4 T_2 - 4 T_1^5 T_2 + 5 T_2^2 - \right. \\
 & \quad 3 T_1 T_2^2 - 26 T_1^2 T_2^2 + 24 T_1^3 T_2^2 - 26 T_1^4 T_2^2 - 3 T_1^5 T_2^2 + 5 T_1^6 T_2^2 - 4 T_2^3 - 3 T_1 T_2^3 + 24 T_1^2 T_2^3 + 4 T_1^3 T_2^3 + \\
 & \quad 4 T_1^4 T_2^3 + 24 T_1^5 T_2^3 - 3 T_1^6 T_2^3 - 4 T_1^7 T_2^3 + T_2^4 + 11 T_1 T_2^4 - 26 T_1^2 T_2^4 + 4 T_1^3 T_2^4 - 30 T_1^4 T_2^4 + 4 T_1^5 T_2^4 - \\
 & \quad 26 T_1^6 T_2^4 + 11 T_1^7 T_2^4 + T_1^8 T_2^4 - 4 T_1 T_2^5 - 3 T_1^2 T_2^5 + 24 T_1^3 T_2^5 + 4 T_1^4 T_2^5 + 4 T_1^5 T_2^5 + 24 T_1^6 T_2^5 - 3 T_1^7 T_2^5 - \\
 & \quad 4 T_1^8 T_2^5 + 5 T_1^2 T_2^6 - 3 T_1^3 T_2^6 - 26 T_1^4 T_2^6 + 24 T_1^5 T_2^6 - 26 T_1^6 T_2^6 - 3 T_1^7 T_2^6 + 5 T_1^8 T_2^6 - 4 T_1^3 T_2^7 + \\
 & \quad \left. 11 T_1^4 T_2^7 - 3 T_1^5 T_2^7 - 3 T_1^6 T_2^7 + 11 T_1^7 T_2^7 - 4 T_1^8 T_2^7 + T_1^4 T_2^8 - 4 T_1^5 T_2^8 + 5 T_1^6 T_2^8 - 4 T_1^7 T_2^8 + T_1^8 T_2^8 \right) \}
 \end{aligned}$$

```
In[]:= DunfieldKnots = ReadList["../People/Dunfield/nmd_random_knots"] /. k_Integer :-> k + 1;
DK[n_] := DunfieldKnots[[n - 2]]
```

```
In[]:= Crossings[DK[576]]
```

```
Out[]:=
576
```

```
In[]:= AbsoluteTiming[Theta[DK[3]]]
```

```
Out[]:=
{0.0061176, { $\frac{1 - T + T^2}{T}$, $\frac{1 - T_1 + T_1^2 - T_2 - T_1^3 T_2 + T_2^2 + T_1^4 T_2^2 - T_1 T_2^3 - T_1^4 T_2^3 + T_1^2 T_2^4 - T_1^3 T_2^4 + T_1^4 T_2^4}{T_1^2 T_2^2}$ }}}
```

```
In[]:= AbsoluteTiming[Theta[DK[30]]];
```

```
Out[]:=
{2.91933, Null}
```

```
In[]:= AbsoluteTiming[Theta[DK[60]]];
```

```
Out[]:=
{27.4555, Null}
```

```
In[]:= AbsoluteTiming[Theta[DK[90]]];
```

```
Out[]:=
{227.389, Null}
```

```
In[]:= AbsoluteTiming[Theta120 = Theta[DK[120]]];
```

```
Out[]:=
{0.0003743, Null}
```

```
In[]:= Put[Theta120, "Theta4DK120.m"]
```

```
In[]:= AbsoluteTiming[Theta[DK[150]]];
```

```
Out[]:=
{2357.39, Null}
```

(during the previous computation I biked home, so the AbsoluteTiming is too much)

```
In[]:= AbsoluteTiming[Theta[DK[180]]];
```

```
Out[]:=
{5391.24, Null}
```

In[\*]:= AbsoluteTiming[ $\theta$ [DK[210]]];]

Out[\*]=  
{9613.68, Null}

In[\*]:= AbsoluteTiming[ $\theta$ [DK[240]]];]

Out[\*]=  
{22462.4, Null}

In[\*]:= AbsoluteTiming[ $\theta$ [DK[270]]];]

Mathematica crashed while trying the above computation.

In[\*]:= AbsoluteTiming[ $\theta$ [DK[300]]];]

In[\*]:= Do[Echo /@ AbsoluteTiming[n  $\rightarrow$   $\theta_{22/7,34/21}$ [DK[n]]], {n, 100, 500, 100}]

» 1.34291

» 100  $\rightarrow$   $\left\{ \left\{ \frac{35\ 388\ 936\ 522\ 490\ 931\ 938\ 908\ 923\ 343\ 364\ 558\ 590\ 414\ 632\ 463\ 375\ 508\ 742\ 089}{264\ 554\ 736\ 545\ 069\ 605\ 885\ 631\ 471\ 128\ 764\ 401\ 339\ 301\ 535\ 744} \right. \right.$   
 $\frac{525\ 106\ 180\ 586\ 933\ 014\ 293\ 865\ 927\ 609\ 379\ 271\ 742\ 972\ 076\ 277\ 257\ 025\ 413\ 914\ 338\ 499}{37\ 324\ 734\ 431\ 368\ 634\ 257\ 516\ 595\ 221\ 111\ 791\ 096\ 751\ 570\ 183\ 668\ 795\ 296\ 664\ 772\ 608}$ ,  
 $50\ 463\ 574\ 955\ 913\ 231\ 815\ 385\ 186\ 261\ 134\ 862\ 814\ 456\ 979\ 779\ 055\ 953\ 806\ 229\ 018\ 368\ 595\ 827\ 102\ 502\ 222\ 063 -$   
 $117\ 299\ 430\ 053\ 887\ 387\ 799\ 738\ 329\ 099\ 644\ 807\ 147\ 011\ 110\ 057\ 363 /$   
 $78\ 995\ 482\ 272\ 843\ 339\ 527\ 758\ 555\ 299\ 340\ 636\ 345\ 228\ 530\ 305\ 737\ 655\ 210\ 586\ 135\ 944\ 082\ 585\ 735\ 874\ 960\ 483 -$   
 $459\ 404\ 024\ 632\ 393\ 880\ 311\ 552\ 802\ 816 \left. \right\} ,$   
 $1\ 528\ 310\ 677\ 820\ 715\ 321\ 034\ 523\ 399\ 570\ 065\ 191\ 062\ 105\ 455\ 458\ 377\ 892\ 190\ 819\ 455\ 810\ 946\ 769\ 247\ 237\ 972\ 364 -$   
 $715\ 885\ 979\ 420\ 470\ 551\ 351\ 869\ 219\ 633\ 193\ 553\ 826\ 417\ 257\ 308\ 347\ 635\ 722\ 740\ 692\ 821\ 508\ 135\ 020\ 135\ 287\ 456 -$   
 $366\ 121\ 034\ 149\ 470\ 683\ 332\ 248\ 166\ 617\ 909\ 950\ 793\ 807\ 487\ 984\ 811\ 798\ 893\ 565\ 093\ 125\ 255\ 348\ 183\ 610\ 375\ 623 -$   
 $605\ 724\ 602\ 385 /$   
 $2\ 021\ 852\ 735\ 124\ 190\ 443\ 601\ 930\ 854\ 750\ 557\ 817\ 293\ 581\ 868\ 158\ 419\ 077\ 273\ 632\ 660\ 683\ 611\ 032\ 469\ 645\ 660 -$   
 $194\ 599\ 371\ 946\ 978\ 382\ 921\ 589\ 937\ 447\ 245\ 226\ 083\ 493\ 038\ 643\ 718\ 762\ 601\ 638\ 877\ 904\ 972\ 518\ 050\ 153\ 046 -$   
 $861\ 446\ 451\ 966\ 589\ 617\ 326\ 519\ 650\ 102\ 159\ 919\ 340\ 894\ 781\ 095\ 877\ 211\ 742\ 319\ 673\ 344 \left. \right\}$

» 6.88529

» 200 →

{ { - (72 941 025 249 230 622 091 769 886 034 332 903 937 878 867 275 035 495 850 289 152 467 601 139 729 946 680 -  
 691 983 449 444 238 470 173 260 899 434 879 455 547 646 677 /  
 79 780 391 006 864 379 747 986 053 920 193 038 680 545 693 079 622 955 011 027 668 359 182 291 645 896 903 -  
 218 461 275 510 571 008) ,  
 13 469 039 288 358 770 844 889 186 746 410 419 403 949 987 382 833 567 787 469 752 570 946 087 488 964 056 464 -  
 083 956 449 441 872 952 430 656 158 262 269 810 083 547 830 189 003 289 443 154 125 /  
 4 240 161 130 043 882 037 823 084 995 205 726 632 691 185 572 237 933 032 456 552 833 243 815 216 744 170 971 -  
 881 548 991 957 331 738 797 061 590 095 303 559 046 326 968 215 750 967 296 ,  
 3 058 236 953 956 402 226 943 593 388 603 713 021 071 954 699 338 326 371 450 792 000 285 430 803 814 324 110 -  
 911 806 690 348 020 780 088 584 382 124 603 092 971 693 299 841 778 094 187 288 377 810 035 496 408 283 188 -  
 130 224 093 352 681 965 580 164 395 682 496 054 504 489 551 954 332 992 465 733 972 977 594 735 369 459 115 -  
 633 590 163 189 798 671 672 600 349 071 866 872 120 468 309 375 /  
 7 389 876 778 587 670 278 409 931 856 936 212 530 694 800 372 408 625 530 583 166 986 417 139 021 654 981 203 -  
 589 910 511 227 601 136 991 125 732 955 086 827 137 765 975 954 473 403 792 833 419 463 344 119 138 486 741 -  
 874 061 457 114 480 552 952 530 491 222 541 669 872 799 328 574 041 719 777 250 405 019 238 495 420 416 } ,  
 - (35 533 798 751 418 160 350 916 090 870 874 408 685 758 076 531 957 553 028 308 354 367 936 952 715 320 377 112 -  
 933 900 291 194 748 021 391 980 122 119 460 697 184 063 729 775 201 344 517 723 397 729 781 282 842 088 707 -  
 536 733 758 752 195 455 093 509 038 015 678 684 681 626 418 035 519 803 604 439 397 416 661 432 511 206 560 -  
 127 326 980 562 590 565 142 398 059 299 186 452 157 584 572 312 347 570 546 167 881 173 768 455 447 102 478 -  
 378 052 565 824 989 035 759 718 349 901 555 797 046 487 367 735 873 953 550 250 292 996 462 075 359 706 165 -  
 962 265 760 112 833 307 407 741 496 584 457 563 023 053 844 158 922 142 850 482 681 009 343 615 561 563 933 -  
 345 073 931 843 736 416 605 341 872 288 994 025 512 080 297 221 469 946 108 375 450 764 191 881 092 403 125 /  
 452 396 514 172 443 948 090 596 720 075 743 969 379 888 907 838 827 526 625 786 124 662 888 374 624 411 285 -  
 068 305 310 109 452 395 752 503 075 302 027 422 247 590 129 306 367 202 635 464 223 536 884 780 523 952 041 -  
 663 218 284 564 278 956 217 013 122 499 393 566 958 337 419 775 741 184 128 728 079 197 011 880 897 341 842 -  
 707 105 674 675 400 895 701 799 815 201 160 823 272 081 093 378 813 312 065 293 550 562 986 036 284 189 802 -  
 691 770 038 253 906 432 703 028 717 411 518 620 619 761 249 698 953 156 705 113 477 290 132 297 634 963 280 -  
 748 846 254 218 110 952 588 168 734 038 774 458 591 745 251 980 483 140 272 217 590 873 158 475 317 248) }

» 81.2757

» 300 →

```
{ { 54 300 428 014 802 247 763 147 703 343 836 297 447 025 108 824 684 772 425 762 525 822 095 039 545 899 375 981 -
953 473 178 602 586 048 430 534 584 880 163 873 723 541 762 115 735 883 067 341 959 560 581 371 283 178 656 -
972 648 408 925 263 946 669 /
6 741 838 682 197 306 940 008 962 116 848 220 280 436 936 971 437 572 995 472 014 771 688 913 708 639 211 514 -
814 195 885 491 758 038 709 972 366 558 512 006 372 340 250 849 089 814 593 530 683 936 627 298 651 512 766 -
464,
1 084 128 382 249 743 436 824 663 986 171 685 150 273 646 351 713 912 937 150 171 700 202 730 323 922 010 700 -
294 161 035 743 289 238 368 194 879 507 950 682 627 574 784 328 439 797 605 967 434 628 113 238 619 877 448 -
933 104 349 915 804 145 167 106 117 098 828 582 214 168 974 179 /
458 816 114 715 914 322 691 410 371 538 510 819 835 906 604 695 828 488 701 592 446 861 566 683 983 329 916 -
364 046 021 667 534 630 113 436 786 891 827 119 466 479 256 930 424 597 743 983 452 685 367 746 981 696 618 -
500 346 273 956 034 473 567 578 882 100 679 941 606 973 898 752,
- (158 777 874 852 495 582 515 909 215 389 994 852 546 352 653 931 705 508 650 307 891 657 053 561 609 520 779 -
186 320 897 348 004 451 340 565 961 074 347 535 242 136 402 407 084 832 097 701 971 876 894 887 835 991 169 -
195 699 017 190 487 685 513 574 819 025 748 109 103 168 978 452 501 811 090 422 603 306 747 210 926 095 970 -
770 670 185 035 477 605 544 327 410 988 587 473 792 754 126 636 018 339 393 952 001 669 899 686 164 600 864 -
484 927 816 109 847 962 066 717 003 302 534 438 301 515 100 500 581 439 281 502 338 168 771 925 334 310 271 -
437 341 818 561 /
8 446 673 524 619 204 540 662 248 188 364 579 654 962 149 362 100 111 349 567 813 607 145 180 164 671 139 -
617 365 814 596 293 558 611 877 467 632 393 708 787 160 491 479 639 500 826 381 376 300 773 027 876 197 -
170 955 833 764 004 216 082 452 919 975 997 020 526 350 495 894 405 720 336 559 612 735 646 735 734 155 -
554 395 961 189 410 159 575 680 771 895 729 613 390 941 354 707 084 783 892 152 666 711 430 746 078 787 -
591 302 278 416 571 017 951 710 864 634 193 356 469 295 526 911 091 658 361 659 195 392) },
- (8 598 040 329 900 132 178 849 810 392 065 575 015 656 948 332 717 228 018 818 196 986 408 406 885 151 173 114 -
729 742 371 657 327 870 129 553 797 167 264 600 601 461 737 612 762 883 778 056 461 125 303 156 682 177 822 -
387 597 941 597 676 133 555 775 929 651 554 558 568 826 851 193 016 325 730 344 539 614 484 324 504 069 552 -
066 916 711 741 608 633 404 825 059 528 743 681 819 722 488 192 923 953 808 035 534 926 597 091 591 375 719 -
708 970 825 214 204 090 352 696 010 613 508 905 010 815 827 512 539 197 920 217 378 414 243 201 536 885 840 -
502 206 086 625 497 141 347 632 796 621 737 879 816 174 936 164 213 142 719 738 496 243 651 312 127 236 569 -
658 634 354 493 624 215 745 814 083 607 554 164 979 886 252 364 586 458 746 627 111 732 001 798 098 411 377 -
469 694 277 623 092 049 862 323 332 740 569 060 876 937 876 842 372 536 879 611 798 159 751 897 313 972 600 -
034 650 421 307 830 475 711 279 585 030 859 630 220 299 694 039 396 624 252 597 049 438 680 591 984 366 793 -
943 898 999 135 971 052 027 731 897 551 216 899 484 829 106 288 612 686 258 405 660 999 234 795 832 964 806 -
965 873 /
812 678 875 896 339 039 067 670 285 655 670 982 811 813 572 505 939 516 526 515 895 464 038 253 655 962 698 -
534 470 158 273 453 471 969 412 225 862 875 287 695 193 281 294 047 490 514 554 993 940 744 345 155 227 044 -
324 133 303 532 650 014 900 112 879 006 801 216 964 275 606 874 592 888 221 306 209 845 338 126 393 770 242 -
035 421 093 568 450 115 306 673 002 230 112 041 735 438 509 121 588 709 168 122 196 789 163 088 049 032 402 -
421 831 930 165 869 146 969 680 446 412 107 255 774 061 135 012 374 209 095 972 722 550 879 186 845 119 609 -
905 916 389 810 258 595 619 812 363 193 227 320 810 658 099 006 534 020 402 904 912 489 467 165 903 133 321 -
063 930 316 828 893 776 965 178 816 926 996 966 709 051 510 488 188 756 691 086 660 277 067 356 140 651 827 -
003 820 730 966 021 344 695 355 788 718 822 044 920 980 326 904 411 394 046 648 037 199 883 563 233 621 627 -
831 014 801 221 912 882 289 230 772 061 896 822 000 174 973 211 954 770 033 843 643 470 936 514 389 292 275 -
081 706 404 767 643 586 504 062 664 704) }
```

Out[ ]=

\$Aborted

The following crashes at n=700:

```
In[*]:= Do[Echo /@ AbsoluteTiming[n → $\Theta_{22/7,34/21}$ [DK[n]]], {n, 500, 1000, 100}]
```