Pensieve header: Proof of invariance of ρ_2 using integration techniques.

Initialization

In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\HigherRank"];
Once[<< KnotTheory`; << Rot.m];
<< FormalGaussianIntegration.m;
```

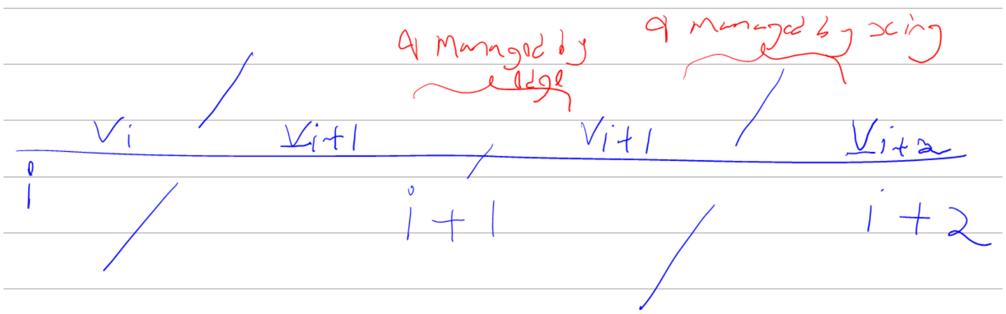
```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
Loading Rot.m from http://drorbn.net/icbs24 to compute rotation numbers.
```

In[∘]:=
```
T2z[p_] := Module[{q = Expand[p], n, c},
   If[q === 0, 0, c = Coefficient[q, T, n = Exponent[q, T]];
    c z^(2 n) + T2z[q - c (T^(1/2) - T^(-1/2))^(2 n)]]];
```

The ρ_2 Integrand

Adopted from pensieve://Talks//Oaxaca-2210/Rho.nb.

Variable convention:



In[ ]:=
```
𝒮 = {x_, p_, x̲_, p̲_};
q[s_, i_, j_] := x_i (p_i − p̲_{i+1}) + x_j (p_j − p̲_{j+1}) + x_i ((1 − T^s) p̲_{i+1} + (T^s − 1) p̲_{j+1});
r₁[s_, i_, j_] :=
  s (−1 + 2 p_i x_i − 2 p_j x_i + (T^s − 1) p_i p_j x_i^2 + (1 − T^s) p_j^2 x_i^2 − 2 p_i p_j x_i x_j + 2 p_j^2 x_i x_j) / 2;
r₂[1, i_, j_] := (−6 p_i x_i + 6 p_j x_i − 3 (−1 + 3 T) p_i p_j x_i^2 + 3 (−1 + 3 T) p_j^2 x_i^2 + 4 (−1 + T) p_i^2 p_j x_i^3 −
     2 (−1 + T) (5 + T) p_i p_j^2 x_i^3 + 2 (−1 + T) (3 + T) p_j^3 x_i^3 + 18 p_i p_j x_i x_j − 18 p_j^2 x_i x_j −
     6 p_i^2 p_j x_i^2 x_j + 6 (2 + T) p_i p_j^2 x_i^2 x_j − 6 (1 + T) p_j^3 x_i^2 x_j − 6 p_i p_j^2 x_i x_j^2 + 6 p_j^3 x_i x_j^2) / 12;
r₂[−1, i_, j_] :=
  (−6 T^2 p_i x_i + 6 T^2 p_j x_i + 3 (−3 + T) T p_i p_j x_i^2 − 3 (−3 + T) T p_j^2 x_i^2 − 4 (−1 + T) T p_i^2 p_j x_i^3 +
     2 (−1 + T) (1 + 5 T) p_i p_j^2 x_i^3 − 2 (−1 + T) (1 + 3 T) p_j^3 x_i^3 + 18 T^2 p_i p_j x_i x_j −
     18 T^2 p_j^2 x_i x_j − 6 T^2 p_i^2 p_j x_i^2 x_j + 6 T (1 + 2 T) p_i p_j^2 x_i^2 x_j −
     6 T (1 + T) p_j^3 x_i^2 x_j − 6 T^2 p_i p_j^2 x_i x_j^2 + 6 T^2 p_j^3 x_i x_j^2) / (12 T^2);
γ₁[φ_, k_] := φ (1 / 2 − x̲_k p̲_k);
γ₂[φ_, k_] := −φ^2 p̲_k x̲_k / 2;
ℒ[s_, i_, j_] := T^(s/2) 𝔼[−q[s, i, j] + ϵ r₁[s, i, j] + ϵ^2 r₂[s, i, j] + O[ϵ]^3];
ℒ[φ_, k_] := T^(φ/2) 𝔼[−x̲_k (p̲_k − p_k) + ϵ γ₁[φ, k] + ϵ^2 γ₂[φ, k] + O[ϵ]^3];
ℒ[Dot, i_] := 𝔼[−x_i (p_i − p̲_{i+1}) + O[ϵ]^3];
ℒ[End, k_] := 𝔼[−x̲_k (p̲_k − p_k) − x_k p_k + O[ϵ]^3];
ℒ[K_] := Module[{Cs, φ, n, c, k, ℰ},
   {Cs, φ} = Rot[K]; n = Length[Cs];
   ℰ = (2 π)^(4 n+2) ℒ[End, 2 n + 1];
   Do[ℰ *= ℒ @@ c, {c, Cs}];
   Do[ℰ *= ℒ[φ[[k]], k], {k, 2 n}];
   CF@ℰ
  ];
vs_i_ := Sequence[x̲_i, p̲_i, x_i, p_i]
ρ2vs[K_] := Union @@ Table[{vs_i}, {i, 2 Crossings[K] + 1}]
```

In[•]:= **ℒ[Knot[3, 1]]**

KnotTheory: Loading precomputed data in PD4Knots`.

Out[•]=

$$\frac{1}{T^2}\,16\,384\,\pi^{14}$$

$$\mathbb{E}\Big[\epsilon\text{Series}\Big[-p_1 x_1 - p_2 x_2 - p_3 x_3 - p_4 x_4 - p_5 x_5 - p_6 x_6 - p_7 x_7 + x_1 \underline{p}_2 + x_3 \underline{p}_4 + \frac{x_4\left(-\underline{p}_2 + T\,\underline{p}_2 + \underline{p}_5\right)}{T} +$$

$$x_5 \underline{p}_6 + \frac{x_2\left(\underline{p}_3 - \underline{p}_6 + T\,\underline{p}_6\right)}{T} + \frac{x_6\left(-\underline{p}_4 + T\,\underline{p}_4 + \underline{p}_7\right)}{T} + p_1 \underline{x}_1 - \underline{p}_1 \underline{x}_1 + p_2 \underline{x}_2 -$$

$$\underline{p}_2 \underline{x}_2 + p_3 \underline{x}_3 - \underline{p}_3 \underline{x}_3 + p_4 \underline{x}_4 - \underline{p}_4 \underline{x}_4 + p_5 \underline{x}_5 - \underline{p}_5 \underline{x}_5 + p_6 \underline{x}_6 - \underline{p}_6 \underline{x}_6 + p_7 \underline{x}_7 - \underline{p}_7 \underline{x}_7,$$

$$1 - p_2 x_2 + p_5 x_2 + \frac{(-1+T)\,p_2 p_5 x_2^2}{2\,T} - \frac{(-1+T)\,p_5^2 x_2^2}{2\,T} + p_1 x_4 - p_4 x_4 - p_1^2 x_1 x_4 +$$

$$p_1 p_4 x_1 x_4 - \frac{(-1+T)\,p_1^2 x_4^2}{2\,T} + \frac{(-1+T)\,p_1 p_4 x_4^2}{2\,T} + p_2 p_5 x_2 x_5 - p_5^2 x_2 x_5 + p_3 x_6 -$$

$$p_6 x_6 - p_3^2 x_3 x_6 + p_3 p_6 x_3 x_6 - \frac{(-1+T)\,p_3^2 x_6^2}{2\,T} + \frac{(-1+T)\,p_3 p_6 x_6^2}{2\,T} + \underline{p}_4 \underline{x}_4,$$

$$-\frac{1}{2} p_2 x_2 + \frac{p_5 x_2}{2} + \frac{(-3+T)\,p_2 p_5 x_2^2}{4\,T} - \frac{(-3+T)\,p_5^2 x_2^2}{4\,T} - \frac{(-1+T)\,p_2^2 p_5 x_2^3}{3\,T} +$$

$$\frac{(-1+T)\,(1+5\,T)\,p_2 p_5^2 x_2^3}{6\,T^2} - \frac{(-1+T)\,(1+3\,T)\,p_5^3 x_2^3}{6\,T^2} + \frac{p_1 x_4}{2} - \frac{p_4 x_4}{2} - \frac{3}{2} p_1^2 x_1 x_4 +$$

$$\frac{3}{2} p_1 p_4 x_1 x_4 + \frac{1}{2} p_1^3 x_1^2 x_4 - \frac{1}{2} p_1^2 p_4 x_1^2 x_4 - \frac{(-3+T)\,p_1^2 x_4^2}{4\,T} + \frac{(-3+T)\,p_1 p_4 x_4^2}{4\,T} -$$

$$\frac{(1+T)\,p_1^3 x_1 x_4^2}{2\,T} + \frac{(1+2\,T)\,p_1^2 p_4 x_1 x_4^2}{2\,T} - \frac{1}{2} p_1 p_4^2 x_1 x_4^2 - \frac{(-1+T)\,(1+3\,T)\,p_1^3 x_4^3}{6\,T^2} +$$

$$\frac{(-1+T)\,(1+5\,T)\,p_1^2 p_4 x_4^3}{6\,T^2} - \frac{(-1+T)\,p_1 p_4^2 x_4^3}{3\,T} + \frac{3}{2} p_2 p_5 x_2 x_5 - \frac{3}{2} p_5^2 x_2 x_5 -$$

$$\frac{1}{2} p_2^2 p_5 x_2^2 x_5 + \frac{(1+2\,T)\,p_2 p_5^2 x_2^2 x_5}{2\,T} - \frac{(1+T)\,p_5^3 x_2^2 x_5}{2\,T} - \frac{1}{2} p_2 p_5^2 x_2 x_5^2 + \frac{1}{2} p_5^3 x_2 x_5^2 +$$

$$\frac{p_3 x_6}{2} - \frac{p_6 x_6}{2} - \frac{3}{2} p_3^2 x_3 x_6 + \frac{3}{2} p_3 p_6 x_3 x_6 + \frac{1}{2} p_3^3 x_3^2 x_6 - \frac{1}{2} p_3^2 p_6 x_3^2 x_6 - \frac{(-3+T)\,p_3^2 x_6^2}{4\,T} +$$

$$\frac{(-3+T)\,p_3 p_6 x_6^2}{4\,T} - \frac{(1+T)\,p_3^3 x_3 x_6^2}{2\,T} + \frac{(1+2\,T)\,p_3^2 p_6 x_3 x_6^2}{2\,T} - \frac{1}{2} p_3 p_6^2 x_3 x_6^2 -$$

$$\frac{(-1+T)\,(1+3\,T)\,p_3^3 x_6^3}{6\,T^2} + \frac{(-1+T)\,(1+5\,T)\,p_3^2 p_6 x_6^3}{6\,T^2} - \frac{(-1+T)\,p_3 p_6^2 x_6^3}{3\,T} - \frac{1}{2} \underline{p}_4 \underline{x}_4\Big]\Big]$$

In[•]:= **ρ2vs[Knot[3, 1]]**

Out[•]=

$$\{p_1, p_2, p_3, p_4, p_5, p_6, p_7, x_1, x_2, x_3, x_4, x_5,$$

$$x_6, x_7, \underline{p}_1, \underline{p}_2, \underline{p}_3, \underline{p}_4, \underline{p}_5, \underline{p}_6, \underline{p}_7, \underline{x}_1, \underline{x}_2, \underline{x}_3, \underline{x}_4, \underline{x}_5, \underline{x}_6, \underline{x}_7\}$$

In[∘]:= K = Knot[3, 1]; ∫ℒ[K] ⅆ(ρ2vs@K)

Out[∘]=

$$\frac{T\,\mathbb{E}\left[\epsilon\text{Series}\left[0,\frac{(-1+T)^2\,(1+T^2)}{(1-T+T^2)^2},-\frac{T^2\,(1-4\,T^2+T^4)}{2\,(1-T+T^2)^4}\right]\right]}{1-T+T^2}$$

In[∘]:= T2z[$T^{-2}\,(1-4\,T^2+T^4)$]

Out[∘]=

$$-2+4\,z^2+z^4$$

In[∘]:= Factor@$(2-4\,z^2+3\,z^4+4\,z^6+z^8)$

Out[∘]=

$$2-4\,z^2+3\,z^4+4\,z^6+z^8$$

In[∘]:= K = Knot[5, 2]; ∫ℒ[K] ⅆ(ρ2vs@K)

Out[∘]=

$$\frac{T\,\mathbb{E}\left[\epsilon\text{Series}\left[0,\frac{(-1+T)^2\,(5-4\,T+5\,T^2)}{(2-3\,T+2\,T^2)^2},\frac{1-4\,T+11\,T^2-44\,T^3+76\,T^4-44\,T^5+11\,T^6-4\,T^7+T^8}{2\,(2-3\,T+2\,T^2)^4}\right]\right]}{2-3\,T+2\,T^2}$$

In[∘]:= T2z[$(1-4\,T+11\,T^2-44\,T^3+76\,T^4-44\,T^5+11\,T^6-4\,T^7+T^8)\,/\,T^4$]

Out[∘]=

$$4-20\,z^2+7\,z^4+4\,z^6+z^8$$

In[∘]:= K = Knot[8, 19]; ∫ℒ[K] ⅆ(ρ2vs@K)

Out[∘]=

$$\frac{1}{1-T+T^3-T^5+T^6}T^3\,\mathbb{E}\left[\epsilon\text{Series}\left[0,-\frac{(-1+T)^2\,(1+T^4)\,(3+4\,T^3+3\,T^6)}{(1-T+T^2)^2\,(1-T^2+T^4)^2},\right.\right.$$
$$-\left(\left(T^3\,(4-7\,T-12\,T^2+62\,T^3-100\,T^4+95\,T^5-60\,T^6-42\,T^7+208\,T^8-306\,T^9+208\,T^{10}-42\,T^{11}-\right.\right.$$
$$\left.\left.\left.\left.60\,T^{12}+95\,T^{13}-100\,T^{14}+62\,T^{15}-12\,T^{16}-7\,T^{17}+4\,T^{18})\right)\,/\,\left(2\,(1-T+T^2)^4\,(1-T^2+T^4)^4\right)\right)\right]\right]$$

Concatenating edges

In[∘]:= lhs = $8\,\pi^3\int\left(\mathbb{E}[\underline{\pi}_i\,\underline{p}_i]\,\mathcal{L}[\varphi 1,i]\,\mathcal{L}[\text{Dot},i]\,\mathcal{L}[\varphi 2,i+1]\right)\,ⅆ\{vs_i,\underline{x}_{i+1},\underline{p}_{i+1}\}$

rhs = $\left(2\,\pi\int\mathbb{E}[\underline{\pi}_i\,\underline{p}_i]\,\mathcal{L}[\varphi 1+\varphi 2,i]\,ⅆ\{\underline{x}_i,\underline{p}_i\}\right)$ /. $p_i\to p_{i+1}$;

lhs == rhs

Out[∘]=

$$-\mathbb{i}\,T^{\frac{\varphi 1}{2}+\frac{\varphi 2}{2}}\,\mathbb{E}\left[\epsilon\text{Series}\left[p_{1+i}\,\underline{\pi}_i,\frac{1}{2}\,(-\varphi 1-\varphi 2)-(\varphi 1+\varphi 2)\,p_{1+i}\,\underline{\pi}_i,\frac{1}{2}\,(\varphi 1+\varphi 2)^2\,p_{1+i}\,\underline{\pi}_i\right]\right]$$

Out[∘]=

True

Invariance Under Reidemeister 3b

In[∘]:=

$$\text{lhs} = \int \left(\mathbb{E}[\pi_i p_i + \pi_j p_j + \pi_k p_k]\, \mathcal{L}[1, i, j]\, \mathcal{L}[1, i+1, k]\, \mathcal{L}[1, j+1, k+1]\, \mathcal{L}[0, i+1]\, \mathcal{L}[0, j+1]\, \mathcal{L}[0, k+1]\right) d\{x_i, x_j, x_k, p_i, p_j, p_k, vs_{i+1}, vs_{j+1}, vs_{k+1}\}$$

$$\text{rhs} = \int \left(\mathbb{E}[\pi_i p_i + \pi_j p_j + \pi_k p_k]\, \mathcal{L}[1, j, k]\, \mathcal{L}[1, i, k+1]\, \mathcal{L}[1, i+1, j+1]\, \mathcal{L}[0, i+1]\, \mathcal{L}[0, j+1]\, \mathcal{L}[0, k+1]\right) d\{x_i, x_j, x_k, p_i, p_j, p_k, vs_{i+1}, vs_{j+1}, vs_{k+1}\};$$

$$\text{lhs} == \text{rhs}$$

Out[∘]=

$$-\frac{1}{512\,\pi^9}$$

$$i\, T^{3/2}\, \mathbb{E}\Big[\epsilon\text{Series}\Big[T^2 \pi_i \underline{p}_{2+i} + T \pi_i \underline{p}_{2+j} - T^2 \pi_i \underline{p}_{2+j} + T \pi_j \underline{p}_{2+j} + \pi_i \underline{p}_{2+k} - T \pi_i \underline{p}_{2+k} + \pi_j \underline{p}_{2+k} - T \pi_j \underline{p}_{2+k} +$$

$$\pi_k \underline{p}_{2+k},\ \frac{1}{2}\Big(-3 + 2 T^2 \pi_i \underline{p}_{2+j} - 2 T \pi_j \underline{p}_{2+j} - T^3 \pi_i^2 \underline{p}_{2+i} \underline{p}_{2+j} + T^4 \pi_i^2 \underline{p}_{2+i} \underline{p}_{2+j} -$$

$$2 T^3 \pi_i \pi_j \underline{p}_{2+i} \underline{p}_{2+j} + T^3 \pi_i^2 \underline{p}_{2+j}^2 - T^4 \pi_i^2 \underline{p}_{2+j}^2 + 2 T^3 \pi_i \pi_j \underline{p}_{2+j}^2 + 2 T \pi_i \underline{p}_{2+k} - 2 \pi_j \underline{p}_{2+k} +$$

$$4 T \pi_j \underline{p}_{2+k} - 4 \pi_k \underline{p}_{2+k} - T^2 \pi_i^2 \underline{p}_{2+i} \underline{p}_{2+k} + T^3 \pi_i^2 \underline{p}_{2+i} \underline{p}_{2+k} - 2 T^2 \pi_i \pi_j \underline{p}_{2+i} \underline{p}_{2+k} +$$

$$2 T^3 \pi_i \pi_j \underline{p}_{2+i} \underline{p}_{2+k} - 2 T^2 \pi_i \pi_k \underline{p}_{2+i} \underline{p}_{2+k} - T \pi_i^2 \underline{p}_{2+j} \underline{p}_{2+k} + 2 T^2 \pi_i^2 \underline{p}_{2+j} \underline{p}_{2+k} - T^3 \pi_i^2 \underline{p}_{2+j} \underline{p}_{2+k} -$$

$$2 T \pi_i \pi_j \underline{p}_{2+j} \underline{p}_{2+k} + 4 T^2 \pi_i \pi_j \underline{p}_{2+j} \underline{p}_{2+k} - 2 T^3 \pi_i \pi_j \underline{p}_{2+j} \underline{p}_{2+k} - T \pi_j^2 \underline{p}_{2+j} \underline{p}_{2+k} + T^2 \pi_j^2 \underline{p}_{2+j} \underline{p}_{2+k} -$$

$$2 T \pi_i \pi_k \underline{p}_{2+j} \underline{p}_{2+k} + 2 T^2 \pi_i \pi_k \underline{p}_{2+j} \underline{p}_{2+k} - 2 T \pi_j \pi_k \underline{p}_{2+j} \underline{p}_{2+k} + T \pi_i^2 \underline{p}_{2+k}^2 - T^2 \pi_i^2 \underline{p}_{2+k}^2 +$$

$$2 T \pi_i \pi_j \underline{p}_{2+k}^2 - 2 T^2 \pi_i \pi_j \underline{p}_{2+k}^2 + T \pi_j^2 \underline{p}_{2+k}^2 - T^2 \pi_j^2 \underline{p}_{2+k}^2 + 2 T \pi_i \pi_k \underline{p}_{2+k}^2 + 2 T \pi_j \pi_k \underline{p}_{2+k}^2\Big),$$

$$\frac{1}{12}\Big(-6 T^2 \pi_i \underline{p}_{2+j} + 6 T \pi_j \underline{p}_{2+j} + 3 T^3 \pi_i^2 \underline{p}_{2+i} \underline{p}_{2+j} - 9 T^4 \pi_i^2 \underline{p}_{2+i} \underline{p}_{2+j} + 18 T^3 \pi_i \pi_j \underline{p}_{2+i} \underline{p}_{2+j} +$$

$$2 T^5 \pi_i^3 \underline{p}_{2+i}^2 \underline{p}_{2+j} - 2 T^6 \pi_i^3 \underline{p}_{2+i}^2 \underline{p}_{2+j} + 6 T^5 \pi_i^2 \pi_j \underline{p}_{2+i}^2 \underline{p}_{2+j} - 9 T^3 \pi_i^2 \underline{p}_{2+j}^2 + 15 T^4 \pi_i^2 \underline{p}_{2+j}^2 -$$

$$30 T^3 \pi_i \pi_j \underline{p}_{2+j}^2 + 2 T^4 \pi_i^3 \underline{p}_{2+i} \underline{p}_{2+j}^2 - 10 T^5 \pi_i^3 \underline{p}_{2+i} \underline{p}_{2+j}^2 + 8 T^6 \pi_i^3 \underline{p}_{2+i} \underline{p}_{2+j}^2 + 6 T^4 \pi_i^2 \pi_j \underline{p}_{2+i} \underline{p}_{2+j}^2 -$$

$$24 T^5 \pi_i^2 \pi_j \underline{p}_{2+i} \underline{p}_{2+j}^2 + 6 T^4 \pi_i \pi_j^2 \underline{p}_{2+i} \underline{p}_{2+j}^2 - 2 T^4 \pi_i^3 \underline{p}_{2+j}^3 + 8 T^5 \pi_i^3 \underline{p}_{2+j}^3 - 6 T^6 \pi_i^3 \underline{p}_{2+j}^3 -$$

$$6 T^4 \pi_i^2 \pi_j \underline{p}_{2+j}^3 + 18 T^5 \pi_i^2 \pi_j \underline{p}_{2+j}^3 - 6 T^4 \pi_i \pi_j^2 \underline{p}_{2+j}^3 - 6 T \pi_i \underline{p}_{2+k} + 6 \pi_j \underline{p}_{2+k} - 24 T \pi_j \underline{p}_{2+k} +$$

$$24 \pi_k \underline{p}_{2+k} + 3 T^2 \pi_i^2 \underline{p}_{2+i} \underline{p}_{2+k} - 9 T^3 \pi_i^2 \underline{p}_{2+i} \underline{p}_{2+k} + 18 T^2 \pi_i \pi_j \underline{p}_{2+i} \underline{p}_{2+k} - 30 T^3 \pi_i \pi_j \underline{p}_{2+i} \underline{p}_{2+k} +$$

$$30 T^2 \pi_i \pi_k \underline{p}_{2+i} \underline{p}_{2+k} + 2 T^4 \pi_i^3 \underline{p}_{2+i}^2 \underline{p}_{2+k} - 2 T^5 \pi_i^3 \underline{p}_{2+i}^2 \underline{p}_{2+k} + 6 T^4 \pi_i^2 \pi_j \underline{p}_{2+i}^2 \underline{p}_{2+k} -$$

$$6 T^5 \pi_i^2 \pi_j \underline{p}_{2+i}^2 \underline{p}_{2+k} + 6 T^4 \pi_i^2 \pi_k \underline{p}_{2+i}^2 \underline{p}_{2+k} + 3 T \pi_i^2 \underline{p}_{2+j} \underline{p}_{2+k} - 18 T^2 \pi_i^2 \underline{p}_{2+j} \underline{p}_{2+k} +$$

$$15 T^3 \pi_i^2 \underline{p}_{2+j} \underline{p}_{2+k} + 18 T \pi_i \pi_j \underline{p}_{2+j} \underline{p}_{2+k} - 60 T^2 \pi_i \pi_j \underline{p}_{2+j} \underline{p}_{2+k} + 42 T^3 \pi_i \pi_j \underline{p}_{2+j} \underline{p}_{2+k} +$$

$$15 T \pi_j^2 \underline{p}_{2+j} \underline{p}_{2+k} - 21 T^2 \pi_j^2 \underline{p}_{2+j} \underline{p}_{2+k} + 30 T \pi_i \pi_k \underline{p}_{2+j} \underline{p}_{2+k} - 42 T^2 \pi_i \pi_k \underline{p}_{2+j} \underline{p}_{2+k} +$$

$$42 T \pi_j \pi_k \underline{p}_{2+j} \underline{p}_{2+k} + 10 T^3 \pi_i^3 \underline{p}_{2+i} \underline{p}_{2+j} \underline{p}_{2+k} - 20 T^4 \pi_i^3 \underline{p}_{2+i} \underline{p}_{2+j} \underline{p}_{2+k} + 10 T^5 \pi_i^3 \underline{p}_{2+i} \underline{p}_{2+j} \underline{p}_{2+k} +$$

$$30 T^3 \pi_i^2 \pi_j \underline{p}_{2+i} \underline{p}_{2+j} \underline{p}_{2+k} - 54 T^4 \pi_i^2 \pi_j \underline{p}_{2+i} \underline{p}_{2+j} \underline{p}_{2+k} + 24 T^5 \pi_i^2 \pi_j \underline{p}_{2+i} \underline{p}_{2+j} \underline{p}_{2+k} +$$

$$24 T^3 \pi_i \pi_j^2 \underline{p}_{2+i} \underline{p}_{2+j} \underline{p}_{2+k} - 24 T^4 \pi_i \pi_j^2 \underline{p}_{2+i} \underline{p}_{2+j} \underline{p}_{2+k} + 24 T^3 \pi_i^2 \pi_k \underline{p}_{2+i} \underline{p}_{2+j} \underline{p}_{2+k} -$$

$$24 T^4 \pi_i^2 \pi_k \underline{p}_{2+i} \underline{p}_{2+j} \underline{p}_{2+k} + 36 T^3 \pi_i \pi_j \pi_k \underline{p}_{2+i} \underline{p}_{2+j} \underline{p}_{2+k} + 2 T^2 \pi_i^3 \underline{p}_{2+j}^2 \underline{p}_{2+k} - 12 T^3 \pi_i^3 \underline{p}_{2+j}^2 \underline{p}_{2+k} +$$

$$18 T^4 \pi_i^3 \underline{p}_{2+j}^2 \underline{p}_{2+k} - 8 T^5 \pi_i^3 \underline{p}_{2+j}^2 \underline{p}_{2+k} + 6 T^2 \pi_i^2 \pi_j \underline{p}_{2+j}^2 \underline{p}_{2+k} - 36 T^3 \pi_i^2 \pi_j \underline{p}_{2+j}^2 \underline{p}_{2+k} +$$

$$48 T^4 \pi_i^2 \pi_j \underline{p}_{2+j}^2 \underline{p}_{2+k} - 18 T^5 \pi_i^2 \pi_j \underline{p}_{2+j}^2 \underline{p}_{2+k} + 6 T^2 \pi_i \pi_j^2 \underline{p}_{2+j}^2 \underline{p}_{2+k} - 30 T^3 \pi_i \pi_j^2 \underline{p}_{2+j}^2 \underline{p}_{2+k} +$$

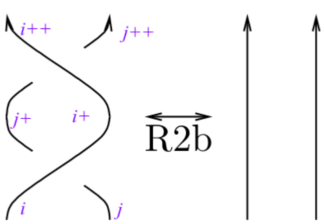
$$24 T^4 \pi_i \pi_j^2 \underline{p}_{2+j}^2 \underline{p}_{2+k} + 2 T^2 \pi_j^3 \underline{p}_{2+j}^2 \underline{p}_{2+k} - 2 T^3 \pi_j^3 \underline{p}_{2+j}^2 \underline{p}_{2+k} + 6 T^2 \pi_i^2 \pi_k \underline{p}_{2+j}^2 \underline{p}_{2+k} -$$

$24\,T^3\,\pi_i^2\,\pi_k\,\underline{p}^2_{2+j}\,\underline{p}_{2+k} + 18\,T^4\,\pi_i^2\,\pi_k\,\underline{p}^2_{2+j}\,\underline{p}_{2+k} + 12\,T^2\,\pi_i\,\pi_j\,\pi_k\,\underline{p}^2_{2+j}\,\underline{p}_{2+k} - 36\,T^3\,\pi_i\,\pi_j\,\pi_k\,\underline{p}^2_{2+j}\,\underline{p}_{2+k} +$
$6\,T^2\,\pi_j^2\,\pi_k\,\underline{p}^2_{2+j}\,\underline{p}_{2+k} - 9\,T\,\pi_i^2\,\underline{p}^2_{2+k} + 15\,T^2\,\pi_i^2\,\underline{p}^2_{2+k} - 30\,T\,\pi_i\,\pi_j\,\underline{p}^2_{2+k} + 42\,T^2\,\pi_i\,\pi_j\,\underline{p}^2_{2+k} - 21\,T\,\pi_j^2\,\underline{p}^2_{2+k} +$
$27\,T^2\,\pi_j^2\,\underline{p}^2_{2+k} - 42\,T\,\pi_i\,\pi_k\,\underline{p}^2_{2+k} - 54\,T\,\pi_j\,\pi_k\,\underline{p}^2_{2+k} + 2\,T^2\,\pi_i^3\,\underline{p}_{2+i}\,\underline{p}^2_{2+k} - 10\,T^3\,\pi_i^3\,\underline{p}_{2+i}\,\underline{p}^2_{2+k} +$
$8\,T^4\,\pi_i^3\,\underline{p}_{2+i}\,\underline{p}^2_{2+k} + 6\,T^2\,\pi_i^2\,\pi_j\,\underline{p}_{2+i}\,\underline{p}^2_{2+k} - 30\,T^3\,\pi_i^2\,\pi_j\,\underline{p}_{2+i}\,\underline{p}^2_{2+k} + 24\,T^4\,\pi_i^2\,\pi_j\,\underline{p}_{2+i}\,\underline{p}^2_{2+k} +$
$6\,T^2\,\pi_i\,\pi_j^2\,\underline{p}_{2+i}\,\underline{p}^2_{2+k} - 24\,T^3\,\pi_i\,\pi_j^2\,\underline{p}_{2+i}\,\underline{p}^2_{2+k} + 18\,T^4\,\pi_i\,\pi_j^2\,\underline{p}_{2+i}\,\underline{p}^2_{2+k} + 6\,T^2\,\pi_i^2\,\pi_k\,\underline{p}_{2+i}\,\underline{p}^2_{2+k} -$
$24\,T^3\,\pi_i^2\,\pi_k\,\underline{p}_{2+i}\,\underline{p}^2_{2+k} + 12\,T^2\,\pi_i\,\pi_j\,\pi_k\,\underline{p}_{2+i}\,\underline{p}^2_{2+k} - 36\,T^3\,\pi_i\,\pi_j\,\pi_k\,\underline{p}_{2+i}\,\underline{p}^2_{2+k} + 6\,T^2\,\pi_i\,\pi_k^2\,\underline{p}_{2+i}\,\underline{p}^2_{2+k} +$
$2\,T\,\pi_i^3\,\underline{p}_{2+j}\,\underline{p}^2_{2+k} - 12\,T^2\,\pi_i^3\,\underline{p}_{2+j}\,\underline{p}^2_{2+k} + 18\,T^3\,\pi_i^3\,\underline{p}_{2+j}\,\underline{p}^2_{2+k} - 8\,T^4\,\pi_i^3\,\underline{p}_{2+j}\,\underline{p}^2_{2+k} + 6\,T\,\pi_i^2\,\pi_j\,\underline{p}_{2+j}\,\underline{p}^2_{2+k} -$
$36\,T^2\,\pi_i^2\,\pi_j\,\underline{p}_{2+j}\,\underline{p}^2_{2+k} + 54\,T^3\,\pi_i^2\,\pi_j\,\underline{p}_{2+j}\,\underline{p}^2_{2+k} - 24\,T^4\,\pi_i^2\,\pi_j\,\underline{p}_{2+j}\,\underline{p}^2_{2+k} + 6\,T\,\pi_i\,\pi_j^2\,\underline{p}_{2+j}\,\underline{p}^2_{2+k} -$
$36\,T^2\,\pi_i\,\pi_j^2\,\underline{p}_{2+j}\,\underline{p}^2_{2+k} + 48\,T^3\,\pi_i\,\pi_j^2\,\underline{p}_{2+j}\,\underline{p}^2_{2+k} - 18\,T^4\,\pi_i\,\pi_j^2\,\underline{p}_{2+j}\,\underline{p}^2_{2+k} + 2\,T\,\pi_j^3\,\underline{p}_{2+j}\,\underline{p}^2_{2+k} -$
$10\,T^2\,\pi_j^3\,\underline{p}_{2+j}\,\underline{p}^2_{2+k} + 8\,T^3\,\pi_j^3\,\underline{p}_{2+j}\,\underline{p}^2_{2+k} + 6\,T\,\pi_i^2\,\pi_k\,\underline{p}_{2+j}\,\underline{p}^2_{2+k} - 30\,T^2\,\pi_i^2\,\pi_k\,\underline{p}_{2+j}\,\underline{p}^2_{2+k} +$
$24\,T^3\,\pi_i^2\,\pi_k\,\underline{p}_{2+j}\,\underline{p}^2_{2+k} + 12\,T\,\pi_i\,\pi_j\,\pi_k\,\underline{p}_{2+j}\,\underline{p}^2_{2+k} - 60\,T^2\,\pi_i\,\pi_j\,\pi_k\,\underline{p}_{2+j}\,\underline{p}^2_{2+k} + 36\,T^3\,\pi_i\,\pi_j\,\pi_k\,\underline{p}_{2+j}\,\underline{p}^2_{2+k} +$
$6\,T\,\pi_j^2\,\pi_k\,\underline{p}_{2+j}\,\underline{p}^2_{2+k} - 24\,T^2\,\pi_j^2\,\pi_k\,\underline{p}_{2+j}\,\underline{p}^2_{2+k} + 6\,T\,\pi_i\,\pi_k^2\,\underline{p}_{2+j}\,\underline{p}^2_{2+k} - 6\,T^2\,\pi_i\,\pi_k^2\,\underline{p}_{2+j}\,\underline{p}^2_{2+k} +$
$6\,T\,\pi_j\,\pi_k^2\,\underline{p}_{2+j}\,\underline{p}^2_{2+k} - 2\,T\,\pi_i^3\,\underline{p}^3_{2+k} + 8\,T^2\,\pi_i^3\,\underline{p}^3_{2+k} - 6\,T^3\,\pi_i^3\,\underline{p}^3_{2+k} - 6\,T\,\pi_i^2\,\pi_j\,\underline{p}^3_{2+k} + 24\,T^2\,\pi_i^2\,\pi_j\,\underline{p}^3_{2+k} -$
$18\,T^3\,\pi_i^2\,\pi_j\,\underline{p}^3_{2+k} - 6\,T\,\pi_i\,\pi_j^2\,\underline{p}^3_{2+k} + 24\,T^2\,\pi_i\,\pi_j^2\,\underline{p}^3_{2+k} - 18\,T^3\,\pi_i\,\pi_j^2\,\underline{p}^3_{2+k} - 2\,T\,\pi_j^3\,\underline{p}^3_{2+k} +$
$8\,T^2\,\pi_j^3\,\underline{p}^3_{2+k} - 6\,T^3\,\pi_j^3\,\underline{p}^3_{2+k} - 6\,T\,\pi_i^2\,\pi_k\,\underline{p}^3_{2+k} + 18\,T^2\,\pi_i^2\,\pi_k\,\underline{p}^3_{2+k} - 12\,T\,\pi_i\,\pi_j\,\pi_k\,\underline{p}^3_{2+k} +$
$36\,T^2\,\pi_i\,\pi_j\,\pi_k\,\underline{p}^3_{2+k} - 6\,T\,\pi_j^2\,\pi_k\,\underline{p}^3_{2+k} + 18\,T^2\,\pi_j^2\,\pi_k\,\underline{p}^3_{2+k} - 6\,T\,\pi_i\,\pi_k^2\,\underline{p}^3_{2+k} - 6\,T\,\pi_j\,\pi_k^2\,\underline{p}^3_{2+k}\big)\big]\big]$

Out[∘]=

```
True
```

Invariance Under Reidemeister 2b



Territory: $(x \mid p)_{(i|j)^+},\ x_{i|j}$.

In[∘]:= $\textbf{lhs} = \int \left(\mathbb{E}[\pi_i\,p_i + \pi_j\,p_j]\ \mathcal{L}[1, i, j]\ \mathcal{L}[-1, i+1, j+1]\ \mathcal{L}[0, i+1]\ \mathcal{L}[0, j+1]\right)$
$d\{x_i, x_j, p_i, p_j, vs_{i+1}, vs_{j+1}\}$

Out[∘]=

$$\frac{\mathbb{E}\left[\in\text{Series}\left[\pi_i\,\underline{p}_{2+i} + \pi_j\,\underline{p}_{2+j},\ 0,\ 0\right]\right]}{64\,\pi^6}$$

In[∘]:=
```
rhs = ∫(𝔼[π_i p_i + π_j p_j] ℒ[0, i + 1] ℒ[0, j + 1]) ℒ[Dot, i]
   ℒ[Dot, j] ℒ[Dot, i + 1] ℒ[Dot, j + 1] ⅆ{x_i, x_j, p_i, p_j, vs_{i+1}, vs_{j+1}}
lhs == rhs
```

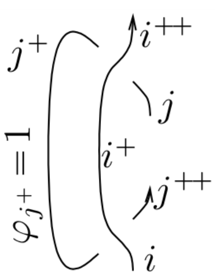
Out[∘]=

$$\frac{\mathbb{E}\left[\in\text{Series}\left[\pi_i\, \underline{p}_{2+i} + \pi_j\, \underline{p}_{2+j},\ 0,\ 0\right]\right]}{64\,\pi^6}$$

Out[∘]=

True

Invariance Under R2c



In[∘]:=
```
lhs = ∫(𝔼[π_i p_i + π_j p_j] ℒ[-1, i, j + 1] ℒ[1, i + 1, j] ℒ[0, i + 1] ℒ[1, j + 1])
   ⅆ{x_i, x_j, p_i, p_j, vs_{i+1}, vs_{j+1}}
```

Out[∘]=

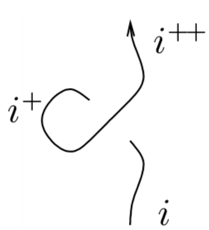
$$\frac{\sqrt{T}\ \mathbb{E}\left[\in\text{Series}\left[\pi_i\, \underline{p}_{2+i} + \pi_j\, \underline{p}_{2+j},\ \frac{1}{2}\left(-1 - 2\,\pi_j\, \underline{p}_{2+j}\right),\ \frac{1}{2}\,\pi_j\, \underline{p}_{2+j}\right]\right]}{64\,\pi^6}$$

In[∘]:=
```
rhs = ∫(𝔼[π_i p_i + π_j p_j] ℒ[Dot, i] ℒ[Dot, j] ℒ[Dot, i + 1] ℒ[Dot, j + 1] ℒ[0, i + 1] ℒ[1, j + 1])
    ⅆ{x_i, x_j, p_i, p_j, vs_{i+1}, vs_{j+1}};
lhs == rhs
```

Out[∘]=

True

Invariance Under R1l



In[∘]:=
```
lhs = 8 π^3 ∫ (𝔼[π_i p_i] ℒ[1, i + 1, i] ℒ[1, i + 1]) ⅆ{x_i, p_i, vs_{i+1}}
rhs = (2 π ∫ (𝔼[π_i p_i] ℒ[Dot, i]) ⅆ{x_i, p_i}) /. p_{i+1} → p_{i+2};
lhs == rhs
```
Out[∘]=

$-\mathbb{i}\,\mathbb{E}\left[\in\text{Series}\left[\pi_i\, \underline{p}_{2+i}, 0, 0\right]\right]$

Out[∘]=

True

Invariance Under R1r

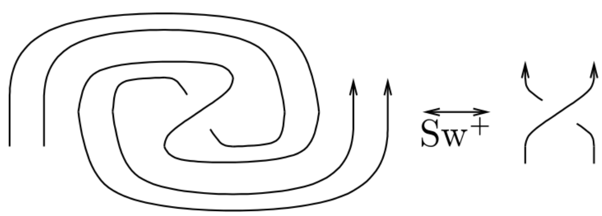
In[∘]:=
```
lhs = 8 π^3 ∫ (𝔼[π_i p_i] ℒ[1, i, i + 1] ℒ[-1, i + 1]) ⅆ{x_i, p_i, vs_{i+1}}
rhs = (2 π ∫ (𝔼[π_i p_i] ℒ[Dot, i]) ⅆ{x_i, p_i}) /. p_{i+1} → p_{i+2};
lhs == rhs
```
Out[∘]=

$-\mathbb{i}\,\mathbb{E}\left[\in\text{Series}\left[\pi_i\, \underline{p}_{2+i}, 0, 0\right]\right]$

Out[∘]=

True

Invariance Under Sw




```
CF /@ {ℒ[1, j], ℒ[1, i, j]}
```

In[∘]:=
$$\text{lhs} = \int \left(\mathbb{E}\left[\underline{\pi}_i\, \underline{p}_i + \underline{\pi}_j\, \underline{p}_j\right] \mathcal{L}[1, i, j]\, \mathcal{L}[-1, i]\, \mathcal{L}[1, i+1]\, \mathcal{L}[-1, j]\, \mathcal{L}[1, j+1]\right) d\left\{vs_i, vs_j, \underline{p}_{i+1}, \underline{p}_{j+1}, \underline{x}_{i+1}, \underline{x}_{j+1}\right\}$$

$$\text{rhs} = \int \left(\mathbb{E}\left[\underline{\pi}_i\, \underline{p}_i + \underline{\pi}_j\, \underline{p}_j\right] \mathcal{L}[1, i, j]\, \mathcal{L}[0, i]\, \mathcal{L}[0, i+1]\, \mathcal{L}[0, j]\, \mathcal{L}[0, j+1]\right) d\left\{vs_i, vs_j, \underline{p}_{i+1}, \underline{p}_{j+1}, \underline{x}_{i+1}, \underline{x}_{j+1}\right\};$$

lhs == rhs

Out[∘]=
$$\frac{1}{64\,\pi^6} \sqrt{T}\, \mathbb{E}\Big[\epsilon\text{Series}\Big[T\, p_{1+i}\, \underline{\pi}_i + p_{1+j}\left(\underline{\pi}_i - T\, \underline{\pi}_i + \underline{\pi}_j\right),$$
$$-\frac{1}{2} + \frac{1}{2} T\, p_{1+i}\, p_{1+j}\, \underline{\pi}_i\left(-\underline{\pi}_i + T\, \underline{\pi}_i - 2\, \underline{\pi}_j\right) - \frac{1}{2} T\, p_{1+j}^2\, \underline{\pi}_i\left(-\underline{\pi}_i + T\, \underline{\pi}_i - 2\, \underline{\pi}_j\right) + p_{1+j}\left(T\, \underline{\pi}_i - \underline{\pi}_j\right),$$
$$\frac{1}{4} T\, p_{1+j}^2\, \underline{\pi}_i\left(-3\, \underline{\pi}_i + 5\, T\, \underline{\pi}_i - 10\, \underline{\pi}_j\right) - \frac{1}{4} T\, p_{1+i}\, p_{1+j}\, \underline{\pi}_i\left(-\underline{\pi}_i + 3\, T\, \underline{\pi}_i - 6\, \underline{\pi}_j\right) -$$
$$\frac{1}{6} T^2\, p_{1+i}^2\, p_{1+j}\, \underline{\pi}_i^2\left(-\underline{\pi}_i + T\, \underline{\pi}_i - 3\, \underline{\pi}_j\right) + \frac{1}{2} p_{1+j}\left(-T\, \underline{\pi}_i + \underline{\pi}_j\right) +$$
$$\frac{1}{6} T\, p_{1+i}\, p_{1+j}^2\, \underline{\pi}_i\left(\underline{\pi}_i^2 - 5\, T\, \underline{\pi}_i^2 + 4\, T^2\, \underline{\pi}_i^2 + 3\, \underline{\pi}_i\, \underline{\pi}_j - 12\, T\, \underline{\pi}_i\, \underline{\pi}_j + 3\, \underline{\pi}_j^2\right) -$$
$$\frac{1}{6} T\, p_{1+j}^3\, \underline{\pi}_i\left(\underline{\pi}_i^2 - 4\, T\, \underline{\pi}_i^2 + 3\, T^2\, \underline{\pi}_i^2 + 3\, \underline{\pi}_i\, \underline{\pi}_j - 9\, T\, \underline{\pi}_i\, \underline{\pi}_j + 3\, \underline{\pi}_j^2\right)\Big]\Big]$$

Out[∘]=
True