Pensieve header: Proof of invariance of $\rho_1$ using integration techniques.

## Initialization

In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\HigherRank"];
Once[<< KnotTheory`; << Rot.m];
<< FormalGaussianIntegration.m;
```

## The $\rho_1$ Integrand

Adopted from pensieve://Projects/APAI/PerturbedGaussianIntegration.nb.

In[∘]:=
```
q[s_, i_, j_] := x_i ((1 - T^s) p_{i+1} + (T^s - 1) p_{j+1});
r_1[s_, i_, j_] :=
  s (-1 + 2 p_i x_i - 2 p_j x_i + (T^s - 1) p_i p_j x_i^2 + (1 - T^s) p_j^2 x_i^2 - 2 p_i p_j x_i x_j + 2 p_j^2 x_i x_j) / 2;
γ_1[φ_, k_] := ϵ φ (1 / 2 - x_k p_k);
ρ1i[s_, i_, j_] := T^(s/2) 𝔼[-q[s, i, j] + ϵ r_1[s, i, j] + O[ϵ]^2];
ρ1i[φ_, k_] := T^(φ/2) 𝔼[-x_k (p_k - p_{k+1}) + γ_1[φ, k] + O[ϵ]^2];
ρ1i[End, k_] := 𝔼[-x_k p_k + O[ϵ]^2];
ρ1i[K_] := Module[{Cs, φ, n, c, k, ℰ},
   {Cs, φ} = Rot[K]; n = Length[Cs];
   ℰ = ρ1i[End, 2 n + 1];
   Do[ℰ *= ρ1i @@ c, {c, Cs}];
   Do[ℰ *= ρ1i[φ⟦k⟧, k], {k, 2 n}];
   CF@ℰ
  ];
ρ1vs[K_] := Union @@ Table[{x_i, p_i}, {i, 2 Crossings[K] + 1}]
```

In[∘]:=
```
ρ1i[Knot[3, 1]]
```

Out[∘]=

$$\frac{1}{T^2}\,\mathbb{E}\Big[\epsilon\text{Series}\Big[-p_1 x_1+p_2 x_1-p_2 x_2+\frac{p_3 x_2}{T}+\frac{(-1+T)\,p_6 x_2}{T}-p_3 x_3+p_4 x_3+\frac{(-1+T)\,p_2 x_4}{T}-p_4 x_4+\frac{p_5 x_4}{T}-p_5 x_5+p_6 x_5+\frac{(-1+T)\,p_4 x_6}{T}-p_6 x_6+\frac{p_7 x_6}{T}-p_7 x_7,\; 1-p_2 x_2+p_5 x_2+\frac{(-1+T)\,p_2 p_5 x_2^2}{2T}-\frac{(-1+T)\,p_5^2 x_2^2}{2T}+p_1 x_4-p_1^2 x_1 x_4+p_1 p_4 x_1 x_4-\frac{(-1+T)\,p_1^2 x_4^2}{2T}+\frac{(-1+T)\,p_1 p_4 x_4^2}{2T}+p_2 p_5 x_2 x_5-p_5^2 x_2 x_5+p_3 x_6-p_6 x_6-p_3^2 x_3 x_6+p_3 p_6 x_3 x_6-\frac{(-1+T)\,p_3^2 x_6^2}{2T}+\frac{(-1+T)\,p_3 p_6 x_6^2}{2T}\Big]\Big]$$

In[∘]:=
```
ρ1vs[Knot[3, 1]]
```

Out[∘]=

$$\{p_1, p_2, p_3, p_4, p_5, p_6, p_7, x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$$

```
In[∘]:= K = Knot[5, 2]; ρ1i[K]
```

Out[∘]=

$$\frac{1}{T^3}\,\mathbb{E}\Big[\epsilon\text{Series}\Big[-p_1 x_1+p_2 x_1-p_2 x_2+\frac{p_3 x_2}{T}+\frac{(-1+T)\,p_8 x_2}{T}-p_3 x_3+p_4 x_3+\frac{(-1+T)\,p_2 x_4}{T}-p_4 x_4+\frac{p_5 x_4}{T}-p_5 x_5+p_6 x_5-p_6 x_6+\frac{p_7 x_6}{T}+\frac{(-1+T)\,p_{10} x_6}{T}-p_7 x_7+p_8 x_7+\frac{(-1+T)\,p_4 x_8}{T}-p_8 x_8+\frac{p_9 x_8}{T}-p_9 x_9+p_{10} x_9+\frac{(-1+T)\,p_6 x_{10}}{T}-p_{10} x_{10}+\frac{p_{11} x_{10}}{T}-p_{11} x_{11},$$
$$2-p_2 x_2+p_7 x_2+\frac{(-1+T)\,p_2 p_7 x_2^2}{2T}-\frac{(-1+T)\,p_7^2 x_2^2}{2T}+p_1 x_4-p_1^2 x_1 x_4+p_1 p_4 x_1 x_4-\frac{(-1+T)\,p_1^2 x_4^2}{2T}+\frac{(-1+T)\,p_1 p_4 x_4^2}{2T}-p_6 x_6+p_9 x_6+\frac{(-1+T)\,p_6 p_9 x_6^2}{2T}-\frac{(-1+T)\,p_9^2 x_6^2}{2T}+p_2 p_7 x_2 x_7-p_7^2 x_2 x_7+p_3 x_8-p_8 x_8-p_3^2 x_3 x_8+p_3 p_8 x_3 x_8-\frac{(-1+T)\,p_3^2 x_8^2}{2T}+\frac{(-1+T)\,p_3 p_8 x_8^2}{2T}-p_9 x_9+p_6 p_9 x_6 x_9-p_9^2 x_6 x_9+p_5 x_{10}-p_5^2 x_5 x_{10}+p_5 p_{10} x_5 x_{10}-\frac{(-1+T)\,p_5^2 x_{10}^2}{2T}+\frac{(-1+T)\,p_5 p_{10} x_{10}^2}{2T}\Big]\Big]$$

```
In[∘]:= K = Knot[5, 2]; ∫ρ1i[K] ⅆ(ρ1vs@K)
```

Out[∘]=

$$-\frac{i\,T\,\mathbb{E}\Big[\epsilon\text{Series}\Big[0,\frac{(-1+T)^2\,(5-4T+5T^2)}{(2-3T+2T^2)^2}\Big]\Big]}{2048\,\pi^{11}\,(2-3T+2T^2)}$$

```
In[∘]:= K = Knot[8, 19]; ∫ρ1i[K] ⅆ(ρ1vs@K)
```

Out[∘]=

$$-\frac{i\,T^3\,\mathbb{E}\Big[\epsilon\text{Series}\Big[0,-\frac{(-1+T)^2\,(1+T^4)\,(3+4T^3+3T^6)}{(1-T+T^2)^2\,(1-T^2+T^4)^2}\Big]\Big]}{131\,072\,\pi^{17}\,(1-T+T^3-T^5+T^6)}$$

## Concatenating edges

```
In[∘]:= lhs = ∫(𝔼[π_i p_i] ρ1i[φ1, i] ρ1i[φ2, i + 1]) ⅆ{x_i, p_i, x_{i+1}, p_{i+1}}
        rhs = ∫(𝔼[π_i p_i] ρ1i[φ1 + φ2, i]) ⅆ{x_i, p_i}
```

Out[∘]=

$$\frac{T^{\frac{\varphi1}{2}+\frac{\varphi2}{2}}\,\mathbb{E}\Big[\epsilon\text{Series}\Big[p_{2+i}\,\pi_i,\ \frac{1}{2}\,(-\varphi1-\varphi2)-(\varphi1+\varphi2)\,p_{2+i}\,\pi_i\Big]\Big]}{4\,\pi^2}$$

Out[∘]=

$$-\frac{i\,T^{\frac{\varphi1}{2}+\frac{\varphi2}{2}}\,\mathbb{E}\Big[\epsilon\text{Series}\Big[p_{1+i}\,\pi_i,\ \frac{1}{2}\,(-\varphi1-\varphi2)-(\varphi1+\varphi2)\,p_{1+i}\,\pi_i\Big]\Big]}{2\,\pi}$$

## Invariance Under Reidemeister 3b

In[∘]:=
```
lhs = ∫(𝔼[πi pi + πj pj + πk pk + O[ϵ]^2] ρ1i[1, i, j] ρ1i[1, i + 1, k] ρ1i[1, j + 1, k + 1]
      ρ1i[0, i] ρ1i[0, j] ρ1i[0, k] ρ1i[0, i + 1] ρ1i[0, j + 1] ρ1i[0, k + 1])
    ⅆ{xi, xj, xk, pi, pj, pk, xi+1, xj+1, xk+1, pi+1, pj+1, pk+1}
rhs = ∫(𝔼[πi pi + πj pj + πk pk + O[ϵ]^2] ρ1i[1, j, k] ρ1i[1, i, k + 1] ρ1i[1, i + 1, j + 1]
      ρ1i[0, i] ρ1i[0, j] ρ1i[0, k] ρ1i[0, i + 1] ρ1i[0, j + 1] ρ1i[0, k + 1])
    ⅆ{xi, xj, xk, pi, pj, pk, xi+1, xj+1, xk+1, pi+1, pj+1, pk+1};
lhs == rhs
```

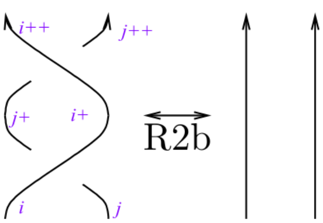
Out[∘]=

$$\frac{1}{64\,\pi^6} T^{3/2}\, \mathbb{E}\Big[\epsilon\text{Series}\Big[T^2 p_{2+i}\,\pi_i - T\,p_{2+j}\,(-\pi_i + T\,\pi_i - \pi_j) + p_{2+k}\,(\pi_i - T\,\pi_i + \pi_j - T\,\pi_j + \pi_k),$$
$$-\frac{3}{2} + \frac{1}{2} T^3 p_{2+i}\, p_{2+j}\,\pi_i\,(-\pi_i + T\,\pi_i - 2\,\pi_j) - \frac{1}{2} T^3 p_{2+j}^2\,\pi_i\,(-\pi_i + T\,\pi_i - 2\,\pi_j) +$$
$$T\,p_{2+j}\,(T\,\pi_i - \pi_j) - \frac{1}{2} T\,p_{2+k}^2\,(\pi_i + \pi_j)\,(-\pi_i + T\,\pi_i - \pi_j + T\,\pi_j - 2\,\pi_k) +$$
$$\frac{1}{2} T^2 p_{2+i}\, p_{2+k}\,\pi_i\,(-\pi_i + T\,\pi_i - 2\,\pi_j + 2\,T\,\pi_j - 2\,\pi_k) + p_{2+k}\,(T\,\pi_i - \pi_j + 2\,T\,\pi_j - 2\,\pi_k) - \frac{1}{2} T\,p_{2+j}\,p_{2+k}$$
$$\left(\pi_i^2 - 2\,T\,\pi_i^2 + T^2\,\pi_i^2 + 2\,\pi_i\,\pi_j - 4\,T\,\pi_i\,\pi_j + 2\,T^2\,\pi_i\,\pi_j + \pi_j^2 - T\,\pi_j^2 + 2\,\pi_i\,\pi_k - 2\,T\,\pi_i\,\pi_k + 2\,\pi_j\,\pi_k\right)\Big]\Big]$$

Out[∘]=

True

## Invariance Under Reidemeister 2b




Territory: $(x \mid p)_{(i|j)^+},\ x_{i|j}$.

In[∘]:=
```
lhs = ∫(𝔼[πi pi + πj pj] ρ1i[1, i, j] ρ1i[-1, i + 1, j + 1] ρ1i[0, i]
      ρ1i[0, j] ρ1i[0, i + 1] ρ1i[0, j + 1]) ⅆ{xi, xj, pi, pj, xi+1, xj+1, pi+1, pj+1}
rhs = ∫(𝔼[πi pi + πj pj] ρ1i[0, i] ρ1i[0, j] ρ1i[0, i + 1] ρ1i[0, j + 1])
    ⅆ{xi, xj, pi, pj, xi+1, xj+1, pi+1, pj+1};
lhs == rhs
```

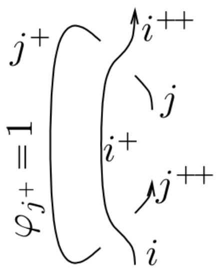
Out[∘]=

$$\frac{\mathbb{E}[\epsilon\text{Series}[p_{2+i}\,\pi_i + p_{2+j}\,\pi_j,\ 0]]}{16\,\pi^4}$$

Out[∘]=

True

## Invariance Under R2c




In[∘]:=
```
lhs = ∫(𝔼[π_i p_i + π_j p_j] ρ1i[-1, i, j + 1] ρ1i[1, i + 1, j] ρ1i[0, i]
    ρ1i[0, j] ρ1i[0, i + 1] ρ1i[1, j + 1]) ⅆ{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}}
```

Out[∘]=

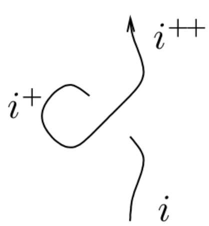
$$\frac{\sqrt{T}\ \mathbb{E}\left[\epsilon\text{Series}\left[p_{2+i}\,\pi_i + p_{2+j}\,\pi_j,\ -\frac{1}{2} - p_{2+j}\,\pi_j\right]\right]}{16\,\pi^4}$$

In[∘]:=
```
rhs = ∫(𝔼[π_i p_i + π_j p_j] ρ1i[0, i] ρ1i[0, j] ρ1i[0, i + 1] ρ1i[1, j + 1])
    ⅆ{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}};
lhs == rhs
```

Out[∘]=

True

## Invariance Under R1l




In[∘]:=
```
lhs = ∫(𝔼[π_i p_i] ρ1i[1, i + 1, i] ρ1i[0, i] ρ1i[1, i + 1]) ⅆ{x_i, p_i, x_{i+1}, p_{i+1}}
```

Out[∘]=

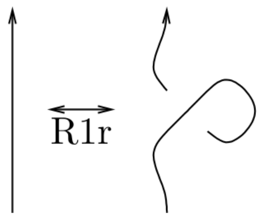
$$\frac{\mathbb{E}[\epsilon\text{Series}[p_{2+i}\,\pi_i,\ 0]]}{4\,\pi^2}$$

In[∘]:=
```
rhs = ∫(𝔼[π_i p_i] ρ1i[0, i] ρ1i[0, i + 1]) ⅆ{x_i, p_i, x_{i+1}, p_{i+1}};
lhs == rhs
```

Out[∘]=

True

## Invariance Under R1r




In[∘]:= lhs = $\int (\mathbb{E}[\pi_i\, p_i]\, \rho 1i[1, i, i+1]\, \rho 1i[0, i]\, \rho 1i[-1, i+1])\, d\{x_i, p_i, x_{i+1}, p_{i+1}\}$

Out[∘]=

$$\frac{\mathbb{E}[\epsilon\text{Series}[p_{2+i}\,\pi_i, 0]]}{4\,\pi^2}$$

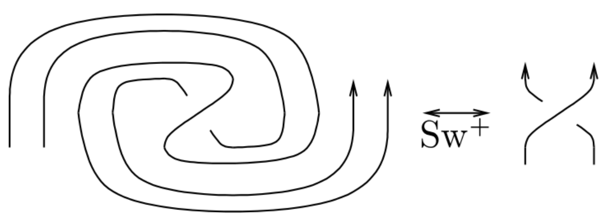
In[∘]:= rhs = $\int (\mathbb{E}[\pi_i\, p_i]\, \rho 1i[0, i]\, \rho 1i[0, i+1])\, d\{x_i, p_i, x_{i+1}, p_{i+1}\}$;

lhs == rhs

Out[∘]=

True

## Invariance Under Sw



In[∘]:=

```
lhs =
  ∫(𝔼[π_i p_i + π_j p_j + ϵ π_{i+1} p_{i+1} + ϵ π_{j+1} p_{j+1} + ξ_{i+1} x_{i+1} + ξ_{j+1} x_{j+1} + O[ϵ]^2] ρ1i[1, i, j] ρ1i[-1, i]
     ρ1i[1, i + 1] ρ1i[-1, j] ρ1i[1, j + 1]) ⅆ{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}}
rhs =
  ∫(𝔼[π_i p_i + π_j p_j + ϵ π_{i+1} p_{i+1} + ϵ π_{j+1} p_{j+1} + ξ_{i+1} x_{i+1} + ξ_{j+1} x_{j+1} + O[ϵ]^2] ρ1i[1, i, j] ρ1i[0, i]
     ρ1i[0, i + 1] ρ1i[0, j] ρ1i[0, j + 1]) ⅆ{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}};
lhs == rhs
```

Out[∘]=

$$\frac{1}{16\pi^4}\sqrt{T}\ \mathbb{E}\Big[\epsilon\text{Series}\Big[T\,p_{2+i}\,\pi_i + p_{2+j}\,(\pi_i - T\,\pi_i + \pi_j) + T\,\pi_i\,\xi_{1+i} + \pi_i\,\xi_{1+j} - T\,\pi_i\,\xi_{1+j} + \pi_j\,\xi_{1+j},$$
$$\frac{1}{2}\,T\,p_{2+i}\,p_{2+j}\,\pi_i\,(-\pi_i + T\,\pi_i - 2\,\pi_j) - \frac{1}{2}\,T\,p_{2+j}^2\,\pi_i\,(-\pi_i + T\,\pi_i - 2\,\pi_j) +$$
$$\frac{1}{2}\,p_{2+i}\,\left(2\,\pi_{1+i} - T\,\pi_i^2\,\xi_{1+j} + T^2\,\pi_i^2\,\xi_{1+j} - 2\,T\,\pi_i\,\pi_j\,\xi_{1+j}\right) + \frac{1}{2}\,p_{2+j}\,\Big(2\,T\,\pi_i - 2\,\pi_j + 2\,\pi_{1+j} -$$
$$T\,\pi_i^2\,\xi_{1+i} + T^2\,\pi_i^2\,\xi_{1+i} - 2\,T\,\pi_i\,\pi_j\,\xi_{1+i} + 2\,T\,\pi_i^2\,\xi_{1+j} - 2\,T^2\,\pi_i^2\,\xi_{1+j} + 4\,T\,\pi_i\,\pi_j\,\xi_{1+j}\Big) +$$
$$\frac{1}{2}\,\Big(-1 + 2\,\pi_{1+i}\,\xi_{1+i} + 2\,T\,\pi_i\,\xi_{1+j} - 2\,\pi_j\,\xi_{1+j} + 2\,\pi_{1+j}\,\xi_{1+j} - T\,\pi_i^2\,\xi_{1+i}\,\xi_{1+j} +$$
$$T^2\,\pi_i^2\,\xi_{1+i}\,\xi_{1+j} - 2\,T\,\pi_i\,\pi_j\,\xi_{1+i}\,\xi_{1+j} + T\,\pi_i^2\,\xi_{1+j}^2 - T^2\,\pi_i^2\,\xi_{1+j}^2 + 2\,T\,\pi_i\,\pi_j\,\xi_{1+j}^2\Big)\Big]\Big]$$

Out[∘]=

True