

Pensieve header: Formal Gaussian integration over an arbitrary “Feynman Ring”.

What must a Feynman Ring F have? (Over some set of labels S with elements x, y, \dots)

- * A vector space over \mathbb{Q} .
- * Has a symmetric linear $Z \mapsto \partial_{x,y} Z$ and a symmetric bilinear $(Z_1, Z_2) \mapsto (\partial_x Z_1)(\partial_y Z_2)$ that satisfy the axioms of (roughly) a connected circuit algebra.
- * Has $q_{x,y} : F \rightarrow \mathbb{Q}$ in some sense dual to some $\theta_{x,y} \in F$.
- * Has $\text{Ev}_{v_s \rightarrow 0} : F \rightarrow F$.

Further axioms must be worked out.

Goals.

- * Define \int .
- * Prove a Fubini theorem.
- * Prove a theorem about the injectivity of the Laplace transform.

Initialization

```
In[*]:= CCF[ $\mathcal{E}$ _] := ExpandDenominator@ExpandNumerator@Together[ $\mathcal{E}$ ];
CCF[ $\mathcal{E}$ _] := Factor[ $\mathcal{E}$ ];
CF[ $\omega$  .  $\mathcal{E}$ _E] := CF[ $\omega$ ] CF /@  $\mathcal{E}$ ;
CF[ $\mathcal{E}$ _List] := CF /@  $\mathcal{E}$ ;
CF[ $\mathcal{E}$ _] := Module[{vs = Cases[ $\mathcal{E}$ , {x | p}_,  $\infty$ ]  $\cup$  {x, p,  $\epsilon$ }, ps, c},
  Total[CoefficientRules[Expand[ $\mathcal{E}$ ], vs] /. (ps_ -> c_) := CCF[c] (Times @@ vsps) ]];
```

The Basic Feynman Ring

```
In[*]:= S = {x, x_, y, z};
qx,y[f_] := (∂x,y f) /. Thread[S -> 0];
θx,y := x y;
f_ ≡ 0 := f === 0;
Evvs_List -> 0[f_] := CF[f /. Thread[vs -> 0]]
```

The ϵ Series Feynman Ring

```

In[*]:= S = {x, y, z, phi, x_, p_, x_, p_};
q_{x,y}[ser_eSeries] := (D_{x,y} ser[[1]]) /. Thread[S -> 0];
theta_{x,y} := x y;
epsilonSeries /: D[ser_eSeries, vs___] := D[#, vs] & /@ ser;
epsilonSeries /: Plus[ss___ epsilonSeries] /; Length[{ss}] > 1 := Module[{l = Min[Length /@ {ss}]},
  epsilonSeries @@ Total[Take[List @@ #, l] & /@ {ss}]]];
epsilonSeries /: t_ + ser_eSeries := MapAt[({# + t}) &, ser, 1];
epsilonSeries /: s1_eSeries * s2_eSeries := epsilonSeries @@ Table[
  Sum[s1[[ii + 1]] s2[[kk - ii + 1]], {ii, 0, kk}], {kk, 0, Min[Length@s1, Length@s2] - 1}];
epsilonSeries /: c_ * ser_eSeries := (c #) & /@ ser;
ser_eSeries == 0 := And @@ ({# == 0} & /@ ser);
epsilonSeries /: Integrate[ser_eSeries, pars__] := epsilonSeries @@ (Integrate[#, pars] & /@ ser);
epsilonSeries /: EV_{vs_List -> 0}[ser_eSeries] := ser /. Thread[vs -> 0];
CF[ser_eSeries] := CF /@ ser;

```

Integration

Using Picard Iteration!

```

In[*]:= E /: E[A_] E[B_] := E[A + B]

```

```

In[*]:= E[sd_SeriesData] /; (List @@ sd) [{1, 2, 4, 6}] == {epsilon, 0, 0, 1} :=
  E[epsilonSeries @@ PadRight[sd[[3], sd[[5], 0]]]

```

pdf

Following a program in Projects/FullDoPeGDO/Engine.nb, we write $Z_\lambda = \sum Z[m] \lambda^m$.

```

Unprotect[Integrate];
Integrate::sing = "How dare you ask me to integrate a singular Gaussian!";
∫ ω_ . E[L_] d(vs_List) := Module[{n, Q, Δ, G, a, b, m, m1, $m}, Clear[Z];
  n = Length@vs;
  Q = Table[q_{vs[[a]], vs[[b]] [L], {a, n}, {b, n}];
  If[(Δ = CF@Det[-Q]) == 0, Message[Integrate::sing]; Return[]];
  G = CF[-Inverse[Q] / 2];
  Z[] = Z[0] = CF[L - Sum[Q[[a, b]] θ_{vs[[a]], vs[[b]]], {a, n}, {b, n}] / 2];
  Z[m_, a_] := Z[m, a] = CF@D[Z[m], vs[[a]]];
  Z[m_, a_, b_] /; a ≤ b := Z[m, a, b] = CF@D[Z[m, a], vs[[b]]];
  Z[m_, a_, b_] /; a > b := Z[m, b, a];
  For[$m = m = 0, m ≤ 2 $m, ++m,
    Z[m + 1] = CF@Sum[Sum[If[G[[a, b]] == 0, 0,
      
$$\frac{G[[a, b]]}{m + 1} (Z[m, a, b] + \text{Sum}[Z[m1, a] Z[m - m1, b], \{m1, 0, m\}])$$
], {a, n}], {b, n}];
    If[!(Z[m + 1] == 0), $m = m + 1; Z[] += Z[m + 1]];
  ];
  PowerExpand@Factor[ω (Δ (2 π)^n)^{-1/2}] E[CF[E_{vs→0}[Z[]]]];
Protect[Integrate];

```

In[*]:= $\int \mathbb{E} \left[\mathbf{i} \lambda \mathbf{x}_1^2 / 2 \right] d\{\mathbf{x}_1\}$

Out[*]= $\frac{(-1)^{1/4} \mathbb{E}[\theta]}{\sqrt{2\pi} \sqrt{\lambda}}$

In[*]:= $\int \mathbb{E} \left[-\mathbf{i} \lambda \mathbf{x}_1^2 / 2 \right] d\{\mathbf{x}_1\}$

Out[*]= $-\frac{(-1)^{3/4} \mathbb{E}[\theta]}{\sqrt{2\pi} \sqrt{\lambda}}$

In[*]:= $\int \mathbb{E} \left[\frac{\mathbf{i}}{2} \{\mathbf{x}_1, \mathbf{x}_2\} \cdot \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{b} & \mathbf{c} \end{pmatrix} \cdot \{\mathbf{x}_1, \mathbf{x}_2\} \right] d\{\mathbf{x}_1, \mathbf{x}_2\}$

Out[*]= $\frac{\mathbb{E}[\theta]}{2 \sqrt{b^2 - a c} \pi}$

$$\text{In[*]:= } \int \mathbb{E} \left[-\lambda \mathbf{x}_1^2 / 2 \right] \mathbb{d} \{ \mathbf{x}_1 \}$$

$$\text{Out[*]= } \frac{\mathbb{E} [\mathbf{0}]}{\sqrt{2 \pi} \sqrt{\lambda}}$$

$$\text{In[*]:= } \text{Clear} [\mathbf{Z}]; \int \mathbb{E} \left[-\mathbf{x}_1^2 / 2 + \xi \mathbf{x}_1 \right] \mathbb{d} \{ \mathbf{x}_1 \}$$

$$\text{Out[*]= } \frac{\mathbb{E} \left[\frac{\xi^2}{2} \right]}{\sqrt{2 \pi}}$$

$$\text{In[*]:= } \int \mathbb{E} \left[-\frac{1}{2} \{ \mathbf{x}_1, \mathbf{x}_2 \} \cdot \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{b} & \mathbf{c} \end{pmatrix} \cdot \{ \mathbf{x}_1, \mathbf{x}_2 \} + \{ \xi_1, \xi_2 \} \cdot \{ \mathbf{x}_1, \mathbf{x}_2 \} \right] \mathbb{d} \{ \mathbf{x}_1, \mathbf{x}_2 \}$$

$$\text{Out[*]= } \frac{\mathbb{E} \left[\frac{\mathbf{c} \xi_1^2 - 2 \mathbf{b} \xi_1 \xi_2 + \mathbf{a} \xi_2^2}{2 (-\mathbf{b}^2 + \mathbf{a} \mathbf{c})} \right]}{2 \sqrt{-\mathbf{b}^2 + \mathbf{a} \mathbf{c}} \pi}$$

$$\text{In[*]:= } \mathbf{I1} = \int \mathbb{E} \left[-\frac{1}{2} \{ \mathbf{x}_1, \mathbf{x}_2 \} \cdot \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{b} & \mathbf{c} \end{pmatrix} \cdot \{ \mathbf{x}_1, \mathbf{x}_2 \} + \{ \xi_1, \xi_2 \} \cdot \{ \mathbf{x}_1, \mathbf{x}_2 \} \right] \mathbb{d} \{ \mathbf{x}_1 \}$$

$$\text{Out[*]= } \frac{\mathbb{E} \left[-\frac{(-\mathbf{b}^2 + \mathbf{a} \mathbf{c}) \mathbf{x}_2^2}{2 \mathbf{a}} + \frac{\xi_1^2}{2 \mathbf{a}} + \frac{\mathbf{x}_2 (-\mathbf{b} \xi_1 + \mathbf{a} \xi_2)}{\mathbf{a}} \right]}{\sqrt{\mathbf{a}} \sqrt{2 \pi}}$$

$$\text{In[*]:= } \int \mathbf{I1} \mathbb{d} \{ \mathbf{x}_2 \}$$

$$\text{Out[*]= } \frac{\mathbb{E} \left[\frac{\mathbf{c} \xi_1^2 - 2 \mathbf{b} \xi_1 \xi_2 + \mathbf{a} \xi_2^2}{2 (-\mathbf{b}^2 + \mathbf{a} \mathbf{c})} \right]}{2 \sqrt{-\mathbf{b}^2 + \mathbf{a} \mathbf{c}} \pi}$$

$$\text{In[*]:= } \int \mathbb{E} \left[\xi \mathbf{x} + \eta \mathbf{y} + \mathbf{z} (\mathbf{x} - \mathbf{y}) + \mathbf{x}^2 \right] \mathbb{d} \{ \mathbf{x}, \mathbf{z} \}$$

$$\text{Out[*]= } -\frac{\mathbf{i} \mathbb{E} [\mathbf{y} (\mathbf{y} + \eta + \xi)]}{2 \pi}$$

Integration of ϵ Series

$$\text{In[*]:= } \int \mathbb{E} \left[-\mathbf{x}^2 / 2 + \epsilon \mathbf{x}^3 / 6 + \mathbf{0} [\epsilon]^7 \right] \mathbb{d} \{ \mathbf{x} \}$$

$$\text{Out[*]= } \frac{\mathbb{E} \left[\epsilon \text{Series} \left[\mathbf{0}, \mathbf{0}, \frac{5}{24}, \mathbf{0}, \frac{5}{16}, \mathbf{0}, \frac{1105}{1152} \right] \right]}{\sqrt{2 \pi}}$$

$$\text{In[*]:= } \int \mathbb{E} \left[-\phi^2 / 2 + \epsilon \phi^4 / 24 + \mathbf{0}[\epsilon]^7 \right] \mathfrak{d}\{\phi\}$$

$$\text{Out[*]= } \frac{\mathbb{E} \left[\epsilon \text{Series} \left[0, \frac{1}{8}, \frac{1}{12}, \frac{11}{96}, \frac{17}{72}, \frac{619}{960}, \frac{709}{324} \right] \right]}{\sqrt{2\pi}}$$

$$\text{In[*]:= } \int \mathbb{E} \left[\mathbf{p} \mathbf{x} + \epsilon \mathbf{p}^2 \mathbf{x} + \mathbf{0}[\epsilon]^5 \right] \mathfrak{d}\{\mathbf{p}, \mathbf{x}\}$$

$$\text{Out[*]= } \frac{\mathfrak{i} \mathbb{E} \left[\epsilon \text{Series} \left[0, 0, 0, 0, 0 \right] \right]}{2\pi}$$

$$\text{In[*]:= } \text{Block} \left[\{ \mathfrak{\pi} = \text{Total} @ \text{Select} \left[\text{MonomialList} \left[\#, \{ \epsilon, \mathbf{x}, \mathbf{p} \} \right], \right. \right. \right. \\ \left. \left. \left. \begin{array}{l} \text{mon} \mapsto \text{And} \left[\right. \right. \right. \\ \left. \left. \left. \begin{array}{l} \text{Exponent} \left[\text{mon}, \epsilon \right] \leq 2, \\ \text{Exponent} \left[\text{mon}, \mathbf{x} \right] = \text{Exponent} \left[\text{mon}, \mathbf{p} \right] \end{array} \right. \right. \right. \\ \left. \left. \left. \right] \right. \right. \\ \left. \left. \left. \right] \&\}, \right. \right. \\ \left. \int \mathbb{E} \left[\mathbf{p} \mathbf{x} + \mathbf{a} \mathbf{x}^2 \mathbf{p} + \epsilon \mathbf{b} \mathbf{x}^3 \mathbf{p}^3 \right] \mathfrak{d}\{\mathbf{p}, \mathbf{x}\} \right]$$

$$\text{Out[*]= } \frac{\mathfrak{i} \mathbb{E} \left[-6 \mathbf{b} \epsilon + 342 \mathbf{b}^2 \epsilon^2 \right]}{2\pi}$$

$$\text{In[*]:= } \text{Block} \left[\{ \mathfrak{\pi} = \text{Total} @ \text{Select} \left[\text{MonomialList} \left[\#, \{ \epsilon, \mathbf{x}, \mathbf{p} \} \right], \right. \right. \right. \\ \left. \left. \left. \begin{array}{l} \text{mon} \mapsto \text{And} \left[\right. \right. \right. \\ \left. \left. \left. \begin{array}{l} \text{Exponent} \left[\text{mon}, \epsilon \right] < 4, \\ \text{Exponent} \left[\text{mon}, \mathbf{x} \right] - \text{Exponent} \left[\text{mon}, \mathbf{p} \right] \leq 3 \end{array} \right. \right. \right. \\ \left. \left. \left. \right] \right. \right. \\ \left. \left. \left. \right] \&\}, \right. \right. \\ \left. \int \mathbb{E} \left[\mathbf{p} \mathbf{x} + \mathbf{a} \mathbf{x}^2 \mathbf{p} + \epsilon \mathbf{b} \mathbf{p}^2 \mathbf{x} \right] \mathfrak{d}\{\mathbf{p}, \mathbf{x}\} \right]$$

$$\text{Out[*]= } \frac{\mathfrak{i} \mathbb{E} \left[-6 \mathbf{a} \mathbf{b} \epsilon + 162 \mathbf{a}^2 \mathbf{b}^2 \epsilon^2 - 9072 \mathbf{a}^3 \mathbf{b}^3 \epsilon^3 \right]}{2\pi}$$

```
In[*]:= Block[{$\pi = Total@Select[MonomialList[#, {\epsilon, x, p}],
    mon \mapsto And[
        Exponent[mon, \epsilon] < 4,
        Exponent[mon, x] - Exponent[mon, p] \le 3 - Exponent[mon, \epsilon]
    ] &},
    \int \mathbb{E}[p x + a x^2 p + \epsilon b p^2 x] d\{p, x\}]
```

Out[*]=

$$-\frac{i \mathbb{E}[-6 a b \epsilon + 162 a^2 b^2 \epsilon^2 - 9072 a^3 b^3 \epsilon^3]}{2 \pi}$$

```
In[*]:= MatrixForm@Table[
    \int \mathbb{E}[x_1 p_2 + x_2 p_3 + x_3 p_1 + \xi_i x_i + \pi_j p_j] d\{x_1, x_2, x_3, p_1, p_2, p_3\},
    {i, 3}, {j, 3}]
```

Out[*]//MatrixForm=

$$\begin{pmatrix} -\frac{i \mathbb{E}[0]}{8 \pi^3} & -\frac{i \mathbb{E}[-\pi_2 \xi_1]}{8 \pi^3} & -\frac{i \mathbb{E}[0]}{8 \pi^3} \\ -\frac{i \mathbb{E}[0]}{8 \pi^3} & -\frac{i \mathbb{E}[0]}{8 \pi^3} & -\frac{i \mathbb{E}[-\pi_3 \xi_2]}{8 \pi^3} \\ -\frac{i \mathbb{E}[-\pi_1 \xi_3]}{8 \pi^3} & -\frac{i \mathbb{E}[0]}{8 \pi^3} & -\frac{i \mathbb{E}[0]}{8 \pi^3} \end{pmatrix}$$