

Pensieve header: Formal Gaussian integration over an arbitrary “Feynman Ring”.

What must a Feynman Ring F have? (Over some set of labels S with elements x, y, \dots)

- * A vector space over \mathbb{Q} .
- * Has a symmetric linear $Z \mapsto \partial_{x,y} Z$ and a symmetric bilinear $(Z_1, Z_2) \mapsto (\partial_x Z_1)(\partial_y Z_2)$ that satisfy the axioms of (roughly) a connected circuit algebra.
- * Has $q_{x,y} : F \rightarrow \mathbb{Q}$ in some sense dual to some $\theta_{x,y} \in F$.
- * Has $\text{Ev}_{v_s \rightarrow 0} : F \rightarrow F$.

Further axioms must be worked out.

Goals.

- * Define \int .
- * Prove a Fubini theorem.
- * Prove a theorem about the injectivity of the Laplace transform.

Initialization

```
In[ ]:= CCF[ $\mathcal{E}$ _] := ExpandDenominator@ExpandNumerator@Together[ $\mathcal{E}$ ];
CCF[ $\mathcal{E}$ _] := Factor[ $\mathcal{E}$ ];
CF[ $\omega$  .  $\mathcal{E}$ _E] := CF[ $\omega$ ] CF /@  $\mathcal{E}$ ;
CF[ $\mathcal{E}$ _List] := CF /@  $\mathcal{E}$ ;
CF[ $\mathcal{E}$ _] := Module[{vs = Cases[ $\mathcal{E}$ , {x | p}_,  $\infty$ ]  $\cup$  {x, p,  $\epsilon$ }, ps, c},
  Total[CoefficientRules[Expand[ $\mathcal{E}$ ], vs] /. (ps_ -> c_) := CCF[c] (Times @@ vsps) ]];
```

The Basic Feynman Ring

```
In[ ]:= S = {x, x_, y, z};
qx,y[f_] := (∂x,y f) /. Thread[S -> 0];
θx,y := x y;
f_ == 0 := f === 0;
Evvs_List -> 0[f_] := CF[f /. Thread[vs -> 0]]
```

The ϵ Series Feynman Ring

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In[*]:= S = {x, y, z,  $\phi$ , x_, p_, x_, p_};
qx,y[ser_eSeries] := (∂x,yser[[1]]) /. Thread[S → 0];
 $\theta_{x,y}$  := x y;
 $\epsilon$ Series /: D[ser_eSeries, vs___] := D[#, vs] & /@ ser;
 $\epsilon$ Series /: s1_eSeries + s2_eSeries :=
   $\epsilon$ Series @@ Table[s1[[kk]] + s2[[kk]], {kk, Min[Length@s1, Length@s2]};
 $\epsilon$ Series /: t_ + ser_eSeries := MapAt[({# + t}) &, ser, 1];
 $\epsilon$ Series /: s1_eSeries * s2_eSeries :=  $\epsilon$ Series @@ Table[
  Sum[s1[[ii + 1]] s2[[kk - ii + 1]], {ii, 0, kk}], {kk, 0, Min[Length@s1, Length@s2] - 1};
 $\epsilon$ Series /: c_ * ser_eSeries := (c #) & /@ ser;
ser_eSeries == 0 := And @@ ({# == 0} & /@ ser);
 $\epsilon$ Series /: Integrate[ser_eSeries, pars_] :=  $\epsilon$ Series @@ (Integrate[#, pars] & /@ ser);
 $\epsilon$ Series /: Evvs_List→0[ser_eSeries] := ser /. Thread[vs → 0];
CF[ser_eSeries] := CF /@ ser;

```

Integration

Using Picard Iteration!

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In[*]:= E /: E[A_] E[B_] := E[A + B]

```

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In[*]:= E[sd_SeriesData] /; (List @@ sd) [{1, 2, 4, 6}] == { $\epsilon$ , 0, 0, 1} :=
  E[ $\epsilon$ Series @@ PadRight[sd[[3]], sd[[5], 0]]

```

```

In[*]:= Unprotect[Integrate];
Integrate::sing = "How dare you ask me to integrate a singular Gaussian!";

$$\int \omega_{-} \cdot \mathbb{E}[L_{-}] \, d(vs\_List) := \text{Module}[\{n, Q, \Delta, G, Z, e, \lambda, a, b\},$$

  n = Length@vs;
  Q = Table[qvs[[a]],vs[[b]][L], {a, n}, {b, n}];
  If[( $\Delta = \text{CF@Det}[-Q]$ ) == 0, Message[Integrate::sing]; Return[]];
  Z = CF[L - Sum[Q[[a, b]] evs[[a]],vs[[b]], {a, n}, {b, n}] / 2];
  G = Inverse[Q] / 2;
  While[e = D[Z,  $\lambda$ ];
    Do[e += G[[a, b]] (D[Z, vs[[a]], vs[[b]] + D[Z, vs[[a]]] D[Z, vs[[b]]]), {a, n}, {b, n}];
    e = CF[e];
    ! (e == 0), Z = CF[Z -  $\int_0^\lambda e \, d\lambda$ ];
  ];
  PowerExpand@Factor[ $\omega (\Delta (2\pi)^n)^{-1/2}$ ] E[CF[Evs→0[Z] /.  $\lambda \rightarrow 1$ ]];
Protect[Integrate];

```

In[*]:= $\int \mathbb{E}[\mathbf{i} \lambda \mathbf{x}_1^2 / 2] \, d\{\mathbf{x}_1\}$

Out[*]=
$$\frac{(-1)^{1/4} \mathbb{E}[0]}{\sqrt{2\pi} \sqrt{\lambda}}$$

In[*]:= $\int \mathbb{E}[-\mathbf{i} \lambda \mathbf{x}_1^2 / 2] \, d\{\mathbf{x}_1\}$

Out[*]=
$$-\frac{(-1)^{3/4} \mathbb{E}[0]}{\sqrt{2\pi} \sqrt{\lambda}}$$

In[*]:= $\int \mathbb{E}\left[\frac{\mathbf{i}}{2} \{\mathbf{x}_1, \mathbf{x}_2\} \cdot \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{b} & \mathbf{c} \end{pmatrix} \cdot \{\mathbf{x}_1, \mathbf{x}_2\}\right] \, d\{\mathbf{x}_1, \mathbf{x}_2\}$

Out[*]=
$$\frac{\mathbb{E}[0]}{2 \sqrt{\mathbf{b}^2 - \mathbf{a} \mathbf{c}} \pi}$$

In[*]:= $\int \mathbb{E}[-\lambda \mathbf{x}_1^2 / 2] \, d\{\mathbf{x}_1\}$

Out[*]=
$$\frac{\mathbb{E}[0]}{\sqrt{2\pi} \sqrt{\lambda}}$$

$$\text{In[*]} := \int \mathbb{E} \left[-\mathbf{x}_1^2 / 2 + \xi \mathbf{x}_1 \right] \mathfrak{d} \{ \mathbf{x}_1 \}$$

$$\text{Out[*]} = \frac{\mathbb{E} \left[\frac{\xi^2}{2} \right]}{\sqrt{2\pi}}$$

$$\text{In[*]} := \int \mathbb{E} \left[-\frac{1}{2} \{ \mathbf{x}_1, \mathbf{x}_2 \} \cdot \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{b} & \mathbf{c} \end{pmatrix} \cdot \{ \mathbf{x}_1, \mathbf{x}_2 \} + \{ \xi_1, \xi_2 \} \cdot \{ \mathbf{x}_1, \mathbf{x}_2 \} \right] \mathfrak{d} \{ \mathbf{x}_1, \mathbf{x}_2 \}$$

$$\text{Out[*]} = \frac{\mathbb{E} \left[\frac{c \xi_1^2 - 2b \xi_1 \xi_2 + a \xi_2^2}{2(-b^2 + ac)} \right]}{2 \sqrt{-b^2 + ac} \pi}$$

$$\text{In[*]} := \mathbf{I1} = \int \mathbb{E} \left[-\frac{1}{2} \{ \mathbf{x}_1, \mathbf{x}_2 \} \cdot \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{b} & \mathbf{c} \end{pmatrix} \cdot \{ \mathbf{x}_1, \mathbf{x}_2 \} + \{ \xi_1, \xi_2 \} \cdot \{ \mathbf{x}_1, \mathbf{x}_2 \} \right] \mathfrak{d} \{ \mathbf{x}_1 \}$$

$$\text{Out[*]} = \frac{\mathbb{E} \left[-\frac{(-b^2 + ac) x_2^2}{2a} + \frac{\xi_1^2}{2a} + \frac{x_2(-b \xi_1 + a \xi_2)}{a} \right]}{\sqrt{a} \sqrt{2\pi}}$$

$$\text{In[*]} := \int \mathbf{I1} \mathfrak{d} \{ \mathbf{x}_2 \}$$

$$\text{Out[*]} = \frac{\mathbb{E} \left[\frac{c \xi_1^2 - 2b \xi_1 \xi_2 + a \xi_2^2}{2(-b^2 + ac)} \right]}{2 \sqrt{-b^2 + ac} \pi}$$

$$\text{In[*]} := \int \mathbb{E} \left[\xi \mathbf{x} + \eta \mathbf{y} + \mathbf{z} (\mathbf{x} - \mathbf{y}) + \mathbf{x}^2 \right] \mathfrak{d} \{ \mathbf{x}, \mathbf{z} \}$$

$$\text{Out[*]} = \frac{i \mathbb{E} [\mathbf{y} (\mathbf{y} + \eta + \xi)]}{2\pi}$$

Integration of ϵ Series

$$\text{In[*]} := \int \mathbb{E} \left[-\mathbf{x}^2 / 2 + \epsilon \mathbf{x}^3 / 6 + \mathbf{0}[\epsilon]^7 \right] \mathfrak{d} \{ \mathbf{x} \}$$

$$\text{Out[*]} = \frac{\mathbb{E} \left[\epsilon \text{Series} \left[0, 0, \frac{5}{24}, 0, \frac{5}{16}, 0, \frac{1105}{1152} \right] \right]}{\sqrt{2\pi}}$$

$$\text{In[*]} := \int \mathbb{E} \left[-\phi^2 / 2 + \epsilon \phi^4 / 24 + \mathbf{0}[\epsilon]^7 \right] \mathfrak{d} \{ \phi \}$$

$$\text{Out[*]} = \frac{\mathbb{E} \left[\epsilon \text{Series} \left[0, \frac{1}{8}, \frac{1}{12}, \frac{11}{96}, \frac{17}{72}, \frac{619}{960}, \frac{709}{324} \right] \right]}{\sqrt{2\pi}}$$

$$\text{In[*]:= } \int \mathbb{E} [p x + \epsilon p^2 x + 0[\epsilon]^5] \, d\{p, x\}$$

$$\text{Out[*]= } \frac{i \mathbb{E} [\epsilon \text{Series}[0, 0, 0, 0, 0]]}{2 \pi}$$

$$\text{In[*]:= } \text{Block} \left[\{ \$\pi = \text{Total}@\text{Select}[\text{MonomialList}[\#, \{\epsilon, x, p\}], \right. \\ \left. \begin{array}{l} \text{mon} \mapsto \text{And}[\\ \quad \text{Exponent}[\text{mon}, \epsilon] \leq 2, \\ \quad \text{Exponent}[\text{mon}, x] = \text{Exponent}[\text{mon}, p] \\ \quad] \\ \quad] \&\}, \\ \int \mathbb{E} [p x + a x^2 p + \epsilon b x^3 p^3] \, d\{p, x\} \end{array} \right]$$

$$\text{Out[*]= } \frac{i \mathbb{E} [-6 b \epsilon + 342 b^2 \epsilon^2]}{2 \pi}$$

$$\text{In[*]:= } \text{Block} \left[\{ \$\pi = \text{Total}@\text{Select}[\text{MonomialList}[\#, \{\epsilon, x, p\}], \right. \\ \left. \begin{array}{l} \text{mon} \mapsto \text{And}[\\ \quad \text{Exponent}[\text{mon}, \epsilon] < 4, \\ \quad \text{Exponent}[\text{mon}, x] - \text{Exponent}[\text{mon}, p] \leq 3 \\ \quad] \\ \quad] \&\}, \\ \int \mathbb{E} [p x + a x^2 p + \epsilon b p^2 x] \, d\{p, x\} \end{array} \right]$$

$$\text{Out[*]= } \frac{i \mathbb{E} [-6 a b \epsilon + 162 a^2 b^2 \epsilon^2 - 9072 a^3 b^3 \epsilon^3]}{2 \pi}$$

$$\text{In[*]:= } \text{Block} \left[\{ \$\pi = \text{Total}@\text{Select}[\text{MonomialList}[\#, \{\epsilon, x, p\}], \right. \\ \left. \begin{array}{l} \text{mon} \mapsto \text{And}[\\ \quad \text{Exponent}[\text{mon}, \epsilon] < 4, \\ \quad \text{Exponent}[\text{mon}, x] - \text{Exponent}[\text{mon}, p] \leq 3 - \text{Exponent}[\text{mon}, \epsilon] \\ \quad] \\ \quad] \&\}, \\ \int \mathbb{E} [p x + a x^2 p + \epsilon b p^2 x] \, d\{p, x\} \end{array} \right]$$

$$\text{Out[*]= } \frac{i \mathbb{E} [-6 a b \epsilon + 162 a^2 b^2 \epsilon^2 - 9072 a^3 b^3 \epsilon^3]}{2 \pi}$$

```
In[ ]:= MatrixForm@Table[
  ∫ E[x1 p2 + x2 p3 + x3 p1 + ξi xi + πj pj] d{x1, x2, x3, p1, p2, p3},
  {i, 3}, {j, 3}]
```

Out[]//MatrixForm=

$$\begin{pmatrix} -\frac{i E[0]}{8 \pi^3} & -\frac{i E[-\pi_2 \xi_1]}{8 \pi^3} & -\frac{i E[0]}{8 \pi^3} \\ -\frac{i E[0]}{8 \pi^3} & -\frac{i E[0]}{8 \pi^3} & -\frac{i E[-\pi_3 \xi_2]}{8 \pi^3} \\ -\frac{i E[-\pi_1 \xi_3]}{8 \pi^3} & -\frac{i E[0]}{8 \pi^3} & -\frac{i E[0]}{8 \pi^3} \end{pmatrix}$$