

Pensieve header: Formal Gaussian integration over an arbitrary “Feynman Ring”.

What must a Feynman Ring  $F$  have? (Over some set of labels  $S$  with elements  $x, y, \dots$ )

- \* A vector space over  $\mathbb{Q}$ .
- \* Has a symmetric linear  $Z \mapsto \partial_{x,y} Z$  and a symmetric bilinear  $(Z_1, Z_2) \mapsto (\partial_x Z_1)(\partial_y Z_2)$  that satisfy the axioms of (roughly) a connected circuit algebra.
- \* Has  $q_{x,y} : F \rightarrow \mathbb{Q}$  in some sense dual to some  $\theta_{x,y} \in F$ .
- \* Has  $\text{Ev}_{v_s \rightarrow 0} : F \rightarrow F$ .

Further axioms must be worked out.

Goals.

- \* Define  $\int$ .
- \* Prove a Fubini theorem.
- \* Prove a theorem about the injectivity of the Laplace transform.

## Initialization

```
In[*]:= CCF[ $\mathcal{E}$ _] := ExpandDenominator@ExpandNumerator@Together[ $\mathcal{E}$ ];
CCF[ $\mathcal{E}$ _] := Factor[ $\mathcal{E}$ ];
CF[ $\omega$  .  $\mathcal{E}$ _E] := CF[ $\omega$ ] CF /@  $\mathcal{E}$ ;
CF[ $\mathcal{E}$ _List] := CF /@  $\mathcal{E}$ ;
CF[ $\mathcal{E}$ _] := Module[{vs = Cases[ $\mathcal{E}$ , {x | p}_,  $\infty$ ]  $\cup$  {x, p,  $\epsilon$ }, ps, c},
  Total[CoefficientRules[Expand[ $\mathcal{E}$ ], vs] /. (ps_ -> c_) := CCF[c] (Times @@ vsps) ]];
```

## The Basic Feynman Ring

```
In[*]:= S = {x, x_, y, z};
qx,y[f_] := (∂x,y f) /. Thread[S -> 0];
θx,y := x y;
f_ ≡ 0 := f === 0;
Evvs_List -> 0[f_] := CF[f /. Thread[vs -> 0]]
```

## The $\epsilon$ Series Feynman Ring

```

In[*]:= S = {x, y, z, phi, x_, p_, x_, p_};
q_{x,y}[ser_eSeries] := (D_{x,y} ser[[1]]) /. Thread[S -> 0];
theta_{x,y} := x y;
epsilonSeries /: D[ser_epsilonSeries, vs___] := D[#, vs] & /@ ser;
epsilonSeries /: s1_epsilonSeries + s2_epsilonSeries :=
  epsilonSeries @@ Table[s1[[kk]] + s2[[kk]], {kk, Min[Length@s1, Length@s2]}];
epsilonSeries /: t_ + ser_epsilonSeries := MapAt[(# + t) &, ser, 1];
epsilonSeries /: s1_epsilonSeries * s2_epsilonSeries := epsilonSeries @@ Table[
  Sum[s1[[ii + 1]] s2[[kk - ii + 1]], {ii, 0, kk}], {kk, 0, Min[Length@s1, Length@s2] - 1};
epsilonSeries /: c_ * ser_epsilonSeries := (c #) & /@ ser;
ser_epsilonSeries == 0 := And@@ (# === 0) & /@ ser;
epsilonSeries /: Integrate[ser_epsilonSeries, pars___] := epsilonSeries @@ (Integrate[#, pars] & /@ ser);
epsilonSeries /: EV_{vs_List -> 0}[ser_epsilonSeries] := ser /. Thread[vs -> 0];
CF[ser_epsilonSeries] := CF /@ ser;
    
```

## Integration

Using Picard Iteration!

```

In[*]:= E /: E[A_] E[B_] := E[A + B]

In[*]:= E[sd_SeriesData] /; (List@@sd)[{1, 2, 4, 6}] === {epsilon, 0, 0, 1} :=
  E[epsilonSeries @@ PadRight[sd[[3]], sd[[5]], 0]]
    
```

pdf

Following a program in Projects/FullDoPeGDO/Engine.nb, we write  $Z_\lambda = \sum Z[m] \lambda^m$ .

```

Unprotect[Integrate];
Integrate::sing = "How dare you ask me to integrate a singular Gaussian!";
∫ ω-. E[L-] d(vs_List) := Module[{n, Q, Δ, G, a, b, S, S1, m, m1, $m}, Clear[Z];
n = Length@vs;
Q = Table[qvs[[a]], vs[[b]] [L], {a, n}, {b, n}];
If[(Δ = CF@Det[-Q]) == 0, Message[Integrate::sing]; Return[]];
G = CF[-Inverse[Q] / 2];
Z[] = Z[0] = CF[L - Sum[Q[[a, b]] evs[[a]], vs[[b]], {a, n}, {b, n}] / 2];
Z[m-, a-] := Z[m, a] = CF@D[Z[m], vs[[a]]];
Z[m-, a-, b-] /; a ≤ b := Z[m, a, b] = CF@D[Z[m, a], vs[[b]]];
Z[m-, a-, b-] /; a > b := Z[m, b, a];
For[$m = m = 0, m ≤ 2 $m, ++m,
S = 0;
Do[If[G[[a, b]] != 0,
S +=  $\frac{G[[a, b]]}{m + 1}$  (Z[m, a, b] + (S1 == 0;
Do[S1 += Z[m1, a] Z[m - m1, b], {m1, 0, m}]; S1))],
{a, n}, {b, n}];
Z[m + 1] = CF[S];
If[!(Z[m + 1] == 0), $m = m + 1; Z[] += Z[m + 1]];
];
PowerExpand@Factor[ω (Δ (2 π)n)-1/2] E[CF[EVvs→0[Z[]]]];
Protect[Integrate];
    
```

In[\*]:= ∫ E [i λ x<sub>1</sub><sup>2</sup> / 2] d{x<sub>1</sub>}

Out[\*]=  $\frac{(-1)^{1/4} E[0]}{\sqrt{2\pi} \sqrt{\lambda}}$

In[\*]:= ∫ E [-i λ x<sub>1</sub><sup>2</sup> / 2] d{x<sub>1</sub>}

Out[\*]=  $\frac{(-1)^{3/4} E[0]}{\sqrt{2\pi} \sqrt{\lambda}}$

In[\*]:= ∫ E [ $\frac{i}{2}$  {x<sub>1</sub>, x<sub>2</sub>} ·  $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$  · {x<sub>1</sub>, x<sub>2</sub>}] d{x<sub>1</sub>, x<sub>2</sub>}

Out[\*]=  $\frac{E[0]}{2 \sqrt{b^2 - a c} \pi}$

$$\text{In[*]} := \int \mathbb{E} \left[ -\lambda \mathbf{x}_1^2 / 2 \right] \text{d} \{ \mathbf{x}_1 \}$$

$$\text{Out[*]} = \frac{\mathbb{E} [ \mathbf{0} ]}{\sqrt{2 \pi} \sqrt{\lambda}}$$

$$\text{In[*]} := \text{Clear} [ \mathbf{Z} ]; \int \mathbb{E} \left[ -\mathbf{x}_1^2 / 2 + \xi \mathbf{x}_1 \right] \text{d} \{ \mathbf{x}_1 \}$$

$$\text{Out[*]} = \frac{\mathbb{E} \left[ \frac{\xi^2}{2} \right]}{\sqrt{2 \pi}}$$

$$\text{In[*]} := \int \mathbb{E} \left[ -\frac{1}{2} \{ \mathbf{x}_1, \mathbf{x}_2 \} \cdot \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{b} & \mathbf{c} \end{pmatrix} \cdot \{ \mathbf{x}_1, \mathbf{x}_2 \} + \{ \xi_1, \xi_2 \} \cdot \{ \mathbf{x}_1, \mathbf{x}_2 \} \right] \text{d} \{ \mathbf{x}_1, \mathbf{x}_2 \}$$

$$\text{Out[*]} = \frac{\mathbb{E} \left[ \frac{\mathbf{c} \xi_1^2 - 2 \mathbf{b} \xi_1 \xi_2 + \mathbf{a} \xi_2^2}{2 (-\mathbf{b}^2 + \mathbf{a} \mathbf{c})} \right]}{2 \sqrt{-\mathbf{b}^2 + \mathbf{a} \mathbf{c}} \pi}$$

$$\text{In[*]} := \mathbf{I1} = \int \mathbb{E} \left[ -\frac{1}{2} \{ \mathbf{x}_1, \mathbf{x}_2 \} \cdot \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{b} & \mathbf{c} \end{pmatrix} \cdot \{ \mathbf{x}_1, \mathbf{x}_2 \} + \{ \xi_1, \xi_2 \} \cdot \{ \mathbf{x}_1, \mathbf{x}_2 \} \right] \text{d} \{ \mathbf{x}_1 \}$$

$$\text{Out[*]} = \frac{\mathbb{E} \left[ -\frac{(-\mathbf{b}^2 + \mathbf{a} \mathbf{c}) \mathbf{x}_2^2}{2 \mathbf{a}} + \frac{\xi_1^2}{2 \mathbf{a}} + \frac{\mathbf{x}_2 (-\mathbf{b} \xi_1 + \mathbf{a} \xi_2)}{\mathbf{a}} \right]}{\sqrt{\mathbf{a}} \sqrt{2 \pi}}$$

$$\text{In[*]} := \int \mathbf{I1} \text{d} \{ \mathbf{x}_2 \}$$

$$\text{Out[*]} = \frac{\mathbb{E} \left[ \frac{\mathbf{c} \xi_1^2 - 2 \mathbf{b} \xi_1 \xi_2 + \mathbf{a} \xi_2^2}{2 (-\mathbf{b}^2 + \mathbf{a} \mathbf{c})} \right]}{2 \sqrt{-\mathbf{b}^2 + \mathbf{a} \mathbf{c}} \pi}$$

$$\text{In[*]} := \int \mathbb{E} \left[ \xi \mathbf{x} + \eta \mathbf{y} + \mathbf{z} (\mathbf{x} - \mathbf{y}) + \mathbf{x}^2 \right] \text{d} \{ \mathbf{x}, \mathbf{z} \}$$

$$\text{Out[*]} = \frac{\mathbf{i} \mathbb{E} [ \mathbf{y} (\mathbf{y} + \eta + \xi) ]}{2 \pi}$$

## Integration of $\epsilon$ Series

$$\text{In[*]} := \int \mathbb{E} \left[ -\mathbf{x}^2 / 2 + \epsilon \mathbf{x}^3 / 6 + \mathbf{0} [\epsilon]^7 \right] \text{d} \{ \mathbf{x} \}$$

$$\text{Out[*]} = \frac{\mathbb{E} \left[ \epsilon \text{Series} \left[ \mathbf{0}, \mathbf{0}, \frac{5}{24}, \mathbf{0}, \frac{5}{16}, \mathbf{0}, \frac{1105}{1152} \right] \right]}{\sqrt{2 \pi}}$$

$$\text{In[*]:= } \int \mathbb{E} \left[ -\phi^2 / 2 + \epsilon \phi^4 / 24 + \mathcal{O}[\epsilon]^7 \right] \mathfrak{d}\{\phi\}$$

$$\text{Out[*]= } \frac{\mathbb{E} \left[ \epsilon \text{Series} \left[ 0, \frac{1}{8}, \frac{1}{12}, \frac{11}{96}, \frac{17}{72}, \frac{619}{960}, \frac{709}{324} \right] \right]}{\sqrt{2\pi}}$$

$$\text{In[*]:= } \int \mathbb{E} \left[ \mathbf{p} \mathbf{x} + \epsilon \mathbf{p}^2 \mathbf{x} + \mathcal{O}[\epsilon]^5 \right] \mathfrak{d}\{\mathbf{p}, \mathbf{x}\}$$

$$\text{Out[*]= } \frac{i \mathbb{E} \left[ \epsilon \text{Series} \left[ 0, 0, 0, 0, 0 \right] \right]}{2\pi}$$

$$\begin{aligned} \text{In[*]:= } & \text{Block} \left[ \{ \$\pi = \text{Total} @ \text{Select} [\text{MonomialList} [\#, \{\epsilon, \mathbf{x}, \mathbf{p}\}], \right. \\ & \quad \text{mon} \mapsto \text{And} [ \\ & \quad \quad \text{Exponent} [\text{mon}, \epsilon] \leq 2, \\ & \quad \quad \text{Exponent} [\text{mon}, \mathbf{x}] = \text{Exponent} [\text{mon}, \mathbf{p}] \\ & \quad \quad ] \\ & \quad \left. \} \&\}, \right. \\ & \int \mathbb{E} \left[ \mathbf{p} \mathbf{x} + \mathbf{a} \mathbf{x}^2 \mathbf{p} + \epsilon \mathbf{b} \mathbf{x}^3 \mathbf{p}^3 \right] \mathfrak{d}\{\mathbf{p}, \mathbf{x}\} \end{aligned}$$

$$\text{Out[*]= } \frac{i \mathbb{E} \left[ -6 \mathbf{b} \epsilon + 342 \mathbf{b}^2 \epsilon^2 \right]}{2\pi}$$

$$\begin{aligned} \text{In[*]:= } & \text{Block} \left[ \{ \$\pi = \text{Total} @ \text{Select} [\text{MonomialList} [\#, \{\epsilon, \mathbf{x}, \mathbf{p}\}], \right. \\ & \quad \text{mon} \mapsto \text{And} [ \\ & \quad \quad \text{Exponent} [\text{mon}, \epsilon] < 4, \\ & \quad \quad \text{Exponent} [\text{mon}, \mathbf{x}] - \text{Exponent} [\text{mon}, \mathbf{p}] \leq 3 \\ & \quad \quad ] \\ & \quad \left. \} \&\}, \right. \\ & \int \mathbb{E} \left[ \mathbf{p} \mathbf{x} + \mathbf{a} \mathbf{x}^2 \mathbf{p} + \epsilon \mathbf{b} \mathbf{p}^2 \mathbf{x} \right] \mathfrak{d}\{\mathbf{p}, \mathbf{x}\} \end{aligned}$$

$$\text{Out[*]= } \frac{i \mathbb{E} \left[ -6 \mathbf{a} \mathbf{b} \epsilon + 162 \mathbf{a}^2 \mathbf{b}^2 \epsilon^2 - 9072 \mathbf{a}^3 \mathbf{b}^3 \epsilon^3 \right]}{2\pi}$$

```
In[*]:= Block[{$\pi = Total@Select[MonomialList[#, {\epsilon, x, p}],
    mon \mapsto And[
        Exponent[mon, \epsilon] < 4,
        Exponent[mon, x] - Exponent[mon, p] \le 3 - Exponent[mon, \epsilon]
    ] &},
    \int \mathbb{E}[p x + a x^2 p + \epsilon b p^2 x] d\{p, x\}]
```

Out[\*]=

$$-\frac{i \mathbb{E}[-6 a b \epsilon + 162 a^2 b^2 \epsilon^2 - 9072 a^3 b^3 \epsilon^3]}{2 \pi}$$

```
In[*]:= MatrixForm@Table[
    \int \mathbb{E}[x_1 p_2 + x_2 p_3 + x_3 p_1 + \xi_i x_i + \pi_j p_j] d\{x_1, x_2, x_3, p_1, p_2, p_3\},
    {i, 3}, {j, 3}]
```

Out[\*]//MatrixForm=

$$\begin{pmatrix} -\frac{i \mathbb{E}[0]}{8 \pi^3} & -\frac{i \mathbb{E}[-\pi_2 \xi_1]}{8 \pi^3} & -\frac{i \mathbb{E}[0]}{8 \pi^3} \\ -\frac{i \mathbb{E}[0]}{8 \pi^3} & -\frac{i \mathbb{E}[0]}{8 \pi^3} & -\frac{i \mathbb{E}[-\pi_3 \xi_2]}{8 \pi^3} \\ -\frac{i \mathbb{E}[-\pi_1 \xi_3]}{8 \pi^3} & -\frac{i \mathbb{E}[0]}{8 \pi^3} & -\frac{i \mathbb{E}[0]}{8 \pi^3} \end{pmatrix}$$