Pensieve header: Finding the A2 $d=1$ invariant using undetermined coefficients.

Initialization

```
In[ ]:=  SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\HigherRank"];
         Once[<< KnotTheory`; << Rot.m];
         << FormalGaussianIntegration.m;
```

Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.

Loading Rot.m from http://drorbn.net/AP/Projects/HigherRank to compute rotation numbers.

The A2 Integrand

Adopted from pensieve://Talks//Oaxaca-2210/Rho.nb.

```
In[ ]:=  T₃ = T₁ T₂;
         𝒮 = {x__, p__};
         q[s_, i_, j_] :=
           Sum[x_{α,i} (p_{α,i} - p_{α,i+1}) + x_{α,j} (p_{α,j} - p_{α,j+1}) + x_{α,i} ((1 - T_α^s) p_{α,i+1} + (T_α^s - 1) p_{α,j+1}), {α, 3}];
         ℒ[X_{i_,j_}[s_]] := T₃^𝒮 𝔼[-q[s, i, j] + ϵ r₁[s, i, j] + O[ϵ]²];
         vs_{i_} := Sequence[p_{1,i}, x_{1,i}, p_{2,i}, x_{2,i}, p_{3,i}, x_{3,i}];
```

The PXX Coefficients from R3

```
In[ ]:=  mons = MonomialList[
           p₃ x₁ x₂ /.
             {(ν : p | x)_{α_} :⧴ ν_{α,i} + ν_{α,j}}
           ] /. c_Integer * mon_ :⧴ mon;
         k = 0;
         r₁[1, i_, j_] := Evaluate[Sum[c_{++k} mon, {mon, mons}]];
         Clear[k];
         r₁[1, j, k]
```

```
Out[ ]=
         c₁ p_{3,j} x_{1,j} x_{2,j} + c₅ p_{3,k} x_{1,j} x_{2,j} + c₃ p_{3,j} x_{1,k} x_{2,j} + c₇ p_{3,k} x_{1,k} x_{2,j} +
          c₂ p_{3,j} x_{1,j} x_{2,k} + c₆ p_{3,k} x_{1,j} x_{2,k} + c₄ p_{3,j} x_{1,k} x_{2,k} + c₈ p_{3,k} x_{1,k} x_{2,k}
```

$In[\cdot]:=$ `{lhs}` = `Cases`$\left[\int \mathbb{E}\left[\pi_{1,i}\, p_{1,i} + \pi_{2,i}\, p_{2,i} + \pi_{1,j}\, p_{1,j} + \pi_{2,j}\, p_{2,j} + \pi_{1,k}\, p_{1,k} + \pi_{2,k}\, p_{2,k} \right] \right.$

$\mathcal{L}$ `/@` $(X_{i,j}[1]\; X_{i+1,k}[1]\; X_{j+1,k+1}[1])$

$d\!d\{vs_i,\, vs_j,\, vs_k,\, vs_{i+1},\, vs_{j+1},\, vs_{k+1}\},\ \epsilon Series[\_,\ \mathcal{E}\_] :\!\!\rightarrow \mathcal{E},\ \infty \Big]$

$Out[\cdot]=$

$\Big\{ 2\, c_1\, T_1^2\, T_2^2\, p_{3,2+i}\, \pi_{1,i}\, \pi_{2,i} -$

$T_1\, T_2\, \big(-2\, c_1 - c_5 + c_1\, T_1 - c_3\, T_1 + c_3\, T_1^2 + c_1\, T_2 - c_2\, T_2 + c_2\, T_1\, T_2 + c_3\, T_1\, T_2 - c_4\, T_1\, T_2 -$

$c_3\, T_1^2\, T_2 + c_4\, T_1^2\, T_2 + c_2\, T_2^2 - c_2\, T_1\, T_2^2 + c_4\, T_1\, T_2^2 - c_4\, T_1^2\, T_2^2 \big)\, p_{3,2+j}\, \pi_{1,i}\, \pi_{2,i} +$

$\big(2\, c_1 + 2\, c_5 - c_1\, T_1 + c_3\, T_1 - c_5\, T_1 + c_7\, T_1 - c_3\, T_1^2 - c_7\, T_1^2 - c_1\, T_2 + c_2\, T_2 - c_5\, T_2 + c_6\, T_2 - c_2\, T_1\, T_2 -$

$c_3\, T_1\, T_2 + c_4\, T_1\, T_2 + c_5\, T_1\, T_2 - c_6\, T_1\, T_2 - c_7\, T_1\, T_2 + c_8\, T_1\, T_2 + c_1\, T_1^2\, T_2 - c_4\, T_1^2\, T_2 + c_7\, T_1^2\, T_2 -$

$c_8\, T_1^2\, T_2 + c_3\, T_1^3\, T_2 - c_2\, T_2^2 - c_6\, T_2^2 + c_1\, T_1\, T_2^2 - c_4\, T_1\, T_2^2 + c_6\, T_1\, T_2^2 - c_8\, T_1\, T_2^2 - 2\, c_1\, T_1^2\, T_2^2 + c_2\, T_1^2\, T_2^2 +$

$c_3\, T_1^2\, T_2^2 + c_8\, T_1^2\, T_2^2 - c_3\, T_1^3\, T_2^2 + c_4\, T_1^3\, T_2^2 + c_2\, T_1\, T_2^3 - c_2\, T_1^2\, T_2^3 + c_4\, T_1^2\, T_2^3 - c_4\, T_1^3\, T_2^3 \big)\, p_{3,2+k}\, \pi_{1,i}\, \pi_{2,i} +$

$c_3\, T_1^2\, T_2^2\, p_{3,2+i}\, \pi_{1,j}\, \pi_{2,i} - T_1\, T_2\, \big(-c_1 - c_3 - c_7 + c_1\, T_2 - c_2\, T_2 + c_3\, T_1\, T_2 + c_2\, T_2^2 \big)\, p_{3,2+j}\, \pi_{1,j}\, \pi_{2,i} +$

$\big(c_1 + c_3 + c_5 + c_7 - c_1\, T_2 + c_2\, T_2 - c_5\, T_2 + c_6\, T_2 - c_1\, T_1\, T_2 - c_3\, T_1\, T_2 -$

$c_7\, T_1\, T_2 - c_2\, T_2^2 - c_6\, T_2^2 + c_1\, T_1\, T_2^2 - c_2\, T_1\, T_2^2 + c_2\, T_1\, T_2^3 \big)\, p_{3,2+k}\, \pi_{1,j}\, \pi_{2,i} +$

$c_3\, T_1\, T_2^2\, p_{3,2+i}\, \pi_{1,k}\, \pi_{2,i} - T_1\, (-1 + T_2)\, T_2\, (c_3 + c_4\, T_2)\, p_{3,2+j}\, \pi_{1,k}\, \pi_{2,i} +$

$\big(c_3 + c_7 + c_4\, T_2 + c_8\, T_2 - c_3\, T_1\, T_2 - c_4\, T_2^2 - c_8\, T_2^2 - c_4\, T_1\, T_2^2 + c_4\, T_1\, T_2^3 \big)\, p_{3,2+k}\, \pi_{1,k}\, \pi_{2,i} +$

$c_2\, T_1^2\, T_2^2\, p_{3,2+i}\, \pi_{1,i}\, \pi_{2,j} - T_1\, T_2\, \big(-c_1 - c_2 - c_6 + c_1\, T_1 - c_3\, T_1 + c_3\, T_1^2 + c_2\, T_1\, T_2 \big)\, p_{3,2+j}\, \pi_{1,i}\, \pi_{2,j} +$

$\big(c_1 + c_2 + c_5 + c_6 - c_1\, T_1 + c_3\, T_1 - c_5\, T_1 + c_7\, T_1 - c_3\, T_1^2 - c_7\, T_1^2 -$

$c_1\, T_1\, T_2 - c_2\, T_1\, T_2 - c_6\, T_1\, T_2 + c_1\, T_1^2\, T_2 - c_3\, T_1^2\, T_2 + c_3\, T_1^3\, T_2 \big)\, p_{3,2+k}\, \pi_{1,i}\, \pi_{2,j} +$

$c_4\, T_1^2\, T_2^2\, p_{3,2+i}\, \pi_{1,j}\, \pi_{2,j} + T_1\, T_2\, \big(c_1 + c_4 + c_8 - c_4\, T_1\, T_2 \big)\, p_{3,2+j}\, \pi_{1,j}\, \pi_{2,j} +$

$\big(c_1 + c_4 + c_5 + c_8 - c_1\, T_1\, T_2 - c_4\, T_1\, T_2 - c_8\, T_1\, T_2 \big)\, p_{3,2+k}\, \pi_{1,j}\, \pi_{2,j} +$

$c_3\, T_1\, T_2\, p_{3,2+j}\, \pi_{1,k}\, \pi_{2,j} + \big(c_3 + c_7 - c_3\, T_1\, T_2 \big)\, p_{3,2+k}\, \pi_{1,k}\, \pi_{2,j} +$

$c_2\, T_1^2\, T_2\, p_{3,2+i}\, \pi_{1,i}\, \pi_{2,k} - (-1 + T_1)\, T_1\, (c_2 + c_4\, T_1)\, T_2\, p_{3,2+j}\, \pi_{1,i}\, \pi_{2,k} +$

$\big(c_2 + c_6 + c_4\, T_1 + c_8\, T_1 - c_4\, T_1^2 - c_8\, T_1^2 - c_2\, T_1\, T_2 - c_4\, T_1^2\, T_2 + c_4\, T_1^3\, T_2 \big)\, p_{3,2+k}\, \pi_{1,i}\, \pi_{2,k} +$

$c_2\, T_1\, T_2\, p_{3,2+j}\, \pi_{1,j}\, \pi_{2,k} + \big(c_2 + c_6 - c_2\, T_1\, T_2 \big)\, p_{3,2+k}\, \pi_{1,j}\, \pi_{2,k} + c_4\, T_1\, T_2\, p_{3,2+i}\, \pi_{1,k}\, \pi_{2,k} +$

$c_4\, T_1\, T_2\, p_{3,2+j}\, \pi_{1,k}\, \pi_{2,k} - 2\, \big(-c_4 - c_8 + c_4\, T_1\, T_2 \big)\, p_{3,2+k}\, \pi_{1,k}\, \pi_{2,k} \Big\}$

$In[\circ]:=$ `{rhs} = Cases[` $\int \mathbb{E}[\pi_{1,i} \, p_{1,i} + \pi_{2,i} \, p_{2,i} + \pi_{1,j} \, p_{1,j} + \pi_{2,j} \, p_{2,j} + \pi_{1,k} \, p_{1,k} + \pi_{2,k} \, p_{2,k}]$

$\mathcal{L}$ `/@` $(X_{j,k}[1] \, X_{i,k+1}[1] \, X_{i+1,j+1}[1])$

$\mathbb{d}\{vs_i, vs_j, vs_k, vs_{i+1}, vs_{j+1}, vs_{k+1}\}, \epsilon$`Series`$[\_, \mathcal{E}\_] \Rightarrow \mathcal{E}, \infty]$

$Out[\circ]=$

$\{2 \, c_1 \, T_1^2 \, T_2^2 \, p_{3,2+i} \, \pi_{1,i} \, \pi_{2,i} - T_1 \, T_2 \, (-2 \, c_1 - c_5 + 2 \, c_1 \, T_1 \, T_2) \, p_{3,2+j} \, \pi_{1,i} \, \pi_{2,i} +$

$(c_1 + c_5 - c_1 \, T_1 \, T_2) \, p_{3,2+k} \, \pi_{1,i} \, \pi_{2,i} - c_3 \, (-2 + T_1) \, T_1^2 \, T_2^2 \, p_{3,2+i} \, \pi_{1,j} \, \pi_{2,i} +$

$T_1 \, T_2 \, (2 \, c_3 + c_7 - c_3 \, T_1 - 2 \, c_3 \, T_1 \, T_2 + c_3 \, T_1^2 \, T_2) \, p_{3,2+j} \, \pi_{1,j} \, \pi_{2,i} +$

$(-1 + T_1) \, (-c_3 - c_7 + c_3 \, T_1 \, T_2) \, p_{3,2+k} \, \pi_{1,j} \, \pi_{2,i} + c_3 \, T_1^2 \, T_2^2 \, p_{3,2+i} \, \pi_{1,k} \, \pi_{2,i} -$

$c_3 \, T_1 \, T_2 \, (-1 + T_1 \, T_2) \, p_{3,2+j} \, \pi_{1,k} \, \pi_{2,i} + (c_3 + c_7 - c_3 \, T_1 \, T_2) \, p_{3,2+k} \, \pi_{1,k} \, \pi_{2,i} -$

$c_2 \, T_1^2 \, (-2 + T_2) \, T_2^2 \, p_{3,2+i} \, \pi_{1,i} \, \pi_{2,j} + T_1 \, T_2 \, (2 \, c_2 + c_6 - c_2 \, T_2 - 2 \, c_2 \, T_1 \, T_2 + c_2 \, T_1 \, T_2^2) \, p_{3,2+j} \, \pi_{1,i} \, \pi_{2,j} +$

$(-1 + T_2) \, (-c_2 - c_6 + c_2 \, T_1 \, T_2) \, p_{3,2+k} \, \pi_{1,i} \, \pi_{2,j} + c_4 \, T_1^2 \, T_2^2 \, (2 - T_1 - T_2 + T_1 \, T_2) \, p_{3,2+i} \, \pi_{1,j} \, \pi_{2,j} +$

$T_1 \, T_2 \, (c_1 + 2 \, c_4 + c_8 - c_4 \, T_1 - c_4 \, T_2 - c_4 \, T_1 \, T_2 + c_4 \, T_1^2 \, T_2 + c_4 \, T_1 \, T_2^2 - c_4 \, T_1^2 \, T_2^2) \, p_{3,2+j} \, \pi_{1,j} \, \pi_{2,j} +$

$(c_1 + c_4 + c_5 + c_8 - c_4 \, T_1 - c_8 \, T_1 - c_4 \, T_2 - c_8 \, T_2 - c_1 \, T_1 \, T_2 + c_8 \, T_1 \, T_2 + c_4 \, T_1^2 \, T_2 + c_4 \, T_1 \, T_2^2 - c_4 \, T_1^2 \, T_2^2)$

$p_{3,2+k} \, \pi_{1,j} \, \pi_{2,j} - c_4 \, T_1^2 \, (-1 + T_2) \, T_2^2 \, p_{3,2+i} \, \pi_{1,k} \, \pi_{2,j} +$

$T_1 \, T_2 \, (c_3 + c_4 - c_4 \, T_2 - c_4 \, T_1 \, T_2 + c_4 \, T_1 \, T_2^2) \, p_{3,2+j} \, \pi_{1,k} \, \pi_{2,j} +$

$(c_3 + c_4 + c_7 + c_8 - c_4 \, T_2 - c_8 \, T_2 - c_3 \, T_1 \, T_2 - c_4 \, T_1 \, T_2 + c_4 \, T_1 \, T_2^2) \, p_{3,2+k} \, \pi_{1,k} \, \pi_{2,j} +$

$c_2 \, T_1^2 \, T_2^2 \, p_{3,2+i} \, \pi_{1,i} \, \pi_{2,k} - c_2 \, T_1 \, T_2 \, (-1 + T_1 \, T_2) \, p_{3,2+j} \, \pi_{1,i} \, \pi_{2,k} + (c_2 + c_6 - c_2 \, T_1 \, T_2) \, p_{3,2+k} \, \pi_{1,i} \, \pi_{2,k} -$

$c_4 \, (-1 + T_1) \, T_1^2 \, T_2^2 \, p_{3,2+i} \, \pi_{1,j} \, \pi_{2,k} + T_1 \, T_2 \, (c_2 + c_4 - c_4 \, T_1 - c_4 \, T_1 \, T_2 + c_4 \, T_1^2 \, T_2) \, p_{3,2+j} \, \pi_{1,j} \, \pi_{2,k} +$

$(c_2 + c_4 + c_6 + c_8 - c_4 \, T_1 - c_8 \, T_1 - c_2 \, T_1 \, T_2 - c_4 \, T_1 \, T_2 + c_4 \, T_1^2 \, T_2) \, p_{3,2+k} \, \pi_{1,j} \, \pi_{2,k} +$

$c_4 \, T_1^2 \, T_2^2 \, p_{3,2+i} \, \pi_{1,k} \, \pi_{2,k} - c_4 \, T_1 \, T_2 \, (-2 + T_1 \, T_2) \, p_{3,2+j} \, \pi_{1,k} \, \pi_{2,k} - 2 \, (-c_4 - c_8 + c_4 \, T_1 \, T_2) \, p_{3,2+k} \, \pi_{1,k} \, \pi_{2,k}\}$

$In[\circ]:=$ **eqn = CF[lhs − rhs]**

$Out[\circ]=$

$$T_1 T_2 \left(-c_1 T_1 + c_3 T_1 - c_3 T_1^2 - c_1 T_2 + c_2 T_2 + 2 c_1 T_1 T_2 - c_2 T_1 T_2 - c_3 T_1 T_2 + \right.$$
$$\left. c_4 T_1 T_2 + c_3 T_1^2 T_2 - c_4 T_1^2 T_2 - c_2 T_2^2 + c_2 T_1 T_2^2 - c_4 T_1 T_2^2 + c_4 T_1^2 T_2^2 \right) p_{3,2+j} \pi_{1,i} \pi_{2,i} +$$
$$\left(c_1 + c_5 - c_1 T_1 + c_3 T_1 - c_5 T_1 + c_7 T_1 - c_3 T_1^2 - c_7 T_1^2 - c_1 T_2 + c_2 T_2 - c_5 T_2 + c_6 T_2 + c_1 T_1 T_2 - \right.$$
$$c_2 T_1 T_2 - c_3 T_1 T_2 + c_4 T_1 T_2 + c_5 T_1 T_2 - c_6 T_1 T_2 - c_7 T_1 T_2 + c_8 T_1 T_2 + c_1 T_1^2 T_2 - c_4 T_1^2 T_2 +$$
$$c_7 T_1^2 T_2 - c_8 T_1^2 T_2 + c_3 T_1^3 T_2 - c_2 T_2^2 - c_6 T_2^2 + c_1 T_1 T_2^2 - c_4 T_1 T_2^2 + c_6 T_1 T_2^2 - c_8 T_1 T_2^2 - 2 c_1 T_1^2 T_2^2 +$$
$$\left. c_2 T_1^2 T_2^2 + c_3 T_1^2 T_2^2 + c_8 T_1^2 T_2^2 - c_3 T_1^3 T_2^2 + c_4 T_1^3 T_2^2 + c_2 T_1 T_2^3 - c_2 T_1^2 T_2^3 + c_4 T_1^2 T_2^3 - c_4 T_1^3 T_2^3 \right)$$
$$p_{3,2+k} \pi_{1,i} \pi_{2,i} + c_3 \left(-1 + T_1 \right) T_1^2 T_2^2 p_{3,2+i} \pi_{1,j} \pi_{2,i} -$$
$$T_1 T_2 \left(-c_1 + c_3 - c_3 T_1 + c_1 T_2 - c_2 T_2 - c_3 T_1 T_2 + c_3 T_1^2 T_2 + c_2 T_2^2 \right) p_{3,2+j} \pi_{1,j} \pi_{2,i} +$$
$$\left(c_1 + c_5 + c_3 T_1 + c_7 T_1 - c_1 T_2 + c_2 T_2 - c_5 T_2 + c_6 T_2 - c_1 T_1 T_2 - \right.$$
$$\left. c_7 T_1 T_2 - c_3 T_1^2 T_2 - c_2 T_2^2 - c_6 T_2^2 + c_1 T_1 T_2^2 - c_2 T_1 T_2^2 + c_2 T_1 T_2^3 \right) p_{3,2+k} \pi_{1,j} \pi_{2,i} -$$
$$c_3 \left(-1 + T_1 \right) T_1 T_2^2 p_{3,2+i} \pi_{1,k} \pi_{2,i} + T_1 T_2^2 \left(-c_3 + c_4 + c_3 T_1 - c_4 T_2 \right) p_{3,2+j} \pi_{1,k} \pi_{2,i} +$$
$$\left(-1 + T_2 \right) T_2 \left(-c_4 - c_8 + c_4 T_1 T_2 \right) p_{3,2+k} \pi_{1,k} \pi_{2,i} +$$
$$c_2 T_1^2 \left(-1 + T_2 \right) T_2^2 p_{3,2+i} \pi_{1,i} \pi_{2,j} -$$
$$T_1 T_2 \left(-c_1 + c_2 + c_1 T_1 - c_3 T_1 + c_3 T_1^2 - c_2 T_2 - c_2 T_1 T_2 + c_2 T_1 T_2^2 \right) p_{3,2+j} \pi_{1,i} \pi_{2,j} +$$
$$\left(c_1 + c_5 - c_1 T_1 + c_3 T_1 - c_5 T_1 + c_7 T_1 - c_3 T_1^2 - c_7 T_1^2 + c_2 T_2 + c_6 T_2 - \right.$$
$$\left. c_1 T_1 T_2 - c_6 T_1 T_2 + c_1 T_1^2 T_2 - c_3 T_1^2 T_2 + c_3 T_1^3 T_2 - c_2 T_1 T_2^2 \right) p_{3,2+k} \pi_{1,i} \pi_{2,j} -$$
$$c_4 \left(-1 + T_1 \right) T_1^2 \left(-1 + T_2 \right) T_2^2 p_{3,2+i} \pi_{1,j} \pi_{2,j} + c_4 \left(-1 + T_1 \right) T_1 \left(-1 + T_2 \right) T_2 \left(-1 + T_1 T_2 \right) p_{3,2+j} \pi_{1,j} \pi_{2,j} +$$
$$\left(c_4 T_1 + c_8 T_1 + c_4 T_2 + c_8 T_2 - c_4 T_1 T_2 - 2 c_8 T_1 T_2 - c_4 T_1^2 T_2 - c_4 T_1 T_2^2 + c_4 T_1^2 T_2^2 \right) p_{3,2+k} \pi_{1,j} \pi_{2,j} +$$
$$c_4 T_1^2 \left(-1 + T_2 \right) T_2^2 p_{3,2+i} \pi_{1,k} \pi_{2,j} -$$
$$c_4 T_1 \left(-1 + T_2 \right) T_2 \left(-1 + T_1 T_2 \right) p_{3,2+j} \pi_{1,k} \pi_{2,j} -$$
$$\left(-1 + T_2 \right) \left(-c_4 - c_8 + c_4 T_1 T_2 \right) p_{3,2+k} \pi_{1,k} \pi_{2,j} -$$
$$c_2 T_1^2 \left(-1 + T_2 \right) T_2 p_{3,2+i} \pi_{1,i} \pi_{2,k} + T_1^2 T_2 \left(-c_2 + c_4 - c_4 T_1 + c_2 T_2 \right) p_{3,2+j} \pi_{1,i} \pi_{2,k} +$$
$$\left(-1 + T_1 \right) T_1 \left(-c_4 - c_8 + c_4 T_1 T_2 \right) p_{3,2+k} \pi_{1,i} \pi_{2,k} +$$
$$c_4 \left(-1 + T_1 \right) T_1^2 T_2^2 p_{3,2+i} \pi_{1,j} \pi_{2,k} - c_4 \left(-1 + T_1 \right) T_1 T_2 \left(-1 + T_1 T_2 \right) p_{3,2+j} \pi_{1,j} \pi_{2,k} -$$
$$\left(-1 + T_1 \right) \left(-c_4 - c_8 + c_4 T_1 T_2 \right) p_{3,2+k} \pi_{1,j} \pi_{2,k} -$$
$$c_4 T_1 T_2 \left(-1 + T_1 T_2 \right) p_{3,2+i} \pi_{1,k} \pi_{2,k} + c_4 T_1 T_2 \left(-1 + T_1 T_2 \right) p_{3,2+j} \pi_{1,k} \pi_{2,k}$$

$In[\circ]:=$ **cvs = Union@Cases[eqn, $p_{\_\_} | \pi_{\_\_}$, ∞]**

$Out[\circ]=$

$$\{ p_{3,2+i}, p_{3,2+j}, p_{3,2+k}, \pi_{1,i}, \pi_{1,j}, \pi_{1,k}, \pi_{2,i}, \pi_{2,j}, \pi_{2,k} \}$$

$In[\circ]:=$ **eqns = CoefficientRules[eqn, cvs] /. (_ → c_) :→ (c == 0)**

$Out[\circ]=$

$\{ -c_2 T_1^2 T_2^2 + c_2 T_1^2 T_2^3 == 0, \; c_2 T_1^2 T_2 - c_2 T_1^2 T_2^2 == 0, \; -c_3 T_1^2 T_2^2 + c_3 T_1^3 T_2^2 == 0,$

$-c_4 T_1^2 T_2^2 + c_4 T_1^3 T_2^2 + c_4 T_1^2 T_2^3 - c_4 T_1^3 T_2^3 == 0, \; -c_4 T_1^2 T_2^2 + c_4 T_1^3 T_2^2 == 0,$

$c_3 T_1 T_2^2 - c_3 T_1^2 T_2^2 == 0, \; -c_4 T_1^2 T_2^2 + c_4 T_1^2 T_2^3 == 0, \; c_4 T_1 T_2 - c_4 T_1^2 T_2^2 == 0,$

$-c_1 T_1^2 T_2 + c_3 T_1^2 T_2 - c_3 T_1^3 T_2 - c_1 T_1 T_2^2 + c_2 T_1 T_2^2 + 2 c_1 T_1^2 T_2^2 - c_2 T_1^2 T_2^2 -$

$\quad c_3 T_1^2 T_2^2 + c_4 T_1^2 T_2^2 + c_3 T_1^3 T_2^2 - c_4 T_1^3 T_2^2 - c_2 T_1 T_2^3 + c_2 T_1^2 T_2^3 - c_4 T_1^2 T_2^3 + c_4 T_1^3 T_2^3 == 0,$

$c_1 T_1 T_2 - c_2 T_1 T_2 - c_1 T_1^2 T_2 + c_3 T_1^2 T_2 - c_3 T_1^3 T_2 + c_2 T_1 T_2^2 + c_2 T_1^2 T_2^2 - c_2 T_1^2 T_2^3 == 0,$

$-c_2 T_1^2 T_2 + c_4 T_1^2 T_2 - c_4 T_1^3 T_2 + c_2 T_1^2 T_2^2 == 0,$

$c_1 T_1 T_2 - c_3 T_1 T_2 + c_3 T_1^2 T_2 - c_1 T_1 T_2^2 + c_2 T_1 T_2^2 + c_3 T_1^2 T_2^2 - c_3 T_1^3 T_2^2 - c_2 T_1 T_2^3 == 0,$

$-c_4 T_1 T_2 + c_4 T_1^2 T_2 + c_4 T_1 T_2^2 - c_4 T_1^3 T_2^2 - c_4 T_1^2 T_2^3 + c_4 T_1^3 T_2^3 == 0,$

$-c_4 T_1 T_2 + c_4 T_1^2 T_2 + c_4 T_1^2 T_2^2 - c_4 T_1^3 T_2^2 == 0, \; -c_3 T_1 T_2^2 + c_4 T_1 T_2^2 + c_3 T_1^2 T_2^2 - c_4 T_1 T_2^3 == 0,$

$-c_4 T_1 T_2 + c_4 T_1 T_2^2 + c_4 T_1^2 T_2^2 - c_4 T_1^2 T_2^3 == 0, \; -c_4 T_1 T_2 + c_4 T_1^2 T_2^2 == 0,$

$c_1 + c_5 - c_1 T_1 + c_3 T_1 - c_5 T_1 + c_7 T_1 - c_3 T_1^2 - c_7 T_1^2 - c_1 T_2 + c_2 T_2 - c_5 T_2 + c_6 T_2 + c_1 T_1 T_2 -$

$\quad c_2 T_1 T_2 - c_3 T_1 T_2 + c_4 T_1 T_2 + c_5 T_1 T_2 - c_6 T_1 T_2 - c_7 T_1 T_2 + c_8 T_1 T_2 + c_1 T_1^2 T_2 - c_4 T_1^2 T_2 +$

$\quad c_7 T_1^2 T_2 - c_8 T_1^2 T_2 + c_3 T_1^3 T_2 - c_2 T_2^2 - c_6 T_2^2 + c_1 T_1 T_2^2 - c_4 T_1 T_2^2 + c_6 T_1 T_2^2 - c_8 T_1 T_2^2 - 2 c_1 T_1^2 T_2^2 +$

$\quad c_2 T_1^2 T_2^2 + c_3 T_1^2 T_2^2 + c_8 T_1^2 T_2^2 - c_3 T_1^3 T_2^2 + c_4 T_1^3 T_2^2 + c_2 T_1 T_2^3 - c_2 T_1^2 T_2^3 + c_4 T_1^2 T_2^3 - c_4 T_1^3 T_2^3 == 0,$

$c_1 + c_5 - c_1 T_1 + c_3 T_1 - c_5 T_1 + c_7 T_1 - c_3 T_1^2 - c_7 T_1^2 + c_2 T_2 + c_6 T_2 - c_1 T_1 T_2 -$

$\quad c_6 T_1 T_2 + c_1 T_1^2 T_2 - c_3 T_1^2 T_2 + c_3 T_1^3 T_2 - c_2 T_1 T_2^2 == 0,$

$c_4 T_1 + c_8 T_1 - c_4 T_1^2 - c_8 T_1^2 - c_4 T_1^2 T_2 + c_4 T_1^3 T_2 == 0, \; c_1 + c_5 + c_3 T_1 + c_7 T_1 - c_1 T_2 + c_2 T_2 -$

$\quad c_5 T_2 + c_6 T_2 - c_1 T_1 T_2 - c_7 T_1 T_2 - c_3 T_1^2 T_2 - c_2 T_2^2 - c_6 T_2^2 + c_1 T_1 T_2^2 - c_2 T_1 T_2^2 + c_2 T_1 T_2^3 == 0,$

$c_4 T_1 + c_8 T_1 + c_4 T_2 + c_8 T_2 - c_4 T_1 T_2 - 2 c_8 T_1 T_2 - c_4 T_1^2 T_2 - c_4 T_1 T_2^2 + c_4 T_1^2 T_2^2 == 0,$

$-c_4 - c_8 + c_4 T_1 + c_8 T_1 + c_4 T_1 T_2 - c_4 T_1^2 T_2 == 0, \; c_4 T_2 + c_8 T_2 - c_4 T_2^2 - c_8 T_2^2 - c_4 T_1 T_2^2 + c_4 T_1 T_2^3 == 0,$

$-c_4 - c_8 + c_4 T_2 + c_8 T_2 + c_4 T_1 T_2 - c_4 T_1 T_2^2 == 0 \}$

$In[\circ]:=$ **vars = Union@Cases[eqn, c_, ∞]**

$Out[\circ]=$

$\{ c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8 \}$

$In[\circ]:=$ **{sol} = Solve[eqns, vars]**

⬤ Solve: Equations may not give solutions for all "solve" variables.

$Out[\circ]=$

$\left\{ \left\{ c_1 \to 0, \; c_2 \to 0, \; c_3 \to 0, \; c_4 \to 0, \; c_7 \to -\dfrac{c_5}{T_1} - \dfrac{c_6 T_2}{T_1}, \; c_8 \to 0 \right\} \right\}$

$In[\circ]:=$ **sol /. (v_ → val_) :→ (v = CF[val])**

$Out[\circ]=$

$\left\{ 0, 0, 0, 0, -\dfrac{c_5 + c_6 T_2}{T_1}, 0 \right\}$

$In[\circ]:=$ **vars = Union@Cases[eqn, c_, ∞]**

$Out[\circ]=$

$\{ c_5, c_6 \}$

$In[\bullet]:=$ $\mathcal{L}[X_{i,j}[1]]$

$Out[\bullet]=$

$T_1 T_2 \mathbb{E}\Big[\epsilon \text{Series}\Big[-p_{1,i} x_{1,i} + T_1 p_{1,1+i} x_{1,i} + p_{1,1+j} x_{1,i} - T_1 p_{1,1+j} x_{1,i} - p_{1,j} x_{1,j} +$

$\qquad p_{1,1+j} x_{1,j} - p_{2,i} x_{2,i} + T_2 p_{2,1+i} x_{2,i} + p_{2,1+j} x_{2,i} - T_2 p_{2,1+j} x_{2,i} - p_{2,j} x_{2,j} + p_{2,1+j} x_{2,j} -$

$\qquad p_{3,i} x_{3,i} + T_1 T_2 p_{3,1+i} x_{3,i} + p_{3,1+j} x_{3,i} - T_1 T_2 p_{3,1+j} x_{3,i} - p_{3,j} x_{3,j} + p_{3,1+j} x_{3,j},$

$\qquad c_5 p_{3,j} x_{1,i} x_{2,i} - \dfrac{(c_5 + c_6 T_2)\ p_{3,j} x_{1,j} x_{2,i}}{T_1} + c_6 p_{3,j} x_{1,i} x_{2,j}\Big]\Big]$

$In[\bullet]:=$ $r_1[1, i, j]$

$Out[\bullet]=$

$c_5 p_{3,j} x_{1,i} x_{2,i} - \dfrac{(c_5 + c_6 T_2)\ p_{3,j} x_{1,j} x_{2,i}}{T_1} + c_6 p_{3,j} x_{1,i} x_{2,j}$

The XPP Coefficients from R3

$In[\bullet]:=$
```
mons = MonomialList[
    x₃ p₁ p₂ /.
     {(v : p | x)_α_ ⧴ v_α,i + v_α,j}
    ] /. c_Integer * mon_ ⧴ mon;
k = 0;
r₁[1, i_, j_] := Evaluate[Sum[d₊₊k mon, {mon, mons}]];
Clear[k];
r₁[1, j, k]
```

$Out[\bullet]=$

$d_1 p_{1,j} p_{2,j} x_{3,j} + d_5 p_{1,k} p_{2,j} x_{3,j} + d_3 p_{1,j} p_{2,k} x_{3,j} + d_7 p_{1,k} p_{2,k} x_{3,j} +$

$\qquad d_2 p_{1,j} p_{2,j} x_{3,k} + d_6 p_{1,k} p_{2,j} x_{3,k} + d_4 p_{1,j} p_{2,k} x_{3,k} + d_8 p_{1,k} p_{2,k} x_{3,k}$

In[ ]:= `{lhs} = Cases[ ∫ 𝔼[π_i p_{3,i} + π_j p_{3,j} + π_k p_{3,k}] ℒ /@ (X_{i,j}[1] X_{i+1,k}[1] X_{j+1,k+1}[1])`

`𝕕{vs_i, vs_j, vs_k, vs_{i+1}, vs_{j+1}, vs_{k+1}}, ϵSeries[_, ℰ_] ⧴ ℰ, ∞]`

Out[ ]=

$$\{d_2\,\pi_k\,T_1\,T_2\,p_{1,2+i}\,p_{2,2+i} + 2\,d_1\,\pi_i\,T_1^2\,T_2^2\,p_{1,2+i}\,p_{2,2+i} + d_2\,\pi_j\,T_1^2\,T_2^2\,p_{1,2+i}\,p_{2,2+i} -$$

$$\pi_i\,T_1\,(-d_1 - d_5 + d_1\,T_1)\,T_2^2\,p_{1,2+j}\,p_{2,2+i} - \pi_j\,T_1\,(-d_2 - d_6 + d_2\,T_1)\,T_2^2\,p_{1,2+j}\,p_{2,2+i} -$$

$$\pi_k\,(-d_2 - d_6 + d_2\,T_1)\,T_2\,p_{1,2+k}\,p_{2,2+i} - (d_2 + d_6)\,\pi_j\,(-1 + T_1)\,T_2^2\,p_{1,2+k}\,p_{2,2+i} -$$

$$\pi_i\,(-d_1 - d_5 + d_1\,T_1^2)\,T_2^2\,p_{1,2+k}\,p_{2,2+i} - \pi_i\,T_1^2\,T_2\,(-d_1 - d_3 + d_1\,T_2)\,p_{1,2+i}\,p_{2,2+j} -$$

$$\pi_j\,T_1^2\,T_2\,(-d_2 - d_4 + d_2\,T_2)\,p_{1,2+i}\,p_{2,2+j} + d_2\,\pi_k\,T_1\,T_2\,p_{1,2+j}\,p_{2,2+j} +$$

$$\pi_j\,T_1\,T_2\,(d_1 + d_2 + d_4 + d_6 + d_8 - d_2\,T_1 - d_4\,T_1 - d_2\,T_2 - d_6\,T_2 + d_2\,T_1\,T_2)\,p_{1,2+j}\,p_{2,2+j} -$$

$$\pi_i\,T_1\,T_2\,(-2\,d_1 - d_3 - d_5 - d_7 + d_1\,T_1 + d_3\,T_1 + d_1\,T_2 + d_5\,T_2 - d_2\,T_1\,T_2 + d_2\,T_1^2\,T_2^2)\,p_{1,2+j}\,p_{2,2+j} -$$

$$\pi_k\,(-d_2 - d_6 + d_2\,T_1)\,T_2\,p_{1,2+k}\,p_{2,2+j} -$$

$$\pi_j\,T_2\,(-d_1 - d_2 - d_4 - d_5 - d_6 - d_8 + d_1\,T_1 + d_2\,T_1 + d_4\,T_1 + d_6\,T_1 + d_8\,T_1 + d_2\,T_2 + d_6\,T_2 - d_2\,T_1\,T_2 - d_6\,T_1\,T_2)$$

$$p_{1,2+k}\,p_{2,2+j} + \pi_i\,T_2\,(2\,d_1 + d_3 + 2\,d_5 + d_7 - 2\,d_1\,T_1 - d_3\,T_1 - d_5\,T_1 - d_7\,T_1 - d_1\,T_2 - d_5\,T_2 +$$

$$d_2\,T_1\,T_2 + d_6\,T_1\,T_2 + d_1\,T_1^2\,T_2 - d_2\,T_1^2\,T_2 - d_2\,T_1^2\,T_2^2 - d_6\,T_1^2\,T_2^2 + d_2\,T_1^3\,T_2^2)\,p_{1,2+k}\,p_{2,2+j} -$$

$$(d_2 + d_4)\,\pi_j\,T_1^2\,(-1 + T_2)\,p_{1,2+i}\,p_{2,2+k} - \pi_k\,T_1\,(-d_2 - d_4 + d_2\,T_2)\,p_{1,2+i}\,p_{2,2+k} -$$

$$\pi_i\,T_1^2\,(-d_1 - d_3 + d_1\,T_2^2)\,p_{1,2+i}\,p_{2,2+k} - \pi_k\,T_1\,(-d_2 - d_4 + d_2\,T_2)\,p_{1,2+j}\,p_{2,2+k} -$$

$$\pi_j\,T_1\,(-d_1 - d_2 - d_3 - d_4 - d_6 - d_8 + d_2\,T_1 + d_4\,T_1 + d_1\,T_2 + d_2\,T_2 + d_4\,T_2 + d_6\,T_2 + d_8\,T_2 - d_2\,T_1\,T_2 - d_4\,T_1\,T_2)$$

$$p_{1,2+j}\,p_{2,2+k} + \pi_i\,T_1\,(2\,d_1 + 2\,d_3 + d_5 + d_7 - d_1\,T_1 - d_3\,T_1 - 2\,d_1\,T_2 - d_3\,T_2 - d_5\,T_2 - d_7\,T_2 +$$

$$d_2\,T_1\,T_2 + d_4\,T_1\,T_2 + d_1\,T_1\,T_2^2 - d_2\,T_1\,T_2^2 - d_2\,T_1^2\,T_2^2 - d_4\,T_1^2\,T_2^2 + d_2\,T_1^2\,T_2^3)\,p_{1,2+j}\,p_{2,2+k} +$$

$$2\,\pi_k\,(d_2 + d_4 + d_6 + d_8 - d_2\,T_1 - d_4\,T_1 - d_2\,T_2 - d_6\,T_2 + d_2\,T_1\,T_2)\,p_{1,2+k}\,p_{2,2+k} +$$

$$\pi_j\,(d_1 + d_2 + d_3 + d_4 + d_5 + d_6 + d_7 + d_8 - d_1\,T_1 - d_2\,T_1 - d_3\,T_1 - d_4\,T_1 - d_6\,T_1 - d_8\,T_1 - d_1\,T_2 - d_2\,T_2 -$$

$$d_4\,T_2 - d_5\,T_2 - d_6\,T_2 - d_8\,T_2 + d_1\,T_1\,T_2 + d_2\,T_1\,T_2 + d_4\,T_1\,T_2 + d_6\,T_1\,T_2 + d_8\,T_1\,T_2)\,p_{1,2+k}\,p_{2,2+k} +$$

$$\pi_i\,(2\,d_1 + 2\,d_3 + 2\,d_5 + 2\,d_7 - 2\,d_1\,T_1 - 2\,d_3\,T_1 - d_5\,T_1 - d_7\,T_1 - 2\,d_1\,T_2 - d_3\,T_2 - 2\,d_5\,T_2 - d_7\,T_2 + 2\,d_1\,T_1\,T_2 +$$

$$d_2\,T_1\,T_2 + d_3\,T_1\,T_2 + d_4\,T_1\,T_2 + d_5\,T_1\,T_2 + d_6\,T_1\,T_2 + d_7\,T_1\,T_2 + d_8\,T_1\,T_2 - d_2\,T_1^2\,T_2 - d_4\,T_1^2\,T_2 - d_2\,T_1\,T_2^2 -$$

$$d_6\,T_1\,T_2^2 - d_4\,T_1^2\,T_2^2 - d_6\,T_1^2\,T_2^2 - d_8\,T_1^2\,T_2^2 + d_2\,T_1^3\,T_2^2 + d_4\,T_1^3\,T_2^2 + d_2\,T_1^2\,T_2^3 + d_6\,T_1^2\,T_2^3 - d_2\,T_1^3\,T_2^3)\,p_{1,2+k}\,p_{2,2+k}\}$$

$In[\circ]:=$ `{rhs} = Cases` $\Big[\int \mathbb{E}[\pi_i\, p_{3,i} + \pi_j\, p_{3,j} + \pi_k\, p_{3,k}]\ \mathcal{L}$ `/@` $(X_{j,k}[1]\, X_{i,k+1}[1]\, X_{i+1,j+1}[1])$

$\mathbb{d}\{vs_i,\, vs_j,\, vs_k,\, vs_{i+1},\, vs_{j+1},\, vs_{k+1}\},\ \epsilon Series[\_,\, \mathcal{E}\_] \mapsto \mathcal{E},\, \infty\Big]$

$Out[\circ]=$

$\Big\{ 2\, d_1\, \pi_i\, T_1^2\, T_2^2\, p_{1,2+i}\, p_{2,2+i} + d_2\, \pi_k\, T_1^2\, T_2^2\, p_{1,2+i}\, p_{2,2+i} - d_2\, \pi_j\, T_1^2\, T_2^2\, (-2 + T_1\, T_2)\, p_{1,2+i}\, p_{2,2+i} -$

$d_2\, \pi_k\, (-1 + T_1)\, T_1\, T_2^2\, p_{1,2+j}\, p_{2,2+i} - \pi_i\, T_1\, (-2\, d_1 - d_5 + 2\, d_1\, T_1)\, T_2^2\, p_{1,2+j}\, p_{2,2+i} +$

$\pi_j\, T_1\, T_2^2\, (2\, d_2 + d_6 - 2\, d_2\, T_1 - d_2\, T_1\, T_2 + d_2\, T_1^2\, T_2)\, p_{1,2+j}\, p_{2,2+i} - \pi_i\, (-d_1 - d_5 + d_1\, T_1)\, T_2^2\, p_{1,2+k}\, p_{2,2+i} -$

$\pi_k\, (-d_2 - d_6 + d_2\, T_1)\, T_2^2\, p_{1,2+k}\, p_{2,2+i} + \pi_j\, (-d_2 - d_6 + d_2\, T_1)\, T_2^2\, (-1 + T_1\, T_2)\, p_{1,2+k}\, p_{2,2+i} -$

$d_2\, \pi_k\, T_1^2\, (-1 + T_2)\, T_2\, p_{1,2+i}\, p_{2,2+j} - \pi_i\, T_1^2\, T_2\, (-2\, d_1 - d_3 + 2\, d_1\, T_2)\, p_{1,2+i}\, p_{2,2+j} +$

$\pi_j\, T_1^2\, T_2\, (2\, d_2 + d_4 - 2\, d_2\, T_2 - d_2\, T_1\, T_2 + d_2\, T_1\, T_2^2)\, p_{1,2+i}\, p_{2,2+j} +$

$d_2\, \pi_k\, T_1\, T_2\, (2 - T_1 - T_2 + T_1\, T_2)\, p_{1,2+j}\, p_{2,2+j} +$

$\pi_i\, T_1\, T_2\, (2\, d_1 + d_3 + d_5 + d_7 - 2\, d_1\, T_1 - d_3\, T_1 - 2\, d_1\, T_2 - d_5\, T_2 + 2\, d_1\, T_1\, T_2)\, p_{1,2+j}\, p_{2,2+j} +$

$\pi_j\, T_1\, T_2\, (d_1 + 2\, d_2 + d_4 + d_6 + d_8 - 2\, d_2\, T_1 - d_4\, T_1 - 2\, d_2\, T_2 - d_6\, T_2 + d_2\, T_1\, T_2 + d_2\, T_1^2\, T_2 + d_2\, T_1\, T_2^2 - d_2\, T_1^2\, T_2^2)$

$p_{1,2+j}\, p_{2,2+j} + \pi_k\, (-d_2 - d_6 + d_2\, T_1)\, (-2 + T_2)\, T_2\, p_{1,2+k}\, p_{2,2+j} +$

$\pi_i\, (-d_1 - d_5 + d_1\, T_1)\, (-1 + T_2)\, T_2\, p_{1,2+k}\, p_{2,2+j} -$

$\pi_j\, T_2\, (-d_1 - d_2 - d_5 - d_6 + d_1\, T_1 + d_2\, T_1 + d_2\, T_2 + d_6\, T_2 + d_6\, T_1\, T_2 - d_2\, T_1^2\, T_2 - d_2\, T_1\, T_2^2 - d_6\, T_1\, T_2^2 + d_2\, T_1^2\, T_2^2)$

$p_{1,2+k}\, p_{2,2+j} - \pi_i\, T_1^2\, (-d_1 - d_3 + d_1\, T_2)\, p_{1,2+i}\, p_{2,2+k} -$

$\pi_k\, T_1^2\, (-d_2 - d_4 + d_2\, T_2)\, p_{1,2+i}\, p_{2,2+k} + \pi_j\, T_1^2\, (-d_2 - d_4 + d_2\, T_2)\, (-1 + T_1\, T_2)\, p_{1,2+i}\, p_{2,2+k} +$

$\pi_i\, (-1 + T_1)\, T_1\, (-d_1 - d_3 + d_1\, T_2)\, p_{1,2+j}\, p_{2,2+k} + \pi_k\, (-2 + T_1)\, T_1\, (-d_2 - d_4 + d_2\, T_2)\, p_{1,2+j}\, p_{2,2+k} -$

$\pi_j\, T_1\, (-d_1 - d_2 - d_3 - d_4 + d_2\, T_1 + d_4\, T_1 + d_1\, T_2 + d_2\, T_2 + d_4\, T_1\, T_2 - d_2\, T_1^2\, T_2 - d_4\, T_1^2\, T_2 - d_2\, T_1\, T_2^2 + d_2\, T_1^2\, T_2^2)$

$p_{1,2+j}\, p_{2,2+k} + \pi_i\, (d_1 + d_3 + d_5 + d_7 - d_1\, T_1 - d_3\, T_1 - d_1\, T_2 - d_5\, T_2 + d_1\, T_1\, T_2)\, p_{1,2+k}\, p_{2,2+k} +$

$2\, \pi_k\, (d_2 + d_4 + d_6 + d_8 - d_2\, T_1 - d_4\, T_1 - d_2\, T_2 - d_6\, T_2 + d_2\, T_1\, T_2)\, p_{1,2+k}\, p_{2,2+k} +$

$\pi_j\, (d_1 + d_2 + d_3 + d_4 + d_5 + d_6 + d_7 + d_8 - d_1\, T_1 - d_2\, T_1 - d_3\, T_1 - d_4\, T_1 - d_1\, T_2 - d_2\, T_2 - d_5\, T_2 - d_6\, T_2 +$

$d_1\, T_1\, T_2 - d_4\, T_1\, T_2 - d_6\, T_1\, T_2 - d_8\, T_1\, T_2 + d_2\, T_1^2\, T_2 + d_4\, T_1^2\, T_2 + d_2\, T_1\, T_2^2 + d_6\, T_1\, T_2^2 - d_2\, T_1^2\, T_2^2)\, p_{1,2+k}\, p_{2,2+k}\Big\}$

$In[\bullet]:=$ **eqn = CF[lhs - rhs]**

$Out[\bullet]=$

$-d_2 \pi_k T_1 T_2 (-1 + T_1 T_2) p_{1,2+i} p_{2,2+i} + d_2 \pi_j T_1^2 T_2^2 (-1 + T_1 T_2) p_{1,2+i} p_{2,2+i} +$

$d_1 \pi_i (-1 + T_1) T_1 T_2^2 p_{1,2+j} p_{2,2+i} + d_2 \pi_k (-1 + T_1) T_1 T_2^2 p_{1,2+j} p_{2,2+i} -$

$d_2 \pi_j (-1 + T_1) T_1 T_2^2 (-1 + T_1 T_2) p_{1,2+j} p_{2,2+i} + \pi_k (-d_2 - d_6 + d_2 T_1) (-1 + T_2) T_2 p_{1,2+k} p_{2,2+i} -$

$d_1 \pi_i (-1 + T_1) T_1 T_2^2 p_{1,2+k} p_{2,2+i} + \pi_j T_1 T_2^2 (-d_6 + d_2 T_2 + d_6 T_2 - d_2 T_1 T_2) p_{1,2+k} p_{2,2+i} +$

$d_1 \pi_i T_1^2 (-1 + T_2) T_2 p_{1,2+i} p_{2,2+j} + d_2 \pi_k T_1^2 (-1 + T_2) T_2 p_{1,2+i} p_{2,2+j} -$

$d_2 \pi_j T_1^2 (-1 + T_2) T_2 (-1 + T_1 T_2) p_{1,2+i} p_{2,2+j} - d_2 \pi_k (-1 + T_1) T_1 (-1 + T_2) T_2 p_{1,2+j} p_{2,2+j} +$

$d_2 \pi_j (-1 + T_1) T_1 (-1 + T_2) T_2 (-1 + T_1 T_2) p_{1,2+j} p_{2,2+j} -$

$\pi_i T_1 T_2 (-d_1 T_1 - d_1 T_2 + 2 d_1 T_1 T_2 - d_2 T_1 T_2 + d_2 T_1^2 T_2^2) p_{1,2+j} p_{2,2+j} -$

$\pi_k (-d_2 - d_6 + d_2 T_1) (-1 + T_2) T_2 p_{1,2+k} p_{2,2+j} -$

$\pi_j T_2 (-d_4 - d_8 + d_4 T_1 + d_6 T_1 + d_8 T_1 - d_2 T_1 T_2 - 2 d_6 T_1 T_2 + d_2 T_1^2 T_2 + d_2 T_1 T_2^2 + d_6 T_1 T_2^2 - d_2 T_1^2 T_2^2)$

$p_{1,2+k} p_{2,2+j} + \pi_i T_2 (d_1 + d_3 + d_5 + d_7 - d_1 T_1 - d_3 T_1 - d_5 T_1 - d_7 T_1 - d_1 T_1 T_2 +$

$d_2 T_1 T_2 + d_6 T_1 T_2 + d_1 T_1^2 T_2 - d_2 T_1^2 T_2 - d_2 T_1^2 T_2^2 - d_6 T_1^2 T_2^2 + d_2 T_1^3 T_2^2) p_{1,2+k} p_{2,2+j} -$

$d_1 \pi_i T_1^2 (-1 + T_2) T_2 p_{1,2+i} p_{2,2+k} + \pi_k (-1 + T_1) T_1 (-d_2 - d_4 + d_2 T_2) p_{1,2+i} p_{2,2+k} +$

$\pi_j T_1^2 T_2 (-d_4 + d_2 T_1 + d_4 T_1 - d_2 T_1 T_2) p_{1,2+i} p_{2,2+k} - \pi_k (-1 + T_1) T_1 (-d_2 - d_4 + d_2 T_2) p_{1,2+j} p_{2,2+k} -$

$\pi_j T_1 (-d_6 - d_8 + d_4 T_2 + d_6 T_2 + d_8 T_2 - d_2 T_1 T_2 - 2 d_4 T_1 T_2 + d_2 T_1^2 T_2 + d_4 T_1^2 T_2 + d_2 T_1 T_2^2 - d_2 T_1^2 T_2^2)$

$p_{1,2+j} p_{2,2+k} + \pi_i T_1 (d_1 + d_3 + d_5 + d_7 - d_1 T_2 - d_3 T_2 - d_5 T_2 - d_7 T_2 - d_1 T_1 T_2 +$

$d_2 T_1 T_2 + d_4 T_1 T_2 + d_1 T_1 T_2^2 - d_2 T_1 T_2^2 - d_2 T_1^2 T_2^2 - d_4 T_1^2 T_2^2 + d_2 T_1^2 T_2^3) p_{1,2+j} p_{2,2+k} +$

$\pi_j (-d_6 T_1 - d_8 T_1 - d_4 T_2 - d_8 T_2 + d_2 T_1 T_2 + 2 d_4 T_1 T_2 + 2 d_6 T_1 T_2 + 2 d_8 T_1 T_2 -$

$d_2 T_1^2 T_2 - d_4 T_1^2 T_2 - d_2 T_1 T_2^2 - d_6 T_1 T_2^2 + d_2 T_1^2 T_2^2) p_{1,2+k} p_{2,2+k} +$

$\pi_i (d_1 + d_3 + d_5 + d_7 - d_1 T_1 - d_3 T_1 - d_5 T_1 - d_7 T_1 - d_1 T_2 - d_3 T_2 - d_5 T_2 - d_7 T_2 + d_1 T_1 T_2 + d_2 T_1 T_2 +$

$d_3 T_1 T_2 + d_4 T_1 T_2 + d_5 T_1 T_2 + d_6 T_1 T_2 + d_7 T_1 T_2 + d_8 T_1 T_2 - d_2 T_1^2 T_2 - d_4 T_1^2 T_2 - d_2 T_1 T_2^2 - d_6 T_1 T_2^2 -$

$d_4 T_1^2 T_2^2 - d_6 T_1^2 T_2^2 - d_8 T_1^2 T_2^2 + d_2 T_1^3 T_2^2 + d_4 T_1^3 T_2^2 + d_2 T_1^2 T_2^3 + d_6 T_1^2 T_2^3 - d_2 T_1^3 T_2^3) p_{1,2+k} p_{2,2+k}$

$In[\bullet]:=$ **cvs = Union@Cases[eqn, p__ | π__, ∞]**

$Out[\bullet]=$

$\{\pi_i, \pi_j, \pi_k, p_{1,2+i}, p_{1,2+j}, p_{1,2+k}, p_{2,2+i}, p_{2,2+j}, p_{2,2+k}\}$

$In[\circ]:=$ **eqns = CoefficientRules[eqn, cvs] /. (_ → c_) :> (c == 0)**

$Out[\circ]=$

$\{-d_1\,T_1^2\,T_2 + d_1\,T_1^2\,T_2^2 == 0,\ d_1\,T_1^2\,T_2 - d_1\,T_1^2\,T_2^2 == 0,$

$-d_1\,T_1\,T_2^2 + d_1\,T_1^2\,T_2^2 == 0,\ d_1\,T_1^2\,T_2 + d_1\,T_1\,T_2^2 - 2\,d_1\,T_1^2\,T_2^2 + d_2\,T_1^2\,T_2^2 - d_2\,T_1^3\,T_2^3 == 0,$

$d_1\,T_1 + d_3\,T_1 + d_5\,T_1 + d_7\,T_1 - d_1\,T_1\,T_2 - d_3\,T_1\,T_2 - d_5\,T_1\,T_2 - d_7\,T_1\,T_2 -$
$\quad d_1\,T_1^2\,T_2 + d_2\,T_1^2\,T_2 + d_4\,T_1^2\,T_2 + d_1\,T_1^2\,T_2^2 - d_2\,T_1^2\,T_2^2 - d_2\,T_1^3\,T_2^2 - d_4\,T_1^3\,T_2^2 + d_2\,T_1^3\,T_2^3 == 0,$

$d_1\,T_1\,T_2^2 - d_1\,T_1^2\,T_2^2 == 0,\ d_1\,T_2 + d_3\,T_2 + d_5\,T_2 + d_7\,T_2 - d_1\,T_1\,T_2 - d_3\,T_1\,T_2 - d_5\,T_1\,T_2 - d_7\,T_1\,T_2 -$
$\quad d_1\,T_1\,T_2^2 + d_2\,T_1\,T_2^2 + d_6\,T_1\,T_2^2 + d_1\,T_1^2\,T_2^2 - d_2\,T_1^2\,T_2^2 - d_2\,T_1^2\,T_2^3 - d_6\,T_1^2\,T_2^3 + d_2\,T_1^3\,T_2^3 == 0,$

$d_1 + d_3 + d_5 + d_7 - d_1\,T_1 - d_3\,T_1 - d_5\,T_1 - d_7\,T_1 - d_1\,T_2 - d_3\,T_2 - d_5\,T_2 - d_7\,T_2 + d_1\,T_1\,T_2 + d_2\,T_1\,T_2 +$
$\quad d_3\,T_1\,T_2 + d_4\,T_1\,T_2 + d_5\,T_1\,T_2 + d_6\,T_1\,T_2 + d_7\,T_1\,T_2 + d_8\,T_1\,T_2 - d_2\,T_1^2\,T_2 - d_4\,T_1^2\,T_2 - d_2\,T_1\,T_2^2 -$
$\quad d_6\,T_1\,T_2^2 - d_4\,T_1^2\,T_2^2 - d_6\,T_1^2\,T_2^2 - d_8\,T_1^2\,T_2^2 + d_2\,T_1^3\,T_2^2 + d_4\,T_1^3\,T_2^2 + d_2\,T_1^2\,T_2^3 + d_6\,T_1^2\,T_2^3 - d_2\,T_1^3\,T_2^3 == 0,$

$-d_2\,T_1^2\,T_2^2 + d_2\,T_1^3\,T_2^3 == 0,\ -d_2\,T_1^2\,T_2 + d_2\,T_1^2\,T_2^2 + d_2\,T_1^3\,T_2^2 - d_2\,T_1^3\,T_2^3 == 0,$

$-d_4\,T_1^2\,T_2 + d_2\,T_1^3\,T_2 + d_4\,T_1^3\,T_2 - d_2\,T_1^3\,T_2^2 == 0,\ -d_2\,T_1\,T_2^2 + d_2\,T_1^2\,T_2^2 + d_2\,T_1^2\,T_2^3 - d_2\,T_1^3\,T_2^3 == 0,$

$-d_2\,T_1\,T_2 + d_2\,T_1^2\,T_2 + d_2\,T_1\,T_2^2 - d_2\,T_1^3\,T_2^2 - d_2\,T_1^2\,T_2^3 + d_2\,T_1^3\,T_2^3 == 0,$

$d_6\,T_1 + d_8\,T_1 - d_4\,T_1\,T_2 - d_6\,T_1\,T_2 - d_8\,T_1\,T_2 + d_2\,T_1^2\,T_2 + 2\,d_4\,T_1^2\,T_2 - d_2\,T_1^3\,T_2 - d_4\,T_1^3\,T_2 - d_2\,T_1^2\,T_2^2 + d_2\,T_1^3\,T_2^2 ==$
$\quad 0,\ -d_6\,T_1\,T_2^2 + d_2\,T_1\,T_2^3 + d_6\,T_1\,T_2^3 - d_2\,T_1^2\,T_2^3 == 0,$

$d_4\,T_2 + d_8\,T_2 - d_4\,T_1\,T_2 - d_6\,T_1\,T_2 - d_8\,T_1\,T_2 + d_2\,T_1\,T_2^2 + 2\,d_6\,T_1\,T_2^2 - d_2\,T_1^2\,T_2^2 - d_2\,T_1\,T_2^3 - d_6\,T_1\,T_2^3 + d_2\,T_1^2\,T_2^3 ==$
$\quad 0,\ -d_6\,T_1 - d_8\,T_1 - d_4\,T_2 - d_8\,T_2 + d_2\,T_1\,T_2 + 2\,d_4\,T_1\,T_2 + 2\,d_6\,T_1\,T_2 + 2\,d_8\,T_1\,T_2 -$
$\quad d_2\,T_1^2\,T_2 - d_4\,T_1^2\,T_2 - d_2\,T_1\,T_2^2 - d_6\,T_1\,T_2^2 + d_2\,T_1^2\,T_2^2 == 0,\ d_2\,T_1\,T_2 - d_2\,T_1^2\,T_2^2 == 0,$

$-d_2\,T_1^2\,T_2 + d_2\,T_1^2\,T_2^2 == 0,\ d_2\,T_1 + d_4\,T_1 - d_2\,T_1^2 - d_4\,T_1^2 - d_2\,T_1\,T_2 + d_2\,T_1^2\,T_2 == 0,$

$-d_2\,T_1\,T_2^2 + d_2\,T_1^2\,T_2^2 == 0,\ -d_2\,T_1\,T_2 + d_2\,T_1^2\,T_2 + d_2\,T_1\,T_2^2 - d_2\,T_1^2\,T_2^2 == 0,$

$-d_2\,T_1 - d_4\,T_1 + d_2\,T_1^2 + d_4\,T_1^2 + d_2\,T_1\,T_2 - d_2\,T_1^2\,T_2 == 0,\ d_2\,T_2 + d_6\,T_2 - d_2\,T_1\,T_2 - d_2\,T_2^2 - d_6\,T_2^2 + d_2\,T_1\,T_2^2 == 0,$

$-d_2\,T_2 - d_6\,T_2 + d_2\,T_1\,T_2 + d_2\,T_2^2 + d_6\,T_2^2 - d_2\,T_1\,T_2^2 == 0\}$

$In[\circ]:=$ **vars = Union@Cases[eqn, d_, ∞]**

$Out[\circ]=$

$\{d_1,\ d_2,\ d_3,\ d_4,\ d_5,\ d_6,\ d_7,\ d_8\}$

$In[\circ]:=$ **{sol} = Solve[eqns, vars]**

⋯ Solve: Equations may not give solutions for all "solve" variables.

$Out[\circ]=$

$\{\{d_1 \to 0,\ d_2 \to 0,\ d_4 \to 0,\ d_6 \to 0,\ d_7 \to -d_3 - d_5,\ d_8 \to 0\}\}$

$In[\circ]:=$ **sol /. (v_ → val_) :> (v = CF[val])**

$Out[\circ]=$

$\{0,\ 0,\ 0,\ 0,\ -d_3 - d_5,\ 0\}$

$In[\circ]:=$ **vars = Union@Cases[eqn, d_, ∞]**

$Out[\circ]=$

$\{d_3,\ d_5\}$

$In[\bullet]:=$ $\mathcal{L}[X_{i,j}[1]]$

$Out[\bullet]=$

$T_1\,T_2\,\mathbb{E}\,[\epsilon\mathsf{Series}\,[-p_{1,i}\,x_{1,i} + T_1\,p_{1,1+i}\,x_{1,i} + p_{1,1+j}\,x_{1,i} - T_1\,p_{1,1+j}\,x_{1,i} - p_{1,j}\,x_{1,j} +$
$\qquad p_{1,1+j}\,x_{1,j} - p_{2,i}\,x_{2,i} + T_2\,p_{2,1+i}\,x_{2,i} + p_{2,1+j}\,x_{2,i} - T_2\,p_{2,1+j}\,x_{2,i} - p_{2,j}\,x_{2,j} + p_{2,1+j}\,x_{2,j} -$
$\qquad p_{3,i}\,x_{3,i} + T_1\,T_2\,p_{3,1+i}\,x_{3,i} + p_{3,1+j}\,x_{3,i} - T_1\,T_2\,p_{3,1+j}\,x_{3,i} - p_{3,j}\,x_{3,j} + p_{3,1+j}\,x_{3,j},$
$\qquad d_5\,p_{1,j}\,p_{2,i}\,x_{3,i} + d_3\,p_{1,i}\,p_{2,j}\,x_{3,i} + (-d_3 - d_5)\,p_{1,j}\,p_{2,j}\,x_{3,i}]\,]$

$In[\bullet]:=$ $r_1[1,\,i,\,j]$

$Out[\bullet]=$

$d_5\,p_{1,j}\,p_{2,i}\,x_{3,i} + d_3\,p_{1,i}\,p_{2,j}\,x_{3,i} + (-d_3 - d_5)\,p_{1,j}\,p_{2,j}\,x_{3,i}$