(Alt) In[ ]:=

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\HigherRank\\Rolands_A2"];
Once[<< KnotTheory`];
<< ../Rot.m
(α_⁺)⁺ := α"++";
(* this is for cosmetic reasons only *)
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

```
Loading Rot.m from http://drorbn.net/AP/Projects/HigherRank to compute rotation numbers.
```

(Alt) In[ ]:=

$$r_0[1, i\_, j\_] := p_{3,j}\, x_{1,i}\, x_{2,i} - \frac{p_{3,j}\, x_{1,j}\, x_{2,i}}{T_1} \quad \text{(*from r0p*)}$$

$$r_0[-1, i\_, j\_] := -\frac{p_{3,j}\, x_{1,i}\, x_{2,i}}{T_1^2\, T_2} + \frac{p_{3,j}\, x_{1,j}\, x_{2,i}}{T_1\, T_2}$$

$$\begin{aligned} r_1[1, i\_, j\_] := {} & \frac{T_2\, p_{1,j}\, p_{2,j}\, x_{1,i}\, x_{2,i}}{-1 + T_1 T_2} - \frac{p_{1,j}\, p_{2,i}\, x_{1,j}\, x_{2,i}}{(-1 + T_1)\, T_1\, (-1 + T_1 T_2)} - \\ & \frac{p_{1,j}\, p_{2,j}\, x_{1,j}\, x_{2,i}}{(-1 + T_1)\, T_1} + \frac{p_{1,i}\, p_{2,j}\, x_{1,i}\, x_{2,j}}{(-1 + T_1)\, (-1 + T_1 T_2)} + p_{1,j}\, p_{2,i}\, x_{3,i} - p_{1,j}\, p_{2,j}\, x_{3,i} + \\ & \frac{p_{3,j}\, x_{3,i}}{T_1\, (-1 + T_1 T_2)} - p_{1,j}\, p_{3,j}\, x_{1,i}\, x_{3,i} + \frac{p_{1,j}\, p_{3,i}\, x_{1,j}\, x_{3,i}}{(-1 + T_1)\, T_1\, (-1 + T_1 T_2)} + \frac{p_{1,j}\, p_{3,j}\, x_{1,j}\, x_{3,i}}{-1 + T_1} - \\ & \frac{T_2\, p_{2,j}\, p_{3,j}\, x_{2,i}\, x_{3,i}}{T_1} - \frac{p_{2,j}\, p_{3,i}\, x_{2,j}\, x_{3,i}}{T_1\, (-1 + T_1 T_2)} - \frac{p_{1,i}\, p_{3,j}\, x_{1,i}\, x_{3,j}}{(-1 + T_1)\, (-1 + T_1 T_2)} + \frac{T_2\, p_{2,j}\, p_{3,j}\, x_{2,i}\, x_{3,j}}{T_1\, (-1 + T_1 T_2)} \end{aligned}$$

$$\begin{aligned} r_1[-1, i\_, j\_] := {} & \\ & \frac{p_{1,j}\, p_{2,i}\, x_{1,i}\, x_{2,i}}{T_1^2\, (-1 + T_1 T_2)} - \frac{(-1 + T_2)\, p_{1,i}\, p_{2,j}\, x_{1,i}\, x_{2,i}}{(-1 + T_1)\, T_2\, (-1 + T_1 T_2)} + \frac{(-T_1 - T_2 + T_1 T_2)\, p_{1,j}\, p_{2,j}\, x_{1,i}\, x_{2,i}}{T_1^2\, T_2\, (-1 + T_1 T_2)} + \\ & \frac{p_{1,j}\, p_{2,i}\, x_{1,j}\, x_{2,i}}{(-1 + T_1)\, T_1\, (-1 + T_1 T_2)} + \frac{p_{1,j}\, p_{2,j}\, x_{1,j}\, x_{2,i}}{T_1\, (-1 + T_1 T_2)} - \frac{p_{1,i}\, p_{2,j}\, x_{1,i}\, x_{2,j}}{(-1 + T_1)\, (-1 + T_1 T_2)} + \frac{p_{1,j}\, p_{2,j}\, x_{1,i}\, x_{2,j}}{T_1\, (-1 + T_1 T_2)} - \\ & \frac{p_{1,j}\, p_{2,i}\, x_{3,i}}{T_1} + \frac{p_{1,j}\, p_{2,j}\, x_{3,i}}{T_1} - \frac{p_{3,j}\, x_{3,i}}{T_1\, (-1 + T_1 T_2)} - \frac{p_{1,j}\, p_{3,i}\, x_{1,i}\, x_{3,i}}{T_1^2\, (-1 + T_1 T_2)} + \frac{p_{1,i}\, p_{3,j}\, x_{1,i}\, x_{3,i}}{(-1 + T_1)\, T_1\, T_2} - \\ & \frac{p_{1,j}\, p_{3,j}\, x_{1,i}\, x_{3,i}}{T_1^2\, T_2} - \frac{p_{1,j}\, p_{3,i}\, x_{1,j}\, x_{3,i}}{(-1 + T_1)\, T_1\, (-1 + T_1 T_2)} + \frac{(-1 + T_2)\, p_{2,j}\, p_{3,i}\, x_{2,i}\, x_{3,i}}{T_1\, T_2\, (-1 + T_1 T_2)} + \\ & \frac{p_{2,i}\, p_{3,j}\, x_{2,i}\, x_{3,i}}{T_1^2\, T_2} - \frac{(-1 + 2\, T_2)\, p_{2,j}\, p_{3,j}\, x_{2,i}\, x_{3,i}}{T_1^2\, T_2^2} + \frac{p_{2,j}\, p_{3,i}\, x_{2,j}\, x_{3,i}}{T_1\, (-1 + T_1 T_2)} - \\ & \frac{p_{2,j}\, p_{3,j}\, x_{2,j}\, x_{3,i}}{T_1^2\, T_2} + \frac{p_{1,i}\, p_{3,j}\, x_{1,i}\, x_{3,j}}{(-1 + T_1)\, (-1 + T_1 T_2)} - \frac{p_{1,j}\, p_{3,j}\, x_{1,i}\, x_{3,j}}{T_1\, (-1 + T_1 T_2)} - \frac{p_{2,j}\, p_{3,j}\, x_{2,i}\, x_{3,j}}{T_1\, (-1 + T_1 T_2)} \end{aligned}$$

(Alt) In[ ]:=

$$\text{g2px}[\mathcal{E}\_] := \text{Module}\left[\{\lambda\},\ \text{Expand}[\mathcal{E}\ /.\ g_{\alpha\_,i\_,j\_} \mapsto \lambda\, p_{\alpha,i}\, x_{\alpha,j}]\ /.\ \lambda^{k\_\cdot} \mapsto 1/k!\right]$$

(Alt) In[ ]:=

```
{p*, x*, π*, ξ*} = {π, ξ, p, x};  (u_i__)* := (u*)_i;
```

(Alt) In[ ]:=

```
Zip_{}[ℰ_] := ℰ;
Zip_{ξ_,ξs___}[ℰ_] := (Collect[ℰ // Zip_{ξs}, ξ] /. f_. ξ^d_. :> (D[f, {ξ*, d}])) /. ξ* → 0
```

(Alt) In[ ]:=

```
px2g[ℰ_] := Module[{ps, xs, Q},
  ps = Union[Cases[ℰ, p__, ∞]];
  xs = Union[Cases[ℰ, x__, ∞]];
  Q = Sum[p0* x0* g_{p0[[2]],x0[[2]],p0[[3]],x0[[3]]}, {p0, ps}, {x0, xs}];
  Expand[Zip_{ps⋃xs}[ℰ e^Q] /. g_{α_,β_,i_,j_} :> If[α === β, g_{α,i,j}, 0]]
 ]
```

(Alt) In[ ]:=

```
R_1[1, i_, j_] := Evaluate[px2g[r_1[1, i, j]] +
   (Coefficient[r_1[1, i, j] /. t : (x | p)__ :> λ t, λ^3] /. x_{3,α_} p_{1,β_} p_{2,γ_} :> y_{α,β,γ})]
R_1[-1, i_, j_] := Evaluate[px2g[r_1[-1, i, j]] +
   (Coefficient[r_1[-1, i, j] /. t : (x | p)__ :> λ t, λ^3] /. x_{3,α_} p_{1,β_} p_{2,γ_} :> y_{α,β,γ})]
Piv_i_ := - 1/(T_1 (-1 + T_1 T_2)) g_{3,i,i} (* - ((-2+T_1+T_2) (-T_1-T_2+2 T_1 T_2) g_{3,i,i})/((-1+T_1) (-1+T_2) (-1+T_1 T_2)) *)
```

(Alt) In[ ]:=

```
θ[1, i_, j_, α_, β_, γ_] :=
  Evaluate[r_0[1, i, j] /. {p_{3,j_} :> g_{3,j,α}, x_{1,i_} :> g_{1,β,i}, x_{2,i_} :> g_{2,γ,i}}] ;
 (* The θ graph with light (pxx) vertex  at (1,i,j) and
 unspecified heavy (xpp) vertex *)
θ[-1, i_, j_, α_, β_, γ_] :=
  Evaluate[r_0[-1, i, j] /. {p_{3,j_} :> g_{3,j,α}, x_{1,i_} :> g_{1,β,i}, x_{2,i_} :> g_{2,γ,i}}] ;
 (* The θ graph with light (pxx) vertex  at (-1,i,j)
 and unspecified heavy (xpp) vertex *)
θ[1, 5, 8, 21, 22, 23]
```

(Alt) Out[ ]=

$$g_{1,22,5}\, g_{2,23,5}\, g_{3,8,21} - \frac{g_{1,22,8}\, g_{2,23,5}\, g_{3,8,21}}{T_1}$$

(Alt) In[ ]:=

```
T₃ = T₁ T₂;
CF[ℰ_] := Factor@Together[ℰ];
λ[K_] := Module[{Cs, φ, n, A, s, i, j, k, Δ, G, gEval, Y, yEval, c, λ1},
   {Cs, φ} = Rot[K]; n = Length[Cs];
   A = IdentityMatrix[2 n + 1];
   Cases[Cs, {s_, i_, j_} ⧴ (A⟦{i, j}, {i + 1, j + 1}⟧ += ( -T^s  T^s - 1
                                                              0     -1   ))];
   Δ = T^((-Total[φ] - Total[Cs⟦All,1⟧])/2) Det[A];
   G = Inverse[A];
   gEval[ℰ_] := CF[ℰ /.
      {g_{1,α_,β_} ⧴ (G⟦α, β⟧ /. T → T₁),
       g_{2,α_,β_} ⧴ (G⟦α, β⟧ /. T → T₂), g_{3,α_,β_} ⧴ (G⟦α, β⟧ /. T → T₃)}];
   Y[α_, β_, γ_] :=
    Y[α, β, γ] = Sum[{s, i, j} = c; (* The expectation value of x_{3,α}p_{1,β}p_{2,γ} *)
      Θ[s, i, j, α, β, γ],
      {c, Cs}];
   yEval[ℰ_] := ℰ /. y_{α_,β_,γ_} ⧴ Y[α, β, γ];
   λ1 = Sum[R₁ @@ Cs⟦k⟧, {k, 1, n}] + Sum[φ⟦k⟧ Piv_k, {k, 1, 2 n}];
   {Δ, (1 - T₃) (Δ /. T → T₁) (Δ /. T → T₂) (Δ /. T → T₃) λ1} // yEval // gEval // Factor
  ];
Θ[K_] := Module[{L = λ[K]},
  {L⟦1⟧, T₁ L⟦2⟧ + (T D[L⟦1⟧, T] /. T → T₃) (L⟦1⟧ /. T → T₁) (L⟦1⟧ /. T → T₂)} // Expand]
```

(Alt) In[ ]:=

```
CF[ℰ_] := Factor@Together[ℰ];
Nλ_{p1_,p2_}[K_] := Module[{G1, G2, G3, Δ1, Δ2, Δ3,
    A1, A2, A3, Cs, φ, n, A, s, i, j, k, Δ, G, gEval, Y, yEval, c, λ1},
   {Cs, φ} = Rot[K]; n = Length[Cs];
   A = IdentityMatrix[2 n + 1];
   Cases[Cs, {s_, i_, j_} ⧴ (A⟦{i, j}, {i + 1, j + 1}⟧ += ( -T^s  T^s - 1
                                                              0    -1    ))];
   A1 = A /. T → p1; A2 = A /. T → p2; A3 = A /. T → p1 p2;
   Δ1 = p1^((-Total[φ]-Total[Cs⟦All,1⟧])/2) Det[A1];
   Δ2 = p2^((-Total[φ]-Total[Cs⟦All,1⟧])/2) Det[A2];
   Δ3 = (p1 p2)^((-Total[φ]-Total[Cs⟦All,1⟧])/2) Det[A3];
   G1 = Inverse[A1]; G2 = Inverse[A2]; G3 = Inverse[A3];
   gEval[ℰ_] := CF[ℰ /.
      {g_{1,α_,β_} ⧴ G1⟦α, β⟧, g_{2,α_,β_} ⧴ G2⟦α, β⟧, g_{3,α_,β_} ⧴ G3⟦α, β⟧}];
   Y[α_, β_, γ_] :=
    Y[α, β, γ] = Sum[{s, i, j} = c; (* The expectation value of x_{3,α}p_{1,β}p_{2,γ} *)
       θ[s, i, j, α, β, γ],
       {c, Cs}] /. {T_1 → p1, T_2 → p2};
   yEval[ℰ_] := ℰ /. y_{α_,β_,γ_} ⧴ Y[α, β, γ];
   λ1 = Σ_{k=1}^{n} R_1 @@ Cs⟦k⟧ + Σ_{k=1}^{2 n} φ⟦k⟧ Piv_k /. {T_1 → p1, T_2 → p2};
   {Δ1, (1 - p1 p2) Δ1 Δ2 Δ3 λ1} // yEval // gEval // Expand
  ];
```

(Alt) In[ ]:=

```
Rrho_1[s_, i_, j_] := s (g_{ji} (g_{j+1,j} + g_{j,j+1} - g_{ij}) - g_{ii} (g_{j,j+1} - 1) - 1 / 2);
ρ[K_] := ρ[K] = Module[{Cs, φ, n, A, s, i, j, k, Δ, G, ρ1},
    {Cs, φ} = Rot[K]; n = Length[Cs];
    A = IdentityMatrix[2 n + 1];
    Cases[Cs, {s_, i_, j_} ⧴ (A⟦{i, j}, {i + 1, j + 1}⟧ += ( -T^s  T^s - 1
                                                               0    -1    ))];
    Δ = T^((-Total[φ]-Total[Cs⟦All,1⟧])/2) Det[A];
    G = Inverse[A];
    ρ1 = Σ_{k=1}^{n} Rrho_1 @@ Cs⟦k⟧ - Σ_{k=1}^{2 n} φ⟦k⟧ (g_{kk} - 1 / 2);
    Expand@Together@{Δ, Δ^2 ρ1 /. g_{α_,β_} ⧴ G⟦α, β⟧}

   ];
```

(Alt) In[ ]:=

```
ColFun[t_] := If[t > 0, {t, 0, 0}, {0, 0, t}]
Renorm[t_] := If[t == 0, 0, Sign[t] Log[Abs[t] + 10]]
Poly2Pic[P_] := Module[{e1 = Exponent[P, T1^-1], e2 = Exponent[P, T2^-1], Mat},
  If[P === 0, P, Mat =
    Map[Renorm, Normal@SparseArray[CoefficientRules[T1^(e1+1) T2^(e2+1) P, {T1, T2}]], {2}];
   MatrixPlot[Mat (*,ColorFunction→(RGBColor[If[#==0,0,1],0,0]&)*)]]
 ]
```

(Alt) In[ ]:=

```
TestSymmetries[K_] := Module[{θ0, θ1},
  {θ0, θ1} = {θ[K]⟦2⟧, θ[Mirror@K]⟦2⟧};
  Simplify@And[
    θ0 == (θ0 /. {T1 → T2, T2 → T1}),
    θ0 == -θ1,
    θ0 == (θ0 /. T_i_ :→ T_i^-1),
    θ0 == (θ0 /. T2 → T1^-1 T2^-1)
   ]
 ]
```

(Alt) In[ ]:=

```
λ[Knot[3, 1]]
```

KnotTheory: Loading precomputed data in PD4Knots`.

(Alt) Out[ ]=

$$\left\{\frac{1-T+T^2}{T},\ -\frac{-1+T_1-T_1^2+T_2-T_1^2\,T_2+2\,T_1^3\,T_2-T_2^2-T_1\,T_2^2+T_1^2\,T_2^2-2\,T_1^3\,T_2^2+2\,T_1\,T_2^3-2\,T_1^2\,T_2^3+2\,T_1^3\,T_2^3}{T_1^2\,T_2}\right\}$$

(Alt) In[ ]:=

```
θ[Knot[3, 1]]
```

(Alt) Out[ ]=

$$\left\{-1+\frac{1}{T}+T,\ -\frac{1}{T_1^2}-T_1^2-\frac{1}{T_2^2}-\frac{1}{T_1^2\,T_2^2}+\frac{1}{T_1\,T_2^2}+\frac{1}{T_1^2\,T_2}+\frac{T_1}{T_2}+\frac{T_2}{T_1}+T_1^2\,T_2-T_2^2+T_1\,T_2^2-T_1^2\,T_2^2\right\}$$

(Alt) In[ ]:=

```
θ[Knot[8, 19]]
```

(Alt) Out[ ]=

$$\Big\{1+\frac{1}{T^3}-\frac{1}{T^2}-T^2+T^3,$$

$$\frac{3}{T_1^6}-\frac{3}{T_1^4}+\frac{4}{T_1^3}-\frac{1}{T_1^2}-T_1^2+4T_1^3-3T_1^4+3T_1^6+\frac{3}{T_2^6}+\frac{3}{T_1^6T_2^6}-\frac{3}{T_1^5T_2^6}+\frac{3}{T_1^3T_2^6}-\frac{3}{T_1T_2^6}-\frac{3}{T_1^6T_2^5}+\frac{3}{T_1^4T_2^5}-\frac{3}{T_1^3T_2^5}-$$
$$\frac{3}{T_1^2T_2^5}+\frac{3}{T_1T_2^5}-\frac{3T_1}{T_2^5}-\frac{3}{T_2^4}+\frac{3}{T_1^5T_2^4}-\frac{3}{T_1^4T_2^4}+\frac{3}{T_1^2T_2^4}+\frac{3T_1}{T_2^4}+\frac{4}{T_2^3}+\frac{3}{T_1^6T_2^3}-\frac{3}{T_1^5T_2^3}+\frac{4}{T_1^3T_2^3}-\frac{2}{T_1^2T_2^3}-$$
$$\frac{2}{T_1T_2^3}-\frac{3T_1^2}{T_2^3}+\frac{3T_1^3}{T_2^3}-\frac{1}{T_2^2}-\frac{3}{T_1^5T_2^2}+\frac{3}{T_1^4T_2^2}-\frac{2}{T_1^3T_2^2}-\frac{1}{T_1^2T_2^2}+\frac{1}{T_1T_2^2}-\frac{2T_1}{T_2^2}+\frac{3T_1^2}{T_2^2}-\frac{3T_1^3}{T_2^2}-\frac{3}{T_1^6T_2}+$$
$$\frac{3}{T_1^5T_2}-\frac{2}{T_1^3T_2}+\frac{1}{T_1^2T_2}+\frac{T_1}{T_2}-\frac{2T_1^2}{T_2}+\frac{3T_1^4}{T_2}-\frac{3T_1^5}{T_2}-\frac{3T_2}{T_1^5}+\frac{3T_2}{T_1^4}-\frac{2T_2}{T_1^2}+\frac{T_2}{T_1}+T_1^2T_2-2T_1^3T_2+$$
$$3T_1^5T_2-3T_1^6T_2-T_2^2-\frac{3T_2^2}{T_1^3}+\frac{3T_2^2}{T_1^2}-\frac{2T_2^2}{T_1}+T_1T_2^2-T_1^2T_2^2-2T_1^3T_2^2+3T_1^4T_2^2-3T_1^5T_2^2+4T_2^3+\frac{3T_2^3}{T_1^3}-$$
$$\frac{3T_2^3}{T_1^2}-2T_1T_2^3-2T_1^2T_2^3+4T_1^3T_2^3-3T_1^5T_2^3+3T_1^6T_2^3-3T_2^4+\frac{3T_2^4}{T_1}+3T_1^2T_2^4-3T_1^4T_2^4+3T_1^5T_2^4-$$
$$\frac{3T_2^5}{T_1}+3T_1T_2^5-3T_1^2T_2^5-3T_1^3T_2^5+3T_1^4T_2^5-3T_1^6T_2^5+3T_2^6-3T_1T_2^6+3T_1^3T_2^6-3T_1^5T_2^6+3T_1^6T_2^6\Big\}$$

(Alt) In[ ]:=

```
TestSymmetries[Knot[3, 1]]
```

(Alt) Out[ ]=

True

(Alt) In[ ]:=

```
Timing[λ[Knot["K11n34"]]]
```

(Alt) Out[ ]=

$$\Big\{0.03125,$$

$$\Big\{1,\ 4+\frac{2}{T_1^6}+\frac{2}{T_1^5}-\frac{2}{T_1^3}+\frac{4}{T_1^2}-\frac{12}{T_1}-2T_1+2T_1^3+2T_1^4-\frac{1}{T_1^5T_2^6}+\frac{2}{T_1^4T_2^6}-\frac{1}{T_1^3T_2^6}+\frac{2}{T_1^6T_2^5}-\frac{2}{T_1^5T_2^5}-\frac{2}{T_1^2T_2^5}+$$
$$\frac{2}{T_1T_2^5}-\frac{2}{T_2^4}-\frac{1}{T_1^7T_2^4}-\frac{2}{T_1^6T_2^4}+\frac{2}{T_1^5T_2^4}+\frac{2}{T_1^3T_2^4}+\frac{2}{T_1T_2^4}-\frac{T_1}{T_2^4}+\frac{2}{T_1^7T_2^3}-\frac{1}{T_1^3T_2^3}-\frac{1}{T_1^2T_2^3}+\frac{2T_1^2}{T_2^3}-\frac{1}{T_2^2}-\frac{1}{T_1^7T_2^2}+$$
$$\frac{2}{T_1^5T_2^2}-\frac{1}{T_1^4T_2^2}-\frac{2}{T_1^3T_2^2}-\frac{2}{T_1T_2^2}+\frac{2T_1}{T_2^2}-\frac{T_1^3}{T_2^2}-\frac{2}{T_1^6T_2}-\frac{1}{T_1^4T_2}+\frac{4}{T_1^2T_2}+\frac{4}{T_1T_2}-\frac{T_1}{T_2}-$$
$$\frac{2T_1^3}{T_2}+4T_2-\frac{2T_2}{T_1^5}-\frac{T_2}{T_1^3}+\frac{4T_2}{T_1}-T_1^2T_2-2T_1^4T_2-\frac{T_2^2}{T_1^5}+\frac{2T_2^2}{T_1^3}-\frac{T_2^2}{T_1^2}-\frac{2T_2^2}{T_1}-2T_1T_2^2-$$
$$T_1^2T_2^2+2T_1^3T_2^2-T_1^5T_2^2-T_2^3+\frac{2T_2^3}{T_1^4}-T_1T_2^3+2T_1^5T_2^3-\frac{T_2^4}{T_1^3}-\frac{2T_2^4}{T_1^2}+\frac{2T_2^4}{T_1}+2T_1T_2^4+$$
$$2T_1^3T_2^4-2T_1^4T_2^4-T_1^5T_2^4-2T_2^5+\frac{2T_2^5}{T_1}-2T_1^3T_2^5+2T_1^4T_2^5-T_1T_2^6+2T_1^2T_2^6-T_1^3T_2^6\Big\}\Big\}$$

(Alt) In[]:=

```
PD[GST48] = PD[X[1, 15, 2, 14], X[29, 2, 30, 3], X[40, 4, 41, 3],
   X[4, 44, 5, 43], X[5, 26, 6, 27], X[95, 7, 96, 6], X[7, 1, 8, 96], X[8, 14, 9, 13],
   X[28, 9, 29, 10], X[41, 11, 42, 10], X[11, 43, 12, 42], X[12, 27, 13, 28],
   X[15, 31, 16, 30], X[61, 16, 62, 17], X[72, 17, 73, 18], X[83, 18, 84, 19],
   X[34, 20, 35, 19], X[20, 89, 21, 90], X[92, 21, 93, 22], X[22, 79, 23, 80],
   X[23, 68, 24, 69], X[24, 57, 25, 58], X[56, 25, 57, 26], X[31, 63, 32, 62],
   X[32, 74, 33, 73], X[33, 85, 34, 84], X[35, 50, 36, 51], X[81, 37, 82, 36],
   X[70, 38, 71, 37], X[59, 39, 60, 38], X[54, 39, 55, 40], X[55, 45, 56, 44],
   X[45, 59, 46, 58], X[46, 70, 47, 69], X[47, 81, 48, 80], X[91, 49, 92, 48],
   X[49, 91, 50, 90], X[82, 52, 83, 51], X[71, 53, 72, 52], X[60, 54, 61, 53],
   X[74, 63, 75, 64], X[85, 64, 86, 65], X[65, 76, 66, 77], X[66, 87, 67, 88],
   X[94, 67, 95, 68], X[86, 75, 87, 76], X[77, 88, 78, 89], X[93, 78, 94, 79]];
```

(Alt) In[]:=

```
AbsoluteTiming[gst48 = λ[GST48]]
```

(Alt) Out[]=

$$\Big\{12.9415,\ \Big\{-\frac{\left(-1+2T-T^2-T^3+2T^4-T^5+T^8\right)\left(-1+T^3-2T^4+T^5+T^6-2T^7+T^8\right)}{T^8},$$

$$\frac{1}{T_1^{21}T_2^{20}}\Big(T_1^5-3T_1^6+4T_1^7-2T_1^8-2T_1^9+4T_1^{10}-2T_1^{11}-2T_1^{12}+4T_1^{13}-3T_1^{14}+T_1^{15}-3T_1^5T_2+6T_1^6T_2-3T_1^7T_2-$$
$$6T_1^8T_2+12T_1^9T_2-6T_1^{10}T_2-6T_1^{11}T_2+12T_1^{12}T_2-6T_1^{13}T_2-3T_1^{14}T_2+6T_1^{15}T_2-3T_1^{16}T_2-T_1^3T_2^2+$$
$$3T_1^4T_2^2-T_1^6T_2^2-4T_1^7T_2^2+9T_1^8T_2^2-7T_1^9T_2^2-3T_1^{10}T_2^2+8T_1^{11}T_2^2-3T_1^{12}T_2^2-7T_1^{13}T_2^2+9T_1^{14}T_2^2-$$
$$4T_1^{15}T_2^2-T_1^{16}T_2^2+3T_1^{18}T_2^2-T_1^{19}T_2^2-T_1^2T_2^3+6T_1^3T_2^3-10T_1^4T_2^3+3T_1^5T_2^3+2T_1^6T_2^3-3T_1^7T_2^3+4T_1^8T_2^3-$$
$$2T_1^9T_2^3+2T_1^{10}T_2^3-T_1^{11}T_2^3-T_1^{12}T_2^3+2T_1^{13}T_2^3-2T_1^{14}T_2^3+4T_1^{15}T_2^3-3T_1^{16}T_2^3+2T_1^{17}T_2^3+3T_1^{18}T_2^3-$$
$$10T_1^{19}T_2^3+6T_1^{20}T_2^3-T_1^{21}T_2^3+3T_1^2T_2^4-10T_1^3T_2^4+3T_1^4T_2^4+17T_1^5T_2^4-19T_1^6T_2^4+10T_1^7T_2^4-7T_1^8T_2^4+$$
$$6T_1^9T_2^4-T_1^{10}T_2^4-18T_1^{11}T_2^4+35T_1^{12}T_2^4-18T_1^{13}T_2^4-T_1^{14}T_2^4+6T_1^{15}T_2^4-7T_1^{16}T_2^4+10T_1^{17}T_2^4-$$
$$19T_1^{18}T_2^4+17T_1^{19}T_2^4+3T_1^{20}T_2^4-10T_1^{21}T_2^4+3T_1^{22}T_2^4+T_2^5-3T_1T_2^5+3T_1^3T_2^5+17T_1^4T_2^5-38T_1^5T_2^5+$$
$$22T_1^6T_2^5+7T_1^7T_2^5-11T_1^8T_2^5-6T_1^9T_2^5+14T_1^{10}T_2^5+11T_1^{11}T_2^5-31T_1^{12}T_2^5+9T_1^{13}T_2^5+T_1^{14}T_2^5+8T_1^{16}T_2^5-$$
$$15T_1^{17}T_2^5+9T_1^{18}T_2^5+16T_1^{19}T_2^5-32T_1^{20}T_2^5+15T_1^{21}T_2^5+3T_1^{22}T_2^5-3T_1^{24}T_2^5+T_1^{25}T_2^5-3T_2^6+6T_1T_2^6-$$
$$T_1^2T_2^6+2T_1^3T_2^6-19T_1^4T_2^6+22T_1^5T_2^6+24T_1^6T_2^6-68T_1^7T_2^6+43T_1^8T_2^6+9T_1^9T_2^6-29T_1^{10}T_2^6+2T_1^{11}T_2^6-$$
$$12T_1^{12}T_2^6+28T_1^{13}T_2^6-42T_1^{14}T_2^6+26T_1^{15}T_2^6-29T_1^{16}T_2^6-T_1^{17}T_2^6+45T_1^{18}T_2^6-64T_1^{19}T_2^6+24T_1^{20}T_2^6+$$
$$18T_1^{21}T_2^6-17T_1^{22}T_2^6+2T_1^{23}T_2^6-T_1^{24}T_2^6+6T_1^{25}T_2^6-3T_1^{26}T_2^6+4T_2^7-3T_1T_2^7-4T_1^2T_2^7-3T_1^3T_2^7+$$
$$10T_1^4T_2^7+7T_1^5T_2^7-68T_1^6T_2^7+74T_1^7T_2^7+14T_1^8T_2^7-56T_1^9T_2^7+14T_1^{10}T_2^7+55T_1^{11}T_2^7-23T_1^{12}T_2^7+$$
$$11T_1^{13}T_2^7+51T_1^{14}T_2^7-33T_1^{15}T_2^7+41T_1^{16}T_2^7+28T_1^{17}T_2^7-60T_1^{18}T_2^7+16T_1^{19}T_2^7+68T_1^{20}T_2^7-$$
$$62T_1^{21}T_2^7+5T_1^{22}T_2^7+10T_1^{23}T_2^7-3T_1^{24}T_2^7-4T_1^{25}T_2^7-3T_1^{26}T_2^7+4T_1^{27}T_2^7-2T_2^8-6T_1T_2^8+9T_1^2T_2^8+$$
$$4T_1^3T_2^8-7T_1^4T_2^8-11T_1^5T_2^8+43T_1^6T_2^8+14T_1^7T_2^8-123T_1^8T_2^8+133T_1^9T_2^8-36T_1^{10}T_2^8-89T_1^{11}T_2^8+$$
$$136T_1^{12}T_2^8-127T_1^{13}T_2^8+31T_1^{14}T_2^8-31T_1^{15}T_2^8+16T_1^{16}T_2^8-33T_1^{17}T_2^8-28T_1^{18}T_2^8+109T_1^{19}T_2^8-$$
$$115T_1^{20}T_2^8+14T_1^{21}T_2^8+51T_1^{22}T_2^8-27T_1^{23}T_2^8+T_1^{24}T_2^8+4T_1^{25}T_2^8+9T_1^{26}T_2^8-6T_1^{27}T_2^8-2T_1^{28}T_2^8-$$
$$2T_2^9+12T_1T_2^9-7T_1^2T_2^9-2T_1^3T_2^9+6T_1^4T_2^9-6T_1^5T_2^9+9T_1^6T_2^9-56T_1^7T_2^9+133T_1^8T_2^9-149T_1^9T_2^9-$$
$$10T_1^{10}T_2^9+224T_1^{11}T_2^9-314T_1^{12}T_2^9+67T_1^{13}T_2^9+111T_1^{14}T_2^9-124T_1^{15}T_2^9+38T_1^{16}T_2^9-49T_1^{17}T_2^9+$$
$$50T_1^{18}T_2^9-38T_1^{19}T_2^9-47T_1^{20}T_2^9+95T_1^{21}T_2^9-68T_1^{22}T_2^9+8T_1^{23}T_2^9+32T_1^{24}T_2^9-19T_1^{25}T_2^9-2T_1^{26}T_2^9-$$
$$7T_1^{27}T_2^9+12T_1^{28}T_2^9-2T_1^{29}T_2^9+4T_2^{10}-6T_1T_2^{10}-3T_1^2T_2^{10}+2T_1^3T_2^{10}-T_1^4T_2^{10}+14T_1^5T_2^{10}-29T_1^6T_2^{10}+$$
$$14T_1^7T_2^{10}-36T_1^8T_2^{10}-10T_1^9T_2^{10}+240T_1^{10}T_2^{10}-314T_1^{11}T_2^{10}+74T_1^{12}T_2^{10}+431T_1^{13}T_2^{10}-386T_1^{14}T_2^{10}+$$
$$200T_1^{15}T_2^{10}+34T_1^{16}T_2^{10}-37T_1^{17}T_2^{10}+186T_1^{18}T_2^{10}-186T_1^{19}T_2^{10}+136T_1^{20}T_2^{10}-22T_1^{21}T_2^{10}-12T_1^{22}T_2^{10}+$$
$$46T_1^{23}T_2^{10}-93T_1^{24}T_2^{10}+30T_1^{25}T_2^{10}+11T_1^{26}T_2^{10}+2T_1^{27}T_2^{10}-3T_1^{28}T_2^{10}-6T_1^{29}T_2^{10}+4T_1^{30}T_2^{10}-2T_2^{11}-$$

$6 T_1 T_2^{11} + 8 T_1^2 T_2^{11} - T_1^3 T_2^{11} - 18 T_1^4 T_2^{11} + 11 T_1^5 T_2^{11} + 2 T_1^6 T_2^{11} + 55 T_1^7 T_2^{11} - 89 T_1^8 T_2^{11} + 224 T_1^9 T_2^{11} -$
$314 T_1^{10} T_2^{11} - 92 T_1^{11} T_2^{11} + 764 T_1^{12} T_2^{11} - 899 T_1^{13} T_2^{11} + 273 T_1^{14} T_2^{11} + 176 T_1^{15} T_2^{11} - 382 T_1^{16} T_2^{11} +$
$391 T_1^{17} T_2^{11} - 420 T_1^{18} T_2^{11} + 75 T_1^{19} T_2^{11} + 212 T_1^{20} T_2^{11} - 156 T_1^{21} T_2^{11} - 46 T_1^{22} T_2^{11} - 6 T_1^{23} T_2^{11} + 65 T_1^{24} T_2^{11} +$
$76 T_1^{25} T_2^{11} - 107 T_1^{26} T_2^{11} + 31 T_1^{27} T_2^{11} - T_1^{28} T_2^{11} + 8 T_1^{29} T_2^{11} - 6 T_1^{30} T_2^{11} - 2 T_1^{31} T_2^{11} - 2 T_2^{12} + 12 T_1 T_2^{12} -$
$3 T_1^2 T_2^{12} - T_1^3 T_2^{12} + 35 T_1^4 T_2^{12} - 31 T_1^5 T_2^{12} - 12 T_1^6 T_2^{12} - 23 T_1^7 T_2^{12} + 136 T_1^8 T_2^{12} - 314 T_1^9 T_2^{12} + 74 T_1^{10} T_2^{12} +$
$764 T_1^{11} T_2^{12} - 1304 T_1^{12} T_2^{12} + 293 T_1^{13} T_2^{12} + 744 T_1^{14} T_2^{12} - 996 T_1^{15} T_2^{12} + 616 T_1^{16} T_2^{12} - 380 T_1^{17} T_2^{12} -$
$68 T_1^{18} T_2^{12} + 589 T_1^{19} T_2^{12} - 596 T_1^{20} T_2^{12} - 72 T_1^{21} T_2^{12} + 294 T_1^{22} T_2^{12} + 38 T_1^{23} T_2^{12} - 64 T_1^{24} T_2^{12} -$
$123 T_1^{25} T_2^{12} + 60 T_1^{26} T_2^{12} + 93 T_1^{27} T_2^{12} - 69 T_1^{28} T_2^{12} - T_1^{29} T_2^{12} - 3 T_1^{30} T_2^{12} + 12 T_1^{31} T_2^{12} - 2 T_1^{32} T_2^{12} + 4 T_2^{13} -$
$6 T_1 T_2^{13} - 7 T_1^2 T_2^{13} + 2 T_1^3 T_2^{13} - 18 T_1^4 T_2^{13} + 9 T_1^5 T_2^{13} + 28 T_1^6 T_2^{13} + 11 T_1^7 T_2^{13} - 127 T_1^8 T_2^{13} + 67 T_1^9 T_2^{13} +$
$431 T_1^{10} T_2^{13} - 899 T_1^{11} T_2^{13} + 293 T_1^{12} T_2^{13} + 1556 T_1^{13} T_2^{13} - 1724 T_1^{14} T_2^{13} + 887 T_1^{15} T_2^{13} + 223 T_1^{16} T_2^{13} -$
$480 T_1^{17} T_2^{13} + 998 T_1^{18} T_2^{13} - 905 T_1^{19} T_2^{13} + 212 T_1^{20} T_2^{13} + 686 T_1^{21} T_2^{13} - 294 T_1^{22} T_2^{13} - 313 T_1^{23} T_2^{13} +$
$146 T_1^{24} T_2^{13} + 24 T_1^{25} T_2^{13} + 123 T_1^{26} T_2^{13} - 238 T_1^{27} T_2^{13} + 65 T_1^{28} T_2^{13} + 45 T_1^{29} T_2^{13} + 2 T_1^{30} T_2^{13} - 7 T_1^{31} T_2^{13} -$
$6 T_1^{32} T_2^{13} + 4 T_1^{33} T_2^{13} - 3 T_2^{14} - 3 T_1 T_2^{14} + 9 T_1^2 T_2^{14} - 2 T_1^3 T_2^{14} - T_1^4 T_2^{14} + T_1^5 T_2^{14} - 42 T_1^6 T_2^{14} + 51 T_1^7 T_2^{14} +$
$31 T_1^8 T_2^{14} + 111 T_1^9 T_2^{14} - 386 T_1^{10} T_2^{14} + 273 T_1^{11} T_2^{14} + 744 T_1^{12} T_2^{14} - 1724 T_1^{13} T_2^{14} + 705 T_1^{14} T_2^{14} +$
$482 T_1^{15} T_2^{14} - 1315 T_1^{16} T_2^{14} + 1061 T_1^{17} T_2^{14} - 855 T_1^{18} T_2^{14} - 140 T_1^{19} T_2^{14} + 809 T_1^{20} T_2^{14} - 758 T_1^{21} T_2^{14} -$
$370 T_1^{22} T_2^{14} + 595 T_1^{23} T_2^{14} + 58 T_1^{24} T_2^{14} - 229 T_1^{25} T_2^{14} + T_1^{26} T_2^{14} + 95 T_1^{27} T_2^{14} + 124 T_1^{28} T_2^{14} - 151 T_1^{29} T_2^{14} +$
$19 T_1^{30} T_2^{14} - 2 T_1^{31} T_2^{14} + 9 T_1^{32} T_2^{14} - 3 T_1^{33} T_2^{14} - 3 T_1^{34} T_2^{14} + T_2^{15} + 6 T_1 T_2^{15} - 4 T_1^2 T_2^{15} + 4 T_1^3 T_2^{15} +$
$6 T_1^4 T_2^{15} + 26 T_1^6 T_2^{15} - 33 T_1^7 T_2^{15} - 31 T_1^8 T_2^{15} - 124 T_1^9 T_2^{15} + 200 T_1^{10} T_2^{15} + 176 T_1^{11} T_2^{15} - 996 T_1^{12} T_2^{15} +$
$887 T_1^{13} T_2^{15} + 482 T_1^{14} T_2^{15} - 1534 T_1^{15} T_2^{15} + 1712 T_1^{16} T_2^{15} - 619 T_1^{17} T_2^{15} - 569 T_1^{18} T_2^{15} + 1420 T_1^{19} T_2^{15} -$
$914 T_1^{20} T_2^{15} - 229 T_1^{21} T_2^{15} + 992 T_1^{22} T_2^{15} - 257 T_1^{23} T_2^{15} - 598 T_1^{24} T_2^{15} + 440 T_1^{25} T_2^{15} - 15 T_1^{26} T_2^{15} -$
$50 T_1^{27} T_2^{15} - 167 T_1^{28} T_2^{15} + 92 T_1^{29} T_2^{15} + 74 T_1^{30} T_2^{15} - 49 T_1^{31} T_2^{15} + 4 T_1^{32} T_2^{15} - 4 T_1^{33} T_2^{15} + 6 T_1^{34} T_2^{15} +$
$T_1^{35} T_2^{15} - 3 T_1 T_2^{16} - T_1^2 T_2^{16} - 3 T_1^3 T_2^{16} - 7 T_1^4 T_2^{16} + 8 T_1^5 T_2^{16} - 29 T_1^6 T_2^{16} + 41 T_1^7 T_2^{16} + 16 T_1^8 T_2^{16} +$
$38 T_1^9 T_2^{16} + 34 T_1^{10} T_2^{16} - 382 T_1^{11} T_2^{16} + 616 T_1^{12} T_2^{16} + 223 T_1^{13} T_2^{16} - 1315 T_1^{14} T_2^{16} + 1712 T_1^{15} T_2^{16} -$
$720 T_1^{16} T_2^{16} - 1180 T_1^{17} T_2^{16} + 2146 T_1^{18} T_2^{16} - 1310 T_1^{19} T_2^{16} - 260 T_1^{20} T_2^{16} + 1108 T_1^{21} T_2^{16} - 545 T_1^{22} T_2^{16} -$
$555 T_1^{23} T_2^{16} + 792 T_1^{24} T_2^{16} - 94 T_1^{25} T_2^{16} - 350 T_1^{26} T_2^{16} + 256 T_1^{27} T_2^{16} - 24 T_1^{28} T_2^{16} + 109 T_1^{29} T_2^{16} -$
$189 T_1^{30} T_2^{16} + 60 T_1^{31} T_2^{16} + 17 T_1^{32} T_2^{16} - 3 T_1^{33} T_2^{16} - T_1^{34} T_2^{16} - 3 T_1^{35} T_2^{16} + 2 T_1^3 T_2^{17} + 10 T_1^4 T_2^{17} -$
$15 T_1^5 T_2^{17} - T_1^6 T_2^{17} + 28 T_1^7 T_2^{17} - 33 T_1^8 T_2^{17} - 49 T_1^9 T_2^{17} - 37 T_1^{10} T_2^{17} + 391 T_1^{11} T_2^{17} - 380 T_1^{12} T_2^{17} -$
$480 T_1^{13} T_2^{17} + 1061 T_1^{14} T_2^{17} - 619 T_1^{15} T_2^{17} - 1180 T_1^{16} T_2^{17} + 2566 T_1^{17} T_2^{17} - 1730 T_1^{18} T_2^{17} - 591 T_1^{19} T_2^{17} +$
$1520 T_1^{20} T_2^{17} - 933 T_1^{21} T_2^{17} - 265 T_1^{22} T_2^{17} + 476 T_1^{23} T_2^{17} + 123 T_1^{24} T_2^{17} - 791 T_1^{25} T_2^{17} + 681 T_1^{26} T_2^{17} -$
$213 T_1^{27} T_2^{17} - 82 T_1^{28} T_2^{17} - 8 T_1^{29} T_2^{17} + 74 T_1^{30} T_2^{17} + 42 T_1^{31} T_2^{17} - 59 T_1^{32} T_2^{17} + 10 T_1^{33} T_2^{17} + 2 T_1^{34} T_2^{17} +$
$3 T_1^2 T_2^{18} + 3 T_1^3 T_2^{18} - 19 T_1^4 T_2^{18} + 9 T_1^5 T_2^{18} + 45 T_1^6 T_2^{18} - 60 T_1^7 T_2^{18} - 28 T_1^8 T_2^{18} + 50 T_1^9 T_2^{18} + 186 T_1^{10} T_2^{18} -$
$420 T_1^{11} T_2^{18} - 68 T_1^{12} T_2^{18} + 998 T_1^{13} T_2^{18} - 855 T_1^{14} T_2^{18} - 569 T_1^{15} T_2^{18} + 2146 T_1^{16} T_2^{18} - 1730 T_1^{17} T_2^{18} -$
$492 T_1^{18} T_2^{18} + 2218 T_1^{19} T_2^{18} - 1372 T_1^{20} T_2^{18} - 146 T_1^{21} T_2^{18} + 878 T_1^{22} T_2^{18} - 163 T_1^{23} T_2^{18} - 695 T_1^{24} T_2^{18} +$
$872 T_1^{25} T_2^{18} - 162 T_1^{26} T_2^{18} - 458 T_1^{27} T_2^{18} + 506 T_1^{28} T_2^{18} - 208 T_1^{29} T_2^{18} + 44 T_1^{30} T_2^{18} - 100 T_1^{31} T_2^{18} +$
$79 T_1^{32} T_2^{18} - 19 T_1^{33} T_2^{18} - 5 T_1^{34} T_2^{18} + 3 T_1^{35} T_2^{18} + 3 T_1^{36} T_2^{18} - T_1^2 T_2^{19} - 10 T_1^3 T_2^{19} + 17 T_1^4 T_2^{19} + 16 T_1^5 T_2^{19} -$
$64 T_1^6 T_2^{19} + 16 T_1^7 T_2^{19} + 109 T_1^8 T_2^{19} - 38 T_1^9 T_2^{19} - 186 T_1^{10} T_2^{19} + 75 T_1^{11} T_2^{19} + 589 T_1^{12} T_2^{19} - 905 T_1^{13} T_2^{19} -$
$140 T_1^{14} T_2^{19} + 1420 T_1^{15} T_2^{19} - 1310 T_1^{16} T_2^{19} - 591 T_1^{17} T_2^{19} + 2218 T_1^{18} T_2^{19} - 2027 T_1^{19} T_2^{19} + 155 T_1^{20} T_2^{19} +$
$1033 T_1^{21} T_2^{19} - 840 T_1^{22} T_2^{19} - 49 T_1^{23} T_2^{19} + 464 T_1^{24} T_2^{19} + 37 T_1^{25} T_2^{19} - 842 T_1^{26} T_2^{19} + 972 T_1^{27} T_2^{19} -$
$412 T_1^{28} T_2^{19} - 44 T_1^{29} T_2^{19} + 150 T_1^{30} T_2^{19} - 21 T_1^{31} T_2^{19} - 10 T_1^{32} T_2^{19} - 42 T_1^{33} T_2^{19} + 50 T_1^{34} T_2^{19} - 13 T_1^{35} T_2^{19} -$
$10 T_1^{36} T_2^{19} - T_1^{37} T_2^{19} + 6 T_1^3 T_2^{20} + 3 T_1^4 T_2^{20} - 32 T_1^5 T_2^{20} + 24 T_1^6 T_2^{20} + 68 T_1^7 T_2^{20} - 115 T_1^8 T_2^{20} - 47 T_1^9 T_2^{20} +$
$136 T_1^{10} T_2^{20} + 212 T_1^{11} T_2^{20} - 596 T_1^{12} T_2^{20} + 212 T_1^{13} T_2^{20} + 809 T_1^{14} T_2^{20} - 914 T_1^{15} T_2^{20} - 260 T_1^{16} T_2^{20} +$
$1520 T_1^{17} T_2^{20} - 1372 T_1^{18} T_2^{20} + 155 T_1^{19} T_2^{20} + 1056 T_1^{20} T_2^{20} - 1291 T_1^{21} T_2^{20} + 674 T_1^{22} T_2^{20} - 128 T_1^{23} T_2^{20} -$
$56 T_1^{24} T_2^{20} - 374 T_1^{25} T_2^{20} + 603 T_1^{26} T_2^{20} - 180 T_1^{27} T_2^{20} - 504 T_1^{28} T_2^{20} + 592 T_1^{29} T_2^{20} - 340 T_1^{30} T_2^{20} +$
$71 T_1^{31} T_2^{20} - 39 T_1^{32} T_2^{20} + 100 T_1^{33} T_2^{20} - 60 T_1^{34} T_2^{20} - 8 T_1^{35} T_2^{20} + 19 T_1^{36} T_2^{20} + 6 T_1^{37} T_2^{20} - T_1^3 T_2^{21} -$
$10 T_1^4 T_2^{21} + 15 T_1^5 T_2^{21} + 18 T_1^6 T_2^{21} - 62 T_1^7 T_2^{21} + 14 T_1^8 T_2^{21} + 95 T_1^9 T_2^{21} - 22 T_1^{10} T_2^{21} - 156 T_1^{11} T_2^{21} -$
$72 T_1^{12} T_2^{21} + 686 T_1^{13} T_2^{21} - 758 T_1^{14} T_2^{21} - 229 T_1^{15} T_2^{21} + 1108 T_1^{16} T_2^{21} - 933 T_1^{17} T_2^{21} - 146 T_1^{18} T_2^{21} +$

$1033 T_1^{19} T_2^{21} - 1291 T_1^{20} T_2^{21} + 891 T_1^{21} T_2^{21} - 152 T_1^{22} T_2^{21} - 395 T_1^{23} T_2^{21} + 328 T_1^{24} T_2^{21} + 152 T_1^{25} T_2^{21} -$
$52 T_1^{26} T_2^{21} - 695 T_1^{27} T_2^{21} + 1069 T_1^{28} T_2^{21} - 559 T_1^{29} T_2^{21} - 14 T_1^{30} T_2^{21} + 166 T_1^{31} T_2^{21} - 35 T_1^{32} T_2^{21} -$
$12 T_1^{33} T_2^{21} - 40 T_1^{34} T_2^{21} + 52 T_1^{35} T_2^{21} - 15 T_1^{36} T_2^{21} - 10 T_1^{37} T_2^{21} - T_1^{38} T_2^{21} + 3 T_1^{4} T_2^{22} + 3 T_1^{5} T_2^{22} -$
$17 T_1^{6} T_2^{22} + 5 T_1^{7} T_2^{22} + 51 T_1^{8} T_2^{22} - 68 T_1^{9} T_2^{22} - 12 T_1^{10} T_2^{22} - 46 T_1^{11} T_2^{22} + 294 T_1^{12} T_2^{22} - 294 T_1^{13} T_2^{22} -$
$370 T_1^{14} T_2^{22} + 992 T_1^{15} T_2^{22} - 545 T_1^{16} T_2^{22} - 265 T_1^{17} T_2^{22} + 878 T_1^{18} T_2^{22} - 840 T_1^{19} T_2^{22} + 674 T_1^{20} T_2^{22} -$
$152 T_1^{21} T_2^{22} - 206 T_1^{22} T_2^{22} + 744 T_1^{23} T_2^{22} - 390 T_1^{24} T_2^{22} + 141 T_1^{25} T_2^{22} - 385 T_1^{26} T_2^{22} + 866 T_1^{27} T_2^{22} -$
$464 T_1^{28} T_2^{22} - 332 T_1^{29} T_2^{22} + 614 T_1^{30} T_2^{22} - 304 T_1^{31} T_2^{22} + 60 T_1^{32} T_2^{22} - 108 T_1^{33} T_2^{22} + 85 T_1^{34} T_2^{22} -$
$23 T_1^{35} T_2^{22} - 3 T_1^{36} T_2^{22} + 3 T_1^{37} T_2^{22} + 3 T_1^{38} T_2^{22} + 2 T_1^{6} T_2^{23} + 10 T_1^{7} T_2^{23} - 27 T_1^{8} T_2^{23} + 8 T_1^{9} T_2^{23} + 46 T_1^{10} T_2^{23} -$
$6 T_1^{11} T_2^{23} + 38 T_1^{12} T_2^{23} - 313 T_1^{13} T_2^{23} + 595 T_1^{14} T_2^{23} - 257 T_1^{15} T_2^{23} - 555 T_1^{16} T_2^{23} + 476 T_1^{17} T_2^{23} -$
$163 T_1^{18} T_2^{23} - 49 T_1^{19} T_2^{23} - 128 T_1^{20} T_2^{23} - 395 T_1^{21} T_2^{23} + 744 T_1^{22} T_2^{23} - 1174 T_1^{23} T_2^{23} + 198 T_1^{24} T_2^{23} +$
$191 T_1^{25} T_2^{23} - 109 T_1^{26} T_2^{23} + 48 T_1^{27} T_2^{23} - 668 T_1^{28} T_2^{23} + 885 T_1^{29} T_2^{23} - 489 T_1^{30} T_2^{23} + 5 T_1^{31} T_2^{23} +$
$19 T_1^{32} T_2^{23} + 92 T_1^{33} T_2^{23} + 51 T_1^{34} T_2^{23} - 71 T_1^{35} T_2^{23} + 10 T_1^{36} T_2^{23} + 2 T_1^{37} T_2^{23} - 3 T_1^{5} T_2^{24} - T_1^{6} T_2^{24} -$
$3 T_1^{7} T_2^{24} + T_1^{8} T_2^{24} + 32 T_1^{9} T_2^{24} - 93 T_1^{10} T_2^{24} + 65 T_1^{11} T_2^{24} - 64 T_1^{12} T_2^{24} + 146 T_1^{13} T_2^{24} + 58 T_1^{14} T_2^{24} -$
$598 T_1^{15} T_2^{24} + 792 T_1^{16} T_2^{24} + 123 T_1^{17} T_2^{24} - 695 T_1^{18} T_2^{24} + 464 T_1^{19} T_2^{24} - 56 T_1^{20} T_2^{24} + 328 T_1^{21} T_2^{24} -$
$390 T_1^{22} T_2^{24} + 198 T_1^{23} T_2^{24} + 404 T_1^{24} T_2^{24} - 140 T_1^{25} T_2^{24} + 75 T_1^{26} T_2^{24} - 655 T_1^{27} T_2^{24} + 968 T_1^{28} T_2^{24} -$
$310 T_1^{29} T_2^{24} - 326 T_1^{30} T_2^{24} + 364 T_1^{31} T_2^{24} - 104 T_1^{32} T_2^{24} + 133 T_1^{33} T_2^{24} - 253 T_1^{34} T_2^{24} + 84 T_1^{35} T_2^{24} +$
$25 T_1^{36} T_2^{24} - 3 T_1^{37} T_2^{24} - T_1^{38} T_2^{24} - 3 T_1^{39} T_2^{24} + T_1^{5} T_2^{25} + 6 T_1^{6} T_2^{25} - 4 T_1^{7} T_2^{25} + 4 T_1^{8} T_2^{25} - 19 T_1^{9} T_2^{25} +$
$30 T_1^{10} T_2^{25} + 76 T_1^{11} T_2^{25} - 123 T_1^{12} T_2^{25} + 24 T_1^{13} T_2^{25} - 229 T_1^{14} T_2^{25} + 440 T_1^{15} T_2^{25} - 94 T_1^{16} T_2^{25} -$
$791 T_1^{17} T_2^{25} + 872 T_1^{18} T_2^{25} + 37 T_1^{19} T_2^{25} - 374 T_1^{20} T_2^{25} + 152 T_1^{21} T_2^{25} + 141 T_1^{22} T_2^{25} + 191 T_1^{23} T_2^{25} -$
$140 T_1^{24} T_2^{25} + 246 T_1^{25} T_2^{25} - 674 T_1^{26} T_2^{25} + 977 T_1^{27} T_2^{25} - 52 T_1^{28} T_2^{25} - 868 T_1^{29} T_2^{25} + 680 T_1^{30} T_2^{25} -$
$120 T_1^{31} T_2^{25} + 5 T_1^{32} T_2^{25} - 257 T_1^{33} T_2^{25} + 142 T_1^{34} T_2^{25} + 104 T_1^{35} T_2^{25} - 74 T_1^{36} T_2^{25} + 4 T_1^{37} T_2^{25} - 4 T_1^{38} T_2^{25} +$
$6 T_1^{39} T_2^{25} + T_1^{40} T_2^{25} - 3 T_1^{6} T_2^{26} - 3 T_1^{7} T_2^{26} + 9 T_1^{8} T_2^{26} - 2 T_1^{9} T_2^{26} + 11 T_1^{10} T_2^{26} - 107 T_1^{11} T_2^{26} + 60 T_1^{12} T_2^{26} +$
$123 T_1^{13} T_2^{26} + T_1^{14} T_2^{26} - 15 T_1^{15} T_2^{26} - 350 T_1^{16} T_2^{26} + 681 T_1^{17} T_2^{26} - 162 T_1^{18} T_2^{26} - 842 T_1^{19} T_2^{26} +$
$603 T_1^{20} T_2^{26} - 52 T_1^{21} T_2^{26} - 385 T_1^{22} T_2^{26} - 109 T_1^{23} T_2^{26} + 75 T_1^{24} T_2^{26} - 674 T_1^{25} T_2^{26} + 707 T_1^{26} T_2^{26} +$
$124 T_1^{27} T_2^{26} - 1276 T_1^{28} T_2^{26} + 1003 T_1^{29} T_2^{26} + 94 T_1^{30} T_2^{26} - 355 T_1^{31} T_2^{26} - 29 T_1^{32} T_2^{26} + 167 T_1^{33} T_2^{26} +$
$226 T_1^{34} T_2^{26} - 259 T_1^{35} T_2^{26} + 31 T_1^{36} T_2^{26} - 2 T_1^{37} T_2^{26} + 9 T_1^{38} T_2^{26} - 3 T_1^{39} T_2^{26} - 3 T_1^{40} T_2^{26} + 4 T_1^{7} T_2^{27} -$
$6 T_1^{8} T_2^{27} - 7 T_1^{9} T_2^{27} + 2 T_1^{10} T_2^{27} + 31 T_1^{11} T_2^{27} + 93 T_1^{12} T_2^{27} - 238 T_1^{13} T_2^{27} + 95 T_1^{14} T_2^{27} - 50 T_1^{15} T_2^{27} +$
$256 T_1^{16} T_2^{27} - 213 T_1^{17} T_2^{27} - 458 T_1^{18} T_2^{27} + 972 T_1^{19} T_2^{27} - 180 T_1^{20} T_2^{27} - 695 T_1^{21} T_2^{27} + 866 T_1^{22} T_2^{27} +$
$48 T_1^{23} T_2^{27} - 655 T_1^{24} T_2^{27} + 977 T_1^{25} T_2^{27} + 124 T_1^{26} T_2^{27} - 1524 T_1^{27} T_2^{27} + 1365 T_1^{28} T_2^{27} + 147 T_1^{29} T_2^{27} -$
$957 T_1^{30} T_2^{27} + 335 T_1^{31} T_2^{27} + 101 T_1^{32} T_2^{27} + 207 T_1^{33} T_2^{27} - 504 T_1^{34} T_2^{27} + 149 T_1^{35} T_2^{27} + 94 T_1^{36} T_2^{27} +$
$2 T_1^{37} T_2^{27} - 7 T_1^{38} T_2^{27} - 6 T_1^{39} T_2^{27} + 4 T_1^{40} T_2^{27} - 2 T_1^{8} T_2^{28} + 12 T_1^{9} T_2^{28} - 3 T_1^{10} T_2^{28} - T_1^{11} T_2^{28} - 69 T_1^{12} T_2^{28} +$
$65 T_1^{13} T_2^{28} + 124 T_1^{14} T_2^{28} - 167 T_1^{15} T_2^{28} - 24 T_1^{16} T_2^{28} - 82 T_1^{17} T_2^{28} + 506 T_1^{18} T_2^{28} - 412 T_1^{19} T_2^{28} -$
$504 T_1^{20} T_2^{28} + 1069 T_1^{21} T_2^{28} - 464 T_1^{22} T_2^{28} - 668 T_1^{23} T_2^{28} + 968 T_1^{24} T_2^{28} - 52 T_1^{25} T_2^{28} - 1276 T_1^{26} T_2^{28} +$
$1365 T_1^{27} T_2^{28} + 204 T_1^{28} T_2^{28} - 1248 T_1^{29} T_2^{28} + 726 T_1^{30} T_2^{28} + 270 T_1^{31} T_2^{28} - 224 T_1^{32} T_2^{28} - 267 T_1^{33} T_2^{28} +$
$196 T_1^{34} T_2^{28} + 189 T_1^{35} T_2^{28} - 173 T_1^{36} T_2^{28} - T_1^{37} T_2^{28} - 3 T_1^{38} T_2^{28} + 12 T_1^{39} T_2^{28} - 2 T_1^{40} T_2^{28} - 2 T_1^{9} T_2^{29} -$
$6 T_1^{10} T_2^{29} + 8 T_1^{11} T_2^{29} - T_1^{12} T_2^{29} + 45 T_1^{13} T_2^{29} - 151 T_1^{14} T_2^{29} + 92 T_1^{15} T_2^{29} + 109 T_1^{16} T_2^{29} - 8 T_1^{17} T_2^{29} -$
$208 T_1^{18} T_2^{29} - 44 T_1^{19} T_2^{29} + 592 T_1^{20} T_2^{29} - 559 T_1^{21} T_2^{29} - 332 T_1^{22} T_2^{29} + 885 T_1^{23} T_2^{29} - 310 T_1^{24} T_2^{29} -$
$868 T_1^{25} T_2^{29} + 1003 T_1^{26} T_2^{29} + 147 T_1^{27} T_2^{29} - 1248 T_1^{28} T_2^{29} + 896 T_1^{29} T_2^{29} + 114 T_1^{30} T_2^{29} - 478 T_1^{31} T_2^{29} +$
$75 T_1^{32} T_2^{29} + 119 T_1^{33} T_2^{29} + 166 T_1^{34} T_2^{29} - 269 T_1^{35} T_2^{29} + 94 T_1^{36} T_2^{29} - T_1^{37} T_2^{29} + 8 T_1^{38} T_2^{29} - 6 T_1^{39} T_2^{29} -$
$2 T_1^{40} T_2^{29} + 4 T_1^{10} T_2^{30} - 6 T_1^{11} T_2^{30} - 3 T_1^{12} T_2^{30} + 2 T_1^{13} T_2^{30} + 19 T_1^{14} T_2^{30} + 74 T_1^{15} T_2^{30} - 189 T_1^{16} T_2^{30} +$
$74 T_1^{17} T_2^{30} + 44 T_1^{18} T_2^{30} + 150 T_1^{19} T_2^{30} - 340 T_1^{20} T_2^{30} - 14 T_1^{21} T_2^{30} + 614 T_1^{22} T_2^{30} - 489 T_1^{23} T_2^{30} -$
$326 T_1^{24} T_2^{30} + 680 T_1^{25} T_2^{30} + 94 T_1^{26} T_2^{30} - 957 T_1^{27} T_2^{30} + 726 T_1^{28} T_2^{30} + 114 T_1^{29} T_2^{30} - 444 T_1^{30} T_2^{30} +$
$138 T_1^{31} T_2^{30} + 68 T_1^{32} T_2^{30} + 106 T_1^{33} T_2^{30} - 253 T_1^{34} T_2^{30} + 90 T_1^{35} T_2^{30} + 31 T_1^{36} T_2^{30} + 2 T_1^{37} T_2^{30} - 3 T_1^{38} T_2^{30} -$
$6 T_1^{39} T_2^{30} + 4 T_1^{40} T_2^{30} - 2 T_1^{11} T_2^{31} + 12 T_1^{12} T_2^{31} - 7 T_1^{13} T_2^{31} - 2 T_1^{14} T_2^{31} - 49 T_1^{15} T_2^{31} + 60 T_1^{16} T_2^{31} +$
$42 T_1^{17} T_2^{31} - 100 T_1^{18} T_2^{31} - 21 T_1^{19} T_2^{31} + 71 T_1^{20} T_2^{31} + 166 T_1^{21} T_2^{31} - 304 T_1^{22} T_2^{31} + 5 T_1^{23} T_2^{31} + 364 T_1^{24} T_2^{31} -$
$120 T_1^{25} T_2^{31} - 355 T_1^{26} T_2^{31} + 335 T_1^{27} T_2^{31} + 270 T_1^{28} T_2^{31} - 478 T_1^{29} T_2^{31} + 138 T_1^{30} T_2^{31} + 173 T_1^{31} T_2^{31} -$

$$59\,T_1^{32}T_2^{31}-112\,T_1^{33}T_2^{31}+41\,T_1^{34}T_2^{31}+98\,T_1^{35}T_2^{31}-74\,T_1^{36}T_2^{31}-2\,T_1^{37}T_2^{31}-7\,T_1^{38}T_2^{31}+12\,T_1^{39}T_2^{31}-$$
$$2\,T_1^{40}T_2^{31}-2\,T_1^{12}T_2^{32}-6\,T_1^{13}T_2^{32}+9\,T_1^{14}T_2^{32}+4\,T_1^{15}T_2^{32}+17\,T_1^{16}T_2^{32}-59\,T_1^{17}T_2^{32}+79\,T_1^{18}T_2^{32}-$$
$$10\,T_1^{19}T_2^{32}-39\,T_1^{20}T_2^{32}-35\,T_1^{21}T_2^{32}+60\,T_1^{22}T_2^{32}+19\,T_1^{23}T_2^{32}-104\,T_1^{24}T_2^{32}+5\,T_1^{25}T_2^{32}-29\,T_1^{26}T_2^{32}+$$
$$101\,T_1^{27}T_2^{32}-224\,T_1^{28}T_2^{32}+75\,T_1^{29}T_2^{32}+68\,T_1^{30}T_2^{32}-59\,T_1^{31}T_2^{32}-31\,T_1^{32}T_2^{32}-10\,T_1^{33}T_2^{32}+87\,T_1^{34}T_2^{32}-$$
$$75\,T_1^{35}T_2^{32}+25\,T_1^{36}T_2^{32}+4\,T_1^{37}T_2^{32}+9\,T_1^{38}T_2^{32}-6\,T_1^{39}T_2^{32}-2\,T_1^{40}T_2^{32}+4\,T_1^{13}T_2^{33}-3\,T_1^{14}T_2^{33}-$$
$$4\,T_1^{15}T_2^{33}-3\,T_1^{16}T_2^{33}+10\,T_1^{17}T_2^{33}-19\,T_1^{18}T_2^{33}-42\,T_1^{19}T_2^{33}+100\,T_1^{20}T_2^{33}-12\,T_1^{21}T_2^{33}-108\,T_1^{22}T_2^{33}+$$
$$92\,T_1^{23}T_2^{33}+133\,T_1^{24}T_2^{33}-257\,T_1^{25}T_2^{33}+167\,T_1^{26}T_2^{33}+207\,T_1^{27}T_2^{33}-267\,T_1^{28}T_2^{33}+119\,T_1^{29}T_2^{33}+$$
$$106\,T_1^{30}T_2^{33}-112\,T_1^{31}T_2^{33}-10\,T_1^{32}T_2^{33}+94\,T_1^{33}T_2^{33}-36\,T_1^{34}T_2^{33}-21\,T_1^{35}T_2^{33}+10\,T_1^{36}T_2^{33}-3\,T_1^{37}T_2^{33}-$$
$$4\,T_1^{38}T_2^{33}-3\,T_1^{39}T_2^{33}+4\,T_1^{40}T_2^{33}-3\,T_1^{14}T_2^{34}+6\,T_1^{15}T_2^{34}-T_1^{16}T_2^{34}+2\,T_1^{17}T_2^{34}-5\,T_1^{18}T_2^{34}+50\,T_1^{19}T_2^{34}-$$
$$60\,T_1^{20}T_2^{34}-40\,T_1^{21}T_2^{34}+85\,T_1^{22}T_2^{34}+51\,T_1^{23}T_2^{34}-253\,T_1^{24}T_2^{34}+142\,T_1^{25}T_2^{34}+226\,T_1^{26}T_2^{34}-$$
$$504\,T_1^{27}T_2^{34}+196\,T_1^{28}T_2^{34}+166\,T_1^{29}T_2^{34}-253\,T_1^{30}T_2^{34}+41\,T_1^{31}T_2^{34}+87\,T_1^{32}T_2^{34}-36\,T_1^{33}T_2^{34}-$$
$$60\,T_1^{34}T_2^{34}+46\,T_1^{35}T_2^{34}-3\,T_1^{36}T_2^{34}+2\,T_1^{37}T_2^{34}-T_1^{38}T_2^{34}+6\,T_1^{39}T_2^{34}-3\,T_1^{40}T_2^{34}+T_1^{15}T_2^{35}-3\,T_1^{16}T_2^{35}+$$
$$3\,T_1^{18}T_2^{35}-13\,T_1^{19}T_2^{35}-8\,T_1^{20}T_2^{35}+52\,T_1^{21}T_2^{35}-23\,T_1^{22}T_2^{35}-71\,T_1^{23}T_2^{35}+84\,T_1^{24}T_2^{35}+104\,T_1^{25}T_2^{35}-$$
$$259\,T_1^{26}T_2^{35}+149\,T_1^{27}T_2^{35}+189\,T_1^{28}T_2^{35}-269\,T_1^{29}T_2^{35}+90\,T_1^{30}T_2^{35}+98\,T_1^{31}T_2^{35}-75\,T_1^{32}T_2^{35}-$$
$$21\,T_1^{33}T_2^{35}+46\,T_1^{34}T_2^{35}-2\,T_1^{35}T_2^{35}-15\,T_1^{36}T_2^{35}+3\,T_1^{37}T_2^{35}-3\,T_1^{39}T_2^{35}+T_1^{40}T_2^{35}+3\,T_1^{18}T_2^{36}-$$
$$10\,T_1^{19}T_2^{36}+19\,T_1^{20}T_2^{36}-15\,T_1^{21}T_2^{36}-3\,T_1^{22}T_2^{36}+10\,T_1^{23}T_2^{36}+25\,T_1^{24}T_2^{36}-74\,T_1^{25}T_2^{36}+31\,T_1^{26}T_2^{36}+$$
$$94\,T_1^{27}T_2^{36}-173\,T_1^{28}T_2^{36}+94\,T_1^{29}T_2^{36}+31\,T_1^{30}T_2^{36}-74\,T_1^{31}T_2^{36}+25\,T_1^{32}T_2^{36}+10\,T_1^{33}T_2^{36}-3\,T_1^{34}T_2^{36}-$$
$$15\,T_1^{35}T_2^{36}+19\,T_1^{36}T_2^{36}-10\,T_1^{37}T_2^{36}+3\,T_1^{38}T_2^{36}-T_1^{19}T_2^{37}+6\,T_1^{20}T_2^{37}-10\,T_1^{21}T_2^{37}+3\,T_1^{22}T_2^{37}+$$
$$2\,T_1^{23}T_2^{37}-3\,T_1^{24}T_2^{37}+4\,T_1^{25}T_2^{37}-2\,T_1^{26}T_2^{37}+2\,T_1^{27}T_2^{37}-T_1^{28}T_2^{37}-T_1^{29}T_2^{37}+2\,T_1^{30}T_2^{37}-2\,T_1^{31}T_2^{37}+$$
$$4\,T_1^{32}T_2^{37}-3\,T_1^{33}T_2^{37}+2\,T_1^{34}T_2^{37}+3\,T_1^{35}T_2^{37}-10\,T_1^{36}T_2^{37}+6\,T_1^{37}T_2^{37}-T_1^{38}T_2^{37}-T_1^{21}T_2^{38}+3\,T_1^{22}T_2^{38}-$$
$$T_1^{24}T_2^{38}-4\,T_1^{25}T_2^{38}+9\,T_1^{26}T_2^{38}-7\,T_1^{27}T_2^{38}-3\,T_1^{28}T_2^{38}+8\,T_1^{29}T_2^{38}-3\,T_1^{30}T_2^{38}-7\,T_1^{31}T_2^{38}+9\,T_1^{32}T_2^{38}-$$
$$4\,T_1^{33}T_2^{38}-T_1^{34}T_2^{38}+3\,T_1^{36}T_2^{38}-T_1^{37}T_2^{38}-3\,T_1^{24}T_2^{39}+6\,T_1^{25}T_2^{39}-3\,T_1^{26}T_2^{39}-6\,T_1^{27}T_2^{39}+12\,T_1^{28}T_2^{39}-$$
$$6\,T_1^{29}T_2^{39}-6\,T_1^{30}T_2^{39}+12\,T_1^{31}T_2^{39}-6\,T_1^{32}T_2^{39}-3\,T_1^{33}T_2^{39}+6\,T_1^{34}T_2^{39}-3\,T_1^{35}T_2^{39}+T_1^{25}T_2^{40}-3\,T_1^{26}T_2^{40}+$$
$$4\,T_1^{27}T_2^{40}-2\,T_1^{28}T_2^{40}-2\,T_1^{29}T_2^{40}+4\,T_1^{30}T_2^{40}-2\,T_1^{31}T_2^{40}-2\,T_1^{32}T_2^{40}+4\,T_1^{33}T_2^{40}-3\,T_1^{34}T_2^{40}+T_1^{35}T_2^{40})\}\}$$

(Alt) In[∘]:=

```
TestSymmetries[GST48]
```

(Alt) Out[∘]=

```
True
```

Relation to ρ_1:

In[∘]:=

```
CheckRelationTorho1[K_] := Module[{th = θ[K]⟦2⟧, rh = ρ[K]⟦2⟧},
  ({th /. {T₁ → 1}, th /. {T₂ → 1}} + rh) /. T_ → T // Together]
```

In[∘]:=

```
CheckRelationTorho1 /@ AllKnots[{3, 8}]
```

Out[∘]=

```
{{0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0},
 {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0},
 {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}}
```

Symmetries

In[∘]:=

```
CheckT12swapsym[K_] := Module[{th = θ[K]⟦2⟧}, {th - (th /. {T₁ → T₂, T₂ → T₁})}]
```

In[∘]:=
```
CheckT12swapsym /@ AllKnots[{3, 8}] // Union
```
Out[∘]=
```
{{0}}
```
In[∘]:=
```
CheckT12swapsym[Knot[11, NonAlternating, 34]]
```
Out[∘]=
```
{0}
```
In[∘]:=
```
CheckMirr[K_] := Module[{th = θ[K][[2]], thm = θ[Mirror@K][[2]]}, {th + thm}]
CheckMirr /@ AllKnots[{3, 7}] // Union
```
Out[∘]=
```
{{0}}
```
In[∘]:=
```
CheckMirr[Knot[11, NonAlternating, 34]]
```
Out[∘]=
```
{0}
```
In[∘]:=
```
CheckT1T2palin[K_] := Module[{th = θ[K][[2]]}, {th - (th /. {T1 → T1^-1, T2 → T2^-1})}]
```
In[∘]:=
```
CheckT1T2palin /@ AllKnots[{3, 8}] // Union
```
Out[∘]=
```
{{0}}
```
In[∘]:=
```
CheckT1T2palin[Knot[11, NonAlternating, 34]]
```
Out[∘]=
```
{0}
```
In[∘]:=
```
CheckT1T2mystsym[K_] := Module[{th = θ[K][[2]]}, {th - (th /. {T1 → T1, T2 → T3^-1})}]
CheckT1T2mystsym /@ AllKnots[{3, 8}]
```
Out[∘]=
```
{{0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0},
 {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}}
```
In[∘]:=
```
CheckT1T2mystsym[Knot[11, NonAlternating, 34]]
```
Out[∘]=
```
{0}
```

Moving to better variables, very similar to Garoufalidis-Kashaev:

$u = T_1 + T_1^{-1} + T_2 + T_2^{-1} + T_3 + T_3^{-1} - 2$

$v = T_1^2 T_2 + T_1^{-2} T_2^{-1} + T_2^2 T_1 + T_2^{-2} T_1^{-1} + T_1 T_2^{-1} - T_1^{-1} T_2 - 2$

In[ ]:=
```
{u - (u /. {s → t, t → s}), v - (v /. {s → t, t → s})}
{u - (u /. {s → s^-1, t → t^-1}), v - (v /. {s → s^-1, t → t^-1})}
{u - (u /. {t → 1 / (s t)}), v - (v /. {t → 1 / (s t)})} // Together
```
Out[ ]=
```
{0, 0}
```
Out[ ]=
```
{0, 0}
```
Out[ ]=
```
{0, 0}
```
(Alt) In[ ]:=
```
pp[x_] := x + x^-1
u = pp[s] + pp[t] + pp[s t] + 1;
v = pp[s^2 t] + pp[s t^2] + pp[s t^-1] + 1;
Monomials_k_[a_, b_] := Flatten@Table[a^m b^n, {m, 0, k}, {n, 0, k - m}]
```

(Alt) In[ ]:=

```
(*This code is not optimal and runs too slowly!*)
ToUV[Q_] :=
 Module[{P = Q /. {T1 → s, T2 → t}, deg, degs, degt, ShiftP, UVMons, Coefs, sol, eqs, cr},
  If[P == 0, Return[0]];
  deg = Exponent[P /. {t → s}, s];
  UVMons = Expand[Monomials_deg[u, v]];
  degs = Exponent[P /. s → 1 / s, s];
  degt = Exponent[P /. t → 1 / t, t];

  degs = Max@Append[Table[Exponent[μ /. s → 1 / s, s], {μ, UVMons}], degs];
  degt = Max@Append[Table[Exponent[μ /. t → 1 / t, t], {μ, UVMons}], degt];
  UVMons = s^degs t^degt UVMons // Expand;
  ShiftP = Expand[P s^degs t^degt];


  Coefs = Table[f_i, {i, 1, Length[UVMons]}];
  cr = CoefficientRules[(UVMons.Coefs - ShiftP), {s, t}];
  eqs = cr /. {(r_ → w_) :→ w == 0};
  {sol} =
   Solve[eqs, Coefs];

  Monomials_deg[U, V].Coefs /. sol
 ]
ToUV[-1/T1^2 - T1^2 - 1/T2^2 - 1/(T1^2 T2^2) + 1/(T1 T2^2) + 1/(T1^2 T2) + T1/T2 + T2/T1 + T1^2 T2 - T2^2 + T1 T2^2 - T1^2 T2^2]
Renorm[t_] := If[t == 0, 0, Sign[t] Log[Abs[t] + 10]]
DrawUVPol[P_] := Module[{Mat},
  If[P === 0, Return[P]],
   Mat = Map[Renorm, Normal@SparseArray[CoefficientRules[U V P, {U, V}]], {2}]];
  MatrixPlot[Mat]
 ]
```

(Alt) Out[ ]=

$4 U - U^2 + 3 V$

Rolfsen table

In[∘]:=
```
UVTable = {#, ToUV[θ[#][[2]]]} & /@ AllKnots[{3, 7}];
Column[% // Factor]
```

Out[∘]=

{Knot[3, 1], $4U - U^2 + 3V$}
{Knot[4, 1], 0}
{Knot[5, 1], $-22U - 11U^2 + 12U^3 - 2U^4 - 13V - 30UV + 10U^2V - 10V^2$}
{Knot[5, 2], $14 + 30U - 9U^2 + 31V$}
{Knot[6, 1], $-28 + 2U + U^2 - 5V$}
{Knot[6, 2], $73U - 18U^2 - 4U^3 + U^4 + 39V + 19UV - 7U^2V + 11V^2$}
{Knot[6, 3], 0}
{Knot[7, 1], $-21 + 29U + 109U^2 - 45U^3 - 44U^4 + 24U^5 - 3U^6 + 2V + 141UV + 58U^2V - 105U^3V + 21U^4V + 47V^2 + 84UV^2 - 42U^2V^2 + 21V^3$}
{Knot[7, 2], $-2(-49 - 54U + 18U^2 - 65V)$}
{Knot[7, 3], $-14 + 267U + 88U^2 - 106U^3 + 17U^4 + 127V + 307UV - 93U^2V + 109V^2$}
{Knot[7, 4], $8(-35 - 28U + 10U^2 - 37V)$}
{Knot[7, 5], $70 - 207U - 153U^2 + 118U^3 - 17U^4 - 76V - 367UV + 101U^2V - 141V^2$}
{Knot[7, 6], $56 + 157U - 67U^2 + 2U^3 + U^4 + 164V + UV - 9U^2V + 19V^2$}
{Knot[7, 7], $56 - 8U - U^2 + 7V$}

In[∘]:=
```
UVTable // Column
```

In[∘]:=
```
{#[[1]], DrawUVPol[#[[2]]]} & /@ UVTable // MatrixForm
```

Ribbon Knot table:

Genus bound:

It appears that $\deg_V \leq g$. Or perhaps $2\deg_V + \deg_U \leq 2g$ is sharper.
See also in the Conway and KT cases below. Conway has genus 3, KT genus 2.

Specific knots

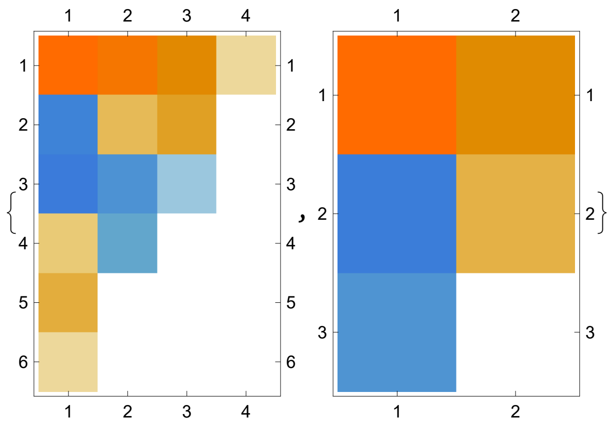
Conway and Kinoshita-Terasaka

In[∘]:=
```
{UVConway = ToUV[θ[Knot[11, NonAlternating, 34]]⟦2⟧],
 UVKT = ToUV[θ[Knot[11, NonAlternating, 42]]⟦2⟧]}
DrawUVPol /@ %
```

Out[∘]=

$\{2856 - 518\,U - 612\,U^2 + 20\,U^3 + 40\,U^4 + 4\,U^5 + 1544\,V + 33\,U\,V - 196\,U^2\,V - 28\,U^3\,V + 224\,V^2 + 44\,U\,V^2 - U^2\,V^2 + 4\,V^3,\ 40 - 6\,U - 4\,U^2 + 8\,V + U\,V\}$

Out[∘]=



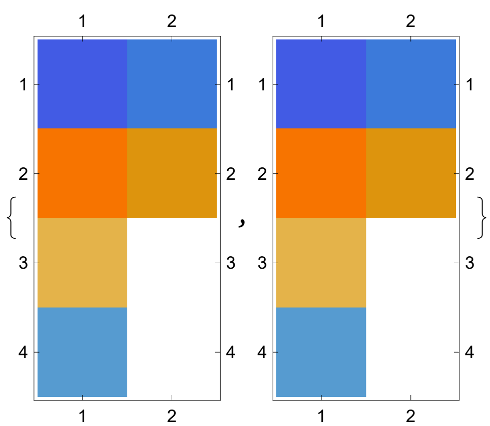
Mutant ninja turtles

In[∘]:=
```
{UVConway = ToUV[θ[Knot[11, NonAlternating, 73]]⟦2⟧],
 UVKT = ToUV[θ[Knot[11, NonAlternating, 74]]⟦2⟧]}
DrawUVPol /@ %
```

Out[∘]=

$\{-88 + 38\,U + 4\,U^2 - 2\,U^3 - 24\,V + 6\,U\,V,\ -88 + 38\,U + 4\,U^2 - 2\,U^3 - 24\,V + 6\,U\,V\}$

Out[∘]=



GST knot.

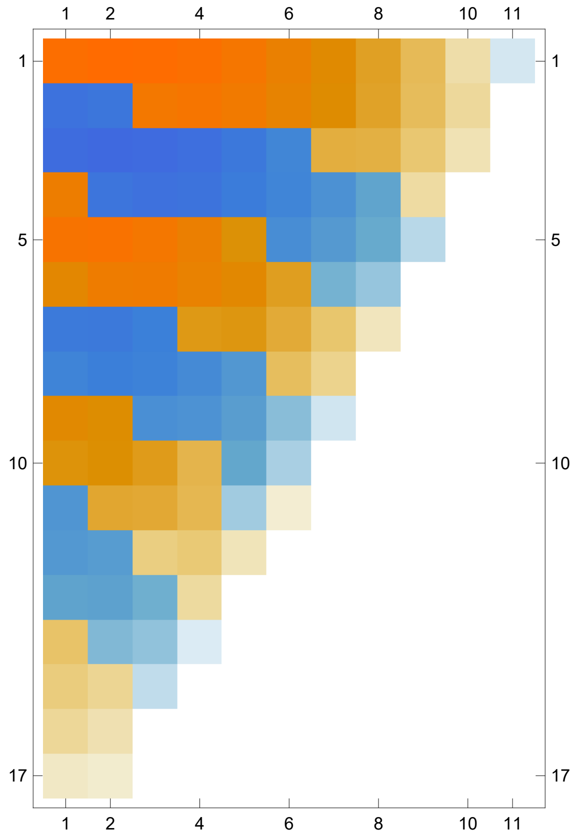
```
In[•]:= PD[GST48] = PD[X[1, 15, 2, 14], X[29, 2, 30, 3], X[40, 4, 41, 3],
   X[4, 44, 5, 43], X[5, 26, 6, 27], X[95, 7, 96, 6], X[7, 1, 8, 96], X[8, 14, 9, 13],
   X[28, 9, 29, 10], X[41, 11, 42, 10], X[11, 43, 12, 42], X[12, 27, 13, 28],
   X[15, 31, 16, 30], X[61, 16, 62, 17], X[72, 17, 73, 18], X[83, 18, 84, 19],
   X[34, 20, 35, 19], X[20, 89, 21, 90], X[92, 21, 93, 22], X[22, 79, 23, 80],
   X[23, 68, 24, 69], X[24, 57, 25, 58], X[56, 25, 57, 26], X[31, 63, 32, 62],
   X[32, 74, 33, 73], X[33, 85, 34, 84], X[35, 50, 36, 51], X[81, 37, 82, 36],
   X[70, 38, 71, 37], X[59, 39, 60, 38], X[54, 39, 55, 40], X[55, 45, 56, 44],
   X[45, 59, 46, 58], X[46, 70, 47, 69], X[47, 81, 48, 80], X[91, 49, 92, 48],
   X[49, 91, 50, 90], X[82, 52, 83, 51], X[71, 53, 72, 52], X[60, 54, 61, 53],
   X[74, 63, 75, 64], X[85, 64, 86, 65], X[65, 76, 66, 77], X[66, 87, 67, 88],
   X[94, 67, 95, 68], X[86, 75, 87, 76], X[77, 88, 78, 89], X[93, 78, 94, 79]];

In[•]:= KGST48 = Θ[PD@GST48];

In[•]:= UVGST48 = ToUV[KGST48[[2]]]
```

Out[•]=

$6230829076 - 1649181286 U - 5550362737 U^2 + 633563170 U^3 + 2149291095 U^4 + 57738350 U^5 - 442863600 U^6 - 68037954 U^7 + 47087638 U^8 + 13742818 U^9 - 1713126 U^{10} - 1133034 U^{11} - 93673 U^{12} + 27628 U^{13} + 7084 U^{14} + 634 U^{15} + 21 U^{16} + 13167733457 V - 742113426 U V - 10317864060 U^2 V - 780044732 U^3 V + 3238407625 U^4 V + 638880245 U^5 V - 474970634 U^6 V - 158493853 U^7 V + 24648280 U^8 V + 16630248 U^9 V + 1117975 U^{10} V - 597951 U^{11} V - 131649 U^{12} V - 6085 U^{13} V + 927 U^{14} V + 120 U^{15} V + 4 U^{16} V + 11869957279 V^2 + 1596094282 U V^2 - 7694098809 U^2 V^2 - 1915654735 U^3 V^2 + 1772355983 U^4 V^2 + 673776096 U^5 V^2 - 139570447 U^6 V^2 - 95990994 U^7 V^2 - 4878592 U^8 V^2 + 4956644 U^9 V^2 + 1012288 U^{10} V^2 + 5355 U^{11} V^2 - 18588 U^{12} V^2 - 2124 U^{13} V^2 - 76 U^{14} V^2 + 5974726186 V^3 + 1846197822 U V^3 - 2937035760 U^2 V^3 - 1250175184 U^3 V^3 + 401371993 U^4 V^3 + 272656716 U^5 V^3 + 6202565 U^6 V^3 - 20912710 U^7 V^3 - 3998030 U^8 V^3 + 181761 U^9 V^3 + 132950 U^{10} V^3 + 14623 U^{11} V^3 + 480 U^{12} V^3 - 5 U^{13} V^3 + 1838914446 V^4 + 858092040 U V^4 - 591691979 U^2 V^4 - 383311959 U^3 V^4 + 15686538 U^4 V^4 + 48517081 U^5 V^4 + 8278217 U^6 V^4 - 1141018 U^7 V^4 - 488295 U^8 V^4 - 48732 U^9 V^4 - 807 U^{10} V^4 + 80 U^{11} V^4 + 354683158 V^5 + 214618897 U V^5 - 52915707 U^2 V^5 - 59477229 U^3 V^5 - 7719781 U^4 V^5 + 3142057 U^5 V^5 + 991283 U^6 V^5 + 74251 U^7 V^5 - 3605 U^8 V^5 - 492 U^9 V^5 + U^{10} V^5 + 41939725 V^6 + 30223366 U V^6 + 486587 U^2 V^6 - 4238868 U^3 V^6 - 1043085 U^4 V^6 - 15128 U^5 V^6 + 18462 U^6 V^6 + 1428 U^7 V^6 - 13 U^8 V^6 + 2800418 V^7 + 2267506 U V^7 + 390623 U^2 V^7 - 87915 U^3 V^7 - 30306 U^4 V^7 - 1835 U^5 V^7 + 63 U^6 V^7 + 84191 V^8 + 74924 U V^8 + 17376 U^2 V^8 + 474 U^3 V^8 - 136 U^4 V^8 + 272 V^9 + 596 U V^9 + 115 U^2 V^9 - 12 V^{10}$

In[ ]:= DrawUVPol[UVGST48]

Out[ ]=



(Alt) In[ ]:=

```
DunfieldKnotList =
  ReadList["C:\\Users\\T15Roland\\Wiskunde\\Bn\\HigherRank\\nmd_random_knots.txt"] /.
   {i_Integer :→ i + 1};
```

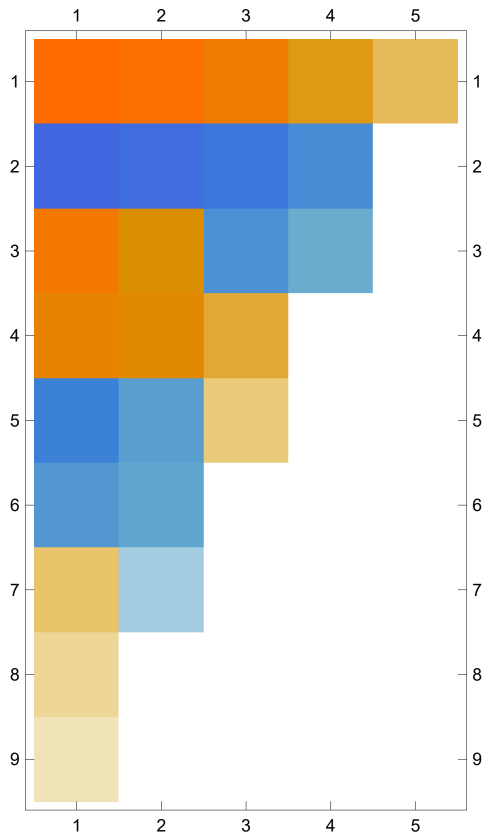
ReadList: Cannot open C:\Users\T15Roland\Wiskunde\Bn\HigherRank\nmd_random_knots.txt.

In[∘]:=
```
ToUV[θ[DunfieldKnotList[[10]]][[2]]]
DrawUVPol@%
```

Out[∘]=

$$99168 - 131978\,U + 31970\,U^2 + 16662\,U^3 - 5055\,U^4 - 1038\,U^5 + 172\,U^6 + 40\,U^7 + 2\,U^8 + 90274\,V - 89599\,U\,V + 7613\,U^2\,V + 10324\,U^3\,V - 648\,U^4\,V - 438\,U^5\,V - 30\,U^6\,V + 30861\,V^2 - 20290\,U\,V^2 - 1512\,U^2\,V^2 + 1496\,U^3\,V^2 + 162\,U^4\,V^2 + 4720\,V^3 - 1542\,U\,V^3 - 364\,U^2\,V^3 + 274\,V^4$$

Out[∘]=

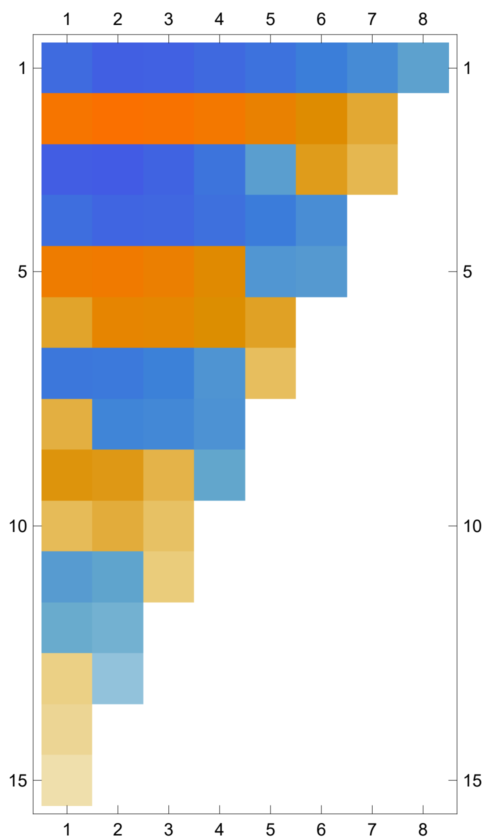


In[∘]:=
```
ToUV[θ[DunfieldKnotList⟦30⟧]⟦2⟧]
DrawUVPol@%
```

Out[∘]=

$$-20\,959\,356\,192 + 82\,648\,870\,670\,U - 61\,420\,204\,654\,U^2 - 12\,889\,058\,040\,U^3 + 21\,952\,491\,586\,U^4 + 75\,909\,790\,U^5 - 3\,467\,252\,696\,U^6 + 32\,343\,128\,U^7 + 314\,994\,260\,U^8 + 11\,593\,600\,U^9 - 15\,968\,084\,U^{10} - 1\,697\,514\,U^{11} + 310\,109\,U^{12} + 64\,702\,U^{13} + 3195\,U^{14} - 49\,508\,478\,050\,V + 147\,417\,992\,421\,U\,V - 79\,306\,207\,340\,U^2\,V - 31\,830\,212\,699\,U^3\,V + 25\,568\,303\,784\,U^4\,V + 3\,664\,498\,263\,U^5\,V - 3\,278\,502\,945\,U^6\,V - 405\,420\,878\,U^7\,V + 203\,012\,405\,U^8\,V + 34\,035\,364\,U^9\,V - 4\,288\,830\,U^{10}\,V - 1\,229\,093\,U^{11}\,V - 70\,217\,U^{12}\,V - 48\,238\,331\,920\,V^2 + 108\,765\,255\,504\,U\,V^2 - 37\,844\,869\,967\,U^2\,V^2 - 25\,625\,045\,308\,U^3\,V^2 + 10\,333\,553\,045\,U^4\,V^2 + 3\,195\,589\,246\,U^5\,V^2 - 902\,413\,026\,U^6\,V^2 - 257\,790\,994\,U^7\,V^2 + 21\,112\,027\,U^8\,V^2 + 9\,629\,496\,U^9\,V^2 + 653\,692\,U^{10}\,V^2 - 25\,424\,737\,904\,V^3 + 42\,535\,474\,929\,U\,V^3 - 7\,222\,442\,748\,U^2\,V^3 - 9\,373\,889\,543\,U^3\,V^3 + 1\,477\,014\,251\,U^4\,V^3 + 933\,076\,873\,U^5\,V^3 - 36\,018\,686\,U^6\,V^3 - 39\,726\,750\,U^7\,V^3 - 3\,330\,993\,U^8\,V^3 - 7\,883\,961\,088\,V^4 + 9\,307\,650\,913\,U\,V^4 - 10\,398\,780\,U^2\,V^4 - 1\,621\,282\,746\,U^3\,V^4 - 32\,734\,022\,U^4\,V^4 + 90\,701\,424\,U^5\,V^4 + 9\,987\,666\,U^6\,V^4 - 1\,444\,915\,816\,V^5 + 1\,081\,283\,525\,U\,V^5 + 172\,871\,586\,U^2\,V^5 - 108\,103\,153\,U^3\,V^5 - 17\,498\,380\,U^4\,V^5 - 145\,376\,287\,V^6 + 52\,130\,232\,U\,V^6 + 16\,396\,920\,U^2\,V^6 - 6\,208\,317\,V^7$$

Out[∘]=




```
(*My ToUV is too slow to handle this*)
```

In[∘]:=
```
DK120 = << Theta4DK120.m;
```

Invariance Proof

(Alt) In[]:=

```
δ_{i_,j_} := If[i === j, 1, 0];
```

(Alt) In[]:=

```
gRules_{s_,i_,j_} := {
    g_{ν_,i,β_} :→ δ_{i,β} + T_ν^s g_{ν,i⁺,β} + (1 - T_ν^s) g_{ν,j⁺,β}, g_{ν_,j,β_} :→ δ_{j,β} + g_{ν,j⁺,β},
    g_{ν_,α_,i} :→ T_ν^{-s} (g_{ν,α,i⁺} - δ_{α,i⁺}), g_{ν_,α_,j} :→ g_{ν,α,j⁺} - (1 - T_ν^s) g_{ν,α,i} - δ_{α,j⁺}
  };
gRules[Cs_List] := Union @@ ((gRules_{Sequence@@#}) & /@ Cs)
```

Invariance of $y_{\alpha\beta\gamma}$ under remote R2bs

In[]:=

```
Clear[i, j];
Cs = {{1, i, j}, {-1, i⁺, j⁺}}
Z = Module[{s, i, j}, Sum[{s, i, j} = c;
    θ[s, i, j, α, β, γ], {c, Cs}]]
```

Out[]=

{{1, i, j}, {-1, i⁺, j⁺}}

Out[]=

$$g_{1,\beta,i}\, g_{2,\gamma,i}\, g_{3,j,\alpha} - \frac{g_{1,\beta,j}\, g_{2,\gamma,i}\, g_{3,j,\alpha}}{T_1} - \frac{g_{1,\beta,i^+}\, g_{2,\gamma,i^+}\, g_{3,j^+,\alpha}}{T_1^2\, T_2} + \frac{g_{1,\beta,j^+}\, g_{2,\gamma,i^+}\, g_{3,j^+,\alpha}}{T_1\, T_2}$$

In[]:=

```
Expand[Z //. gRules_{1,i,j} ⋃ gRules_{-1,i⁺,j⁺} /. _If → 0]
```

Out[]=

0

Invariance of $y_{\alpha\beta\gamma}$ under remote R3s

In[∘]:=
```
Clear[i, j, k];
Cs = {{1, i, j}, {1, i⁺, k}, {1, j⁺, k⁺}}
Z = Module[{s, i, j}, Sum[{s, i, j} = c;
    {s, i, j} = c; θ[s, i, j, α, β, γ],
    {c, Cs}]]
lhs = Simplify[Z //. gRules[Cs] /. _If → 0]
```

Out[∘]=

$\{\{1, i, j\}, \{1, i^+, k\}, \{1, j^+, k^+\}\}$

Out[∘]=

$$g_{1,\beta,i}\, g_{2,\gamma,i}\, g_{3,j,\alpha} - \frac{g_{1,\beta,j}\, g_{2,\gamma,i}\, g_{3,j,\alpha}}{T_1} - \frac{g_{1,\beta,k}\, g_{2,\gamma,i^+}\, g_{3,k,\alpha}}{T_1} + g_{1,\beta,i^+}\, g_{2,\gamma,i^+}\, g_{3,k,\alpha} + g_{1,\beta,j^+}\, g_{2,\gamma,j^+}\, g_{3,k^+,\alpha} - \frac{g_{1,\beta,k^+}\, g_{2,\gamma,j^+}\, g_{3,k^+,\alpha}}{T_1}$$

Out[∘]=

$$\frac{1}{T_1^3 T_2^2} \Big(g_{1,\beta,i^{++}}\, g_{2,\gamma,i^{++}}\, g_{3,k^{++},\alpha} - T_1^2 T_2 \left(g_{1,\beta,j^{++}}\, g_{2,\gamma,i^{++}}\, g_{3,j^{++},\alpha} + g_{1,\beta,k^{++}} \left(g_{2,\gamma,i^{++}} + g_{2,\gamma,j^{++}}\right) g_{3,k^{++},\alpha}\right) + T_1 \left(-g_{1,\beta,j^{++}}\, g_{2,\gamma,i^{++}}\, g_{3,k^{++},\alpha} + T_2 \left(g_{1,\beta,i^{++}}\, g_{2,\gamma,i^{++}}\, g_{3,j^{++},\alpha} + g_{1,\beta,j^{++}} \left(g_{2,\gamma,i^{++}} + g_{2,\gamma,j^{++}}\right) g_{3,k^{++},\alpha}\right)\right)\Big)$$

In[∘]:=
```
Clear[i, j, k];
Cs = {{1, j, k}, {1, i, k⁺}, {1, i⁺, j⁺}}
Z = Module[{s, i, j}, Sum[{s, i, j} = c;
    {s, i, j} = c; θ[s, i, j, α, β, γ],
    {c, Cs}]]
rhs = Simplify[Z //. gRules[Cs] /. _If → 0]
```

Out[∘]=

$\{\{1, j, k\}, \{1, i, k^+\}, \{1, i^+, j^+\}\}$

Out[∘]=

$$g_{1,\beta,j}\, g_{2,\gamma,j}\, g_{3,k,\alpha} - \frac{g_{1,\beta,k}\, g_{2,\gamma,j}\, g_{3,k,\alpha}}{T_1} + g_{1,\beta,i^+}\, g_{2,\gamma,i^+}\, g_{3,j^+,\alpha} - \frac{g_{1,\beta,j^+}\, g_{2,\gamma,i^+}\, g_{3,j^+,\alpha}}{T_1} + g_{1,\beta,i}\, g_{2,\gamma,i}\, g_{3,k^+,\alpha} - \frac{g_{1,\beta,k^+}\, g_{2,\gamma,i}\, g_{3,k^+,\alpha}}{T_1}$$

Out[∘]=

$$\frac{1}{T_1^3 T_2^2} \Big(g_{1,\beta,i^{++}}\, g_{2,\gamma,i^{++}}\, g_{3,k^{++},\alpha} - T_1^2 T_2 \left(g_{1,\beta,j^{++}}\, g_{2,\gamma,i^{++}}\, g_{3,j^{++},\alpha} + g_{1,\beta,k^{++}} \left(g_{2,\gamma,i^{++}} + g_{2,\gamma,j^{++}}\right) g_{3,k^{++},\alpha}\right) + T_1 \left(-g_{1,\beta,j^{++}}\, g_{2,\gamma,i^{++}}\, g_{3,k^{++},\alpha} + T_2 \left(g_{1,\beta,i^{++}}\, g_{2,\gamma,i^{++}}\, g_{3,j^{++},\alpha} + g_{1,\beta,j^{++}} \left(g_{2,\gamma,i^{++}} + g_{2,\gamma,j^{++}}\right) g_{3,k^{++},\alpha}\right)\right)\Big)$$

In[∘]:=
```
lhs == rhs
```

Out[∘]=

True

Invariance of $y_{\alpha\beta\gamma}$ under remote R2cs

In[∘]:=
```
Clear[i, j];
Cs = {{1, i⁺, j}, {-1, i, j⁺}};
Z = Module[{s, i, j}, Sum[{s, i, j} = c; θ[s, i, j, α, β, γ], {c, Cs}]]
Expand[Z //. gRules_{1,i⁺,j} ⋃ gRules_{-1,i,j⁺} /. _If → 0]
```

Out[∘]=

$$-\frac{g_{1,\beta,j}\, g_{2,\gamma,i^+}\, g_{3,j,\alpha}}{T_1} + g_{1,\beta,i^+}\, g_{2,\gamma,i^+}\, g_{3,j,\alpha} - \frac{g_{1,\beta,i}\, g_{2,\gamma,i}\, g_{3,j^+,\alpha}}{T_1^2\, T_2} + \frac{g_{1,\beta,j^+}\, g_{2,\gamma,i}\, g_{3,j^+,\alpha}}{T_1\, T_2}$$

Out[∘]=

0

Invariance under R2b

(Alt) In[∘]:=
```
Y[α_, β_, γ_] := Module[{s, i, j}, Sum[{s, i, j} = c;
    θ[s, i, j, α, β, γ], {c, Cs}]];
yEval[ℰ_] := ℰ /. y_{α_,β_,γ_} :→ Y[α, β, γ];
```

In[∘]:=
```
Clear[i, j];
Cs = {{1, i, j}, {-1, i⁺, j⁺}}
Expand@Together[(Total[R₁ @@@ Cs] // yEval) //. gRules[Cs]]
```

Out[∘]=

{{1, i, j}, {-1, i⁺, j⁺}}

Out[∘]=

0

Invariance under R3b

In[∘]:=
```
Clear[i, j, k];
Cs = {{1, i, j}, {1, i⁺, k}, {1, j⁺, k⁺}}
lhs = Expand@Together[(Total[R₁ @@@ Cs] // yEval) //. gRules[Cs]]
```

Out[∘]=

{{1, i, j}, {1, i⁺, k}, {1, j⁺, k⁺}}

Out[∘]=

$$\frac{g_{1,j^{++},i^{++}}\, g_{2,i^{++},i^{++}}}{(-1+T_1)\, T_1^2\, (-1+T_1 T_2)} - \frac{g_{1,j^{++},i^{++}}\, g_{2,i^{++},i^{++}}}{(-1+T_1)\, T_1\, (-1+T_1 T_2)} - \frac{g_{1,j^{++},j^{++}}\, g_{2,i^{++},i^{++}}}{(-1+T_1)\, T_1\, (-1+T_1 T_2)} + \cdots 775 \cdots + \frac{g_{2,k^{++},i^{++}}\, g_{3,k^{++},k^{++}}}{(-1+T_1)\, (-1+T_1 T_2)} - \frac{g_{2,k^{++},i^{++}}\, g_{3,k^{++},k^{++}}}{(-1+T_1)\, T_1\, (-1+T_1 T_2)} + \frac{g_{2,k^{++},j^{++}}\, g_{3,k^{++},k^{++}}}{(-1+T_1)\, (-1+T_1 T_2)} - \frac{g_{2,k^{++},j^{++}}\, g_{3,k^{++},k^{++}}}{(-1+T_1)\, T_1\, (-1+T_1 T_2)}$$

large output | show less | show more | show all | set size limit...

In[∘]:=
```
Clear[i, j, k];
Cs = {{1, j, k}, {1, i, k⁺}, {1, i⁺, j⁺}}
rhs = Expand@Together[(Total[R₁ @@@ Cs] // yEval) //. gRules[Cs]]
```

Out[∘]=

$\{\{1, j, k\}, \{1, i, k^+\}, \{1, i^+, j^+\}\}$

Out[∘]=

$$\frac{g_{1,j^{++},i^{++}}\, g_{2,i^{++},i^{++}}}{(-1+T_1)\, T_1^2\, (-1+T_1 T_2)} - \frac{g_{1,j^{++},i^{++}}\, g_{2,i^{++},i^{++}}}{(-1+T_1)\, T_1\, (-1+T_1 T_2)} - \frac{g_{1,j^{++},j^{++}}\, g_{2,i^{++},i^{++}}}{(-1+T_1)\, T_1\, (-1+T_1 T_2)} + \cdots 775 \cdots + \frac{g_{2,k^{++},i^{++}}\, g_{3,k^{++},k^{++}}}{(-1+T_1)\, (-1+T_1 T_2)} - \frac{g_{2,k^{++},i^{++}}\, g_{3,k^{++},k^{++}}}{(-1+T_1)\, T_1\, (-1+T_1 T_2)} + \frac{g_{2,k^{++},j^{++}}\, g_{3,k^{++},k^{++}}}{(-1+T_1)\, (-1+T_1 T_2)} - \frac{g_{2,k^{++},j^{++}}\, g_{3,k^{++},k^{++}}}{(-1+T_1)\, T_1\, (-1+T_1 T_2)}$$

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In[∘]:=
```
lhs == rhs
```

Out[∘]=

True

Invariance under R2c

In[∘]:=
```
Clear[i, j];
Cs = {{1, i⁺, j}, {-1, i, j⁺}};
lhs = Expand@Together[(Total[R₁ @@@ Cs] + Piv_{j⁺} // yEval) //. gRules[Cs]]
rhs = Piv_{j"++"};
lhs == rhs // FullSimplify
```

Out[∘]=

$$-\frac{g_{3,j^{++},j^{++}}}{T_1\, (-1 + T_1 T_2)}$$

Out[∘]=

True

In[∘]:=
```
Solve[1 + h T₁ (-1 + T₁ T₂) == 0, h]
```

Out[∘]=

$$\left\{\left\{h \to -\frac{1}{T_1\, (-1 + T_1 T_2)}\right\}\right\}$$

Invariance under R1l

In[∘]:=
```
Cs = {{1, i⁺, i}};
gRules[Cs]
```

Out[∘]=

$$\{g_{\nu\$\_,i,\beta\$\_} :\to \delta_{i,\beta\$} + g_{\nu\$,i^+,\beta\$},\ g_{\nu\$\_,\alpha\$\_,i} :\to g_{\nu\$,\alpha\$,i^+} - (1 - T^1_{\nu\$})\, g_{\nu\$,\alpha\$,i^+} - \delta_{\alpha\$,i^+},$$
$$g_{\nu\$\_,\alpha\$\_,i^+} :\to T^{-1}_{\nu\$}\, (g_{\nu\$,\alpha\$,(i^+)^+} - \delta_{\alpha\$,(i^+)^+}),\ g_{\nu\$\_,i^+,\beta\$\_} :\to \delta_{i^+,\beta\$} + T^1_{\nu\$}\, g_{\nu\$,(i^+)^+,\beta\$} + (1 - T^1_{\nu\$})\, g_{\nu\$,i^+,\beta\$}\}$$

In[∘]:= **gr1lRules** = $\{g_{\nu\$\_,i,\beta\$\_} \rightarrow \delta_{i,\beta\$} + g_{\nu\$,i^+,\beta\$},$
$g_{\nu\$\_,\alpha\$\_,i} \rightarrow g_{\nu\$,\alpha\$,i^+} - (1 - T^1_{\nu\$})\, g_{\nu\$,\alpha\$,i^+} - \delta_{\alpha\$,i^+},$
$g_{\nu\$\_,\alpha\$\_,i^+} \rightarrow T^{-1}_{\nu\$}\, (g_{\nu\$,\alpha\$,(i^+)^+} - \delta_{\alpha\$,(i^+)^+}),$
$g_{\nu\$\_,i^+,\beta\$\_} \rightarrow T^{-1}_{\nu\$}\, (\delta_{i^+,\beta\$} + T^1_{\nu\$}\, g_{\nu\$,(i^+)^+,\beta\$})\}$

Out[∘]=

$$\left\{g_{\nu\$\_,i,\beta\$\_} \rightarrow \delta_{i,\beta\$} + g_{\nu\$,i^+,\beta\$},\ g_{\nu\$\_,\alpha\$\_,i} \rightarrow g_{\nu\$,\alpha\$,i^+} - (1 - T^1_{\nu\$})\, g_{\nu\$,\alpha\$,i^+} - \delta_{\alpha\$,i^+},\ g_{\nu\$\_,\alpha\$\_,i^+} \rightarrow \frac{g_{\nu\$,\alpha\$,(i^+)^+} - \delta_{\alpha\$,(i^+)^+}}{T_{\nu\$}},\ g_{\nu\$\_,i^+,\beta\$\_} \rightarrow \frac{\delta_{i^+,\beta\$} + T^1_{\nu\$}\, g_{\nu\$,(i^+)^+,\beta\$}}{T_{\nu\$}}\right\}$$

In[∘]:= **Total[R₁ @@@ Cs]**

In[∘]:= **(Total[R₁ @@@ Cs] + Piv$_{i^+}$ // yEval) //. gr1lRules // Simplify**

Out[∘]=

0

Invariance under R1r

In[∘]:= **Cs = {{1, i, i⁺}};**
gRules[Cs]

Out[∘]=

$$\{g_{\nu\$\_,i,\beta\$\_} \rightarrow \delta_{i,\beta\$} + T^1_{\nu\$}\, g_{\nu\$,i^+,\beta\$} + (1 - T^1_{\nu\$})\, g_{\nu\$,(i^+)^+,\beta\$},\ g_{\nu\$\_,\alpha\$\_,i} \rightarrow T^{-1}_{\nu\$}\, (g_{\nu\$,\alpha\$,i^+} - \delta_{\alpha\$,i^+}),\ g_{\nu\$\_,\alpha\$\_,i^+} \rightarrow g_{\nu\$,\alpha\$,(i^+)^+} - (1 - T^1_{\nu\$})\, g_{\nu\$,\alpha\$,i} - \delta_{\alpha\$,(i^+)^+},\ g_{\nu\$\_,i^+,\beta\$\_} \rightarrow \delta_{i^+,\beta\$} + g_{\nu\$,(i^+)^+,\beta\$}\}$$

In[∘]:= **gr1rRules** = $\{$
$g_{\nu\$\_,i,\beta\$\_} \rightarrow \delta_{i,\beta\$} + T^1_{\nu\$}\, g_{\nu\$,i^+,\beta\$} + (1 - T^1_{\nu\$})\, g_{\nu\$,(i^+)^+,\beta\$},$
$g_{\nu\$\_,\alpha\$\_,i} \rightarrow T^{-1}_{\nu\$}\, (g_{\nu\$,\alpha\$,i^+} - \delta_{\alpha\$,i^+}),$
$g_{\nu\$\_,\alpha\$\_,i} \rightarrow (1 - T^1_{\nu\$})^{-1}\, (-g_{\nu\$,\alpha\$,i^+} + g_{\nu\$,\alpha\$,(i^+)^+} - \delta_{\alpha\$,(i^+)^+}),$
$g_{\nu\$\_,i^+,\beta\$\_} \rightarrow \delta_{i^+,\beta\$} + g_{\nu\$,(i^+)^+,\beta\$}\};$

In[∘]:= **Total[R₁ @@@ Cs]**

In[∘]:= **(Total[R₁ @@@ Cs] - Piv$_{i^+}$ // yEval) //. gr1rRules // Simplify**

Out[∘]=

0

Invariance under Swirl

In[∘]:= **Cs = {{1, i, j}};**
gRules[Cs]

Out[∘]=

$$\{g_{\nu\$\_,i,\beta\$\_} \rightarrow \delta_{i,\beta\$} + T^1_{\nu\$}\, g_{\nu\$,i^+,\beta\$} + (1 - T^1_{\nu\$})\, g_{\nu\$,j^+,\beta\$},\ g_{\nu\$\_,j,\beta\$\_} \rightarrow \delta_{j,\beta\$} + g_{\nu\$,j^+,\beta\$},\ g_{\nu\$\_,\alpha\$\_,i} \rightarrow T^{-1}_{\nu\$}\, (g_{\nu\$,\alpha\$,i^+} - \delta_{\alpha\$,i^+}),\ g_{\nu\$\_,\alpha\$\_,j} \rightarrow g_{\nu\$,\alpha\$,j^+} - (1 - T^1_{\nu\$})\, g_{\nu\$,\alpha\$,i} - \delta_{\alpha\$,j^+}\}$$

In[◦]:=
```
rhs = (Total[R₁ @@@ Cs] + Piv_i + Piv_j − Piv_{i⁺} − Piv_{j⁺} // yEval) //. gRules[Cs] // Simplify
lhs = (Total[R₁ @@@ Cs] // yEval) //. gRules[Cs];
lhs − rhs // Simplify;
```

Out[◦]=

$$
\begin{aligned}
&\frac{1}{(-1+T_1)\, T_1^3\, T_2^2\, (-1+T_1 T_2)} \\
&\Big(T_2\, g_{1,j^+,i^+} (-1 + g_{2,i^+,i^+} - g_{2,j^+,i^+})\, g_{3,j^+,i^+} + T_1 \Big(-g_{2,j^+,i^+}\, g_{3,j^+,i^+} + T_2 (-1 + g_{2,j^+,i^+} + \\
&\qquad g_{1,j^+,j^+} (1 - g_{2,i^+,i^+} + g_{2,j^+,i^+}) + g_{1,j^+,i^+} (2 - g_{2,i^+,i^+} + g_{2,j^+,i^+}) + g_{2,j^+,j^+})\, g_{3,j^+,i^+} + \\
&\quad T_2^2\, g_{1,j^+,i^+} (-g_{3,i^+,i^+} - g_{2,i^+,i^+} (-1 + g_{3,j^+,i^+}) + g_{2,j^+,i^+} (-1 + g_{3,j^+,i^+}) + g_{3,j^+,i^+})\Big) + \\
&T_1^2 \Big(g_{2,j^+,i^+}\, g_{3,j^+,i^+} - T_2 ((-1 - g_{1,i^+,i^+} - g_{1,j^+,j^+} (-1 + g_{2,i^+,i^+}) + g_{2,j^+,j^+})\, g_{3,j^+,i^+} + \\
&\qquad g_{1,j^+,i^+} (g_{2,j^+,i^+} + g_{3,j^+,i^+}) + g_{2,j^+,i^+} (g_{3,i^+,i^+} + g_{1,j^+,j^+}\, g_{3,j^+,i^+})) + T_2^2 (g_{2,j^+,i^+}\, g_{3,i^+,i^+} + \\
&\qquad g_{2,j^+,j^+}\, g_{3,i^+,i^+} + g_{1,j^+,j^+} (g_{3,i^+,i^+} + g_{2,i^+,i^+} (-1 + g_{3,j^+,i^+}) - g_{2,j^+,i^+} (-1 + g_{3,j^+,i^+}) - g_{3,j^+,i^+}) - \\
&\qquad g_{2,j^+,i^+}\, g_{3,j^+,i^+} - g_{2,j^+,j^+}\, g_{3,j^+,i^+} + g_{1,j^+,i^+} (g_{2,j^+,j^+} + g_{3,i^+,i^+} - g_{2,j^+,i^+} (-2 + g_{3,j^+,i^+}) + \\
&\qquad\quad g_{2,i^+,i^+} (-1 + g_{3,j^+,i^+}) - 2\, g_{3,j^+,i^+} - g_{3,j^+,j^+}) - g_{2,j^+,i^+}\, g_{3,j^+,j^+})\Big) + \\
&T_1^3\, T_2 (-T_2\, g_{1,j^+,j^+}\, g_{2,j^+,i^+} + g_{2,j^+,i^+}\, g_{3,i^+,i^+} - T_2\, g_{2,j^+,i^+}\, g_{3,i^+,i^+} - T_2\, g_{2,j^+,j^+}\, g_{3,i^+,i^+} + \\
&\quad T_2\, g_{1,j^+,j^+}\, g_{3,j^+,i^+} - T_2\, g_{1,j^+,j^+}\, g_{2,i^+,i^+}\, g_{3,j^+,i^+} - g_{2,j^+,i^+}\, g_{3,j^+,i^+} + \\
&\quad T_2\, g_{2,j^+,i^+}\, g_{3,j^+,i^+} + T_2\, g_{1,j^+,j^+}\, g_{2,j^+,i^+}\, g_{3,j^+,i^+} + T_2\, g_{2,j^+,j^+}\, g_{3,j^+,i^+} + \\
&\quad g_{1,i^+,i^+} ((-1 + T_2)\, g_{2,j^+,i^+} + T_2 (g_{2,j^+,j^+} - g_{3,j^+,i^+} - g_{3,j^+,j^+})) + T_2\, g_{2,j^+,i^+}\, g_{3,j^+,j^+} + \\
&\quad g_{1,j^+,i^+} (-((-1 + T_2)\, g_{2,j^+,i^+}) + T_2 (-g_{2,j^+,j^+} + g_{3,j^+,i^+} + g_{3,j^+,j^+}))))\Big)
\end{aligned}
$$