In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\HigherRank\\Rolands_A2"];
Once[<< KnotTheory`];
<< ../Rot.m
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

```
Loading Rot.m from http://drorbn.net/AP/Projects/HigherRank to compute rotation numbers.
```

In[∘]:=

$$r_0[1, i\_, j\_] := 2\,(-T_1+T_2+T_1 T_2)\, p_{3,j}\, x_{1,i}\, x_{2,i} - 2\,(-1+T_2)\, p_{3,j}\, x_{1,j}\, x_{2,i} - 2\, p_{3,j}\, x_{1,i}\, x_{2,j}$$

(*from r0p*)

$$r_0[-1, i\_, j\_] := -\frac{2\, p_{3,j}\, x_{1,i}\, x_{2,i}}{T_1} + \frac{2\,(-1+T_2)\, p_{3,j}\, x_{1,j}\, x_{2,i}}{T_2} + \frac{2\, p_{3,j}\, x_{1,i}\, x_{2,j}}{T_1}$$

$$r_1[1, i\_, j\_] :=$$

$$-\frac{3}{2} - 2\, p_{1,i}\, x_{1,i} - 2\, p_{1,j}\, x_{1,i} - p_{1,i}\, p_{1,j}\, x_{1,i}^2 - \frac{1}{2}\,(-1+T_1)\,(2+T_1)\, p_{1,j}^2\, x_{1,i}^2 + \frac{4\, p_{1,j}\, x_{1,j}}{T_1} -$$

$$p_{1,i}\, p_{1,j}\, x_{1,i}\, x_{1,j} + \frac{1}{2}\,(3+T_1)\, p_{1,j}^2\, x_{1,i}\, x_{1,j} - 2\, p_{2,i}\, x_{2,i} + 2\, p_{2,j}\, x_{2,i} - 2\, p_{1,j}\, p_{2,i}\, x_{1,i}\, x_{2,i} +$$

$$\left(\left(1-3T_1+2T_2-4T_1T_2+9T_1^2T_2-4T_2^2+11T_1T_2^2-18T_1^2T_2^2-4T_1T_2^3+9T_1^2T_2^3\right) p_{1,i}\, p_{2,j}\, x_{1,i}\, x_{2,i}\right) /$$

$$\left((-1+T_1)\,(-1+2T_2)\right) + \left(1+4T_1-6T_2+9T_1T_2-9T_1^2T_2-4T_1T_2^2+9T_1^2T_2^2\right) p_{1,j}\, p_{2,j}\, x_{1,i}\, x_{2,i} +$$

$$\frac{\left(-4+3T_1+16T_1T_2-9T_1^2T_2+8T_2^2-22T_1T_2^2+9T_1^2T_2^2\right) p_{1,j}\, p_{2,i}\, x_{1,j}\, x_{2,i}}{(-1+T_1)\,(-1+2T_2)} -$$

$$\frac{(-1+T_2)\left(2+T_1-11T_1T_2-4T_1T_2^2+9T_1^2T_2^2\right) p_{1,j}\, p_{2,j}\, x_{1,j}\, x_{2,i}}{(-1+T_1)\,(-1+2T_2)} -$$

$$2\, p_{2,i}\, p_{2,j}\, x_{2,i}^2 - \frac{1}{2}\,(-1+T_2)\,(4+T_2)\, p_{2,j}^2\, x_{2,i}^2 + 2\, p_{1,i}\, p_{2,j}\, x_{1,i}\, x_{2,j} -$$

$$\frac{\left(3T_1-4T_2+7T_1T_2-9T_1^2T_2-4T_1T_2^2+9T_1^2T_2^2\right) p_{1,j}\, p_{2,j}\, x_{1,i}\, x_{2,j}}{-1+2T_2} - p_{2,i}\, p_{2,j}\, x_{2,i}\, x_{2,j} +$$

$$\frac{1}{2}\,(5+T_2)\, p_{2,j}^2\, x_{2,i}\, x_{2,j} + \frac{\left(T_1+4T_2-11T_1T_2-4T_1T_2^2+9T_1^2T_2^2\right) p_{1,j}\, p_{2,i}\, x_{3,i}}{2\,(-1+2T_2)} +$$

$$\frac{1}{2}\, p_{1,i}\, p_{2,j}\, x_{3,i} - \frac{\left(-1+T_1+6T_2-11T_1T_2-4T_1T_2^2+9T_1^2T_2^2\right) p_{1,j}\, p_{2,j}\, x_{3,i}}{2\,(-1+2T_2)} -$$

$$2\, p_{3,i}\, x_{3,i} + p_{3,j}\, x_{3,i} + 2\, p_{1,j}\, p_{3,i}\, x_{1,i}\, x_{3,i} - 2\, p_{1,i}\, p_{3,j}\, x_{1,i}\, x_{3,i} - \frac{1}{-1+2T_2}$$

$$\left(-3T_1+4T_2-5T_1T_2+10T_1^2T_2-4T_1T_2^2+2T_1^2T_2^2-9T_1^3T_2^2-4T_1^2T_2^3+9T_1^3T_2^3\right) p_{1,j}\, p_{3,j}\, x_{1,i}\, x_{3,i} +$$

$$\frac{\left(6-4T_1+T_1^2+4T_2-13T_1T_2-4T_1T_2^2+9T_1^2T_2^2\right) p_{1,j}\, p_{3,i}\, x_{1,j}\, x_{3,i}}{(-1+T_1)\,(-1+T_1T_2)} + p_{1,i}\, p_{3,j}\, x_{1,j}\, x_{3,i} +$$

$$\left(\left(-3+T_1-T_1^2+6T_2-4T_1T_2+12T_1^2T_2-7T_1^2T_2^2-9T_1^3T_2^2-4T_1^2T_2^3+9T_1^3T_2^3\right) p_{1,j}\, p_{3,j}\, x_{1,j}\, x_{3,i}\right) /$$

$$\left((-1+T_1)\,(-1+2T_2)\right) + p_{2,j}\, p_{3,i}\, x_{2,i}\, x_{3,i} - p_{2,i}\, p_{3,j}\, x_{2,i}\, x_{3,i} -$$

$$\frac{1}{(-1+T_1)\,(-1+2T_2)}\left(-1+T_1+5T_2-4T_1T_2+2T_2^2-20T_1T_2^2+10T_1^2T_2^2-8T_2^3+\right.$$

$$\left.18T_1T_2^3+11T_1^2T_2^3-9T_1^3T_2^3+8T_1T_2^4-22T_1^2T_2^4+9T_1^3T_2^4\right) p_{2,j}\, p_{3,j}\, x_{2,i}\, x_{3,i} -$$

$$\Big(\big(-3+2T_1-T_2+21T_1T_2-11T_1^2T_2+14T_2^2-44T_1T_2^2+T_1^2T_2^2+9T_1^3T_2^2-8T_2^3+10T_1T_2^3+33T_1^2T_2^3-18T_1^3T_2^3+8T_1T_2^4-22T_1^2T_2^4+9T_1^3T_2^4\big)\,p_{2,j}\,p_{3,i}\,x_{2,j}\,x_{3,i}\Big)\Big/\big((-1+T_1)(-1+T_2)(-1+2T_2)(-1+T_1T_2)\big)+\frac{\big(T_1+4T_2-11T_1T_2-4T_1T_2^2+9T_1^2T_2^2\big)\,p_{2,i}\,p_{3,j}\,x_{2,j}\,x_{3,i}}{(-1+T_1)(-1+2T_2)}-$$

$$\Big(\big(-1-3T_2+12T_1T_2-T_1^2T_2+6T_2^2-11T_1T_2^2-7T_1^2T_2^2-4T_1T_2^3+9T_1^2T_2^3\big)\,p_{2,j}\,p_{3,j}\,x_{2,j}\,x_{3,i}\Big)\Big/\big((-1+T_1)(-1+T_2)(-1+2T_2)\big)-p_{3,i}\,p_{3,j}\,x_{3,i}^2+\frac{1}{2(-1+T_1)(-1+2T_2)}(-1+T_1T_2)\big(-2+2T_1+4T_2-7T_1T_2+2T_1^2T_2-2T_1T_2^2+18T_1^2T_2^2-9T_1^3T_2^2+8T_1T_2^3-22T_1^2T_2^3+9T_1^3T_2^3\big)\,p_{3,j}^2\,x_{3,i}^2+\frac{(-4+9T_1)\,p_{3,j}\,x_{3,j}}{T_1}-$$

$$\Big(\big(-4+3T_1-T_1^2+12T_2-17T_1T_2+12T_1^2T_2-4T_1T_2^2+2T_1^2T_2^2-9T_1^3T_2^2-4T_1^2T_2^3+9T_1^3T_2^3\big)\,p_{1,i}\,p_{3,j}\,x_{1,i}\,x_{3,j}\Big)\Big/\big((-1+T_1)(-1+2T_2)(-1+T_1T_2)\big)-$$

$$\frac{\big(2-4T_1+T_1^2\big)\,p_{1,j}\,p_{3,j}\,x_{1,i}\,x_{3,j}}{-1+T_1T_2}+\frac{(-1-T_2+T_1T_2)\,p_{2,i}\,p_{3,j}\,x_{2,i}\,x_{3,j}}{(-1+T_2)(-1+T_1T_2)}+$$

$$\Big((-1+T_2)\big(1-T_1-2T_2+T_1^2T_2-8T_2^2+26T_1T_2^2-11T_1^2T_2^2+8T_1T_2^3-22T_1^2T_2^3+9T_1^3T_2^3\big)\,p_{2,j}\,p_{3,j}\,x_{2,i}\,x_{3,j}\Big)\Big/\big((-1+T_1)(-1+2T_2)(-1+T_1T_2)\big)+$$

$$\frac{\big(-3+2T_1-2T_2+18T_1T_2-9T_1^2T_2+8T_2^2-22T_1T_2^2+9T_1^2T_2^2\big)\,p_{3,i}\,p_{3,j}\,x_{3,i}\,x_{3,j}}{(-1+T_1)(-1+2T_2)}-$$

$$\frac{1}{2(-1+T_1)(-1+2T_2)}\big(-5+4T_1+2T_2+11T_1T_2-7T_1^2T_2+8T_2^2-24T_1T_2^2+27T_1^2T_2^2-9T_1^3T_2^2+8T_1T_2^3-22T_1^2T_2^3+9T_1^3T_2^3\big)\,p_{3,j}^2\,x_{3,i}\,x_{3,j}$$

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r1[-1, i_, j_] :=
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$$\frac{3}{2}+2p_{1,i}\,x_{1,i}-\frac{2\big(-2+T_1^2\big)\,p_{1,j}\,x_{1,i}}{T_1^2}+p_{1,i}\,p_{1,j}\,x_{1,i}^2-\frac{(-1+T_1)(1+2T_1)\,p_{1,j}^2\,x_{1,i}^2}{2T_1^2}-$$

$$\frac{4p_{1,j}\,x_{1,j}}{T_1}+p_{1,i}\,p_{1,j}\,x_{1,i}\,x_{1,j}-\frac{(1+3T_1)\,p_{1,j}^2\,x_{1,i}\,x_{1,j}}{2T_1}+2p_{2,i}\,x_{2,i}-2p_{2,j}\,x_{2,i}-$$

$$\frac{\big(-2+3T_1-4T_2+16T_1T_2-9T_1^2T_2+8T_2^2-22T_1T_2^2+9T_1^2T_2^2\big)\,p_{1,j}\,p_{2,i}\,x_{1,i}\,x_{2,i}}{T_1(-1+2T_2)}-$$

$$\Big(\big(-1-T_1+8T_2-10T_1T_2+9T_1^2T_2-8T_2^2+15T_1T_2^2-18T_1^2T_2^2-4T_1T_2^3+9T_1^2T_2^3\big)\,p_{1,i}\,p_{2,j}\,x_{1,i}\,x_{2,i}\Big)\Big/\big((-1+T_1)T_2(-1+2T_2)\big)+\frac{1}{T_1T_2(-1+2T_2)}\big(-4T_1+8T_2-14T_1T_2+18T_1^2T_2-12T_2^2+35T_1T_2^2-36T_1^2T_2^2+8T_2^3-26T_1T_2^3+18T_1^2T_2^3\big)\,p_{1,j}\,p_{2,j}\,x_{1,i}\,x_{2,i}-\frac{\big(-4+3T_1+16T_1T_2-9T_1^2T_2+8T_2^2-22T_1T_2^2+9T_1^2T_2^2\big)\,p_{1,j}\,p_{2,i}\,x_{1,j}\,x_{2,i}}{(-1+T_1)(-1+2T_2)}+$$

$$\frac{(-1+T_2)\big(2-4T_1+4T_2-13T_1T_2+9T_1^2T_2\big)\,p_{1,j}\,p_{2,j}\,x_{1,j}\,x_{2,i}}{(-1+T_1)T_2}+\frac{(1+T_2)\,p_{2,i}\,p_{2,j}\,x_{2,i}^2}{T_2}-$$

$$\frac{(-1+T_2)(3+2T_2)\,p_{2,j}^2\,x_{2,i}^2}{2T_2^2}-2p_{1,i}\,p_{2,j}\,x_{1,i}\,x_{2,j}+$$

$$\frac{\left(2+T_1-8\,T_2+11\,T_1\,T_2-9\,T_1^2\,T_2-4\,T_1\,T_2^2+9\,T_1^2\,T_2^2\right)\,p_{1,j}\,p_{2,j}\,x_{1,i}\,x_{2,j}}{T_1\,(-1+2\,T_2)}+p_{2,i}\,p_{2,j}\,x_{2,i}\,x_{2,j}-$$

$$\frac{3\,(1+T_2)\,p_{2,j}^2\,x_{2,i}\,x_{2,j}}{2\,T_2}-\frac{\left(T_1+4\,T_2-11\,T_1\,T_2-4\,T_1\,T_2^2+9\,T_1^2\,T_2^2\right)\,p_{1,j}\,p_{2,i}\,x_{3,i}}{2\,T_1\,(-1+2\,T_2)}-\frac{p_{1,i}\,p_{2,j}\,x_{3,i}}{2\,T_2}+$$

$$\frac{\left(-T_1+3\,T_1\,T_2+4\,T_2^2-11\,T_1\,T_2^2-4\,T_1\,T_2^3+9\,T_1^2\,T_2^3\right)\,p_{1,j}\,p_{2,j}\,x_{3,i}}{2\,T_1\,T_2\,(-1+2\,T_2)}+2\,p_{3,i}\,x_{3,i}-\frac{1}{T_1^2\,T_2\,(-1+2\,T_2)}$$

$$\left(-4+9\,T_1-T_1^2+8\,T_2-18\,T_1\,T_2+T_1^2\,T_2-4\,T_1\,T_2^2+15\,T_1^2\,T_2^2-9\,T_1^3\,T_2^2-4\,T_1^2\,T_2^3+9\,T_1^3\,T_2^3\right)\,p_{3,j}\,x_{3,i}-$$

$$\frac{\left(4-4\,T_1+T_1^2+4\,T_2-11\,T_1\,T_2-4\,T_1\,T_2^2+9\,T_1^2\,T_2^2\right)\,p_{1,j}\,p_{3,i}\,x_{1,i}\,x_{3,i}}{T_1\,(-1+T_1\,T_2)}+$$

$$\left(\left(-2\,T_1+4\,T_2-7\,T_1\,T_2+10\,T_1^2\,T_2-4\,T_1\,T_2^2+2\,T_1^2\,T_2^2-9\,T_1^3\,T_2^2-4\,T_1^2\,T_2^3+9\,T_1^3\,T_2^3\right)\,p_{1,i}\,p_{3,j}\,x_{1,i}\,x_{3,i}\right)\Big/$$

$$\left((-1+T_1)\,T_1\,T_2\,(-1+2\,T_2)\right)-\frac{1}{T_1^2\,T_2\,(-1+2\,T_2)}$$

$$\left(2-6\,T_1+T_1^2+T_1\,T_2+8\,T_1^2\,T_2-4\,T_1\,T_2^2+2\,T_1^2\,T_2^2-9\,T_1^3\,T_2^2-4\,T_1^2\,T_2^3+9\,T_1^3\,T_2^3\right)\,p_{1,j}\,p_{3,j}\,x_{1,i}\,x_{3,i}-$$

$$\frac{\left(6-4\,T_1+T_1^2+4\,T_2-13\,T_1\,T_2-4\,T_1\,T_2^2+9\,T_1^2\,T_2^2\right)\,p_{1,j}\,p_{3,i}\,x_{1,j}\,x_{3,i}}{(-1+T_1)\,(-1+T_1\,T_2)}-\frac{p_{1,i}\,p_{3,j}\,x_{1,j}\,x_{3,i}}{T_2}+$$

$$\left(\left(-4+6\,T_1-2\,T_1^2+8\,T_2-11\,T_1\,T_2+4\,T_1^2\,T_2+4\,T_2^2-11\,T_1\,T_2^2-4\,T_1\,T_2^3+9\,T_1^2\,T_2^3\right)\,p_{1,j}\,p_{3,j}\,x_{1,j}\,x_{3,i}\right)\Big/$$

$$\left((-1+T_1)\,T_1\,T_2\,(-1+2\,T_2)\right)+\left(\left(-2+T_1-3\,T_2+22\,T_1\,T_2-10\,T_1^2\,T_2+14\,T_2^2-42\,T_1\,T_2^2-\right.\right.$$

$$\left.T_1^2\,T_2^2+9\,T_1^3\,T_2^2-8\,T_2^3+10\,T_1\,T_2^3+33\,T_1^2\,T_2^3-18\,T_1^3\,T_2^3+8\,T_1\,T_2^4-22\,T_1^2\,T_2^4+9\,T_1^3\,T_2^4\right)$$

$$\left.p_{2,j}\,p_{3,i}\,x_{2,i}\,x_{3,i}\right)\Big/\left((-1+T_1)\,T_2\,(-1+2\,T_2)\,(-1+T_1\,T_2)\right)+$$

$$\left((-2+T_1)\,\left(T_1+3\,T_2-12\,T_1\,T_2-6\,T_2^2+17\,T_1\,T_2^2+9\,T_1^2\,T_2^2+4\,T_2^3-3\,T_1\,T_2^3-18\,T_1^2\,T_2^3-4\,T_1\,T_2^4+9\,T_1^2\,T_2^4\right)\right.$$

$$\left.p_{2,i}\,p_{3,j}\,x_{2,i}\,x_{3,i}\right)\Big/\left((-1+T_1)\,T_1\,(-1+T_2)\,T_2\,(-1+2\,T_2)\right)-$$

$$\frac{1}{(-1+T_1)\,T_1\,T_2^2}\,\left(1+T_1-T_1^2+5\,T_2-24\,T_1\,T_2+11\,T_1^2\,T_2-8\,T_2^2+18\,T_1\,T_2^2+11\,T_1^2\,T_2^2-\right.$$

$$\left.9\,T_1^3\,T_2^2+8\,T_1\,T_2^3-22\,T_1^2\,T_2^3+9\,T_1^3\,T_2^3\right)\,p_{2,j}\,p_{3,j}\,x_{2,i}\,x_{3,i}+$$

$$\left(\left(-3+2\,T_1-T_2+21\,T_1\,T_2-11\,T_1^2\,T_2+14\,T_2^2-44\,T_1\,T_2^2+T_1^2\,T_2^2+9\,T_1^3\,T_2^2-8\,T_2^3+10\,T_1\,T_2^3+\right.\right.$$

$$\left.\left.33\,T_1^2\,T_2^3-18\,T_1^3\,T_2^3+8\,T_1\,T_2^4-22\,T_1^2\,T_2^4+9\,T_1^3\,T_2^4\right)\,p_{2,j}\,p_{3,i}\,x_{2,j}\,x_{3,i}\right)\Big/\left((-1+T_1)\,(-1+T_2)\right.$$

$$\left.(-1+2\,T_2)\,(-1+T_1\,T_2)\right)-\frac{\left(T_1+4\,T_2-11\,T_1\,T_2-4\,T_1\,T_2^2+9\,T_1^2\,T_2^2\right)\,p_{2,i}\,p_{3,j}\,x_{2,j}\,x_{3,i}}{(-1+T_1)\,T_1\,(-1+2\,T_2)}-$$

$$\frac{1}{(-1+T_1)\,T_1\,T_2\,(-1+2\,T_2)}\,\left(1+6\,T_2-25\,T_1\,T_2+10\,T_1^2\,T_2-12\,T_2^2+29\,T_1\,T_2^2+\right.$$

$$\left.11\,T_1^2\,T_2^2-9\,T_1^3\,T_2^2+12\,T_1\,T_2^3-31\,T_1^2\,T_2^3+9\,T_1^3\,T_2^3\right)\,p_{2,j}\,p_{3,j}\,x_{2,j}\,x_{3,i}-$$

$$\left(\left(2-T_1+4\,T_2-23\,T_1\,T_2+11\,T_1^2\,T_2-8\,T_2^2+20\,T_1\,T_2^2+9\,T_1^2\,T_2^2-9\,T_1^3\,T_2^2+8\,T_1\,T_2^3-22\,T_1^2\,T_2^3+9\,T_1^3\,T_2^3\right)\right.$$

$$\left.p_{3,i}\,p_{3,j}\,x_{3,i}^2\right)\Big/\left((-1+T_1)\,T_1\,T_2\,(-1+2\,T_2)\right)+$$

$$\left((-1+T_1\,T_2)\,\left(1+6\,T_2-28\,T_1\,T_2+13\,T_1^2\,T_2-8\,T_2^2+18\,T_1\,T_2^2+27\,T_1^2\,T_2^2-18\,T_1^3\,T_2^2+16\,T_1\,T_2^3-\right.\right.$$

$$\left.\left.44\,T_1^2\,T_2^3+18\,T_1^3\,T_2^3\right)\,p_{3,j}^2\,x_{3,i}^2\right)\Big/\left(2\,(-1+T_1)\,T_1^2\,T_2^2\,(-1+2\,T_2)\right)-\frac{(-4+9\,T_1)\,p_{3,j}\,x_{3,j}}{T_1}+$$

$$\left(\left(-4+3\,T_1-T_1^2+12\,T_2-17\,T_1\,T_2+12\,T_1^2\,T_2-4\,T_1\,T_2^2+2\,T_1^2\,T_2^2-9\,T_1^3\,T_2^2-4\,T_1^2\,T_2^3+9\,T_1^3\,T_2^3\right)\right.$$

$$\left.p_{1,i}\,p_{3,j}\,x_{1,i}\,x_{3,j}\right)\Big/\left((-1+T_1)\,(-1+2\,T_2)\,(-1+T_1\,T_2)\right)-$$

$$\frac{\left(2+T_1-8 T_2+11 T_1 T_2-9 T_1^2 T_2-4 T_1 T_2^2+9 T_1^2 T_2^2\right) p_{1,j} p_{3,j} x_{1,i} x_{3,j}}{T_1 \left(-1+2 T_2\right)}-$$
$$\frac{\left(-1-T_2+T_1 T_2\right) p_{2,i} p_{3,j} x_{2,i} x_{3,j}}{\left(-1+T_2\right)\left(-1+T_1 T_2\right)}-$$
$$\frac{\left(-2+T_1\right)\left(1+2 T_2-9 T_1 T_2-4 T_2^2+9 T_1 T_2^2\right) p_{2,j} p_{3,j} x_{2,i} x_{3,j}}{\left(-1+T_1\right)\left(-1+2 T_2\right)}-$$
$$\frac{\left(-3+2 T_1-2 T_2+18 T_1 T_2-9 T_1^2 T_2+8 T_2^2-22 T_1 T_2^2+9 T_1^2 T_2^2\right) p_{3,i} p_{3,j} x_{3,i} x_{3,j}}{\left(-1+T_1\right)\left(-1+2 T_2\right)}+$$
$$\left(\left(1+6 T_2-31 T_1 T_2+15 T_1^2 T_2-8 T_2^2+16 T_1 T_2^2+45 T_1^2 T_2^2-27 T_1^3 T_2^2+24 T_1 T_2^3-66 T_1^2 T_2^3+27 T_1^3 T_2^3\right) p_{3,j}^2 x_{3,i} x_{3,j}\right) \Big/ \left(2\left(-1+T_1\right) T_1 T_2\left(-1+2 T_2\right)\right)$$

In[∘]:=
```
g2px[ℰ_] := Module[{λ}, Expand[ℰ /. g_{α_,i_,j_} :→ λ p_{α,i} x_{α,j}] /. λ^{k_.} :→ 1 / k!]
```

In[∘]:=
```
{p*, x*, π*, ξ*} = {π, ξ, p, x};  (u_{i__})* := (u*)_i;
```

In[∘]:=
```
Zip_{}[ℰ_] := ℰ;
Zip_{ζ_,ζs___}[ℰ_] := (Collect[ℰ // Zip_{ζs}, ζ] /. f_. ζ^{d_.} :→ (D[f, {ζ*, d}])) /. ζ* → 0
```

In[∘]:=
```
px2g[ℰ_] := Module[{ps, xs, Q},
  ps = Union[Cases[ℰ, p__, ∞]];
  xs = Union[Cases[ℰ, x__, ∞]];
  Q = Sum[p0* x0* g_{p0⟦2⟧,x0⟦2⟧,p0⟦3⟧,x0⟦3⟧}, {p0, ps}, {x0, xs}];
  Expand[Zip_{ps⋃xs}[ℰ e^Q] /. g_{α_,β_,i_,j_} :→ If[α == β, g_{α,i,j}, 0]]
 ]
```

In[∘]:=
```
Coefficient[r_1[1, i, j] /. t : (x | p)__ :→ λ t, λ^3] /. x_{3,α_} p_{1,β_} p_{2,γ_} :→ y_{α,β,γ}
```

Out[∘]=

$$\frac{1}{2} y_{i,i,j}+\frac{\left(T_1+4 T_2-11 T_1 T_2-4 T_1 T_2^2+9 T_1^2 T_2^2\right) y_{i,j,i}}{2\left(-1+2 T_2\right)}-$$
$$\frac{\left(-1+T_1+6 T_2-11 T_1 T_2-4 T_1 T_2^2+9 T_1^2 T_2^2\right) y_{i,j,j}}{2\left(-1+2 T_2\right)}$$

In[∘]:=
```
R_1[1, i_, j_] := Evaluate[px2g[r_1[1, i, j]] +
   (Coefficient[r_1[1, i, j] /. t : (x | p)__ :→ λ t, λ^3] /. x_{3,α_} p_{1,β_} p_{2,γ_} :→ y_{α,β,γ})]
R_1[-1, i_, j_] := Evaluate[px2g[r_1[-1, i, j]] +
   (Coefficient[r_1[-1, i, j] /. t : (x | p)__ :→ λ t, λ^3] /. x_{3,α_} p_{1,β_} p_{2,γ_} :→ y_{α,β,γ})]
```

In[∘]:=
```
θ[1, i_, j_, α_, β_, γ_] :=
  Evaluate[r0[1, i, j] /. {p3,j_ ⧴ g3,α,j, x1,i_ ⧴ g1,i,β, x2,i_ ⧴ g2,i,γ}];
 (* The θ graph with light (pxx) vertex  at (1,i,j) and
 unspecified heavy (xpp) vertex *)
θ[-1, i_, j_, α_, β_, γ_] :=
  Evaluate[r0[-1, i, j] /. {p3,j_ ⧴ g3,α,j, x1,i_ ⧴ g1,i,β, x2,i_ ⧴ g2,i,γ}];
 (* The θ graph with light (pxx) vertex  at (-1,i,j)
 and unspecified heavy (xpp) vertex *)
θ[1, 5, 8, 21, 22, 23]
```

Out[∘]=

$$2\,(-T_1+T_2+T_1 T_2)\, g_{1,5,22}\, g_{2,5,23}\, g_{3,21,8} - 2\,(-1+T_2)\, g_{1,8,22}\, g_{2,5,23}\, g_{3,21,8} - 2\, g_{1,5,22}\, g_{2,8,23}\, g_{3,21,8}$$

In[∘]:=
```
T3 = T1 T2;
CF[ℰ_] := Factor@Together[ℰ];
λ[K_] := Module[{Cs, φ, n, A, s, i, j, k, Δ, G, gEval, Y, yEval, c, λ1},
   {Cs, φ} = Rot[K]; n = Length[Cs];
   A = IdentityMatrix[2 n + 1];
   Cases[Cs, {s_, i_, j_} ⧴ (A⟦{i, j}, {i + 1, j + 1}⟧ += ( -T^s  T^s - 1
                                                              0    -1     ))];
   Δ = T^((-Total[φ]-Total[Cs⟦All,1⟧])/2) Det[A];
   G = Inverse[A];
   gEval[ℰ_] := CF[ℰ /.
      {g1,α_,β_ ⧴ (G⟦α, β⟧ /. T → T1),
       g2,α_,β_ ⧴ (G⟦α, β⟧ /. T → T2), g3,α_,β_ ⧴ (G⟦α, β⟧ /. T → T3)}];
   Y[α_, β_, γ_] :=
    Y[α, β, γ] = Sum[{s, i, j} = c; (* The expectation value of x3,α p1,β p2,γ *)
      θ[s, i, j, α, β, γ],
      {c, Cs}];
   yEval[ℰ_] := CF[ℰ /. yα_,β_,γ_ ⧴ Y[α, β, γ]];
   λ1 = Sum[R1 @@ Cs⟦k⟧, {k, 1, n}] - Sum[φ⟦k⟧ (g1,k,k + g2,k,k + g3,k,k - 3 / 2), {k, 1, 2 n}];
   {Δ, (Δ^2 /. T → T1) (Δ^2 /. T → T2) (Δ^2 /. T → T3) λ1} // yEval // gEval
  ];
```

In[•]:=
```
Timing[λ[Knot[3, 1]]]
```

KnotTheory: Loading precomputed data in PD4Knots`.

Out[•]=

$$
\begin{aligned}
&\Big\{1.15625,\\
&\Big\{\frac{1-T+T^2}{T},\ \frac{1}{T_1^6\,T_2^6\,(-1+2\,T_2)}\,\big(-2\,T_1^2+8\,T_1^3-15\,T_1^4+15\,T_1^5-9\,T_1^6+2\,T_1^7+13\,T_1^2\,T_2-44\,T_1^3\,T_2+70\,T_1^4\,T_2-\\
&50\,T_1^5\,T_2+13\,T_1^6\,T_2+13\,T_1^7\,T_2-6\,T_1^8\,T_2+7\,T_1\,T_2^2-70\,T_1^2\,T_2^2+171\,T_1^3\,T_2^2-214\,T_1^4\,T_2^2+90\,T_1^5\,T_2^2+\\
&18\,T_1^6\,T_2^2-58\,T_1^7\,T_2^2-2\,T_1^8\,T_2^2+9\,T_1^9\,T_2^2-30\,T_1\,T_2^3+175\,T_1^2\,T_2^3-285\,T_1^3\,T_2^3+177\,T_1^4\,T_2^3+222\,T_1^5\,T_2^3-\\
&268\,T_1^6\,T_2^3+155\,T_1^7\,T_2^3+86\,T_1^8\,T_2^3-24\,T_1^9\,T_2^3-7\,T_1^{10}\,T_2^3+4\,T_2^4+48\,T_1\,T_2^4-214\,T_1^2\,T_2^4+97\,T_1^3\,T_2^4+\\
&349\,T_1^4\,T_2^4-888\,T_1^5\,T_2^4+304\,T_1^6\,T_2^4+107\,T_1^7\,T_2^4-434\,T_1^8\,T_2^4-15\,T_1^9\,T_2^4+28\,T_1^{10}\,T_2^4+3\,T_1^{11}\,T_2^4-16\,T_2^5-\\
&24\,T_1\,T_2^5+77\,T_1^2\,T_2^5+366\,T_1^3\,T_2^5-768\,T_1^4\,T_2^5+707\,T_1^5\,T_2^5+870\,T_1^6\,T_2^5-871\,T_1^7\,T_2^5+690\,T_1^8\,T_2^5+\\
&385\,T_1^9\,T_2^5-30\,T_1^{10}\,T_2^5-17\,T_1^{11}\,T_2^5+16\,T_2^6+49\,T_1\,T_2^6-50\,T_1^2\,T_2^6-438\,T_1^3\,T_2^6+372\,T_1^4\,T_2^6+201\,T_1^5\,T_2^6-\\
&1688\,T_1^6\,T_2^6+256\,T_1^7\,T_2^6+185\,T_1^8\,T_2^6-1089\,T_1^9\,T_2^6-188\,T_1^{10}\,T_2^6+43\,T_1^{11}\,T_2^6-12\,T_2^7-31\,T_1\,T_2^7-99\,T_1^2\,T_2^7+\\
&489\,T_1^3\,T_2^7+105\,T_1^4\,T_2^7-837\,T_1^5\,T_2^7+1520\,T_1^6\,T_2^7+877\,T_1^7\,T_2^7-661\,T_1^8\,T_2^7+888\,T_1^9\,T_2^7+836\,T_1^{10}\,T_2^7-\\
&24\,T_1^{11}\,T_2^7+48\,T_1\,T_2^8+18\,T_1^2\,T_2^8-25\,T_1^3\,T_2^8-689\,T_1^4\,T_2^8+628\,T_1^5\,T_2^8+81\,T_1^6\,T_2^8-2316\,T_1^7\,T_2^8+798\,T_1^8\,T_2^8-\\
&251\,T_1^9\,T_2^8-1191\,T_1^{10}\,T_2^8-181\,T_1^{11}\,T_2^8-100\,T_1^2\,T_2^9+81\,T_1^3\,T_2^9+204\,T_1^4\,T_2^9+293\,T_1^5\,T_2^9-821\,T_1^6\,T_2^9+\\
&1273\,T_1^7\,T_2^9+874\,T_1^8\,T_2^9-975\,T_1^9\,T_2^9+1144\,T_1^{10}\,T_2^9+388\,T_1^{11}\,T_2^9+18\,T_1^{12}\,T_2^9+128\,T_1^3\,T_2^{10}-212\,T_1^4\,T_2^{10}-\\
&228\,T_1^5\,T_2^{10}+422\,T_1^6\,T_2^{10}-172\,T_1^7\,T_2^{10}-1039\,T_1^8\,T_2^{10}+511\,T_1^9\,T_2^{10}-191\,T_1^{10}\,T_2^{10}-571\,T_1^{11}\,T_2^{10}-\\
&18\,T_1^{12}\,T_2^{10}-104\,T_1^4\,T_2^{11}+258\,T_1^5\,T_2^{11}-479\,T_1^7\,T_2^{11}+815\,T_1^8\,T_2^{11}-134\,T_1^9\,T_2^{11}-214\,T_1^{10}\,T_2^{11}+394\,T_1^{11}\,T_2^{11}+\\
&27\,T_1^{12}\,T_2^{11}+52\,T_1^5\,T_2^{12}-175\,T_1^6\,T_2^{12}+164\,T_1^7\,T_2^{12}+52\,T_1^8\,T_2^{12}-397\,T_1^9\,T_2^{12}+377\,T_1^{10}\,T_2^{12}-213\,T_1^{11}\,T_2^{12}-\\
&9\,T_1^{12}\,T_2^{12}-12\,T_1^6\,T_2^{13}+51\,T_1^7\,T_2^{13}-90\,T_1^8\,T_2^{13}+101\,T_1^9\,T_2^{13}-53\,T_1^{10}\,T_2^{13}+14\,T_1^{11}\,T_2^{13}+9\,T_1^{12}\,T_2^{13}\big)\Big\}\Big\}
\end{aligned}
$$

In[•]:=
```
Timing[λ[Knot[4, 1]]]
```

Out[•]=

$$
\begin{aligned}
&\Big\{2.29688,\\
&\Big\{-\frac{1-3\,T+T^2}{T},\ -\frac{1}{T_1^4\,T_2^4\,(-1+2\,T_2)}\,\big(1-3\,T_1+T_1^2\big)\,\big(1-3\,T_2+T_2^2\big)\,\big(1-3\,T_1\,T_2+T_1^2\,T_2^2\big)\,\big(12-18\,T_1-T_1^2+\\
&12\,T_1^3-5\,T_1^4+T_1^5-79\,T_2+133\,T_1\,T_2-6\,T_1^2\,T_2-72\,T_1^3\,T_2+13\,T_1^4\,T_2-3\,T_1^5\,T_2+151\,T_2^2-203\,T_1\,T_2^2-\\
&124\,T_1^2\,T_2^2+178\,T_1^3\,T_2^2+27\,T_1^4\,T_2^2+50\,T_1^5\,T_2^2-5\,T_1^6\,T_2^2-4\,T_1^7\,T_2^2-138\,T_2^3+217\,T_1\,T_2^3+51\,T_1^2\,T_2^3-\\
&62\,T_1^3\,T_2^3-130\,T_1^4\,T_2^3-54\,T_1^5\,T_2^3-206\,T_1^6\,T_2^3+78\,T_1^7\,T_2^3+T_1^8\,T_2^3+64\,T_2^4-109\,T_1\,T_2^4-161\,T_1^2\,T_2^4+\\
&309\,T_1^3\,T_2^4+128\,T_1^4\,T_2^4-282\,T_1^5\,T_2^4+805\,T_1^6\,T_2^4-125\,T_1^7\,T_2^4-52\,T_1^8\,T_2^4-20\,T_2^5+22\,T_1\,T_2^5+\\
&264\,T_1^2\,T_2^5-578\,T_1^3\,T_2^5+128\,T_1^4\,T_2^5+232\,T_1^5\,T_2^5-759\,T_1^6\,T_2^5-341\,T_1^7\,T_2^5+203\,T_1^8\,T_2^5+9\,T_1^9\,T_2^5+\\
&4\,T_2^6+13\,T_1\,T_2^6-194\,T_1^2\,T_2^6+322\,T_1^3\,T_2^6+163\,T_1^4\,T_2^6-483\,T_1^5\,T_2^6+366\,T_1^6\,T_2^6+755\,T_1^7\,T_2^6-\\
&208\,T_1^8\,T_2^6-45\,T_1^9\,T_2^6-8\,T_1\,T_2^7+44\,T_1^2\,T_2^7+23\,T_1^3\,T_2^7-315\,T_1^4\,T_2^7+325\,T_1^5\,T_2^7+102\,T_1^6\,T_2^7-\\
&552\,T_1^7\,T_2^7+6\,T_1^8\,T_2^7+72\,T_1^9\,T_2^7+4\,T_1^2\,T_2^8-37\,T_1^3\,T_2^8+82\,T_1^4\,T_2^8-10\,T_1^5\,T_2^8-142\,T_1^6\,T_2^8+\\
&131\,T_1^7\,T_2^8+81\,T_1^8\,T_2^8-45\,T_1^9\,T_2^8+4\,T_1^4\,T_2^9-21\,T_1^5\,T_2^9+35\,T_1^6\,T_2^9-10\,T_1^7\,T_2^9-22\,T_1^8\,T_2^9+9\,T_1^9\,T_2^9\big)\Big\}\Big\}
\end{aligned}
$$

Invariance Proof

In[•]:=
```
δi_,j_ := If[i == j, 1, 0];
```

In[•]:=
```
gRules_{s_,i_,j_} := {
  g_{ν_,i,β_} :→ δ_{i,β} + T_ν^s g_{ν,i⁺,β} + (1 - T_ν^s) g_{ν,j⁺,β}, g_{ν_,j,β_} :→ δ_{j,β} + g_{ν,j⁺,β},
  g_{ν_,α_,i} :→ T_ν^{-s} (g_{ν,α,i⁺} - δ_{α,i⁺}), g_{ν_,α_,j} :→ g_{ν,α,j⁺} - (1 - T_ν^s) g_{ν,α,i} - δ_{α,j⁺}
 }
```

Invariance of $y_{\alpha\beta\gamma}$ under remote R2s

In[•]:=
```
Clear[i, j];
Cs = {{1, i, j}, {-1, i⁺, j⁺}}
Z = Module[{s, i, j}, Sum[{s, i, j} = c;
    θ[s, i, j, α, β, γ], {c, Cs}]]
```

Out[•]=

$\{\{1, i, j\}, \{-1, i^+, j^+\}\}$

Out[•]=

$$2\,(-T_1 + T_2 + T_1 T_2)\, g_{1,i,\beta}\, g_{2,i,\gamma}\, g_{3,\alpha,j} - 2\,(-1 + T_2)\, g_{1,j,\beta}\, g_{2,i,\gamma}\, g_{3,\alpha,j} - 2\, g_{1,i,\beta}\, g_{2,j,\gamma}\, g_{3,\alpha,j} - \frac{2\, g_{1,i^+,\beta}\, g_{2,i^+,\gamma}\, g_{3,\alpha,j^+}}{T_1} + \frac{2\,(-1 + T_2)\, g_{1,j^+,\beta}\, g_{2,i^+,\gamma}\, g_{3,\alpha,j^+}}{T_2} + \frac{2\, g_{1,i^+,\beta}\, g_{2,j^+,\gamma}\, g_{3,\alpha,j^+}}{T_1}$$

In[∘]:= $\text{Expand}\left[Z\ //.\ \text{gRules}_{1,i,j}\bigcup\text{gRules}_{-1,i^+,j^+}\ /.\ \_\text{If}\to 0\right]$

Out[∘]=

$$\frac{2\, g_{1,(i^+)^+,\beta}\, g_{2,(i^+)^+,\gamma}\, g_{3,\alpha,(i^+)^+}}{T_1} - \frac{2\, g_{1,(i^+)^+,\beta}\, g_{2,(i^+)^+,\gamma}\, g_{3,\alpha,(i^+)^+}}{T_1^2\, T_2} + 2\, g_{1,(j^+)^+,\beta}\, g_{2,(i^+)^+,\gamma}\, g_{3,\alpha,(i^+)^+} -$$

$$\frac{2\, g_{1,(j^+)^+,\beta}\, g_{2,(i^+)^+,\gamma}\, g_{3,\alpha,(i^+)^+}}{T_1} - 2\, T_1\, g_{1,(j^+)^+,\beta}\, g_{2,(i^+)^+,\gamma}\, g_{3,\alpha,(i^+)^+} - \frac{2\, g_{1,(j^+)^+,\beta}\, g_{2,(i^+)^+,\gamma}\, g_{3,\alpha,(i^+)^+}}{T_2^2} +$$

$$\frac{2\, g_{1,(j^+)^+,\beta}\, g_{2,(i^+)^+,\gamma}\, g_{3,\alpha,(i^+)^+}}{T_2} + \frac{2\, g_{1,(j^+)^+,\beta}\, g_{2,(i^+)^+,\gamma}\, g_{3,\alpha,(i^+)^+}}{T_1^2\, T_2} - \frac{2\, g_{1,(j^+)^+,\beta}\, g_{2,(i^+)^+,\gamma}\, g_{3,\alpha,(i^+)^+}}{T_1\, T_2} +$$

$$\frac{2\, T_1\, g_{1,(j^+)^+,\beta}\, g_{2,(i^+)^+,\gamma}\, g_{3,\alpha,(i^+)^+}}{T_2} - \frac{2\, g_{1,(i^+)^+,\beta}\, g_{2,(j^+)^+,\gamma}\, g_{3,\alpha,(i^+)^+}}{T_1} + \frac{2\, g_{1,(i^+)^+,\beta}\, g_{2,(j^+)^+,\gamma}\, g_{3,\alpha,(i^+)^+}}{T_1^2\, T_2} +$$

$$\frac{2\, g_{1,(j^+)^+,\beta}\, g_{2,(j^+)^+,\gamma}\, g_{3,\alpha,(i^+)^+}}{T_1} + 4\, T_1\, g_{1,(j^+)^+,\beta}\, g_{2,(j^+)^+,\gamma}\, g_{3,\alpha,(i^+)^+} + \frac{2\, g_{1,(j^+)^+,\beta}\, g_{2,(j^+)^+,\gamma}\, g_{3,\alpha,(i^+)^+}}{T_2^2} -$$

$$\frac{4\, g_{1,(j^+)^+,\beta}\, g_{2,(j^+)^+,\gamma}\, g_{3,\alpha,(i^+)^+}}{T_2} - \frac{2\, g_{1,(j^+)^+,\beta}\, g_{2,(j^+)^+,\gamma}\, g_{3,\alpha,(i^+)^+}}{T_1^2\, T_2} + \frac{2\, g_{1,(j^+)^+,\beta}\, g_{2,(j^+)^+,\gamma}\, g_{3,\alpha,(i^+)^+}}{T_1\, T_2} -$$

$$\frac{2\, T_1\, g_{1,(j^+)^+,\beta}\, g_{2,(j^+)^+,\gamma}\, g_{3,\alpha,(i^+)^+}}{T_2} - 2\, T_1\, T_2\, g_{1,(j^+)^+,\beta}\, g_{2,(j^+)^+,\gamma}\, g_{3,\alpha,(i^+)^+} -$$

$$2\, T_1\, g_{1,(i^+)^+,\beta}\, g_{2,(i^+)^+,\gamma}\, g_{3,\alpha,(j^+)^+} - \frac{2\, g_{1,(i^+)^+,\beta}\, g_{2,(i^+)^+,\gamma}\, g_{3,\alpha,(j^+)^+}}{T_1^2\, T_2} + 2\, T_2\, g_{1,(i^+)^+,\beta}\, g_{2,(i^+)^+,\gamma}\, g_{3,\alpha,(j^+)^+} +$$

$$2\, T_1\, T_2\, g_{1,(i^+)^+,\beta}\, g_{2,(i^+)^+,\gamma}\, g_{3,\alpha,(j^+)^+} + 2\, g_{1,(j^+)^+,\beta}\, g_{2,(i^+)^+,\gamma}\, g_{3,\alpha,(j^+)^+} -$$

$$\frac{2\, g_{1,(j^+)^+,\beta}\, g_{2,(i^+)^+,\gamma}\, g_{3,\alpha,(j^+)^+}}{T_2^2} + \frac{2\, g_{1,(j^+)^+,\beta}\, g_{2,(i^+)^+,\gamma}\, g_{3,\alpha,(j^+)^+}}{T_2} + \frac{2\, g_{1,(j^+)^+,\beta}\, g_{2,(i^+)^+,\gamma}\, g_{3,\alpha,(j^+)^+}}{T_1^2\, T_2} -$$

$$\frac{2\, g_{1,(j^+)^+,\beta}\, g_{2,(i^+)^+,\gamma}\, g_{3,\alpha,(j^+)^+}}{T_1\, T_2} - 2\, T_2\, g_{1,(j^+)^+,\beta}\, g_{2,(i^+)^+,\gamma}\, g_{3,\alpha,(j^+)^+} - 2\, g_{1,(i^+)^+,\beta}\, g_{2,(j^+)^+,\gamma}\, g_{3,\alpha,(j^+)^+} +$$

$$\frac{2\, g_{1,(i^+)^+,\beta}\, g_{2,(j^+)^+,\gamma}\, g_{3,\alpha,(j^+)^+}}{T_1^2\, T_2} + 2\, g_{1,(j^+)^+,\beta}\, g_{2,(j^+)^+,\gamma}\, g_{3,\alpha,(j^+)^+} + \frac{2\, g_{1,(j^+)^+,\beta}\, g_{2,(j^+)^+,\gamma}\, g_{3,\alpha,(j^+)^+}}{T_2^2} -$$

$$\frac{4\, g_{1,(j^+)^+,\beta}\, g_{2,(j^+)^+,\gamma}\, g_{3,\alpha,(j^+)^+}}{T_2} - \frac{2\, g_{1,(j^+)^+,\beta}\, g_{2,(j^+)^+,\gamma}\, g_{3,\alpha,(j^+)^+}}{T_1^2\, T_2} + \frac{2\, g_{1,(j^+)^+,\beta}\, g_{2,(j^+)^+,\gamma}\, g_{3,\alpha,(j^+)^+}}{T_1\, T_2}$$