

```
In[*]:= SetDirectory["C:\\Users\\T15Roland\\Wiskunde\\Bn\\HigherRank"];
Once[<< KnotTheory`];
<< Rot.m
<< FormalGaussianIntegration.m
```

ParentDirectory: Argument File should be a positive machine-size integer, a nonempty string, or a File specification.

ParentDirectory: Argument File should be a positive machine-size integer, a nonempty string, or a File specification.

ToFileName: String or list of strings expected at position 1 in ToFileName[{File, WikiLink, mathematica}].

ToFileName: String or list of strings expected at position 1 in ToFileName[{File, QuantumGroups}].

Loading KnotTheory` version of September 6, 2014, 13:37:37.2841.

Read more at <http://katlas.org/wiki/KnotTheory>.

Loading Rot.m from <http://drorbn.net/AP/Projects/HigherRank> to compute rotation numbers.

```
In[*]:= (*The R3 solutions from UC4A242 (written hard coded below the fold):*)
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Solve: Equations may not give solutions for all "solve" variables.

```
In[*]:= r0p /. Sub // CF
r0m /. Sub // CF
```

Out[\*]=

$$2(-T_1 + T_2 + T_1 T_2) p_{3,j} x_{1,i} x_{2,i} - 2(-1 + T_2) p_{3,j} x_{1,j} x_{2,i} - 2 p_{3,j} x_{1,i} x_{2,j}$$

Out[\*]=

$$-\frac{2 p_{3,j} x_{1,i} x_{2,i}}{T_1} + \frac{2(-1 + T_2) p_{3,j} x_{1,j} x_{2,i}}{T_2} + \frac{2 p_{3,j} x_{1,i} x_{2,j}}{T_1}$$

```
In[*]:= r1p /. Sub // CF
```

Out[\*]=

$$-\frac{3}{2} - 2 p_{1,i} x_{1,i} - 2 p_{1,j} x_{1,i} - p_{1,i} p_{1,j} x_{1,i}^2 - \frac{1}{2}(-1 + T_1)(2 + T_1) p_{1,j}^2 x_{1,i}^2 + \frac{4 p_{1,j} x_{1,j}}{T_1} -$$

$$p_{1,i} p_{1,j} x_{1,i} x_{1,j} + \frac{1}{2}(3 + T_1) p_{1,j}^2 x_{1,i} x_{1,j} - 2 p_{2,i} x_{2,i} + 2 p_{2,j} x_{2,i} - 2 p_{1,j} p_{2,i} x_{1,i} x_{2,i} +$$

$$\frac{(1 - 3 T_1 + 2 T_2 - 4 T_1 T_2 + 9 T_1^2 T_2 - 4 T_2^2 + 11 T_1 T_2^2 - 18 T_1^2 T_2^2 - 4 T_1 T_2^3 + 9 T_1^2 T_2^3) p_{1,i} p_{2,j} x_{1,i} x_{2,i}}{(-1 + T_1)(-1 + 2 T_2)} +$$

$$\frac{(1 + 4 T_1 - 6 T_2 + 9 T_1 T_2 - 9 T_1^2 T_2 - 4 T_1 T_2^2 + 9 T_1^2 T_2^2) p_{1,j} p_{2,j} x_{1,i} x_{2,i} +}{(-1 + T_1)(-1 + 2 T_2)}$$

$$\frac{(-4 + 3 T_1 + 16 T_1 T_2 - 9 T_1^2 T_2 + 8 T_2^2 - 22 T_1 T_2^2 + 9 T_1^2 T_2^2) p_{1,j} p_{2,i} x_{1,j} x_{2,i}}{(-1 + T_1)(-1 + 2 T_2)} -$$

$$\frac{(-1 + T_2)(2 + T_1 - 11 T_1 T_2 - 4 T_1 T_2^2 + 9 T_1^2 T_2^2) p_{1,j} p_{2,j} x_{1,j} x_{2,i}}{(-1 + T_1)(-1 + 2 T_2)} -$$

$$2 p_{2,i} p_{2,j} x_{2,i}^2 - \frac{1}{2}(-1 + T_2)(4 + T_2) p_{2,j}^2 x_{2,i}^2 + 2 p_{1,i} p_{2,j} x_{1,i} x_{2,j} -$$

$$\frac{(3 T_1 - 4 T_2 + 7 T_1 T_2 - 9 T_1^2 T_2 - 4 T_1 T_2^2 + 9 T_1^2 T_2^2) p_{1,j} p_{2,j} x_{1,i} x_{2,j}}{-1 + 2 T_2} - p_{2,i} p_{2,j} x_{2,i} x_{2,j} +$$

$$\begin{aligned}
& \frac{1}{2} (5 + T_2) p_{2,j}^2 x_{2,i} x_{2,j} + \frac{(T_1 + 4 T_2 - 11 T_1 T_2 - 4 T_1 T_2^2 + 9 T_1^2 T_2^2) p_{1,j} p_{2,i} x_{3,i}}{2 (-1 + 2 T_2)} + \\
& \frac{1}{2} p_{1,i} p_{2,j} x_{3,i} - \frac{(-1 + T_1 + 6 T_2 - 11 T_1 T_2 - 4 T_1 T_2^2 + 9 T_1^2 T_2^2) p_{1,j} p_{2,j} x_{3,i}}{2 (-1 + 2 T_2)} - \\
& \frac{2 p_{3,i} x_{3,i} + p_{3,j} x_{3,i} + 2 p_{1,j} p_{3,i} x_{1,i} x_{3,i} - 2 p_{1,i} p_{3,j} x_{1,i} x_{3,i} -}{(-3 T_1 + 4 T_2 - 5 T_1 T_2 + 10 T_1^2 T_2 - 4 T_1 T_2^2 + 2 T_1^2 T_2^2 - 9 T_1^3 T_2 - 4 T_1^2 T_2^3 + 9 T_1^3 T_2^3) p_{1,j} p_{3,j} x_{1,i} x_{3,i}} + \\
& \frac{(6 - 4 T_1 + T_1^2 + 4 T_2 - 13 T_1 T_2 - 4 T_1 T_2^2 + 9 T_1^2 T_2^2) p_{1,j} p_{3,i} x_{1,j} x_{3,i}}{(-1 + T_1) (-1 + T_1 T_2)} + p_{1,i} p_{3,j} x_{1,j} x_{3,i} + \\
& \frac{(-3 + T_1 - T_1^2 + 6 T_2 - 4 T_1 T_2 + 12 T_1^2 T_2 - 7 T_1^2 T_2^2 - 9 T_1^3 T_2 - 4 T_1^2 T_2^3 + 9 T_1^3 T_2^3) p_{1,j} p_{3,j} x_{1,j} x_{3,i}}{(-1 + T_1) (-1 + 2 T_2)} + \\
& p_{2,j} p_{3,i} x_{2,i} x_{3,i} - p_{2,i} p_{3,j} x_{2,i} x_{3,i} - \frac{1}{(-1 + T_1) (-1 + 2 T_2)} (-1 + T_1 + 5 T_2 - 4 T_1 T_2 + 2 T_2^2 - 20 T_1 T_2^2 + \\
& \quad 10 T_1^2 T_2^2 - 8 T_2^3 + 18 T_1 T_2^3 + 11 T_1^2 T_2^3 - 9 T_1^3 T_2^3 + 8 T_1 T_2^4 - 22 T_1^2 T_2^4 + 9 T_1^3 T_2^4) p_{2,j} p_{3,j} x_{2,i} x_{3,i} - \\
& \quad ((-3 + 2 T_1 - T_2 + 21 T_1 T_2 - 11 T_1^2 T_2 + 14 T_2^2 - 44 T_1 T_2^2 + T_1^2 T_2^2 + 9 T_1^3 T_2^2 - 8 T_2^3 + 10 T_1 T_2^3 + \\
& \quad 33 T_1^2 T_2^3 - 18 T_1^3 T_2^3 + 8 T_1 T_2^4 - 22 T_1^2 T_2^4 + 9 T_1^3 T_2^4) p_{2,j} p_{3,i} x_{2,j} x_{3,i}) / ((-1 + T_1) (-1 + T_2) \\
& \quad (-1 + 2 T_2) (-1 + T_1 T_2)) + \frac{(T_1 + 4 T_2 - 11 T_1 T_2 - 4 T_1 T_2^2 + 9 T_1^2 T_2^2) p_{2,i} p_{3,j} x_{2,j} x_{3,i}}{(-1 + T_1) (-1 + 2 T_2)} - \\
& \frac{(-1 - 3 T_2 + 12 T_1 T_2 - T_1^2 T_2 + 6 T_2^2 - 11 T_1 T_2^2 - 7 T_1^2 T_2^2 - 4 T_1 T_2^3 + 9 T_1^2 T_2^3) p_{2,j} p_{3,j} x_{2,j} x_{3,i}}{(-1 + T_1) (-1 + T_2) (-1 + 2 T_2)} - \\
& \frac{1}{2 (-1 + T_1) (-1 + 2 T_2)} p_{3,i} p_{3,j} x_{3,i}^2 + \\
& \frac{(-1 + T_1 T_2) (-2 + 2 T_1 + 4 T_2 - 7 T_1 T_2 + 2 T_1^2 T_2 - 2 T_1 T_2^2 + 18 T_1^2 T_2^2 - 9 T_1^3 T_2^2 + 8 T_1 T_2^3 - 22 T_1^2 T_2^3 + 9 T_1^3 T_2^3)}{p_{3,j}^2 x_{3,i}^2} + \frac{(-4 + 9 T_1) p_{3,j} x_{3,j}}{T_1} - \\
& \frac{(-4 + 3 T_1 - T_1^2 + 12 T_2 - 17 T_1 T_2 + 12 T_1^2 T_2 - 4 T_1 T_2^2 + 2 T_1^2 T_2^2 - 9 T_1^3 T_2^2 - 4 T_1^2 T_2^3 + 9 T_1^3 T_2^3) p_{1,i} p_{3,j} x_{1,i} x_{3,j}}{(-1 + T_1) (-1 + 2 T_2) (-1 + T_1 T_2)} + \\
& - \frac{(2 - 4 T_1 + T_1^2) p_{1,j} p_{3,j} x_{1,i} x_{3,j}}{-1 + T_1 T_2} + \frac{(-1 - T_2 + T_1 T_2) p_{2,i} p_{3,j} x_{2,i} x_{3,j}}{(-1 + T_2) (-1 + T_1 T_2)} + \\
& \frac{(-1 + T_2) (1 - T_1 - 2 T_2 + T_1^2 T_2 - 8 T_2^2 + 26 T_1 T_2^2 - 11 T_1^2 T_2^2 + 8 T_1 T_2^3 - 22 T_1^2 T_2^3 + 9 T_1^3 T_2^3) p_{2,j} p_{3,j} x_{2,i} x_{3,j}}{(-1 + T_1) (-1 + 2 T_2) (-1 + T_1 T_2)} + \\
& \frac{(-3 + 2 T_1 - 2 T_2 + 18 T_1 T_2 - 9 T_1^2 T_2 + 8 T_2^2 - 22 T_1 T_2^2 + 9 T_1^2 T_2^2) p_{3,i} p_{3,j} x_{3,i} x_{3,j}}{(-1 + T_1) (-1 + 2 T_2)} - \\
& \frac{1}{2 (-1 + T_1) (-1 + 2 T_2)} (-5 + 4 T_1 + 2 T_2 + 11 T_1 T_2 - 7 T_1^2 T_2 + \\
& \quad 8 T_2^2 - 24 T_1 T_2^2 + 27 T_1^2 T_2^2 - 9 T_1^3 T_2^2 + 8 T_1 T_2^3 - 22 T_1^2 T_2^3 + 9 T_1^3 T_2^3) p_{3,j}^2 x_{3,i} x_{3,j}
\end{aligned}$$

In[\*]:= **r1m / . Sub // CF**

Out[\*]=

$$\frac{3}{2} + 2 p_{1,i} x_{1,i} - \frac{2 (-2 + T_1^2) p_{1,j} x_{1,i}}{T_1^2} + p_{1,i} p_{1,j} x_{1,i}^2 - \frac{(-1 + T_1) (1 + 2 T_1) p_{1,j}^2 x_{1,i}^2}{2 T_1^2} -$$

$$\begin{aligned}
& \frac{4 p_{1,j} x_{1,j}}{T_1} + p_{1,i} p_{1,j} x_{1,i} x_{1,j} - \frac{(1 + 3 T_1) p_{1,j}^2 x_{1,i} x_{1,j}}{2 T_1} + 2 p_{2,i} x_{2,i} - 2 p_{2,j} x_{2,i} - \\
& \frac{(-2 + 3 T_1 - 4 T_2 + 16 T_1 T_2 - 9 T_1^2 T_2 + 8 T_2^2 - 22 T_1 T_2^2 + 9 T_1^2 T_2^2) p_{1,j} p_{2,i} x_{1,i} x_{2,i}}{T_1 (-1 + 2 T_2)} - \\
& \frac{(-1 - T_1 + 8 T_2 - 10 T_1 T_2 + 9 T_1^2 T_2 - 8 T_2^2 + 15 T_1 T_2^2 - 18 T_1^2 T_2^2 - 4 T_1 T_2^3 + 9 T_1^2 T_2^3) p_{1,i} p_{2,j} x_{1,i} x_{2,i}}{(-1 + T_1) T_2 (-1 + 2 T_2)} + \\
& \frac{1}{T_1 T_2 (-1 + 2 T_2)} (-4 T_1 + 8 T_2 - 14 T_1 T_2 + 18 T_1^2 T_2 - 12 T_2^2 + 35 T_1 T_2^2 - 36 T_1^2 T_2^2 + 8 T_2^3 - 26 T_1 T_2^3 + 18 T_1^2 T_2^3) \\
& p_{1,j} p_{2,j} x_{1,i} x_{2,i} - \frac{(-4 + 3 T_1 + 16 T_1 T_2 - 9 T_1^2 T_2 + 8 T_2^2 - 22 T_1 T_2^2 + 9 T_1^2 T_2^2) p_{1,j} p_{2,i} x_{1,j} x_{2,i}}{(-1 + T_1) (-1 + 2 T_2)} + \\
& \frac{(-1 + T_2) (2 - 4 T_1 + 4 T_2 - 13 T_1 T_2 + 9 T_1^2 T_2) p_{1,j} p_{2,j} x_{1,j} x_{2,i}}{(-1 + T_1) T_2} + \\
& \frac{(1 + T_2) p_{2,i} p_{2,j} x_{2,i}^2}{T_2} - \frac{(-1 + T_2) (3 + 2 T_2) p_{2,j}^2 x_{2,i}^2}{2 T_2^2} - 2 p_{1,i} p_{2,j} x_{1,i} x_{2,j} + \\
& \frac{(2 + T_1 - 8 T_2 + 11 T_1 T_2 - 9 T_1^2 T_2 - 4 T_1 T_2^2 + 9 T_1^2 T_2^2) p_{1,j} p_{2,j} x_{1,i} x_{2,j}}{T_1 (-1 + 2 T_2)} + p_{2,i} p_{2,j} x_{2,i} x_{2,j} - \\
& \frac{3 (1 + T_2) p_{2,j}^2 x_{2,i} x_{2,j}}{2 T_2} - \frac{(T_1 + 4 T_2 - 11 T_1 T_2 - 4 T_1 T_2^2 + 9 T_1^2 T_2^2) p_{1,j} p_{2,i} x_{3,i}}{2 T_1 (-1 + 2 T_2)} - \\
& \frac{p_{1,i} p_{2,j} x_{3,i}}{2 T_2} + \frac{(-T_1 + 3 T_1 T_2 + 4 T_2^2 - 11 T_1 T_2^2 - 4 T_1 T_2^3 + 9 T_1^2 T_2^3) p_{1,j} p_{2,j} x_{3,i}}{2 T_1 T_2 (-1 + 2 T_2)} + 2 p_{3,i} x_{3,i} - \\
& \frac{(-4 + 9 T_1 - T_1^2 + 8 T_2 - 18 T_1 T_2 + T_1^2 T_2 - 4 T_1 T_2^2 + 15 T_1^2 T_2^2 - 9 T_1^3 T_2^2 - 4 T_1^2 T_2^3 + 9 T_1^3 T_2^3) p_{3,j} x_{3,i}}{T_1^2 T_2 (-1 + 2 T_2)} - \\
& \frac{(4 - 4 T_1 + T_1^2 + 4 T_2 - 11 T_1 T_2 - 4 T_1 T_2^2 + 9 T_1^2 T_2^2) p_{1,j} p_{3,i} x_{1,i} x_{3,i}}{T_1 (-1 + T_1 T_2)} + \\
& \frac{(-2 T_1 + 4 T_2 - 7 T_1 T_2 + 10 T_1^2 T_2 - 4 T_1 T_2^2 + 2 T_1^2 T_2^2 - 9 T_1^3 T_2^2 - 4 T_1^2 T_2^3 + 9 T_1^3 T_2^3) p_{1,i} p_{3,j} x_{1,i} x_{3,i}}{(-1 + T_1) T_1 T_2 (-1 + 2 T_2)} - \\
& \frac{(2 - 6 T_1 + T_1^2 + T_1 T_2 + 8 T_1^2 T_2 - 4 T_1 T_2^2 + 2 T_1^2 T_2^2 - 9 T_1^3 T_2^2 - 4 T_1^2 T_2^3 + 9 T_1^3 T_2^3) p_{1,j} p_{3,j} x_{1,i} x_{3,i}}{T_1^2 T_2 (-1 + 2 T_2)} - \\
& \frac{(6 - 4 T_1 + T_1^2 + 4 T_2 - 13 T_1 T_2 - 4 T_1 T_2^2 + 9 T_1^2 T_2^2) p_{1,j} p_{3,i} x_{1,j} x_{3,i}}{(-1 + T_1) (-1 + T_1 T_2)} - \frac{p_{1,i} p_{3,j} x_{1,j} x_{3,i}}{T_2} + \\
& \frac{(-4 + 6 T_1 - 2 T_1^2 + 8 T_2 - 11 T_1 T_2 + 4 T_1^2 T_2 + 4 T_2^2 - 11 T_1 T_2^2 - 4 T_1 T_2^3 + 9 T_1^2 T_2^3) p_{1,j} p_{3,j} x_{1,j} x_{3,i}}{(-1 + T_1) T_1 T_2 (-1 + 2 T_2)} + \\
& \left( (-2 + T_1 - 3 T_2 + 22 T_1 T_2 - 10 T_1^2 T_2 + 14 T_2^2 - 42 T_1 T_2^2 - T_1^2 T_2^2 + 9 T_1^3 T_2^2 - 8 T_2^3 + 10 T_1 T_2^3 + 33 T_1^2 T_2^3 - 18 T_1^3 T_2^3 \right. \\
& \left. + 8 T_1 T_2^4 - 22 T_1^2 T_2^4 + 9 T_1^3 T_2^4) p_{2,j} p_{3,i} x_{2,i} x_{3,i} \right) / \left( (-1 + T_1) T_2 (-1 + 2 T_2) (-1 + T_1 T_2) \right) + \\
& \left( (-2 + T_1) (T_1 + 3 T_2 - 12 T_1 T_2 - 6 T_2^2 + 17 T_1 T_2^2 + 9 T_1^2 T_2^2 + 4 T_2^3 - 3 T_1 T_2^3 - 18 T_1^2 T_2^3 - 4 T_1 T_2^4 + 9 T_1^2 T_2^4) \right. \\
& \left. p_{2,i} p_{3,j} x_{2,i} x_{3,i} \right) / \left( (-1 + T_1) T_1 (-1 + 2 T_2) T_2 (-1 + 2 T_2) \right) - \frac{1}{(-1 + T_1) T_1 T_2^2}
\end{aligned}$$

$$\begin{aligned}
 & \left( (1 + T_1 - T_1^2 + 5 T_2 - 24 T_1 T_2 + 11 T_1^2 T_2 - 8 T_2^2 + 18 T_1 T_2^2 + 11 T_1^2 T_2^2 - 9 T_1^3 T_2^2 + 8 T_1 T_2^3 - 22 T_1^2 T_2^3 + 9 T_1^3 T_2^3) \right. \\
 & \quad p_{2,j} p_{3,j} x_{2,i} x_{3,i} + \left( (-3 + 2 T_1 - T_2 + 21 T_1 T_2 - 11 T_1^2 T_2 + 14 T_2^2 - 44 T_1 T_2^2 + T_1^2 T_2^2 + 9 T_1^3 T_2^2 - \right. \\
 & \quad \quad \left. 8 T_2^3 + 10 T_1 T_2^3 + 33 T_1^2 T_2^3 - 18 T_1^3 T_2^3 + 8 T_1 T_2^4 - 22 T_1^2 T_2^4 + 9 T_1^3 T_2^4) p_{2,j} p_{3,i} x_{2,j} x_{3,i} \right) / \\
 & \quad \left( (-1 + T_1) (-1 + T_2) (-1 + 2 T_2) (-1 + T_1 T_2) \right) - \\
 & \quad \frac{(T_1 + 4 T_2 - 11 T_1 T_2 - 4 T_1 T_2^2 + 9 T_1^2 T_2^2) p_{2,i} p_{3,j} x_{2,j} x_{3,i}}{(-1 + T_1) T_1 (-1 + 2 T_2)} - \frac{1}{(-1 + T_1) T_1 T_2 (-1 + 2 T_2)} \\
 & \quad \left( (1 + 6 T_2 - 25 T_1 T_2 + 10 T_1^2 T_2 - 12 T_2^2 + 29 T_1 T_2^2 + 11 T_1^2 T_2^2 - 9 T_1^3 T_2^2 + 12 T_1 T_2^3 - 31 T_1^2 T_2^3 + 9 T_1^3 T_2^3) \right. \\
 & \quad \quad p_{2,j} p_{3,j} x_{2,j} x_{3,i} - \\
 & \quad \left( (2 - T_1 + 4 T_2 - 23 T_1 T_2 + 11 T_1^2 T_2 - 8 T_2^2 + 20 T_1 T_2^2 + 9 T_1^2 T_2^2 - 9 T_1^3 T_2^2 + 8 T_1 T_2^3 - 22 T_1^2 T_2^3 + 9 T_1^3 T_2^3) \right. \\
 & \quad \quad \left. p_{3,i} p_{3,j} x_{3,i}^2 \right) / \left( (-1 + T_1) T_1 T_2 (-1 + 2 T_2) \right) + \\
 & \quad \left( (-1 + T_1 T_2) (1 + 6 T_2 - 28 T_1 T_2 + 13 T_1^2 T_2 - 8 T_2^2 + 18 T_1 T_2^2 + 27 T_1^2 T_2^2 - 18 T_1^3 T_2^2 + 16 T_1 T_2^3 - \right. \\
 & \quad \quad \left. 44 T_1^2 T_2^3 + 18 T_1^3 T_2^3) p_{3,j}^2 x_{3,i}^2 \right) / \left( 2 (-1 + T_1) T_1^2 T_2^2 (-1 + 2 T_2) \right) - \frac{(-4 + 9 T_1) p_{3,j} x_{3,j}}{T_1} + \\
 & \quad \frac{(-4 + 3 T_1 - T_1^2 + 12 T_2 - 17 T_1 T_2 + 12 T_1^2 T_2 - 4 T_1 T_2^2 + 2 T_1^2 T_2^2 - 9 T_1^3 T_2^2 - 4 T_1^2 T_2^3 + 9 T_1^3 T_2^3) p_{1,i} p_{3,j} x_{1,i} x_{3,j}}{(-1 + T_1) (-1 + 2 T_2) (-1 + T_1 T_2)} \\
 & \quad \frac{(2 + T_1 - 8 T_2 + 11 T_1 T_2 - 9 T_1^2 T_2 - 4 T_1 T_2^2 + 9 T_1^2 T_2^2) p_{1,j} p_{3,j} x_{1,i} x_{3,j}}{T_1 (-1 + 2 T_2)} - \\
 & \quad \frac{(-1 - T_2 + T_1 T_2) p_{2,i} p_{3,j} x_{2,i} x_{3,j}}{(-1 + T_2) (-1 + T_1 T_2)} - \frac{(-2 + T_1) (1 + 2 T_2 - 9 T_1 T_2 - 4 T_2^2 + 9 T_1 T_2^2) p_{2,j} p_{3,j} x_{2,i} x_{3,j}}{(-1 + T_1) (-1 + 2 T_2)} - \\
 & \quad \frac{(-3 + 2 T_1 - 2 T_2 + 18 T_1 T_2 - 9 T_1^2 T_2 + 8 T_2^2 - 22 T_1 T_2^2 + 9 T_1^2 T_2^2) p_{3,i} p_{3,j} x_{3,i} x_{3,j}}{(-1 + T_1) (-1 + 2 T_2)} + \\
 & \quad \frac{\left( (1 + 6 T_2 - 31 T_1 T_2 + 15 T_1^2 T_2 - 8 T_2^2 + 16 T_1 T_2^2 + 45 T_1^2 T_2^2 - 27 T_1^3 T_2^2 + 24 T_1 T_2^3 - 66 T_1^2 T_2^3 + 27 T_1^3 T_2^3) \right. \\
 & \quad \quad \left. p_{3,j}^2 x_{3,i} x_{3,j} \right) / \left( 2 (-1 + T_1) T_1 T_2 (-1 + 2 T_2) \right)}{(-1 + T_1) (-1 + 2 T_2)}
 \end{aligned}$$

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In[*]:= r0[1, i_, j_] := 2 (-T1 + T2 + T1 T2) p3,j x1,i x2,i - 2 (-1 + T2) p3,j x1,j x2,i - 2 p3,j x1,i x2,j
(*from r0p*)
r0[-1, i_, j_] := - 2 p3,j x1,i x2,i / T1 + 2 (-1 + T2) p3,j x1,j x2,i / T2 + 2 p3,j x1,i x2,j / T1
r1[1, i_, j_] :=
- 3/2 - 2 p1,i x1,i - 2 p1,j x1,i - p1,i p1,j x1,i^2 - 1/2 (-1 + T1) (2 + T1) p1,j^2 x1,i^2 + 4 p1,j x1,j / T1 -
p1,i p1,j x1,i x1,j + 1/2 (3 + T1) p1,j^2 x1,i x1,j - 2 p2,i x2,i + 2 p2,j x2,i - 2 p1,j p2,i x1,i x2,i +
(1 - 3 T1 + 2 T2 - 4 T1 T2 + 9 T1^2 T2 - 4 T2^2 + 11 T1 T2^2 - 18 T1^2 T2^2 - 4 T1 T2^3 + 9 T1^2 T2^3) p1,i p2,j x1,i x2,i /
(-1 + T1) (-1 + 2 T2) +
(1 + 4 T1 - 6 T2 + 9 T1 T2 - 9 T1^2 T2 - 4 T1 T2^2 + 9 T1^2 T2^2) p1,j p2,j x1,i x2,i +
(-4 + 3 T1 + 16 T1 T2 - 9 T1^2 T2 + 8 T2^2 - 22 T1 T2^2 + 9 T1^2 T2^2) p1,j p2,i x1,j x2,i /
(-1 + T1) (-1 + 2 T2) -
(-1 + T2) (2 + T1 - 11 T1 T2 - 4 T1 T2^2 + 9 T1^2 T2^2) p1,j p2,j x1,j x2,i /
(-1 + T1) (-1 + 2 T2)

```

$$\begin{aligned}
 & 2 p_{2,i} p_{2,j} x_{2,i}^2 - \frac{1}{2} (-1 + T_2) (4 + T_2) p_{2,j}^2 x_{2,i}^2 + 2 p_{1,i} p_{2,j} x_{1,i} x_{2,j} - \\
 & \frac{(3 T_1 - 4 T_2 + 7 T_1 T_2 - 9 T_1^2 T_2 - 4 T_1 T_2^2 + 9 T_1^2 T_2^2) p_{1,j} p_{2,j} x_{1,i} x_{2,j}}{-1 + 2 T_2} - p_{2,i} p_{2,j} x_{2,i} x_{2,j} + \\
 & \frac{1}{2} (5 + T_2) p_{2,j}^2 x_{2,i} x_{2,j} + \frac{(T_1 + 4 T_2 - 11 T_1 T_2 - 4 T_1 T_2^2 + 9 T_1^2 T_2^2) p_{1,j} p_{2,i} x_{3,i}}{2 (-1 + 2 T_2)} + \\
 & \frac{1}{2} p_{1,i} p_{2,j} x_{3,i} - \frac{(-1 + T_1 + 6 T_2 - 11 T_1 T_2 - 4 T_1 T_2^2 + 9 T_1^2 T_2^2) p_{1,j} p_{2,j} x_{3,i}}{2 (-1 + 2 T_2)} - \\
 & 2 p_{3,i} x_{3,i} + p_{3,j} x_{3,i} + 2 p_{1,j} p_{3,i} x_{1,i} x_{3,i} - 2 p_{1,i} p_{3,j} x_{1,i} x_{3,i} - \\
 & \frac{(-3 T_1 + 4 T_2 - 5 T_1 T_2 + 10 T_1^2 T_2 - 4 T_1 T_2^2 + 2 T_1^2 T_2^2 - 9 T_1^3 T_2^2 - 4 T_1^2 T_2^3 + 9 T_1^3 T_2^3) p_{1,j} p_{3,j} x_{1,i} x_{3,i}}{-1 + 2 T_2} + \\
 & \frac{(6 - 4 T_1 + T_1^2 + 4 T_2 - 13 T_1 T_2 - 4 T_1 T_2^2 + 9 T_1^2 T_2^2) p_{1,j} p_{3,i} x_{1,j} x_{3,i}}{(-1 + T_1) (-1 + T_1 T_2)} + p_{1,i} p_{3,j} x_{1,j} x_{3,i} + \\
 & \frac{(-3 + T_1 - T_1^2 + 6 T_2 - 4 T_1 T_2 + 12 T_1^2 T_2 - 7 T_1^2 T_2^2 - 9 T_1^3 T_2^2 - 4 T_1^2 T_2^3 + 9 T_1^3 T_2^3) p_{1,j} p_{3,j} x_{1,j} x_{3,i}}{(-1 + T_1) (-1 + 2 T_2)} + \\
 & p_{2,j} p_{3,i} x_{2,i} x_{3,i} - p_{2,i} p_{3,j} x_{2,i} x_{3,i} - \\
 & \frac{1}{(-1 + T_1) (-1 + 2 T_2)} (-1 + T_1 + 5 T_2 - 4 T_1 T_2 + 2 T_2^2 - 20 T_1 T_2^2 + 10 T_1^2 T_2^2 - 8 T_2^3 + \\
 & 18 T_1 T_2^3 + 11 T_1^2 T_2^3 - 9 T_1^3 T_2^3 + 8 T_1 T_2^4 - 22 T_1^2 T_2^4 + 9 T_1^3 T_2^4) p_{2,j} p_{3,j} x_{2,i} x_{3,i} - \\
 & \left( (-3 + 2 T_1 - T_2 + 21 T_1 T_2 - 11 T_1^2 T_2 + 14 T_2^2 - 44 T_1 T_2^2 + T_1^2 T_2^2 + 9 T_1^3 T_2^2 - 8 T_2^3 + 10 T_1 T_2^3 + \right. \\
 & \left. 33 T_1^2 T_2^3 - 18 T_1^3 T_2^3 + 8 T_1 T_2^4 - 22 T_1^2 T_2^4 + 9 T_1^3 T_2^4) p_{2,j} p_{3,i} x_{2,j} x_{3,i} \right) / ((-1 + T_1) (-1 + T_2) \\
 & (-1 + 2 T_2) (-1 + T_1 T_2)) + \frac{(T_1 + 4 T_2 - 11 T_1 T_2 - 4 T_1 T_2^2 + 9 T_1^2 T_2^2) p_{2,i} p_{3,j} x_{2,j} x_{3,i}}{(-1 + T_1) (-1 + 2 T_2)} - \\
 & \frac{(-1 - 3 T_2 + 12 T_1 T_2 - T_1^2 T_2 + 6 T_2^2 - 11 T_1 T_2^2 - 7 T_1^2 T_2^2 - 4 T_1 T_2^3 + 9 T_1^2 T_2^3) p_{2,j} p_{3,j} x_{2,j} x_{3,i}}{(-1 + T_1) (-1 + T_2) (-1 + 2 T_2)} - \\
 & p_{3,i} p_{3,j} x_{3,i}^2 + \frac{1}{2 (-1 + T_1) (-1 + 2 T_2)} (-1 + T_1 T_2) \\
 & (-2 + 2 T_1 + 4 T_2 - 7 T_1 T_2 + 2 T_1^2 T_2 - 2 T_1 T_2^2 + 18 T_1^2 T_2^2 - 9 T_1^3 T_2^2 + 8 T_1 T_2^3 - 22 T_1^2 T_2^3 + 9 T_1^3 T_2^3) \\
 & p_{3,j}^2 x_{3,i}^2 + \frac{(-4 + 9 T_1) p_{3,j} x_{3,j}}{T_1} - \\
 & \left( (-4 + 3 T_1 - T_1^2 + 12 T_2 - 17 T_1 T_2 + 12 T_1^2 T_2 - 4 T_1 T_2^2 + 2 T_1^2 T_2^2 - 9 T_1^3 T_2^2 - 4 T_1^2 T_2^3 + 9 T_1^3 T_2^3) \right. \\
 & \left. p_{1,i} p_{3,j} x_{1,i} x_{3,j} \right) / ((-1 + T_1) (-1 + 2 T_2) (-1 + T_1 T_2)) - \\
 & \frac{(2 - 4 T_1 + T_1^2) p_{1,j} p_{3,j} x_{1,i} x_{3,j}}{-1 + T_1 T_2} + \frac{(-1 - T_2 + T_1 T_2) p_{2,i} p_{3,j} x_{2,i} x_{3,j}}{(-1 + T_2) (-1 + T_1 T_2)} + \\
 & \left( (-1 + T_2) (1 - T_1 - 2 T_2 + T_1^2 T_2 - 8 T_2^2 + 26 T_1 T_2^2 - 11 T_1^2 T_2^2 + 8 T_1 T_2^3 - 22 T_1^2 T_2^3 + 9 T_1^3 T_2^3) \right. \\
 & \left. p_{2,j} p_{3,j} x_{2,i} x_{3,j} \right) / ((-1 + T_1) (-1 + 2 T_2) (-1 + T_1 T_2)) + \\
 & \frac{(-3 + 2 T_1 - 2 T_2 + 18 T_1 T_2 - 9 T_1^2 T_2 + 8 T_2^2 - 22 T_1 T_2^2 + 9 T_1^2 T_2^2) p_{3,i} p_{3,j} x_{3,i} x_{3,j}}{(-1 + T_1) (-1 + 2 T_2)} -
 \end{aligned}$$

$$\begin{aligned}
 & \frac{1}{2(-1+T_1)(-1+2T_2)} \left( -5+4T_1+2T_2+11T_1T_2-7T_1^2T_2+8T_2^2- \right. \\
 & \quad \left. 24T_1T_2^2+27T_1^2T_2^2-9T_1^3T_2^2+8T_1T_2^3-22T_1^2T_2^3+9T_1^3T_2^3 \right) p_{3,j}^2 x_{3,i} x_{3,j} \\
 r_1[-1, i_-, j_-] := & \frac{3}{2} + 2p_{1,i} x_{1,i} - \frac{2(-2+T_1^2)p_{1,j} x_{1,i}}{T_1^2} + p_{1,i} p_{1,j} x_{1,i}^2 - \frac{(-1+T_1)(1+2T_1)p_{1,j}^2 x_{1,i}^2}{2T_1^2} - \\
 & \frac{4p_{1,j} x_{1,j}}{T_1} + p_{1,i} p_{1,j} x_{1,i} x_{1,j} - \frac{(1+3T_1)p_{1,j}^2 x_{1,i} x_{1,j}}{2T_1} + 2p_{2,i} x_{2,i} - 2p_{2,j} x_{2,i} - \\
 & \frac{(-2+3T_1-4T_2+16T_1T_2-9T_1^2T_2+8T_2^2-22T_1T_2^2+9T_1^2T_2^2)p_{1,j} p_{2,i} x_{1,i} x_{2,i}}{T_1(-1+2T_2)} - \\
 & \frac{(-1-T_1+8T_2-10T_1T_2+9T_1^2T_2-8T_2^2+15T_1T_2^2-18T_1^2T_2^2-4T_1T_2^3+9T_1^2T_2^3)p_{1,i} p_{2,j} x_{1,i} x_{2,i}}{(-1+T_1)T_2(-1+2T_2)} + \\
 & \frac{1}{T_1T_2(-1+2T_2)} \\
 & \quad \left( -4T_1+8T_2-14T_1T_2+18T_1^2T_2-12T_2^2+35T_1T_2^2-36T_1^2T_2^2+8T_2^3-26T_1T_2^3+18T_1^2T_2^3 \right) p_{1,j} \\
 & \quad p_{2,j} x_{1,i} x_{2,i} - \frac{(-4+3T_1+16T_1T_2-9T_1^2T_2+8T_2^2-22T_1T_2^2+9T_1^2T_2^2)p_{1,j} p_{2,i} x_{1,j} x_{2,i}}{(-1+T_1)(-1+2T_2)} + \\
 & \frac{(-1+T_2)(2-4T_1+4T_2-13T_1T_2+9T_1^2T_2)p_{1,j} p_{2,j} x_{1,j} x_{2,i}}{(-1+T_1)T_2} + \frac{(1+T_2)p_{2,i} p_{2,j} x_{2,i}^2}{T_2} - \\
 & \frac{(-1+T_2)(3+2T_2)p_{2,j}^2 x_{2,i}^2}{2T_2^2} - 2p_{1,i} p_{2,j} x_{1,i} x_{2,j} + \\
 & \frac{(2+T_1-8T_2+11T_1T_2-9T_1^2T_2-4T_1T_2^2+9T_1^2T_2^2)p_{1,j} p_{2,j} x_{1,i} x_{2,j}}{T_1(-1+2T_2)} + p_{2,i} p_{2,j} x_{2,i} x_{2,j} - \\
 & \frac{3(1+T_2)p_{2,j}^2 x_{2,i} x_{2,j}}{2T_2} - \frac{(T_1+4T_2-11T_1T_2-4T_1T_2^2+9T_1^2T_2^2)p_{1,j} p_{2,i} x_{3,i}}{2T_1(-1+2T_2)} - \\
 & \frac{p_{1,i} p_{2,j} x_{3,i}}{2T_2} + \frac{(-T_1+3T_1T_2+4T_2^2-11T_1T_2^2-4T_1T_2^3+9T_1^2T_2^3)p_{1,j} p_{2,j} x_{3,i}}{2T_1T_2(-1+2T_2)} + 2p_{3,i} x_{3,i} - \\
 & \frac{(-4+9T_1-T_1^2+8T_2-18T_1T_2+T_1^2T_2-4T_1T_2^2+15T_1^2T_2^2-9T_1^3T_2^2-4T_1^2T_2^3+9T_1^3T_2^3)p_{3,j} x_{3,i}}{T_1^2T_2(-1+2T_2)} - \\
 & \frac{(4-4T_1+T_1^2+4T_2-11T_1T_2-4T_1T_2^2+9T_1^2T_2^2)p_{1,j} p_{3,i} x_{1,i} x_{3,i}}{T_1(-1+T_1T_2)} + \\
 & \frac{(-2T_1+4T_2-7T_1T_2+10T_1^2T_2-4T_1T_2^2+2T_1^2T_2^2-9T_1^3T_2^2-4T_1^2T_2^3+9T_1^3T_2^3)p_{1,i} p_{3,j} x_{1,i} x_{3,i}}{(-1+T_1)T_1T_2(-1+2T_2)} - \\
 & \frac{(2-6T_1+T_1^2+T_1T_2+8T_1^2T_2-4T_1T_2^2+2T_1^2T_2^2-9T_1^3T_2^2-4T_1^2T_2^3+9T_1^3T_2^3)p_{1,j} p_{3,j} x_{1,i} x_{3,i}}{T_1^2T_2(-1+2T_2)} - \\
 & \frac{(6-4T_1+T_1^2+4T_2-13T_1T_2-4T_1T_2^2+9T_1^2T_2^2)p_{1,j} p_{3,i} x_{1,j} x_{3,i}}{(-1+T_1)(-1+T_1T_2)} - \frac{p_{1,i} p_{3,j} x_{1,j} x_{3,i}}{T_2} +
 \end{aligned}$$

$$\begin{aligned}
& \frac{(-4 + 6 T_1 - 2 T_1^2 + 8 T_2 - 11 T_1 T_2 + 4 T_1^2 T_2 + 4 T_2^2 - 11 T_1 T_2^2 - 4 T_1 T_2^3 + 9 T_1^2 T_2^3) p_{1,j} p_{3,j} x_{1,j} x_{3,i}}{(-1 + T_1) T_1 T_2 (-1 + 2 T_2)} + \\
& \left( (-2 + T_1 - 3 T_2 + 22 T_1 T_2 - 10 T_1^2 T_2 + 14 T_2^2 - 42 T_1 T_2^2 - T_1^2 T_2^2 + \right. \\
& \quad \left. 9 T_1^3 T_2^2 - 8 T_2^3 + 10 T_1 T_2^3 + 33 T_1^2 T_2^3 - 18 T_1^3 T_2^3 + 8 T_1 T_2^4 - 22 T_1^2 T_2^4 + 9 T_1^3 T_2^4) \right. \\
& \quad \left. p_{2,j} p_{3,i} x_{2,i} x_{3,i} \right) / ((-1 + T_1) T_2 (-1 + 2 T_2) (-1 + T_1 T_2)) + \\
& \left( (-2 + T_1) (T_1 + 3 T_2 - 12 T_1 T_2 - 6 T_2^2 + 17 T_1 T_2^2 + 9 T_1^2 T_2^2 + 4 T_2^3 - 3 T_1 T_2^3 - 18 T_1^2 T_2^3 - 4 T_1 T_2^4 + 9 T_1^2 T_2^4) \right. \\
& \quad \left. p_{2,i} p_{3,j} x_{2,i} x_{3,i} \right) / ((-1 + T_1) T_1 (-1 + T_2) T_2 (-1 + 2 T_2)) - \\
& \frac{1}{(-1 + T_1) T_1 T_2^2} (1 + T_1 - T_1^2 + 5 T_2 - 24 T_1 T_2 + 11 T_1^2 T_2 - 8 T_2^2 + 18 T_1 T_2^2 + 11 T_1^2 T_2^2 - \\
& \quad 9 T_1^3 T_2^2 + 8 T_1 T_2^3 - 22 T_1^2 T_2^3 + 9 T_1^3 T_2^3) p_{2,j} p_{3,j} x_{2,i} x_{3,i} + \\
& \left( (-3 + 2 T_1 - T_2 + 21 T_1 T_2 - 11 T_1^2 T_2 + 14 T_2^2 - 44 T_1 T_2^2 + T_1^2 T_2^2 + 9 T_1^3 T_2^2 - 8 T_2^3 + 10 T_1 T_2^3 + \right. \\
& \quad \left. 33 T_1^2 T_2^3 - 18 T_1^3 T_2^3 + 8 T_1 T_2^4 - 22 T_1^2 T_2^4 + 9 T_1^3 T_2^4) p_{2,j} p_{3,i} x_{2,j} x_{3,i} \right) / ((-1 + T_1) (-1 + T_2) \\
& \quad (-1 + 2 T_2) (-1 + T_1 T_2)) - \frac{(T_1 + 4 T_2 - 11 T_1 T_2 - 4 T_1 T_2^2 + 9 T_1^2 T_2^2) p_{2,i} p_{3,j} x_{2,j} x_{3,i}}{(-1 + T_1) T_1 (-1 + 2 T_2)} - \\
& \frac{1}{(-1 + T_1) T_1 T_2 (-1 + 2 T_2)} (1 + 6 T_2 - 25 T_1 T_2 + 10 T_1^2 T_2 - 12 T_2^2 + 29 T_1 T_2^2 + \\
& \quad 11 T_1^2 T_2^2 - 9 T_1^3 T_2^2 + 12 T_1 T_2^3 - 31 T_1^2 T_2^3 + 9 T_1^3 T_2^3) p_{2,j} p_{3,j} x_{2,j} x_{3,i} - \\
& \left( (2 - T_1 + 4 T_2 - 23 T_1 T_2 + 11 T_1^2 T_2 - 8 T_2^2 + 20 T_1 T_2^2 + 9 T_1^2 T_2^2 - 9 T_1^3 T_2^2 + 8 T_1 T_2^3 - 22 T_1^2 T_2^3 + 9 T_1^3 T_2^3) \right. \\
& \quad \left. p_{3,i} p_{3,j} x_{3,i}^2 \right) / ((-1 + T_1) T_1 T_2 (-1 + 2 T_2)) + \\
& \left( (-1 + T_1 T_2) (1 + 6 T_2 - 28 T_1 T_2 + 13 T_1^2 T_2 - 8 T_2^2 + 18 T_1 T_2^2 + 27 T_1^2 T_2^2 - 18 T_1^3 T_2^2 + 16 T_1 T_2^3 - \right. \\
& \quad \left. 44 T_1^2 T_2^3 + 18 T_1^3 T_2^3) p_{3,j}^2 x_{3,i}^2 \right) / (2 (-1 + T_1) T_1^2 T_2^2 (-1 + 2 T_2)) - \frac{(-4 + 9 T_1) p_{3,j} x_{3,j}}{T_1} + \\
& \left( (-4 + 3 T_1 - T_1^2 + 12 T_2 - 17 T_1 T_2 + 12 T_1^2 T_2 - 4 T_1 T_2^2 + 2 T_1^2 T_2^2 - 9 T_1^3 T_2^2 - 4 T_1^2 T_2^3 + 9 T_1^3 T_2^3) \right. \\
& \quad \left. p_{1,i} p_{3,j} x_{1,i} x_{3,j} \right) / ((-1 + T_1) (-1 + 2 T_2) (-1 + T_1 T_2)) - \\
& \frac{(2 + T_1 - 8 T_2 + 11 T_1 T_2 - 9 T_1^2 T_2 - 4 T_1 T_2^2 + 9 T_1^2 T_2^2) p_{1,j} p_{3,j} x_{1,i} x_{3,j}}{T_1 (-1 + 2 T_2)} - \\
& \frac{(-1 - T_2 + T_1 T_2) p_{2,i} p_{3,j} x_{2,i} x_{3,j}}{(-1 + T_2) (-1 + T_1 T_2)} - \\
& \frac{(-2 + T_1) (1 + 2 T_2 - 9 T_1 T_2 - 4 T_2^2 + 9 T_1 T_2^2) p_{2,j} p_{3,j} x_{2,i} x_{3,j}}{(-1 + T_1) (-1 + 2 T_2)} - \\
& \frac{(-3 + 2 T_1 - 2 T_2 + 18 T_1 T_2 - 9 T_1^2 T_2 + 8 T_2^2 - 22 T_1 T_2^2 + 9 T_1^2 T_2^2) p_{3,i} p_{3,j} x_{3,i} x_{3,j}}{(-1 + T_1) (-1 + 2 T_2)} + \\
& \left( (1 + 6 T_2 - 31 T_1 T_2 + 15 T_1^2 T_2 - 8 T_2^2 + 16 T_1 T_2^2 + 45 T_1^2 T_2^2 - 27 T_1^3 T_2^2 + 24 T_1 T_2^3 - 66 T_1^2 T_2^3 + 27 T_1^3 T_2^3) \right. \\
& \quad \left. p_{3,j}^2 x_{3,i} x_{3,j} \right) / (2 (-1 + T_1) T_1 T_2 (-1 + 2 T_2))
\end{aligned}$$

```

In[*]:= T3 = T1 T2;
S = {x_, p_};
q[s_, i_, j_] :=
  Sum[x $\alpha$ , i (p $\alpha$ , i - p $\alpha$ , i+1) + x $\alpha$ , j (p $\alpha$ , j - p $\alpha$ , j+1) + x $\alpha$ , i ((1 - T $\alpha^S$ ) p $\alpha$ , i+1 + (T $\alpha^S$  - 1) p $\alpha$ , j+1), { $\alpha$ , 3}];
 $\gamma_1[\varphi, k] := \varphi (3/2 - x_{1,k} p_{1,k} - x_{2,k} p_{2,k} - x_{3,k} p_{3,k})$ ;
 $\mathcal{L}[X_{i,j}[S]] := T_3^S \mathbb{E}[-q[S, i, j] + r_0[S, i, j] + \epsilon r_1[S, i, j] + O[\epsilon]^2]$ ;
 $\mathcal{L}[C_{k_\varphi}] :=$ 
  T $_3^\varphi \mathbb{E}[-x_{1,k} (p_{1,k} - p_{1,k+1}) - x_{2,k} (p_{2,k} - p_{2,k+1}) - x_{3,k} (p_{3,k} - p_{3,k+1}) + \epsilon \gamma_1[\varphi, k] + O[\epsilon]^2]$ ;
 $\mathcal{L}[K_] := (2\pi)^{-\text{Features}[K][[1]]} \text{CF}[\mathcal{L}/@ \text{Features}[K][[2]]]$ ;
vs $_i := \text{Sequence}[p_{1,i}, x_{1,i}, p_{2,i}, x_{2,i}, p_{3,i}, x_{3,i}]$ ;
vs[K_] := Union@@Table[{vs $_i$ }, {i, Features[K][[1]]}]

```

```

In[*]:= Features[Knot[3, 1]]

```

```

Out[*]=
Features[7, C4[-1] X2,6[-1] X5,1[-1] X7,3[-1]]

```

```

In[*]:= FKnot31nullm =

```

```

Features[16, X16,14[1] X13,11[1] X10,8[1] C15[1] C12[1] C9[1] C4[-1] X2,6[-1] X5,1[-1] X7,3[-1]]

```

```

Out[*]=

```

```

Features[16, C4[-1] C9[1] C12[1] C15[1] X2,6[-1] X5,1[-1] X7,3[-1] X10,8[1] X13,11[1] X16,14[1]]

```

```

In[*]:= Short[ $\mathcal{L}/@ \text{FKnot31nullm}[[2]]$ , 10]

```

```

Out[*]//Short=

```

$$\begin{aligned}
& T_1^2 T_2^2 \mathbb{E} \left[ \epsilon \text{Series} \left[ \langle\langle 1 \rangle\rangle, 3 - \frac{4 p_{1,1} x_{1,1}}{T_1} + 2 p_{1,2} x_{1,2} - \frac{2(-2 + T_1^2) p_{1,6} x_{1,2}}{T_1^2} + \langle\langle 441 \rangle\rangle + \right. \right. \\
& \quad \left. \frac{(-3 + 2 T_1 - 2 T_2 + 18 T_1 T_2 - 9 T_1^2 T_2 + 8 T_2^2 - 22 T_1 T_2^2 + 9 T_1^2 T_2^2) p_{3,14} p_{3,16} x_{3,14} x_{3,16}}{(-1 + T_1)(-1 + 2 T_2)} + \right. \\
& \quad \left. \frac{1}{2(-1 + T_1)(-1 + 2 T_2)} (-1 + T_1 T_2) (-2 + 2 T_1 + 4 T_2 - 7 T_1 T_2 + 2 T_1^2 T_2 - 2 T_1 T_2^2 + \right. \\
& \quad \left. \left. 18 T_1^2 T_2^2 - 9 T_1^3 T_2^2 + 8 T_1 T_2^3 - 22 T_1^2 T_2^3 + 9 T_1^3 T_2^3) p_{3,14}^2 x_{3,16}^2 - p_{3,14} p_{3,16} x_{3,16}^2 \right] \right]
\end{aligned}$$

```

In[*]:= Short[ $\mathcal{L}[\text{Knot}[3, 1]]$ , 10]

```

```

In[*]:= vs[Knot[3, 1]]

```

```

Out[*]=

```

```

{p1,1, p1,2, p1,3, p1,4, p1,5, p1,6, p1,7, p2,1, p2,2, p2,3, p2,4, p2,5, p2,6,
 p2,7, p3,1, p3,2, p3,3, p3,4, p3,5, p3,6, p3,7, x1,1, x1,2, x1,3, x1,4, x1,5, x1,6, x1,7,
 x2,1, x2,2, x2,3, x2,4, x2,5, x2,6, x2,7, x3,1, x3,2, x3,3, x3,4, x3,5, x3,6, x3,7}

```

```

In[*]:= FKnot31nullm[[1]]

```

```

Out[*]=

```

```

16

```



$$\text{In[*]:= } \int \text{CF}[\mathcal{L} / @ \text{FKnot31nullm}[[2]]] \text{ d Union} @@ \text{Table}[\{\text{vs}_i\}, \{i, \text{FKnot31nullm}[[1]]\}]$$

Out[\*]=

$$\begin{aligned} & (281474976710656 \pi^{48} T_1^2 T_2^2 \\ & \mathbb{E}[\text{eSeries}[0, (-7 + 21 T_1 - 29 T_1^2 + 23 T_1^3 - 9 T_1^4 + 35 T_2 - 90 T_1 T_2 + 107 T_1^2 T_2 - 66 T_1^3 T_2 + \\ & 9 T_1^4 T_2 + 12 T_1^5 T_2 - 79 T_2^2 + 157 T_1 T_2^2 - 139 T_1^2 T_2^2 + 30 T_1^3 T_2^2 + 52 T_1^4 T_2^2 - 38 T_1^5 T_2^2 - \\ & 7 T_1^6 T_2^2 + 109 T_2^3 - 142 T_1 T_2^3 + 56 T_1^2 T_2^3 + 75 T_1^3 T_2^3 - 73 T_1^4 T_2^3 - 28 T_1^5 T_2^3 + 70 T_1^6 T_2^3 - \\ & 9 T_1^7 T_2^3 - 91 T_2^4 + 19 T_1 T_2^4 + 91 T_1^2 T_2^4 - 108 T_1^3 T_2^4 - 83 T_1^4 T_2^4 + 232 T_1^5 T_2^4 - 167 T_1^6 T_2^4 + \\ & 12 T_1^7 T_2^4 + 3 T_1^8 T_2^4 + 46 T_2^5 + 78 T_1 T_2^5 - 97 T_1^2 T_2^5 - 58 T_1^3 T_2^5 + 334 T_1^4 T_2^5 - 376 T_1^5 T_2^5 + \\ & 160 T_1^6 T_2^5 + 22 T_1^7 T_2^5 - 9 T_1^8 T_2^5 - 8 T_2^6 - 84 T_1 T_2^6 + 4 T_1^2 T_2^6 + 196 T_1^3 T_2^6 - 370 T_1^4 T_2^6 + \\ & 270 T_1^5 T_2^6 - 36 T_1^6 T_2^6 - 58 T_1^7 T_2^6 + 7 T_1^8 T_2^6 + 24 T_1 T_2^7 + 72 T_1^2 T_2^7 - 170 T_1^3 T_2^7 + 172 T_1^4 T_2^7 - \\ & 30 T_1^5 T_2^7 - 85 T_1^6 T_2^7 + 61 T_1^7 T_2^7 + T_1^8 T_2^7 - 32 T_1^2 T_2^8 + 22 T_1^3 T_2^8 + 30 T_1^4 T_2^8 - 88 T_1^5 T_2^8 + 78 T_1^6 T_2^8 - \\ & 21 T_1^7 T_2^8 - 9 T_1^8 T_2^8 + 16 T_1^3 T_2^9 - 30 T_1^4 T_2^9 + 30 T_1^5 T_2^9 - 10 T_1^6 T_2^9 - 6 T_1^7 T_2^9 + 6 T_1^8 T_2^9) / \\ & ((1 - T_1 + T_1^2)^2 (-1 + 2 T_2) (1 - T_2 + T_2^2)^2 (1 - T_1 T_2 + T_1^2 T_2^2)^2)]] / \\ & ((1 - T_1 + T_1^2) (1 - T_2 + T_2^2) (1 - T_1 T_2 + T_1^2 T_2^2)) \end{aligned}$$

$$\text{In[*]:= } \int \mathcal{L}[\text{Knot}[3, 1]] \text{ d vs}[\text{Knot}[3, 1]]$$

Out[\*]=

$$\begin{aligned} & - ((16384 i \pi^{14} T_1^2 T_2^2 \\ & \mathbb{E}[\text{eSeries}[0, (-7 + 21 T_1 - 29 T_1^2 + 23 T_1^3 - 9 T_1^4 + 35 T_2 - 90 T_1 T_2 + 107 T_1^2 T_2 - 66 T_1^3 T_2 + \\ & 9 T_1^4 T_2 + 12 T_1^5 T_2 - 79 T_2^2 + 157 T_1 T_2^2 - 139 T_1^2 T_2^2 + 30 T_1^3 T_2^2 + 52 T_1^4 T_2^2 - 38 T_1^5 T_2^2 - \\ & 7 T_1^6 T_2^2 + 109 T_2^3 - 142 T_1 T_2^3 + 56 T_1^2 T_2^3 + 75 T_1^3 T_2^3 - 73 T_1^4 T_2^3 - 28 T_1^5 T_2^3 + 70 T_1^6 T_2^3 - \\ & 9 T_1^7 T_2^3 - 91 T_2^4 + 19 T_1 T_2^4 + 91 T_1^2 T_2^4 - 108 T_1^3 T_2^4 - 83 T_1^4 T_2^4 + 232 T_1^5 T_2^4 - 167 T_1^6 T_2^4 + \\ & 12 T_1^7 T_2^4 + 3 T_1^8 T_2^4 + 46 T_2^5 + 78 T_1 T_2^5 - 97 T_1^2 T_2^5 - 58 T_1^3 T_2^5 + 334 T_1^4 T_2^5 - 376 T_1^5 T_2^5 + \\ & 160 T_1^6 T_2^5 + 22 T_1^7 T_2^5 - 9 T_1^8 T_2^5 - 8 T_2^6 - 84 T_1 T_2^6 + 4 T_1^2 T_2^6 + 196 T_1^3 T_2^6 - 370 T_1^4 T_2^6 + \\ & 270 T_1^5 T_2^6 - 36 T_1^6 T_2^6 - 58 T_1^7 T_2^6 + 7 T_1^8 T_2^6 + 24 T_1 T_2^7 + 72 T_1^2 T_2^7 - 170 T_1^3 T_2^7 + 172 T_1^4 T_2^7 - \\ & 30 T_1^5 T_2^7 - 85 T_1^6 T_2^7 + 61 T_1^7 T_2^7 + T_1^8 T_2^7 - 32 T_1^2 T_2^8 + 22 T_1^3 T_2^8 + 30 T_1^4 T_2^8 - 88 T_1^5 T_2^8 + 78 T_1^6 T_2^8 - \\ & 21 T_1^7 T_2^8 - 9 T_1^8 T_2^8 + 16 T_1^3 T_2^9 - 30 T_1^4 T_2^9 + 30 T_1^5 T_2^9 - 10 T_1^6 T_2^9 - 6 T_1^7 T_2^9 + 6 T_1^8 T_2^9) / \\ & ((1 - T_1 + T_1^2)^2 (-1 + 2 T_2) (1 - T_2 + T_2^2)^2 (1 - T_1 T_2 + T_1^2 T_2^2)^2)]] / \\ & ((1 - T_1 + T_1^2) (1 - T_2 + T_2^2) (1 - T_1 T_2 + T_1^2 T_2^2)) \end{aligned}$$

$$\text{In[*]:= } \text{Timing} \left[ \int \mathcal{L}[\text{Knot}[4, 1]] \text{ d vs}[\text{Knot}[4, 1]] \right]$$

Out[\*]=

$$\begin{aligned} & \{582.469, \\ & (1048576 \pi^{20} T_1^2 T_2^2 \mathbb{E}[\text{eSeries}[0, (12 - 17 T_1 + 6 T_1^2 - 57 T_2 + 96 T_1 T_2 - 42 T_1^2 T_2 + 54 T_2^2 - 54 T_1 T_2^2 - \\ & 48 T_1^2 T_2^2 + 54 T_1^3 T_2^2 - 6 T_1^4 T_2^2 - 12 T_2^3 - 24 T_1 T_2^3 + 102 T_1^2 T_2^3 - 72 T_1^3 T_2^3 + 9 T_1^4 T_2^3 + 8 T_1 T_2^4 - \\ & 18 T_1^2 T_2^4 + 9 T_1^3 T_2^4) / ((1 - 3 T_1 + T_1^2) (1 - 3 T_2 + T_2^2) (1 - 3 T_1 T_2 + T_1^2 T_2^2))]] / \\ & ((1 - 3 T_1 + T_1^2) (1 - 3 T_2 + T_2^2) (1 - 3 T_1 T_2 + T_1^2 T_2^2)) \} \end{aligned}$$

$$\text{In[*]:= } \text{Features}[\text{Knot}[3, 1]]$$

Out[\*]=

$$\text{Features}[7, C_4[-1] X_{2,6}[-1] X_{5,1}[-1] X_{7,3}[-1]]$$

And now the mutants!

```
In[*]:= KTKnot = Knot[11, NonAlternating, 42];
ConwayKnot = Knot[11, NonAlternating, 34];
Alexander /@ {KTKnot, ConwayKnot}

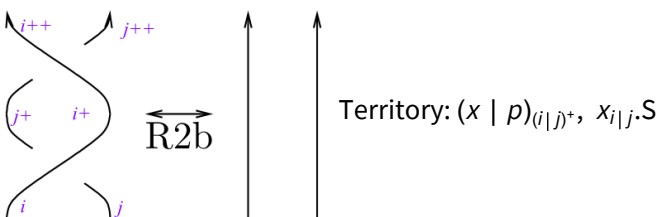
Out[*]=
{1 &, 1 &}

In[*]:= A2Conway = Timing [ ∫ ℒ[ConwayKnot] dℓ vs[ConwayKnot] ]

Out[*]=
$Aborted

A2KT = Timing [ ∫ ℒ[KTKnot] dℓ vs[KTKnot] ]
```

### Reidemeister 2b



```
In[*]:= {lhs2b} = Cases [
∫ ℰ[Sum[π_{α,i} p_{α,i} + π_{α,j} p_{α,j}, {α, 3}]] ℒ /@ (X_{i,j}[1] X_{i+1,j+1}[-1]) dℓ {vs_i, vs_j, vs_{i+1}, vs_{j+1}},
eSeries[ε_-, ℱ_-] ⇒ ℱ, ∞]

Out[*]=
```

$$\left\{ 2 c_1 + c_5 + c_7 + c_{10} + \dots 356 \dots + \frac{(d_{77} + c_{77} T_1^2 T_2^2) p_{3,2+i}^2 \pi_{3,j}^2}{T_1^2 T_2^2} + \frac{(-2 d_{77} + 2 d_{77} T_1 T_2 + d_{80} T_1 T_2 + c_{80} T_1^2 T_2^2) p_{3,2+i} p_{3,2+j} \pi_{3,j}^2}{T_1^2 T_2^2} + \frac{(d_{77} - 2 d_{77} T_1 T_2 - d_{80} T_1 T_2 + c_{85} T_1^2 T_2^2 + d_{77} T_1^2 T_2^2 + d_{80} T_1^2 T_2^2 + d_{85} T_1^2 T_2^2) p_{3,2+j}^2 \pi_{3,j}^2}{T_1^2 T_2^2} \right\}$$

large output    show less    show more    show all    set size limit...

```
In[*]:= {rhs2b} = Cases [ ∫ ℰ[Sum[π_{α,i} p_{α,i} + π_{α,j} p_{α,j}, {α, 3}]]
ℒ /@ (C_i[0] C_{i+1}[0] C_j[0] C_{j+1}[0]) dℓ {vs_i, vs_j, vs_{i+1}, vs_{j+1}}, eSeries[ε_-, ℱ_-] ⇒ ℱ, ∞ ]

Out[*]=
{0}
```

```
In[*]:= eqn2b =
CF[CF[CF[lhs2b - rhs2b] /. {$A $C → α_1, $A $D → α_2, $B $C → α_3, $B $D → α_4}] /. {$A | $B → 0}]

In[*]:= cvs2b = Union@Cases[eqn2b, p_ | π_., ∞];

In[*]:= eqns2b = CoefficientRules[eqn2b, cvs2b] /. (_ → c_) ⇒ (c == 0);
```

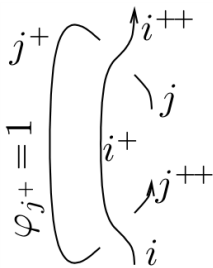
```

In[*]:= vars2b = Union@Cases[eqn2b, d_, ∞];
In[*]:= eqns2b // Column
In[*]:= {sol2b} = Solve[eqns2b, vars2b]
In[*]:= Cases[sol2b, α_, ∞]
Out[*]=
{α4, α1, α3, α3, α3, α4, α1, α3, α1, α3, α1, α2, α2, α3, α4, α4, α4, α4, α1, α3, α4}

In[*]:= eqn2b /. sol2b // CF
Out[*]=
0
    
```

## Verification of Invariance Under Reidemeister 3b

### Invariance Under R2c



```

In[*]:= lhs2c = ∫ ℰ [Sum[π_{α,i} p_{α,i} + π_{α,j} p_{α,j}, {α, 3}]]
           ℒ /@ (X_{i+1,j}[1] X_{i,j+2}[-1] C_{j+1}[1]) d[{v_{S_i}, v_{S_j}, v_{S_{i+1}}, v_{S_{j+1}}, v_{S_{j+2}}}]

In[*]:= rhs2c = ∫ ℰ [Sum[π_{α,i} p_{α,i} + π_{α,j} p_{α,j}, {α, 3}]]
           ℒ /@ (C_i[0] C_{i+1}[0] C_j[0] C_{j+1}[1] C_{j+2}[0]) d[{v_{S_i}, v_{S_j}, v_{S_{i+1}}, v_{S_{j+1}}, v_{S_{j+2}}}]

Out[*]=
-32768 i π^{15} T_1 T_2 ℰ [Series[p_{1,2+i} π_{1,i} + p_{1,3+j} π_{1,j} + p_{2,2+i} π_{2,i} + p_{2,3+j} π_{2,j} + p_{3,2+i} π_{3,i} + p_{3,3+j} π_{3,j},
  - 3/2 - p_{1,3+j} π_{1,j} - p_{2,3+j} π_{2,j} - p_{3,3+j} π_{3,j}]]

In[*]:= Cases[Expand[rhs2c], $A $C, ∞]
Out[*]=
{}

In[*]:= eqn2c =
CF[CF[CF[Cases[lhs2c, eSeries[ε_, ℱ_] := ℱ, ∞] - Cases[rhs2c, eSeries[ε_, ℱ_] := ℱ, ∞]] /.
  {$A $C → α_1, $A $D → α_2, $B $C → α_3, $B $D → α_4}] /. {$A | $B → 0}]

In[*]:= Cases[eqn2c, α_, ∞]
Out[*]=
{α_1, α_3, α_4, α_4, α_1, α_3, α_4, α_1, α_3, α_1, α_3, α_1, α_2, α_2, α_3, α_4, α_4}
    
```

```

In[*]:= eqn2c /. sol2b // CF

In[*]:= eqn2cred = eqn2c /. sol2b /. sol3b /. sol3bα /. σα3 // CF

Out[*]=

$$\left\{ \frac{(1 + c_5) (-1 + T_1) p_{1,3+j} \pi_{1,i}}{T_1} + \frac{(1 + c_{50}) (-1 + T_2) p_{2,3+j} \pi_{2,i}}{T_2} + \right.$$


$$\left. \left( (-1 + T_1 T_2) \left( -c_9 c_{22} + c_{10} c_{22} + c_{22} c_{28} + c_{22} T_1 + 2 c_9 c_{22} T_1 - 2 c_{10} c_{22} T_1 - c_{22} c_{28} T_1 + c_{22} c_{79} T_1 - \right. \right. \right.$$


$$c_{22} T_1^2 - c_9 c_{22} T_1^2 + c_{10} c_{22} T_1^2 - c_{22} c_{79} T_1^2 - c_{10} c_{22} T_2 - c_{22} c_{28} T_2 - c_{22} T_1 T_2 + 2 c_{10} c_{22} T_1 T_2 +$$


$$c_{22} c_{28} T_1 T_2 - c_{22} c_{79} T_1 T_2 + c_{22} T_1^2 T_2 - c_{10} c_{22} T_1^2 T_2 + c_{22} c_{79} T_1^2 T_2 + \alpha_4 + c_9 \alpha_4 - c_{10} \alpha_4 -$$


$$c_{28} \alpha_4 + c_{79} \alpha_4 - T_1 \alpha_4 - c_9 T_1 \alpha_4 + c_{10} T_1 \alpha_4 - c_{79} T_1 \alpha_4 + c_{10} T_2 \alpha_4 + c_{28} T_2 \alpha_4 - c_{10} T_1 T_2 \alpha_4 \left. \right) p_{3,3+j} \pi_{3,i} \Big/ \left( (-1 + T_1) T_1 T_2 (-c_{22} T_1 + c_{22} T_1 T_2 - \alpha_4) \right) \Big\}$$


In[*]:= cvs2c = Union@Cases[eqn2cred, p__ | π__, ∞]

Out[*]=
{p1,3+j, p2,3+j, p3,3+j, π1,i, π2,i, π3,i}

In[*]:= {eqns2c} = CoefficientRules[eqn2cred, cvs2c] /. (_ -> c_) :-> (c == 0)

In[*]:= vars2b = Union@Cases[eqn2cred, c_, ∞]

Out[*]=
{c5, c9, c10, c22, c28, c50, c79}

In[*]:= {sol2c} = Solve[eqns2c, {c5, c50, c79}] // Simplify

Out[*]=

$$\left\{ \left\{ c_5 \rightarrow -1, c_{50} \rightarrow -1, \right. \right.$$


$$c_{79} \rightarrow \left( -c_{22} c_{28} - c_{22} T_1 + c_{22} c_{28} T_1 + c_{22} T_1^2 + c_{22} c_{28} T_2 + c_{22} T_1 T_2 - c_{22} c_{28} T_1 T_2 - c_{22} T_1^2 T_2 - \right.$$


$$\alpha_4 + c_{28} \alpha_4 + T_1 \alpha_4 - c_{28} T_2 \alpha_4 + c_9 (-1 + T_1) (c_{22} (-1 + T_1) + \alpha_4) +$$

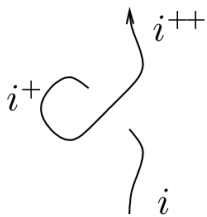

$$c_{10} (-1 + T_1) (-1 + T_2) (c_{22} (-1 + T_1) + \alpha_4) \Big/ \left( (-1 + T_1) (c_{22} T_1 (-1 + T_2) - \alpha_4) \right) \Big\}$$


In[*]:= eqn2c /. sol2b /. sol3b /. sol3bα /. σα3 /. sol2c // CF

Out[*]=
{0}

```

## Invariance Under R1I



```

In[*]:= {lhs11} = Cases[

$$\int \mathbb{E}[\text{Sum}[\pi_{\alpha,i} p_{\alpha,i}, \{\alpha, 3\}]] \mathcal{L} / @ (X_{i+2,i}[1] C_{i+1}[1]) d\{\mathbf{vs}_i, \mathbf{vs}_{i+1}, \mathbf{vs}_{i+2}\},$$

eSeries[ε_, ℱ_] :-> ℱ, ∞]

```

$$\text{In[*]:= rhs11} = \int \mathbb{E}[\text{Sum}[\pi_{\alpha,i} p_{\alpha,i}, \{\alpha, 3\}]] \mathcal{L} / @ (C_i[0] C_{i+1}[0] C_{i+2}[0]) \mathbb{d}\{v_{s_i}, v_{s_{i+1}}, v_{s_{i+2}}\}$$

Out[\*]=

$$-512 i \pi^9 \mathbb{E}[\text{Series}[p_{1,3+i} \pi_{1,i} + p_{2,3+i} \pi_{2,i} + p_{3,3+i} \pi_{3,i}, 0]]$$

$$\text{In[*]:= eqn11} = \text{CF}[\text{CF}[\text{CF}[\text{lhs11}]] /. \{\$A \$C \rightarrow \alpha_1, \$A \$D \rightarrow \alpha_2, \$B \$C \rightarrow \alpha_3, \$B \$D \rightarrow \alpha_4\}] /. \{\$A | \$B \rightarrow 0\};$$

$$\text{In[*]:= eqn11red} = \text{eqn11} /. \text{sol2b} /. \text{sol3b} /. \text{sol3ba} /. \text{sa3} /. \text{sol2c} // \text{CF}$$

Out[\*]=

$$\frac{1}{2} (3 + 2 c_{88})$$

$$\text{In[*]:= \{sol11\} = Solve[eqn11red == 0, c_{88}]$$

Out[\*]=

$$\left\{ \left\{ c_{88} \rightarrow -\frac{3}{2} \right\} \right\}$$

### Invariance Under R1r



$$\text{In[*]:= \{lhs1r\} = Cases\left[\int \mathbb{E}[\text{Sum}[\pi_{\alpha,i} p_{\alpha,i}, \{\alpha, 3\}]] \mathcal{L} / @ (X_{i,i+2}[1] C_{i+1}[-1]) \mathbb{d}\{v_{s_i}, v_{s_{i+1}}, v_{s_{i+2}}\}, \text{Series}[\mathcal{E}_-, \mathcal{F}_-] \Rightarrow \mathcal{F}, \infty\right]$$

$$\text{In[*]:= rhs1r} = \int \mathbb{E}[\text{Sum}[\pi_{\alpha,i} p_{\alpha,i}, \{\alpha, 3\}]] \mathcal{L} / @ (C_i[0] C_{i+1}[0] C_{i+2}[0]) \mathbb{d}\{v_{s_i}, v_{s_{i+1}}, v_{s_{i+2}}\}$$

Out[\*]=

$$-512 i \pi^9 \mathbb{E}[\text{Series}[p_{1,3+i} \pi_{1,i} + p_{2,3+i} \pi_{2,i} + p_{3,3+i} \pi_{3,i}, 0]]$$

$$\text{In[*]:= eqn1r} = \text{CF}[\text{CF}[\text{CF}[\text{lhs1r}]] /. \{\$A \$C \rightarrow \alpha_1, \$A \$D \rightarrow \alpha_2, \$B \$C \rightarrow \alpha_3, \$B \$D \rightarrow \alpha_4\}] /. \{\$A | \$B \rightarrow 0\};$$

```
In[*]:= eqn1rred = eqn1r /. sol2b /. sol3b /. sol3ba /. sa3 /. sol2c /. sol11 // CF
```

Out[\*]=

$$\begin{aligned} & (C_9 C_{22} - C_{10} C_{22} - C_{22} C_{28} - C_9 C_{22} T_1 + C_{10} C_{22} T_1 + C_{22}^2 T_1 - C_{22} C_{81} T_1 - C_{22} C_{86} T_1 - C_{22} C_{56} T_1^2 - C_{22} C_{69} T_1^2 - \\ & C_9 C_{22} T_2 + 2 C_{10} C_{22} T_2 + 2 C_{22} C_{28} T_2 + C_9 C_{22} T_1 T_2 - 2 C_{10} C_{22} T_1 T_2 - C_{15} C_{22} T_1 T_2 - 2 C_{22}^2 T_1 T_2 - \\ & C_{22} C_{36} T_1 T_2 + 2 C_{22} C_{81} T_1 T_2 + 2 C_{22} C_{86} T_1 T_2 - 3 C_{22} T_1^2 T_2 + C_{15} C_{22} T_1^2 T_2 + 3 C_{22} C_{56} T_1^2 T_2 + \\ & 2 C_{22} C_{69} T_1^2 T_2 + C_{22} C_{81} T_1^2 T_2 - C_{10} C_{22} T_2^2 - C_{22} C_{28} T_2^2 + C_{10} C_{22} T_1 T_2^2 + 2 C_{15} C_{22} T_1 T_2^2 + C_{22}^2 T_1 T_2^2 + \\ & 2 C_{22} C_{36} T_1 T_2^2 - C_{22} C_{81} T_1 T_2^2 - C_{22} C_{86} T_1 T_2^2 + 6 C_{22} T_1^2 T_2^2 - 2 C_{15} C_{22} T_1^2 T_2^2 - 3 C_{22} C_{56} T_1^2 T_2^2 - \\ & C_{22} C_{69} T_1^2 T_2^2 - 2 C_{22} C_{81} T_1^2 T_2^2 - C_{15} C_{22} T_1 T_2^3 - C_{22} C_{36} T_1 T_2^3 - 3 C_{22} T_1^2 T_2^3 + C_{15} C_{22} T_1^2 T_2^3 + C_{22} C_{56} T_1^2 T_2^3 + \\ & C_{22} C_{81} T_1^2 T_2^3 + C_{22} \alpha_4 - C_{81} \alpha_4 - C_{86} \alpha_4 + C_{22} T_1 \alpha_4 - C_{56} T_1 \alpha_4 - C_{69} T_1 \alpha_4 + C_9 T_2 \alpha_4 - C_{10} T_2 \alpha_4 - C_{15} T_2 \alpha_4 - \\ & C_{22} T_2 \alpha_4 - C_{28} T_2 \alpha_4 - C_{36} T_2 \alpha_4 + C_{81} T_2 \alpha_4 + C_{86} T_2 \alpha_4 - 3 T_1 T_2 \alpha_4 - C_9 T_1 T_2 \alpha_4 + C_{10} T_1 T_2 \alpha_4 + \\ & C_{15} T_1 T_2 \alpha_4 - C_{22} T_1 T_2 \alpha_4 + 2 C_{56} T_1 T_2 \alpha_4 + C_{69} T_1 T_2 \alpha_4 + C_{81} T_1 T_2 \alpha_4 + C_{10} T_2^2 \alpha_4 + C_{15} T_2^2 \alpha_4 + \\ & C_{28} T_2^2 \alpha_4 + C_{36} T_2^2 \alpha_4 + 3 T_1 T_2^2 \alpha_4 - C_{10} T_1 T_2^2 \alpha_4 - C_{15} T_1 T_2^2 \alpha_4 - C_{56} T_1 T_2^2 \alpha_4 - C_{81} T_1 T_2^2 \alpha_4 + \alpha_4^2) / \\ & (T_1 (-1 + T_2) T_2 (-C_{22} T_1 + C_{22} T_1 T_2 - \alpha_4)) \end{aligned}$$

```
In[*]:= {sol1r} = Solve[eqn1rred == 0, {c9}] // Simplify
```

Out[\*]=

$$\left\{ \left\{ c_9 \rightarrow \left( -C_{22}^2 T_1 (-1 + T_2)^2 + \right. \right. \right.$$

$$\left. \left. \left( C_{56} T_1 + C_{69} T_1 - C_{86} (-1 + T_2) + C_{15} T_2 + C_{28} T_2 + C_{36} T_2 + 3 T_1 T_2 - C_{15} T_1 T_2 - 2 C_{56} T_1 T_2 - C_{69} T_1 T_2 - \right. \right. \right.$$

$$\left. \left. C_{15} T_2^2 - C_{28} T_2^2 - C_{36} T_2^2 - 3 T_1 T_2^2 + C_{15} T_1 T_2^2 + C_{56} T_1 T_2^2 + C_{81} (-1 + T_2) (-1 + T_1 T_2) - \alpha_4 \right) \alpha_4 + \right.$$

$$\left. C_{22} (-1 + T_2) \left( -C_{86} T_1 - C_{56} T_1^2 - C_{69} T_1^2 + C_{28} (-1 + T_2) - C_{15} T_1 T_2 - C_{36} T_1 T_2 + \right. \right.$$

$$\left. C_{86} T_1 T_2 - 3 T_1^2 T_2 + C_{15} T_1^2 T_2 + 2 C_{56} T_1^2 T_2 + C_{69} T_1^2 T_2 + C_{15} T_1 T_2^2 + C_{36} T_1 T_2^2 + \right.$$

$$\left. 3 T_1^2 T_2^2 - C_{15} T_1^2 T_2^2 - C_{56} T_1^2 T_2^2 - C_{81} T_1 (-1 + T_2) (-1 + T_1 T_2) + \alpha_4 + T_1 \alpha_4 \right) - \left. \right.$$

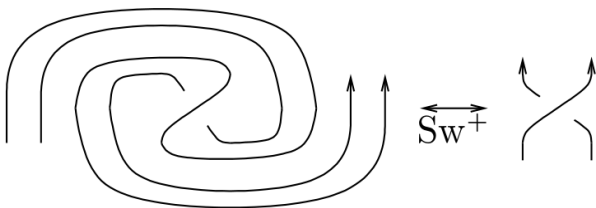
$$\left. C_{10} (-1 + T_1) (-1 + T_2) (C_{22} (-1 + T_2) - T_2 \alpha_4) \right) / \left( (-1 + T_1) (C_{22} (-1 + T_2) - T_2 \alpha_4) \right) \left. \right\}$$

```
In[*]:= eqn1r /. sol2b /. sol3b /. sol3ba /. sa3 /. sol2c /. sol11 /. sol1r // CF
```

Out[\*]=

0

### Invariance Under Sw



$$\text{In[*]:= lhssw} = \int \mathbb{E}[\text{Sum}[\pi_{\alpha,i} p_{\alpha,i} + \pi_{\alpha,j} p_{\alpha,j}, \{\alpha, 3\}]]$$

$$\mathcal{L} / @ (\mathbf{X}_{i+1,j+1} [1] \mathbf{C}_i [-1] \mathbf{C}_j [-1] \mathbf{C}_{i+2} [1] \mathbf{C}_{j+2} [1]) \mathcal{d} \{\mathbf{v}_{S_i}, \mathbf{v}_{S_j}, \mathbf{v}_{S_{i+1}}, \mathbf{v}_{S_{j+1}}, \mathbf{v}_{S_{i+2}}, \mathbf{v}_{S_{j+2}}\}$$

Out[\*]=

262144  $\pi^{18}$   $T_1 T_2$

$$\mathbb{E} \left[ \text{Series} \left[ T_1 p_{1,3+i} \pi_{1,i} + (1 - T_1) p_{1,3+j} \pi_{1,i} + p_{1,3+j} \pi_{1,j} + T_2 p_{2,3+i} \pi_{2,i} + \dots 8 \dots + \right. \right.$$

$$\left. T_1 T_2 p_{3,3+i} \pi_{3,i} + (1 - T_1 T_2) p_{3,3+j} \pi_{3,i} + p_{3,3+j} \pi_{3,j}, 2 C_1 + C_5 + \dots 367 \dots + \dots 1 \dots \right]$$

large output    show less    show more    show all    set size limit...

$$\text{In[*]:= rhssw} = \int \mathbb{E}[\text{Sum}[\pi_{\alpha,i} p_{\alpha,i} + \pi_{\alpha,j} p_{\alpha,j}, \{\alpha, 3\}]]$$

$$\mathcal{L} / @ (\mathbf{X}_{i+1,j+1} [1] \mathbf{C}_i [0] \mathbf{C}_j [0] \mathbf{C}_{i+2} [0] \mathbf{C}_{j+2} [0]) \mathcal{d} \{\mathbf{v}_{S_i}, \mathbf{v}_{S_j}, \mathbf{v}_{S_{i+1}}, \mathbf{v}_{S_{j+1}}, \mathbf{v}_{S_{i+2}}, \mathbf{v}_{S_{j+2}}\}$$

Out[\*]=

262144  $\pi^{18}$   $T_1 T_2$

$$\mathbb{E} \left[ \text{Series} \left[ T_1 p_{1,3+i} \pi_{1,i} + (1 - T_1) p_{1,3+j} \pi_{1,i} + p_{1,3+j} \pi_{1,j} + T_2 p_{2,3+i} \pi_{2,i} + \dots 8 \dots + \right. \right.$$

$$\left. T_1 T_2 p_{3,3+i} \pi_{3,i} + (1 - T_1 T_2) p_{3,3+j} \pi_{3,i} + p_{3,3+j} \pi_{3,j}, 2 C_1 + C_5 + \dots 367 \dots + \dots 1 \dots \right]$$

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$$\text{In[*]:= eqnsw} = \text{CF}[\text{CF}[\text{CF}[\text{Cases}[\text{lhssw}, \text{eSeries}[\mathcal{E}_-, \mathcal{F}_-] \Rightarrow \mathcal{F}, \infty] - \text{Cases}[\text{rhssw}, \text{eSeries}[\mathcal{E}_-, \mathcal{F}_-] \Rightarrow \mathcal{F}, \infty]]]]$$

Out[\*]=

$$\left\{ \$A p_{3,3+j} \pi_{1,i} \pi_{2,i} - \frac{\$A p_{3,3+j} \pi_{1,j} \pi_{2,i}}{T_1} - \frac{\$B T_2 p_{3,3+j} \pi_{1,j} \pi_{2,i}}{T_1} + \$B p_{3,3+j} \pi_{1,i} \pi_{2,j} \right\}$$

(\*single \$A and \$B never arise so sw equation holds.\*)

$$\text{In[*]:= eqnsw} = \text{CF}[\text{CF}[\text{CF}[\text{Cases}[\text{lhssw}, \text{eSeries}[\mathcal{E}_-, \mathcal{F}_-] \Rightarrow \mathcal{F}, \infty] - \text{Cases}[\text{rhssw}, \text{eSeries}[\mathcal{E}_-, \mathcal{F}_-] \Rightarrow \mathcal{F}, \infty]]] /. \{ \$A \$C \rightarrow \alpha_1, \$A \$D \rightarrow \alpha_2, \$B \$C \rightarrow \alpha_3, \$B \$D \rightarrow \alpha_4 \} /. \{ \$A | \$B \rightarrow 0 \}]$$

Out[\*]=

$$\{0\}$$

In conclusion: here are the values for the R-matrix we found:

$$\text{In[*]:= } \{r_{0p}, r_{0m}\} = \{r_{0[1, i, j]}, r_{0[-1, i, j]}\} /. \text{sol2b} /. \text{sol3b} /. \text{sol3ba} /. \alpha_3 /. \text{sol2c} /. \text{sol11} /. \text{sol1r} // \text{CF}$$

Out[\*]=

$$\left\{ \frac{\$B (-C_{22} T_1 + C_{22} T_1 T_2 - T_2 \alpha_4) p_{3,j} x_{1,i} x_{2,i}}{\alpha_4} - \frac{\$B C_{22} (-1 + T_2) p_{3,j} x_{1,j} x_{2,i}}{\alpha_4} + \$B p_{3,j} x_{1,i} x_{2,j}, \right.$$

$$\left. - \frac{\$B (-C_{22} + C_{22} T_2 - \alpha_4) p_{3,j} x_{1,i} x_{2,i}}{T_1 T_2 \alpha_4} + \frac{\$B C_{22} (-1 + T_2) p_{3,j} x_{1,j} x_{2,i}}{T_2 \alpha_4} - \frac{\$B p_{3,j} x_{1,i} x_{2,j}}{T_1} \right\}$$

In[\*]:= **r1p** = CF[CF[r<sub>1</sub>[1, i, j] /. sol3b /. sol3ba] /. sa3 /. sol2c /. sol11] /. sol1r // CF

Out[\*]=

$$-\frac{\dots 5600 \dots + \$B T_1^3 T_2^4 \alpha_4^2 p_{3,j}^2 x_{3,i} x_{3,j}}{2 \$B (-1+T_1) T_1 (-1+T_2) T_2 (-1+T_1 T_2) (-C_{22}+C_{22} T_2 - T_2 \alpha_4)}$$

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In[\*]:= **r1m** =

CF[CF[CF[r<sub>1</sub>[-1, i, j] /. sol2b] /. sol3b /. sol3ba] /. sa3 /. sol2c /. sol11] /. sol1r // CF

Out[\*]=

$$\frac{\dots 8699 \dots + 3 \$B T_1^3 T_2^4 \alpha_4^2 p_{3,j}^2 x_{3,i} x_{3,j}}{2 \$B (-1+T_1) T_1^2 (-1+T_2) T_2^2 (-1+T_1 T_2) (-C_{22}+C_{22} T_2 - T_2 \alpha_4)}$$

large output    show less    show more    show all    set size limit...

In[\*]:= **Union@Cases[r1m, c\_, ∞]**

Out[\*]=

{C<sub>4</sub>, C<sub>10</sub>, C<sub>13</sub>, C<sub>15</sub>, C<sub>22</sub>, C<sub>28</sub>, C<sub>32</sub>, C<sub>36</sub>, C<sub>49</sub>, C<sub>54</sub>, C<sub>56</sub>, C<sub>65</sub>, C<sub>69</sub>, C<sub>78</sub>, C<sub>81</sub>, C<sub>86</sub>}

In[\*]:= **RandomChoice[{-2, -1, 1, 2}]**

Out[\*]=

2

In[\*]:= **Choices = {-2, -1, 1, 2};**

**Sub = {c<sub>4</sub> → RandomChoice[Choices], c<sub>10</sub> → RandomChoice[Choices],**  
**c<sub>13</sub> → RandomChoice[Choices], c<sub>15</sub> → RandomChoice[Choices],**  
**c<sub>22</sub> → RandomChoice[Choices], c<sub>28</sub> → RandomChoice[{-2, -1, 1, 2}],**  
**c<sub>32</sub> → RandomChoice[Choices], c<sub>36</sub> → RandomChoice[Choices], c<sub>49</sub> → RandomChoice[Choices],**  
**c<sub>54</sub> → RandomChoice[Choices], c<sub>56</sub> → RandomChoice[Choices], c<sub>65</sub> → RandomChoice[Choices],**  
**c<sub>69</sub> → RandomChoice[Choices], c<sub>78</sub> → RandomChoice[Choices], c<sub>81</sub> → RandomChoice[Choices],**  
**c<sub>86</sub> → RandomChoice[Choices], α<sub>4</sub> → RandomChoice[Choices], \$B → RandomChoice[Choices]}**

Out[\*]=

{C<sub>4</sub> → -1, C<sub>10</sub> → 2, C<sub>13</sub> → -2, C<sub>15</sub> → -2, C<sub>22</sub> → 1, C<sub>28</sub> → -2, C<sub>32</sub> → 2, C<sub>36</sub> → -2, C<sub>49</sub> → -2,  
 C<sub>54</sub> → -1, C<sub>56</sub> → -2, C<sub>65</sub> → 1, C<sub>69</sub> → 2, C<sub>78</sub> → -1, C<sub>81</sub> → -2, C<sub>86</sub> → 1, α<sub>4</sub> → -1, \$B → -2}

In[\*]:= **r0p** /. Sub // CF

**r0m** /. Sub // CF

Out[\*]=

2 (-T<sub>1</sub> + T<sub>2</sub> + T<sub>1</sub> T<sub>2</sub>) p<sub>3,j</sub> x<sub>1,i</sub> x<sub>2,i</sub> - 2 (-1 + T<sub>2</sub>) p<sub>3,j</sub> x<sub>1,j</sub> x<sub>2,i</sub> - 2 p<sub>3,j</sub> x<sub>1,i</sub> x<sub>2,j</sub>

Out[\*]=

$$-\frac{2 p_{3,j} x_{1,i} x_{2,i}}{T_1} + \frac{2 (-1 + T_2) p_{3,j} x_{1,j} x_{2,i}}{T_2} + \frac{2 p_{3,j} x_{1,i} x_{2,j}}{T_1}$$

In[\*]:= **r1p** /. Sub // CF

In[\*]:= **r1m** /. Sub // CF

Out[\*]=

$$\frac{3}{2} + 2 p_{1,i} x_{1,i} - \frac{2 (-2 + T_1^2) p_{1,j} x_{1,i}}{T_1^2} + p_{1,i} p_{1,j} x_{1,i}^2 - \frac{(-1 + T_1) (1 + 2 T_1) p_{1,j}^2 x_{1,i}^2}{2 T_1^2} -$$

[https://drorbn.net/AcademicPensieve/Projects/HigherRank/Rolands\\_A2/MathematicaNotebooks](https://drorbn.net/AcademicPensieve/Projects/HigherRank/Rolands_A2/MathematicaNotebooks)



$$\begin{aligned}
& \frac{4 p_{1,j} x_{1,j}}{T_1} + p_{1,i} p_{1,j} x_{1,i} x_{1,j} - \frac{(1 + 3 T_1) p_{1,j}^2 x_{1,i} x_{1,j}}{2 T_1} + 2 p_{2,i} x_{2,i} - 2 p_{2,j} x_{2,i} - \\
& \frac{(-2 + 3 T_1 - 4 T_2 + 16 T_1 T_2 - 9 T_1^2 T_2 + 8 T_2^2 - 22 T_1 T_2^2 + 9 T_1^2 T_2^2) p_{1,j} p_{2,i} x_{1,i} x_{2,i}}{T_1 (-1 + 2 T_2)} - \\
& \frac{(-1 - T_1 + 8 T_2 - 10 T_1 T_2 + 9 T_1^2 T_2 - 8 T_2^2 + 15 T_1 T_2^2 - 18 T_1^2 T_2^2 - 4 T_1 T_2^3 + 9 T_1^2 T_2^3) p_{1,i} p_{2,j} x_{1,i} x_{2,i}}{(-1 + T_1) T_2 (-1 + 2 T_2)} + \\
& \frac{1}{T_1 T_2 (-1 + 2 T_2)} (-4 T_1 + 8 T_2 - 14 T_1 T_2 + 18 T_1^2 T_2 - 12 T_2^2 + 35 T_1 T_2^2 - 36 T_1^2 T_2^2 + 8 T_2^3 - 26 T_1 T_2^3 + 18 T_1^2 T_2^3) \\
& p_{1,j} p_{2,j} x_{1,i} x_{2,i} - \frac{(-4 + 3 T_1 + 16 T_1 T_2 - 9 T_1^2 T_2 + 8 T_2^2 - 22 T_1 T_2^2 + 9 T_1^2 T_2^2) p_{1,j} p_{2,i} x_{1,j} x_{2,i}}{(-1 + T_1) (-1 + 2 T_2)} + \\
& \frac{(-1 + T_2) (2 - 4 T_1 + 4 T_2 - 13 T_1 T_2 + 9 T_1^2 T_2) p_{1,j} p_{2,j} x_{1,j} x_{2,i}}{(-1 + T_1) T_2} + \\
& \frac{(1 + T_2) p_{2,i} p_{2,j} x_{2,i}^2}{T_2} - \frac{(-1 + T_2) (3 + 2 T_2) p_{2,j}^2 x_{2,i}^2}{2 T_2^2} - 2 p_{1,i} p_{2,j} x_{1,i} x_{2,j} + \\
& \frac{(2 + T_1 - 8 T_2 + 11 T_1 T_2 - 9 T_1^2 T_2 - 4 T_1 T_2^2 + 9 T_1^2 T_2^2) p_{1,j} p_{2,j} x_{1,i} x_{2,j}}{T_1 (-1 + 2 T_2)} + p_{2,i} p_{2,j} x_{2,i} x_{2,j} - \\
& \frac{3 (1 + T_2) p_{2,j}^2 x_{2,i} x_{2,j}}{2 T_2} - \frac{(T_1 + 4 T_2 - 11 T_1 T_2 - 4 T_1 T_2^2 + 9 T_1^2 T_2^2) p_{1,j} p_{2,i} x_{3,i}}{2 T_1 (-1 + 2 T_2)} - \\
& \frac{p_{1,i} p_{2,j} x_{3,i}}{2 T_2} + \frac{(-T_1 + 3 T_1 T_2 + 4 T_2^2 - 11 T_1 T_2^2 - 4 T_1 T_2^3 + 9 T_1^2 T_2^3) p_{1,j} p_{2,j} x_{3,i}}{2 T_1 T_2 (-1 + 2 T_2)} + 2 p_{3,i} x_{3,i} - \\
& \frac{(-4 + 9 T_1 - T_1^2 + 8 T_2 - 18 T_1 T_2 + T_1^2 T_2 - 4 T_1 T_2^2 + 15 T_1^2 T_2^2 - 9 T_1^3 T_2^2 - 4 T_1^2 T_2^3 + 9 T_1^3 T_2^3) p_{3,j} x_{3,i}}{T_1^2 T_2 (-1 + 2 T_2)} - \\
& \frac{(4 - 4 T_1 + T_1^2 + 4 T_2 - 11 T_1 T_2 - 4 T_1 T_2^2 + 9 T_1^2 T_2^2) p_{1,j} p_{3,i} x_{1,i} x_{3,i}}{T_1 (-1 + T_1 T_2)} + \\
& \frac{(-2 T_1 + 4 T_2 - 7 T_1 T_2 + 10 T_1^2 T_2 - 4 T_1 T_2^2 + 2 T_1^2 T_2^2 - 9 T_1^3 T_2^2 - 4 T_1^2 T_2^3 + 9 T_1^3 T_2^3) p_{1,i} p_{3,j} x_{1,i} x_{3,i}}{(-1 + T_1) T_1 T_2 (-1 + 2 T_2)} - \\
& \frac{(2 - 6 T_1 + T_1^2 + T_1 T_2 + 8 T_1^2 T_2 - 4 T_1 T_2^2 + 2 T_1^2 T_2^2 - 9 T_1^3 T_2^2 - 4 T_1^2 T_2^3 + 9 T_1^3 T_2^3) p_{1,j} p_{3,j} x_{1,i} x_{3,i}}{T_1^2 T_2 (-1 + 2 T_2)} - \\
& \frac{(6 - 4 T_1 + T_1^2 + 4 T_2 - 13 T_1 T_2 - 4 T_1 T_2^2 + 9 T_1^2 T_2^2) p_{1,j} p_{3,i} x_{1,j} x_{3,i}}{(-1 + T_1) (-1 + T_1 T_2)} - \frac{p_{1,i} p_{3,j} x_{1,j} x_{3,i}}{T_2} + \\
& \frac{(-4 + 6 T_1 - 2 T_1^2 + 8 T_2 - 11 T_1 T_2 + 4 T_1^2 T_2 + 4 T_2^2 - 11 T_1 T_2^2 - 4 T_1 T_2^3 + 9 T_1^2 T_2^3) p_{1,j} p_{3,j} x_{1,j} x_{3,i}}{(-1 + T_1) T_1 T_2 (-1 + 2 T_2)} + \\
& \left( (-2 + T_1 - 3 T_2 + 22 T_1 T_2 - 10 T_1^2 T_2 + 14 T_2^2 - 42 T_1 T_2^2 - T_1^2 T_2^2 + 9 T_1^3 T_2^2 - 8 T_2^3 + 10 T_1 T_2^3 + 33 T_1^2 T_2^3 - 18 T_1^3 T_2^3 \right. \\
& \quad \left. + 8 T_1 T_2^4 - 22 T_1^2 T_2^4 + 9 T_1^3 T_2^4) p_{2,j} p_{3,i} x_{2,i} x_{3,i} \right) / \left( (-1 + T_1) T_2 (-1 + 2 T_2) (-1 + T_1 T_2) \right) + \\
& \left( (-2 + T_1) (T_1 + 3 T_2 - 12 T_1 T_2 - 6 T_2^2 + 17 T_1 T_2^2 + 9 T_1^2 T_2^2 + 4 T_2^3 - 3 T_1 T_2^3 - 18 T_1^2 T_2^3 - 4 T_1 T_2^4 + 9 T_1^2 T_2^4) \right. \\
& \quad \left. p_{2,i} p_{3,j} x_{2,i} x_{3,i} \right) / \left( (-1 + T_1) T_1 (-1 + T_2) T_2 (-1 + 2 T_2) \right) - \frac{1}{(-1 + T_1) T_1 T_2^2}
\end{aligned}$$

$$\begin{aligned}
& \left( (1 + T_1 - T_1^2 + 5 T_2 - 24 T_1 T_2 + 11 T_1^2 T_2 - 8 T_2^2 + 18 T_1 T_2^2 + 11 T_1^2 T_2^2 - 9 T_1^3 T_2^2 + 8 T_1 T_2^3 - 22 T_1^2 T_2^3 + 9 T_1^3 T_2^3) \right. \\
& \quad \rho_{2,j} \rho_{3,j} x_{2,i} x_{3,i} + \left( (-3 + 2 T_1 - T_2 + 21 T_1 T_2 - 11 T_1^2 T_2 + 14 T_2^2 - 44 T_1 T_2^2 + T_1^2 T_2^2 + 9 T_1^3 T_2^2 - \right. \\
& \quad \quad \left. 8 T_2^3 + 10 T_1 T_2^3 + 33 T_1^2 T_2^3 - 18 T_1^3 T_2^3 + 8 T_1 T_2^4 - 22 T_1^2 T_2^4 + 9 T_1^3 T_2^4) \rho_{2,j} \rho_{3,i} x_{2,j} x_{3,i} \right) / \\
& \quad \left( (-1 + T_1) (-1 + T_2) (-1 + 2 T_2) (-1 + T_1 T_2) \right) - \\
& \quad \frac{\left( T_1 + 4 T_2 - 11 T_1 T_2 - 4 T_1 T_2^2 + 9 T_1^2 T_2^2 \right) \rho_{2,i} \rho_{3,j} x_{2,j} x_{3,i}}{\left( -1 + T_1 \right) T_1 \left( -1 + 2 T_2 \right)} - \frac{1}{\left( -1 + T_1 \right) T_1 T_2 \left( -1 + 2 T_2 \right)} \\
& \quad \left( 1 + 6 T_2 - 25 T_1 T_2 + 10 T_1^2 T_2 - 12 T_2^2 + 29 T_1 T_2^2 + 11 T_1^2 T_2^2 - 9 T_1^3 T_2^2 + 12 T_1 T_2^3 - 31 T_1^2 T_2^3 + 9 T_1^3 T_2^3 \right) \\
& \quad \rho_{2,j} \rho_{3,j} x_{2,j} x_{3,i} - \\
& \quad \left( \left( 2 - T_1 + 4 T_2 - 23 T_1 T_2 + 11 T_1^2 T_2 - 8 T_2^2 + 20 T_1 T_2^2 + 9 T_1^2 T_2^2 - 9 T_1^3 T_2^2 + 8 T_1 T_2^3 - 22 T_1^2 T_2^3 + 9 T_1^3 T_2^3 \right) \right. \\
& \quad \left. \rho_{3,i} \rho_{3,j} x_{3,i}^2 \right) / \left( \left( -1 + T_1 \right) T_1 T_2 \left( -1 + 2 T_2 \right) \right) + \\
& \quad \left( -1 + T_1 T_2 \right) \left( 1 + 6 T_2 - 28 T_1 T_2 + 13 T_1^2 T_2 - 8 T_2^2 + 18 T_1 T_2^2 + 27 T_1^2 T_2^2 - 18 T_1^3 T_2^2 + 16 T_1 T_2^3 - \right. \\
& \quad \quad \left. 44 T_1^2 T_2^3 + 18 T_1^3 T_2^3 \right) \rho_{3,j}^2 x_{3,i}^2 / \left( 2 \left( -1 + T_1 \right) T_1^2 T_2^2 \left( -1 + 2 T_2 \right) \right) - \frac{\left( -4 + 9 T_1 \right) \rho_{3,j} x_{3,j}}{T_1} + \\
& \quad \frac{\left( -4 + 3 T_1 - T_1^2 + 12 T_2 - 17 T_1 T_2 + 12 T_1^2 T_2 - 4 T_1 T_2^2 + 2 T_1^2 T_2^2 - 9 T_1^3 T_2^2 - 4 T_1^2 T_2^3 + 9 T_1^3 T_2^3 \right) \rho_{1,i} \rho_{3,j} x_{1,i} x_{3,j}}{\left( -1 + T_1 \right) \left( -1 + 2 T_2 \right) \left( -1 + T_1 T_2 \right)} \\
& \quad - \frac{\left( 2 + T_1 - 8 T_2 + 11 T_1 T_2 - 9 T_1^2 T_2 - 4 T_1 T_2^2 + 9 T_1^2 T_2^2 \right) \rho_{1,j} \rho_{3,j} x_{1,i} x_{3,j}}{T_1 \left( -1 + 2 T_2 \right)} - \\
& \quad \frac{\left( -1 - T_2 + T_1 T_2 \right) \rho_{2,i} \rho_{3,j} x_{2,i} x_{3,j}}{\left( -1 + T_2 \right) \left( -1 + T_1 T_2 \right)} - \frac{\left( -2 + T_1 \right) \left( 1 + 2 T_2 - 9 T_1 T_2 - 4 T_2^2 + 9 T_1 T_2^2 \right) \rho_{2,j} \rho_{3,j} x_{2,i} x_{3,j}}{\left( -1 + T_1 \right) \left( -1 + 2 T_2 \right)} - \\
& \quad \frac{\left( -3 + 2 T_1 - 2 T_2 + 18 T_1 T_2 - 9 T_1^2 T_2 + 8 T_2^2 - 22 T_1 T_2^2 + 9 T_1^2 T_2^2 \right) \rho_{3,i} \rho_{3,j} x_{3,i} x_{3,j}}{\left( -1 + T_1 \right) \left( -1 + 2 T_2 \right)} + \\
& \quad \left( 1 + 6 T_2 - 31 T_1 T_2 + 15 T_1^2 T_2 - 8 T_2^2 + 16 T_1 T_2^2 + 45 T_1^2 T_2^2 - 27 T_1^3 T_2^2 + 24 T_1 T_2^3 - 66 T_1^2 T_2^3 + 27 T_1^3 T_2^3 \right) \\
& \quad \rho_{3,j}^2 x_{3,i} x_{3,j} / \left( 2 \left( -1 + T_1 \right) T_1 T_2 \left( -1 + 2 T_2 \right) \right)
\end{aligned}$$