

Pensieve header: Finding the A_2 $\mathcal{S}d=1$ invariant using undetermined coefficients.

Searching for $Q + p_{xx} + \epsilon(ppx + 1 + px + ppxx)$ solutions.

Initialization

```
In[*]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\HigherRank"];
Once[<< KnotTheory` ; << Rot.m];
<< FormalGaussianIntegration.m;
i_+ := i + 1;
```

Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.

Read more at <http://katlas.org/wiki/KnotTheory>.

Loading Rot.m from <http://drorbn.net/AP/Projects/HigherRank> to compute rotation numbers.

```
In[*]:= Features[Knot[8, 17]]
```

 KnotTheory: Loading precomputed data in PD4Knots`.

```
Out[*]=
```

```
Features[18,
C6[-1] C14[-1] X1,7[1] X3,9[-1] X5,13[-1] X8,16[1] X10,4[-1] X12,18[1] X15,2[-1] X17,11[1]]
```

```
In[*]:= T3 = T1 T2;
S = {x_, p_};
q[s_, i_, j_] := Sum[
  xv,i (pv,i+ - pv,i) + xv,j (pv,j+ - pv,j) + (T3^S - 1) xv,i (pv,i+ - pv,j+),
  {v, 3}];
L[Xi_,j_[s_]] :=
  T3^S E[q[s, i, j] + B^-1 r0[s, i, j] + E B r1[s, i, j] + E r42[s, i, j] + O[epsilon]^2];
(*gamma1[phi_,k_] := phi(3/2 - X1,k p1,k - X2,k p2,k - X3,k p3,k); *)
L[Ck_[0]] := E[Sum[xv,k (pv,k+ - pv,k), {v, 3}] + O[epsilon]^2];
L[Ck_[phi_]] :=
  T3^phi E[Sum[xv,k (pv,k+ - pv,k), {v, 3}] + B^-1 gamma0[phi, k] + E B gamma1[phi, k] + E gamma42[phi, k] + O[epsilon]^2];
ps_i := Sequence[p1,i, p2,i, p3,i];
xs_i := Sequence[x1,i, x2,i, x3,i];
vs_i := Sequence[ps_i, xs_i];
F[is___] := E[Sum[pi,v,i pv,i, {i, {is}}, {v, 3}]];
L[K_] := (2 pi)^-Features[K][[1]] CF[L/@Features[K][[2]]];
vs[K_] := Union@@Table[{vs_i}, {i, Features[K][[1]]}]
```

```
In[*]:= vs_i
```

```
Out[*]=
```

```
Sequence[p1,i, p2,i, p3,i, x1,i, x2,i, x3,i]
```

The Various Terms (r_0)

The pxx Terms (r_0)

```
In[*]:=  $\mathbf{x} = \mathbf{0}$ ;
 $r_0[1, \mathbf{i}_-, \mathbf{j}_-]$  := Evaluate[Sum[
   $a_{++\kappa} p_{3,k3} x_{1,k1} x_{2,k2}$ ,
  { $\mathbf{k1}$ , { $\mathbf{i}$ ,  $\mathbf{j}$ }}, { $\mathbf{k2}$ , { $\mathbf{i}$ ,  $\mathbf{j}$ }}, { $\mathbf{k3}$ , { $\mathbf{i}$ ,  $\mathbf{j}$ }}
]];
 $r_0[1, \mathbf{i}, \mathbf{j}]$ 

Out[*]=
 $a_1 p_{3,i} x_{1,i} x_{2,i} + a_2 p_{3,j} x_{1,i} x_{2,i} + a_5 p_{3,i} x_{1,j} x_{2,i} + a_6 p_{3,j} x_{1,j} x_{2,i} +$ 
 $a_3 p_{3,i} x_{1,i} x_{2,j} + a_4 p_{3,j} x_{1,i} x_{2,j} + a_7 p_{3,i} x_{1,j} x_{2,j} + a_8 p_{3,j} x_{1,j} x_{2,j}$ 
```

```
In[*]:=  $\mathbf{x} = \mathbf{0}$ ;
 $r_0[-1, \mathbf{i}_-, \mathbf{j}_-]$  := Evaluate[Sum[
   $d_{++\kappa} p_{3,k3} x_{1,k1} x_{2,k2}$ ,
  { $\mathbf{k1}$ , { $\mathbf{i}$ ,  $\mathbf{j}$ }}, { $\mathbf{k2}$ , { $\mathbf{i}$ ,  $\mathbf{j}$ }}, { $\mathbf{k3}$ , { $\mathbf{i}$ ,  $\mathbf{j}$ }}
]];
 $r_0[-1, \mathbf{i}, \mathbf{j}]$ 

Out[*]=
 $d_1 p_{3,i} x_{1,i} x_{2,i} + d_2 p_{3,j} x_{1,i} x_{2,i} + d_5 p_{3,i} x_{1,j} x_{2,i} + d_6 p_{3,j} x_{1,j} x_{2,i} +$ 
 $d_3 p_{3,i} x_{1,i} x_{2,j} + d_4 p_{3,j} x_{1,i} x_{2,j} + d_7 p_{3,i} x_{1,j} x_{2,j} + d_8 p_{3,j} x_{1,j} x_{2,j}$ 
```

The ppx Terms (r_1)

```
In[*]:=  $\mathbf{x} = \mathbf{0}$ ;
 $r_1[1, \mathbf{i}_-, \mathbf{j}_-]$  := Evaluate[Sum[
   $b_{++\kappa} x_{3,k3} p_{1,k1} p_{2,k2}$ ,
  { $\mathbf{k1}$ , { $\mathbf{i}$ ,  $\mathbf{j}$ }}, { $\mathbf{k2}$ , { $\mathbf{i}$ ,  $\mathbf{j}$ }}, { $\mathbf{k3}$ , { $\mathbf{i}$ ,  $\mathbf{j}$ }}
]];
 $r_1[1, \mathbf{i}, \mathbf{j}]$ 

Out[*]=
 $b_1 p_{1,i} p_{2,i} x_{3,i} + b_5 p_{1,j} p_{2,i} x_{3,i} + b_3 p_{1,i} p_{2,j} x_{3,i} + b_7 p_{1,j} p_{2,j} x_{3,i} +$ 
 $b_2 p_{1,i} p_{2,i} x_{3,j} + b_6 p_{1,j} p_{2,i} x_{3,j} + b_4 p_{1,i} p_{2,j} x_{3,j} + b_8 p_{1,j} p_{2,j} x_{3,j}$ 
```

```
In[*]:=  $\mathbf{x} = \mathbf{0}$ ;
 $r_1[-1, \mathbf{i}_-, \mathbf{j}_-]$  := Evaluate[Sum[
   $e_{++\kappa} x_{3,k3} p_{1,k1} p_{2,k2}$ ,
  { $\mathbf{k1}$ , { $\mathbf{i}$ ,  $\mathbf{j}$ }}, { $\mathbf{k2}$ , { $\mathbf{i}$ ,  $\mathbf{j}$ }}, { $\mathbf{k3}$ , { $\mathbf{i}$ ,  $\mathbf{j}$ }}
]];
 $r_1[-1, \mathbf{i}, \mathbf{j}]$ 

Out[*]=
 $e_1 p_{1,i} p_{2,i} x_{3,i} + e_5 p_{1,j} p_{2,i} x_{3,i} + e_3 p_{1,i} p_{2,j} x_{3,i} + e_7 p_{1,j} p_{2,j} x_{3,i} +$ 
 $e_2 p_{1,i} p_{2,i} x_{3,j} + e_6 p_{1,j} p_{2,i} x_{3,j} + e_4 p_{1,i} p_{2,j} x_{3,j} + e_8 p_{1,j} p_{2,j} x_{3,j}$ 
```

The ppx Terms (r_{42})

```
In[*]:= x = 0;
Short[r42[1, i_, j_] = Evaluate[Plus[
  Sum[
    C+++ Xv1,k1 Pv1,k2 Xv2,k3 Pv2,k4,
    {k1, {i, j}}, {k2, {i, j}}, {k3, {i, j}}, {k4, {i, j}}, {v1, 2}, {v2, v1 + 1, 3}
  ],
  Sum[
    C+++ Xv,k1 Pv,k2,
    {k1, {i, j}}, {k2, {i, j}}, {v, 3}
  ]
]]]
```

```
Out[*]//Short=
C49 p1,i X1,i + C52 p1,j X1,i + C55 p1,i X1,j + <<54>> +
C45 p2,j p3,i X2,j X3,j + C36 p2,i p3,j X2,j X3,j + C48 p2,j p3,j X2,j X3,j
```

```
In[*]:= x = 0;
Short[r42[-1, i_, j_] = Evaluate[Plus[
  Sum[
    f+++ Xv1,k1 Pv1,k2 Xv2,k3 Pv2,k4,
    {k1, {i, j}}, {k2, {i, j}}, {k3, {i, j}}, {k4, {i, j}}, {v1, 2}, {v2, v1 + 1, 3}
  ],
  Sum[
    f+++ Xv,k1 Pv,k2,
    {k1, {i, j}}, {k2, {i, j}}, {v, 3}
  ]
]]]
```

```
Out[*]//Short=
f49 p1,i X1,i + f52 p1,j X1,i + f55 p1,i X1,j + <<54>> +
f45 p2,j p3,i X2,j X3,j + f36 p2,i p3,j X2,j X3,j + f48 p2,j p3,j X2,j X3,j
```

The γ Terms ($\gamma_0, \gamma_1, \gamma_{42}$)

```
In[*]:= x = 0;
 $\gamma_0[1, k_] := Evaluate[g+++ p3,k X1,k X2,k];$ 
 $\gamma_1[1, k_] := Evaluate[g+++ x3,k p1,k p2,k];$ 
 $\gamma_{42}[1, k_] := Evaluate[Plus[
  Sum[g+++ Xv,k Pv,k, {v, 3}],
  Sum[g+++ Xv1,k Pv1,k Xv2,k Pv2,k, {v1, 2}, {v2, v1 + 1, 3}]
]];
{ $\gamma_0[1, k]$ ,  $\gamma_0[1, k]$ ,  $\gamma_{42}[1, k]$ }$ 
```

```
Out[*]=
{g1 p3,k X1,k X2,k, g1 p3,k X1,k X2,k,
g3 p1,k X1,k + g4 p2,k X2,k + g6 p1,k p2,k X1,k X2,k + g5 p3,k X3,k + g7 p1,k p3,k X1,k X3,k + g8 p2,k p3,k X2,k X3,k}
```

```
In[*]:= x = 0;
γ₀[-1, k_] := Evaluate[h+++ p3,k x1,k x2,k];
γ₁[-1, k_] := Evaluate[h+++ x3,k p1,k p2,k];
γ42[-1, k_] := Evaluate[Plus[
  Sum[h+++ xv,k pv,k, {v, 3}],
  Sum[h+++ xv1,k pv1,k xv2,k pv2,k, {v1, 2}, {v2, v1 + 1, 3}]
]];
{γ₀[-1, k], γ₀[-1, k], γ42[-1, k]}
```

```
Out[*]=
{h₁ p3,k x1,k x2,k, h₁ p3,k x1,k x2,k,
h₃ p1,k x1,k + h₄ p2,k x2,k + h₆ p1,k p2,k x1,k x2,k + h₅ p3,k x3,k + h₇ p1,k p3,k x1,k x3,k + h₈ p2,k p3,k x2,k x3,k}}
```

Reidemeister 3b

```
In[*]:= Timing[{LeftR3b} =
Cases[∫ ℱ[i, j, k] × ℒ / @ (Xi,j[1] Xi+,k[1] Xj+,k+[1]) d[{vSi, vSj, vSk, vSi+}, vSj+}, vSk+}]},
E[ε_] := ε, ∞]]
```

```
Out[*]=
{3.23438,
{eSeries[T₁² p1,2+i π1,i - (-1 + T₁) T₁ p1,2+j π1,i + (1 - T₁) p1,2+k π1,i + T₁ p1,2+j π1,j + (1 - T₁) p1,2+k π1,j + p1,2+k π1,k +
T₂² p2,2+i π2,i - (-1 + T₂) T₂ p2,2+j π2,i + (1 - T₂) p2,2+k π2,i + ... 33 ... + a₃ T₁ T₂ p3,2+j π1,j π2,k -
(-a₃ - a₄ + a₃ T₁ T₂) p3,2+k π1,j π2,k + a₇ T₁ T₂ p3,2+i π1,k π2,k + a₇ T₁ T₂ p3,2+j π1,k π2,k - 2(-a₇ - a₈ + a₇ T₁ T₂) p3,2+k π1,k π2,k + T₁² T₂² p3,2+i π3,i -
T₁ T₂ (-1 + T₁ T₂) p3,2+j π3,i + (1 - T₁ T₂) p3,2+k π3,i + T₁ T₂ p3,2+j π3,j + (1 - T₁ T₂) p3,2+k π3,j + p3,2+k π3,k,
3 (a₁ b₁ + a₂ b₂ + a₃ b₃ + a₄ b₄ + a₅ b₅ + a₆ b₆ + a₇ b₇ + a₈ b₈ + c₁ + c₂ + c₃ + c₁₀ + c₁₁ + c₁₂ +
c₃₇ + c₃₈ + c₃₉ + c₄₆ + c₄₇ + c₄₈ + c₄₉ + c₅₀ + c₅₁ + c₅₈ + c₅₉ + c₆₀) + ... 406 ... +
(2 a₃ b₂ + 2 a₄ b₂ + a₇ b₂ + a₈ b₂ + 2 a₃ b₄ + 2 a₄ b₄ + a₇ b₄ + a₈ b₄ + 3 a₇ b₆ + 3 a₈ b₆ + 3 a₇ b₈ + 3 a₈ b₈ + 2 c₃₃ + ... 42 ... +
a₇ b₂ T₁² T₂ + a₇ b₄ T₁² T₂ + 2 a₃ b₂ T₁ T₂² + a₇ b₂ T₁ T₂² + 3 a₇ b₆ T₁ T₂² + 2 c₃₃ T₁ T₂² - a₇ b₂ T₁² T₂²) p2,2+k p3,2+k π2,k π3,k]}]}
```

Full expression not available (original memory size: 3.6 MB)

```
In[*]:= Timing[{RightR3b} =
Cases[∫ ℱ[i, j, k] × ℒ / @ (Xj,k[1] Xi,k+[1] Xi+,j+[1]) d[{vSi, vSj, vSk, vSi+}, vSj+}, vSk+}]},
E[ε_] := ε, ∞]];
```

```
Out[*]=
{1.98438, Null}
```

```
In[*]:= Short[eqn = CF[LeftR3b[[1]] - RightR3b[[1]]]
cvs = Union@Cases[eqn, p__ | π__, ∞]
vars = Union@Cases[r_0[1, i, j], a_, ∞]
Short[eqns = CoefficientRules[eqn, cvs] /. (_ -> c_) :-> (c == 0), 3]
{sol} = Solve[eqns, vars]
```

Out[*]//Short=

$$\frac{T_1 T_2 (-a_1 T_1 + \ll 20 \gg + a_7 T_1^2 T_2^2) p_{3,2+j} \pi_{1,i} \pi_{2,i}}{B} - \frac{\ll 1 \gg}{B} + \frac{a_5 \ll 5 \gg \pi_{\ll 1 \gg}}{B} - \frac{\ll 1 \gg}{B} + \ll 29 \gg + \frac{a_7 T_1 T_2 (-1 + T_1 T_2) p_{3,2+j} \pi_{1,k} \pi_{2,k}}{B}$$

Out[*]=

$$\{p_{3,2+i}, p_{3,2+j}, p_{3,2+k}, \pi_{1,i}, \pi_{1,j}, \pi_{1,k}, \pi_{2,i}, \pi_{2,j}, \pi_{2,k}\}$$

Out[*]=

$$\{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}$$

Out[*]//Short=

$$\left\{ \begin{aligned} -\frac{a_3 T_1^2 T_2^2}{B} + \frac{a_3 T_1^2 T_2^3}{B} &== 0, & \frac{a_3 T_1^2 T_2}{B} - \frac{a_3 T_1^2 T_2^2}{B} &== 0, \\ -\frac{a_5 T_1^2 T_2^2}{B} + \frac{a_5 T_1^3 T_2^2}{B} &== 0, & -\frac{a_7 T_1^2 T_2^2}{B} + \frac{a_7 T_1^3 T_2^2}{B} + \frac{a_7 T_1^2 T_2^3}{B} - \frac{a_7 T_1^3 T_2^3}{B} &== 0, \\ \ll 18 \gg, & -\frac{a_7}{B} - \frac{a_8}{B} + \frac{a_7 T_1}{B} + \frac{a_8 T_1}{B} + \frac{a_7 T_1 T_2}{B} - \frac{a_7 T_1^2 T_2}{B} &== 0, \\ \frac{a_7 T_2}{B} + \frac{a_8 T_2}{B} - \frac{a_7 T_2^2}{B} - \frac{a_8 T_2^2}{B} - \frac{a_7 T_1 T_2^2}{B} + \frac{a_7 T_1 T_2^3}{B} &== 0, \\ -\frac{a_7}{B} - \frac{a_8}{B} + \frac{a_7 T_2}{B} + \frac{a_8 T_2}{B} + \frac{a_7 T_1 T_2}{B} - \frac{a_7 T_1 T_2^2}{B} &== 0 \end{aligned} \right\}$$

Solve: Equations may not give solutions for all "solve" variables.

Out[*]=

$$\left\{ \left\{ a_1 \rightarrow 0, a_3 \rightarrow 0, a_5 \rightarrow 0, a_6 \rightarrow -\frac{a_2}{T_1} - \frac{a_4 T_2}{T_1}, a_7 \rightarrow 0, a_8 \rightarrow 0 \right\} \right\}$$

```
In[*]:= sol /. (v_ -> val_) :-> (v = CF[val]);
r_0[1, i, j]
```

Out[*]=

$$a_2 p_{3,j} x_{1,i} x_{2,i} - \frac{(a_2 + a_4 T_2) p_{3,j} x_{1,j} x_{2,i}}{T_1} + a_4 p_{3,j} x_{1,i} x_{2,j}$$

```

In[*]:= Short[eqn = CF[Coefficient[
    LeftR3b[[2]] - RightR3b[[2]] /. v : (π | p) __ => μ v,
    μ^3
  ]], 5]
cvs = Union@Cases[eqn, p__ | π__, ∞]
vars = Union@Cases[r1[1, i, j], b_, ∞]
Short[eqns = CoefficientRules[eqn, cvs] /. (_ -> c_) => (c == 0), 3]
{sol} = Solve[eqns, vars]

Out[*]//Short=
B b1 (-1 + T1) T1 T2^2 p1,2+j p2,2+i π3,i -
B b1 (-1 + T1) T1 T2^2 p1,2+k p2,2+i π3,i + B b1 T1^2 (-1 + T2) T2 p1,2+i p2,2+j π3,i -
B T1 T2 (-b1 T1 - b1 T2 + 2 b1 T1 T2 - b2 T1 T2 + b2 T1^2 T2^2) p1,2+j p2,2+j π3,i + <<23>> +
B b2 T1^2 (-1 + T2) T2 p1,2+i p2,2+j π3,k - B b2 (-1 + T1) T1 (-1 + T2) T2 p1,2+j p2,2+j π3,k -
B (-b2 - b6 + b2 T1) (-1 + T2) T2 p1,2+k p2,2+j π3,k +
B (-1 + T1) T1 (-b2 - b4 + b2 T2) p1,2+i p2,2+k π3,k - B (-1 + T1) T1 (-b2 - b4 + b2 T2) p1,2+j p2,2+k π3,k

Out[*]=
{p1,2+i, p1,2+j, p1,2+k, p2,2+i, p2,2+j, p2,2+k, π3,i, π3,j, π3,k}

Out[*]=
{b1, b2, b3, b4, b5, b6, b7, b8}

Out[*]//Short=
{-B b2 T1^2 T2^2 + B b2 T1^3 T2^3 == 0, B b2 T1 T2 - B b2 T1^2 T2^2 == 0, <<22>>,
 -B b6 T1 - B b8 T1 - B b4 T2 - B b8 T2 + B b2 T1 T2 + 2 B b4 T1 T2 + 2 B b6 T1 T2 +
 2 B b8 T1 T2 - B b2 T1^2 T2 - B b4 T1^2 T2 - B b2 T1 T2^2 - B b6 T1 T2^2 + B b2 T1^2 T2^2 == 0}

Solve: Equations may not give solutions for all "solve" variables. ⓘ

Out[*]=
{{b1 -> 0, b2 -> 0, b4 -> 0, b6 -> 0, b7 -> -b3 - b5, b8 -> 0}}

In[*]:= sol /. (v_ -> val_) => (v = CF[val]);
r1[1, i, j]

Out[*]=
b5 p1,j p2,i x3,i + b3 p1,i p2,j x3,i + (-b3 - b5) p1,j p2,j x3,i

```

```
In[*]:= Short[eqn = CF[LeftR3b[[2]] - RightR3b[[2]], 5]
cvs = Union@Cases[eqn, p_ | π_, ∞]
vars = Union@Cases[r42[1, i, j], c_, ∞]
Short[eqns = CoefficientRules[eqn, cvs] /. (_ -> c_) :-> (c == 0), 3]
Short[{sol} = Solve[eqns, vars]]
```

Out[*]//Short=

$$- \left((C_{25} + C_{26} + C_{34} + C_{35} + C_{55}) (-1 + T_1) T_1^2 p_{1,2+j} \pi_{1,i} \right) -$$

$$(-1 + T_1) \left(C_1 + C_2 + C_{10} + C_{11} + C_{13} + C_{14} + C_{22} + C_{23} + C_{49} + C_{52} + C_{25} T_1 + C_{26} T_1 + C_{34} T_1 + C_{35} T_1 + C_{37} T_1 + \right.$$

$$C_{38} T_1 + C_{46} T_1 + C_{47} T_1 + C_{55} T_1 + C_{58} T_1 - C_{25} T_1^2 - C_{26} T_1^2 - C_{34} T_1^2 - C_{35} T_1^2 - C_{55} T_1^2 \left. \right) p_{1,2+k} \pi_{1,i} +$$

$$\llcorner 374 \gg + (-1 + T_2) T_2 (-C_{33} - C_{36} + C_{33} T_1 T_2) p_{2,2+i} p_{3,2+k} \pi_{2,k} \pi_{3,k} -$$

$$(-1 + T_2) T_2 (-C_{33} - C_{36} + C_{33} T_1 T_2) p_{2,2+j} p_{3,2+k} \pi_{2,k} \pi_{3,k}$$

Out[*]=

$$\{p_{1,2+i}, p_{1,2+j}, p_{1,2+k}, p_{2,2+i}, p_{2,2+j}, p_{2,2+k}, p_{3,2+i},$$

$$p_{3,2+j}, p_{3,2+k}, \pi_{1,i}, \pi_{1,j}, \pi_{1,k}, \pi_{2,i}, \pi_{2,j}, \pi_{2,k}, \pi_{3,i}, \pi_{3,j}, \pi_{3,k}\}$$

Out[*]=

$$\{C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8, C_9, C_{10}, C_{11}, C_{12}, C_{13}, C_{14}, C_{15}, C_{16}, C_{17}, C_{18}, C_{19}, C_{20}, C_{21}, C_{22},$$

$$C_{23}, C_{24}, C_{25}, C_{26}, C_{27}, C_{28}, C_{29}, C_{30}, C_{31}, C_{32}, C_{33}, C_{34}, C_{35}, C_{36}, C_{37}, C_{38}, C_{39}, C_{40}, C_{41},$$

$$C_{42}, C_{43}, C_{44}, C_{45}, C_{46}, C_{47}, C_{48}, C_{49}, C_{50}, C_{51}, C_{52}, C_{53}, C_{54}, C_{55}, C_{56}, C_{57}, C_{58}, C_{59}, C_{60}\}$$

Out[*]//Short=

$$\{-C_7 T_1^2 T_2^2 + C_7 T_1^2 T_2^3 == 0, C_7 T_1^2 T_2 - C_7 T_1^2 T_2^2 == 0,$$

$$\llcorner 249 \gg, C_8 T_1 T_2 + C_9 T_1 T_2 + C_{44} T_1 T_2 + C_{45} T_1 T_2 + C_{57} T_1 T_2 -$$

$$C_8 T_1^2 T_2^2 - C_9 T_1^2 T_2^2 - C_{44} T_1^2 T_2^2 - C_{45} T_1^2 T_2^2 - C_{57} T_1^2 T_2^2 == 0\}$$

 Solve: Equations may not give solutions for all "solve" variables. 

Out[*]//Short=

$$\left\{ \left\{ C_1 \rightarrow 0, C_2 \rightarrow 0, C_3 \rightarrow 0, \llcorner 42 \gg, C_{58} \rightarrow -\llcorner 1 \gg - \llcorner 1 \gg, C_{59} \rightarrow -\frac{C_{50}}{T_2} - \frac{C_{53}}{T_2}, \right. \right.$$

$$C_{60} \rightarrow -\frac{C_{51}}{T_1 T_2} - \frac{C_{54}}{T_1 T_2} - \frac{-a_2 b_5 - a_2 b_3 T_1 + a_4 b_3 T_1 - a_4 \llcorner 1 \gg \llcorner 1 \gg \llcorner 1 \gg - a_4 b_3 T_1 T_2 + a_4 b_5 T_1 T_2}{T_1^2 T_2 (-1 + T_1 T_2)} \left. \right\}$$

```
In[*]:= sol /. (v_ -> val_) :-> (v = CF[val]);
```

```
In[*]:= Short[CF[r42[1, i, j]], 20]
```

Out[*]//Short=

$$C_{49} p_{1,i} x_{1,i} + C_{52} p_{1,j} x_{1,i} - \frac{(C_{49} + C_{52}) p_{1,j} x_{1,j}}{T_1} + C_{50} p_{2,i} x_{2,i} +$$

$$C_{53} p_{2,j} x_{2,i} + C_{13} p_{1,j} p_{2,i} x_{1,i} x_{2,i} + C_4 p_{1,i} p_{2,j} x_{1,i} x_{2,i} + \frac{1}{-1 + T_1 T_2}$$

$$\left(a_2 b_3 + C_4 + C_{13} - C_4 T_1 + a_4 b_3 T_2 + a_2 b_5 T_2 - a_4 b_5 T_2 - C_{13} T_2 - a_4 b_3 T_1 T_2 - C_4 T_1 T_2 - C_{13} T_1 T_2 + \right.$$

$$C_4 T_1^2 T_2 + a_4 b_5 T_2^2 + C_{13} T_1 T_2^2 \left. \right) p_{1,j} p_{2,j} x_{1,i} x_{2,i} + \frac{1}{(-1 + T_1) T_1 (-1 + T_2) (-1 + T_1 T_2)}$$

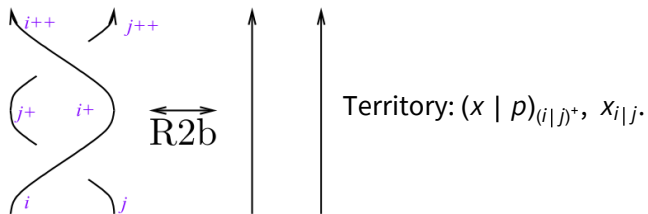
$$\left(-a_2 b_3 + a_2 b_5 + a_2 b_3 T_1 + C_4 T_1 - C_{13} T_1 - C_4 T_1^2 - a_4 b_3 T_2 - a_2 b_5 T_2 + a_4 b_5 T_2 + 2 a_4 b_3 T_1 T_2 - \right.$$

$$a_4 b_5 T_1 T_2 + C_{13} T_1 T_2 - a_4 b_3 T_1^2 T_2 - C_4 T_1^2 T_2 + C_{13} T_1^2 T_2 + C_4 T_1^3 T_2 - a_4 b_5 T_2^2 + a_4 b_5 T_1 T_2^2 - C_{13} T_1^2 T_2^2 \left. \right)$$

$$p_{1,j} p_{2,i} x_{1,j} x_{2,i} - \frac{(a_2 b_5 - C_{13} T_1 + a_4 b_5 T_2 + C_{13} T_1 T_2) p_{1,j} p_{2,j} x_{1,j} x_{2,i}}{(-1 + T_1) T_1} -$$

$$\begin{aligned}
 & \frac{(c_{50} + c_{53}) p_{2,j} x_{2,j}}{T_2} - \frac{1}{(-1 + T_1) (-1 + T_2) (-1 + T_1 T_2)} \\
 & (-a_2 b_3 + a_4 b_3 + a_2 b_5 - a_4 b_5 + c_4 - c_{13} + a_2 b_3 T_1 - a_4 b_3 T_1 - c_4 T_1 - a_4 b_3 T_2 - a_2 b_5 T_2 + 2 a_4 b_5 T_2 + \\
 & \quad c_{13} T_2 + a_4 b_3 T_1 T_2 - c_4 T_1 T_2 + c_{13} T_1 T_2 + c_4 T_1^2 T_2 - a_4 b_5 T_2^2 - c_{13} T_1 T_2^2) p_{1,i} p_{2,j} x_{1,i} x_{2,j} + \\
 & \frac{(a_4 b_3 + c_4 - c_4 T_1) p_{1,j} p_{2,j} x_{1,i} x_{2,j}}{-1 + T_2} + c_{51} p_{3,i} x_{3,i} + \ll 5 \gg + \frac{b_3 (a_2 + a_4 T_2) p_{1,i} p_{3,j} x_{1,j} x_{3,i}}{T_1 (-1 + T_2)} - \\
 & \frac{1}{(-1 + T_1) T_1 (-1 + T_2)} \\
 & (-a_2 b_3 + a_2 b_3 T_1 + a_2 b_5 T_1 + c_{14} T_1 - a_4 b_3 T_2 + a_4 b_3 T_1 T_2 - a_2 b_5 T_1 T_2 + \\
 & \quad a_4 b_5 T_1 T_2 - c_{14} T_1 T_2 - c_{14} T_1^2 T_2 - a_4 b_5 T_1 T_2^2 + c_{14} T_1^2 T_2^2) p_{1,j} p_{3,j} x_{1,j} x_{3,i} + \\
 & c_{15} p_{2,j} p_{3,i} x_{2,i} x_{3,i} + c_6 p_{2,i} p_{3,j} x_{2,i} x_{3,i} - \frac{1}{(-1 + T_1) T_1} \\
 & (-a_2 b_3 + a_2 b_3 T_1 - c_6 T_1 - c_{15} T_1 + c_6 T_1^2 + c_{15} T_1^2 - a_4 b_3 T_2 - a_2 b_5 T_2 + 2 a_4 b_3 T_1 T_2 + a_2 b_5 T_1 T_2 - \\
 & \quad a_4 b_5 T_1 T_2 + c_6 T_1 T_2 - a_4 b_3 T_1^2 T_2 - c_6 T_1^2 T_2 + c_{15} T_1^2 T_2 - c_{15} T_1^3 T_2 - a_4 b_5 T_2^2 + 2 a_4 b_5 T_1 T_2^2) \\
 & \quad p_{2,j} p_{3,j} x_{2,i} x_{3,i} + \frac{1}{(-1 + T_1) T_1 (-1 + T_2) (-1 + T_1 T_2)} \\
 & (-a_2 b_5 - a_2 b_3 T_1 + a_4 b_3 T_1 + a_2 b_5 T_1 - a_4 b_5 T_1 + c_6 T_1 - c_{15} T_1 + a_2 b_3 T_1^2 - a_4 b_3 T_1^2 - c_6 T_1^2 + c_{15} T_1^2 + \\
 & \quad a_2 b_5 T_2 - a_4 b_5 T_2 - a_4 b_3 T_1 T_2 - a_2 b_5 T_1 T_2 + 3 a_4 b_5 T_1 T_2 - c_6 T_1 T_2 + a_4 b_3 T_1^2 T_2 + c_6 T_1^2 T_2 + \\
 & \quad c_{15} T_1^2 T_2 - c_{15} T_1^3 T_2 + a_4 b_5 T_2^2 - 2 a_4 b_5 T_1 T_2^2) p_{2,j} p_{3,i} x_{2,j} x_{3,i} - \frac{a_4 b_5 p_{2,i} p_{3,j} x_{2,j} x_{3,i}}{-1 + T_1} + \\
 & \frac{(-a_4 b_5 - c_{15} + c_{15} T_1 + a_4 b_3 T_2 + a_4 b_5 T_2 - a_4 b_3 T_1 T_2 + c_{15} T_1 T_2 - c_{15} T_1^2 T_2) p_{2,j} p_{3,j} x_{2,j} x_{3,i}}{(-1 + T_1) (-1 + T_2)} - \\
 & \frac{1}{T_1^2 T_2 (-1 + T_1 T_2)} \\
 & (-a_2 b_5 - a_2 b_3 T_1 + a_4 b_3 T_1 - c_{51} T_1 - c_{54} T_1 - a_4 b_5 T_2 - a_4 b_3 T_1 T_2 + a_4 b_5 T_1 T_2 + c_{51} T_1^2 T_2 + c_{54} T_1^2 T_2) \\
 & p_{3,j} x_{3,j} - \frac{(a_2 b_5 - a_4 b_5 - c_5 + c_{14} + c_5 T_1 + a_4 b_5 T_2 - c_{14} T_1 T_2) p_{1,i} p_{3,j} x_{1,i} x_{3,j}}{(-1 + T_1) (-1 + T_1 T_2)} - \\
 & \frac{(-a_2 b_3 + c_5 + a_2 b_3 T_1 - a_4 b_3 T_1 - c_5 T_1 - a_4 b_3 T_2 - c_5 T_2 + 2 a_4 b_3 T_1 T_2 + c_5 T_1 T_2) p_{1,j} p_{3,j} x_{1,i} x_{3,j}}{(-1 + T_2) (-1 + T_1 T_2)} - \\
 & \frac{(a_2 b_3 - c_6 T_1 + c_{15} T_1 + a_4 b_3 T_2 - a_4 b_3 T_1 T_2 + c_6 T_1 T_2 - c_{15} T_1^2 T_2) p_{2,i} p_{3,j} x_{2,i} x_{3,j}}{T_1 (-1 + T_2) (-1 + T_1 T_2)} - \\
 & \frac{1}{(-1 + T_1) T_1 (-1 + T_1 T_2)} \\
 & (c_6 T_1 - c_6 T_1^2 + a_2 b_5 T_2 - a_2 b_5 T_1 T_2 + a_4 b_5 T_1 T_2 - c_6 T_1 T_2 + c_6 T_1^2 T_2 + a_4 b_5 T_2^2 - 2 a_4 b_5 T_1 T_2^2) \\
 & p_{2,j} p_{3,j} x_{2,i} x_{3,j}
 \end{aligned}$$

Reidemeister 2b



```
In[*]:= Timing[ Short[ LeftR2b = ( ( Integrate[ F[i, j] * L / @ ( X_{i,j}[1] X_{i+,j+}[-1] ) d[ {vs_i, vs_j, vs_{i+}, vs_{j+}} ] ) [[1]] ] ] ] ]
```

```
Out[*]= { 0.234375, Series[ p_{1,2+i} pi_{1,i} + p_{1,2+j} pi_{1,j} + p_{2,2+i} pi_{2,i} + <<10>> + p_{3,2+i} pi_{3,i} + p_{3,2+j} pi_{3,j}, { -1 + T_1 } T_1^2 { -1 + T_2 } T_2 + <<1>> + <<74>> + <<1>> + ( <<1>> ) <<3>> pi_{<<1>>} ] ] }
```

```
In[*]:= RightR2b = Series[ p_{1,2+i} pi_{1,i} + p_{1,2+j} pi_{1,j} + p_{2,2+i} pi_{2,i} + p_{2,2+j} pi_{2,j} + p_{3,2+i} pi_{3,i} + p_{3,2+j} pi_{3,j}, 0 ]
```

```
Out[*]= Series[ p_{1,2+i} pi_{1,i} + p_{1,2+j} pi_{1,j} + p_{2,2+i} pi_{2,i} + p_{2,2+j} pi_{2,j} + p_{3,2+i} pi_{3,i} + p_{3,2+j} pi_{3,j}, 0 ]
```

```
In[*]:= Short[eqn = CF[LeftR2b[[1]] - RightR2b[[1]]]
cvs = Union@Cases[eqn, p_ | π_, ∞]
vars = Union@Cases[r_[-1, i, j], d_, ∞]
Short[eqns = CoefficientRules[eqn, cvs] /. (_ -> c_) :-> (c == 0), 3]
{sol} = Solve[eqns, vars]
```

Out[*]//Short=

$$\frac{(d_7 + d_3 T_1 - d_7 T_1 + d_5 T_2 - d_7 T_2 + d_1 T_1 T_2 - d_3 T_1 T_2 - d_5 T_1 T_2 + d_7 T_1 T_2) \ll 2 \gg \pi_{\ll 1 \gg}}{B T_1 T_2} +$$

$$\frac{(\ll 1 \gg) \ll 2 \gg \ll 1 \gg}{B T_1 T_2} + \ll 5 \gg + \frac{\ll 1 \gg}{\ll 1 \gg} + \frac{(-d_7 + d_7 T_1 T_2 + d_8 T_1 T_2) \ll 2 \gg \pi_{\ll 1 \gg}}{B T_1 T_2}$$

Out[*]=

$$\{p_{3,2+i}, p_{3,2+j}, \pi_{1,i}, \pi_{1,j}, \pi_{2,i}, \pi_{2,j}\}$$

Out[*]=

$$\{d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8\}$$

Out[*]//Short=

$$\left\{ \begin{aligned} \frac{d_1}{B} - \frac{d_3}{B} - \frac{d_5}{B} + \frac{d_7}{B} + \frac{d_5}{B T_1} - \frac{d_7}{B T_1} + \frac{d_3}{B T_2} - \frac{d_7}{B T_2} + \frac{d_7}{B T_1 T_2} &= 0, \frac{d_3}{B T_2} - \frac{d_7}{B T_2} + \frac{d_7}{B T_1 T_2} = 0, \\ \frac{d_5}{B T_1} - \frac{d_7}{B T_1} + \frac{d_7}{B T_1 T_2} = 0, \frac{d_7}{B T_1 T_2} = 0, \frac{a_2}{B} - \frac{d_1}{B} + \frac{d_3}{B} + \ll 32 \gg + \frac{d_7 T_1 T_2}{B} + \frac{d_8 T_1 T_2}{B} &= 0, \\ \frac{a_4}{B} + \frac{d_7}{B} + \frac{d_8}{B} + \frac{d_3 T_1}{B} + \frac{d_4 T_1}{B} - \frac{d_7 T_1}{B} - \frac{d_8 T_1}{B} - \frac{d_3}{B T_2} + \frac{d_7}{B T_2} - \frac{d_7}{B T_1 T_2} &= 0, \\ \frac{d_7}{B} + \frac{d_8}{B} - \frac{a_2}{B T_1} - \frac{d_5}{B T_1} + \frac{d_7}{B T_1} - \frac{d_7}{B T_1 T_2} + \frac{d_5 T_2}{B} + \frac{d_6 T_2}{B} - \frac{d_7 T_2}{B} - \frac{d_8 T_2}{B} - \frac{a_4 T_2}{B T_1} &= 0, \\ \frac{d_7}{B} + \frac{d_8}{B} - \frac{d_7}{B T_1 T_2} &= 0 \end{aligned} \right\}$$

Out[*]=

$$\left\{ \left\{ d_1 \rightarrow 0, d_2 \rightarrow -\frac{a_2 - a_4 T_1 + a_4 T_2}{T_1^2 T_2}, d_3 \rightarrow 0, d_4 \rightarrow -\frac{a_4}{T_1}, d_5 \rightarrow 0, d_6 \rightarrow -\frac{-a_2 - a_4 T_2}{T_1 T_2}, d_7 \rightarrow 0, d_8 \rightarrow 0 \right\} \right\}$$

```
In[*]:= sol /. (v_ -> val_) :-> (v = CF[val]);
r_[-1, i, j]
```

Out[*]=

$$\frac{(-a_2 + a_4 T_1 - a_4 T_2) p_{3,j} x_{1,i} x_{2,i}}{T_1^2 T_2} + \frac{(a_2 + a_4 T_2) p_{3,j} x_{1,j} x_{2,i}}{T_1 T_2} - \frac{a_4 p_{3,j} x_{1,i} x_{2,j}}{T_1}$$

```
In[*]:= Short[eqn = CF[LeftR2b[[2]] - RightR2b[[2]]]
cvs = Union@Cases[eqn, p__ | π__, ∞]
vars = Union@Cases[r1[-1, i, j] + r42[-1, i, j], e_ | f_, ∞]
Short[eqns = CoefficientRules[eqn, cvs] /. (_ -> c_) :-> (c == 0), 3]
Short[{sol} = Solve[eqns, vars]]
```

Out[*]//Short=

$$\ll<85>> + \frac{\ll<1>>}{\ll<1>>} + \frac{(f_{33} + \ll<16>> + f_{48} T_1 T_2^2) \ll<3>> \pi_{\ll<1>>}}{T_1 T_2^2}$$

Out[*]=

$$\{p_{1,2+i}, p_{1,2+j}, p_{2,2+i}, p_{2,2+j}, p_{3,2+i}, p_{3,2+j}, \pi_{1,i}, \pi_{1,j}, \pi_{2,i}, \pi_{2,j}, \pi_{3,i}, \pi_{3,j}\}$$

Out[*]=

$$\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}, f_{11}, f_{12}, f_{13}, f_{14}, f_{15}, f_{16}, f_{17}, f_{18}, f_{19}, f_{20}, f_{21}, f_{22}, f_{23}, f_{24}, f_{25}, f_{26}, f_{27}, f_{28}, f_{29}, f_{30}, f_{31}, f_{32}, f_{33}, f_{34}, f_{35}, f_{36}, f_{37}, f_{38}, f_{39}, f_{40}, f_{41}, f_{42}, f_{43}, f_{44}, f_{45}, f_{46}, f_{47}, f_{48}, f_{49}, f_{50}, f_{51}, f_{52}, f_{53}, f_{54}, f_{55}, f_{56}, f_{57}, f_{58}, f_{59}, f_{60}\}$$

Out[*]//Short=

$$\left\{ f_1 - f_7 - f_{25} + f_{31} + \frac{f_{25}}{T_1} - \frac{f_{31}}{T_1} + \frac{f_7}{T_2} - \frac{f_{31}}{T_2} + \frac{f_{31}}{T_1 T_2} = 0, \right.$$

$$\left. \frac{f_7}{T_2} - \frac{f_{31}}{T_2} + \frac{f_{31}}{T_1 T_2} = 0, \ll<1>> = 0, \ll<63>>, \ll<1>> = 0, \ll<1>> = 0, \right.$$

$$\left. \frac{2 a_4 b_3}{(1 - T_1) (1 - T_2)} + \frac{2 c_{49}}{(1 - T_1) (1 - T_2)} + \frac{2 c_{50}}{(1 - T_1) (1 - T_2)} + \frac{c_{52}}{(1 - T_1) (1 - T_2)} + \frac{c_{53}}{(1 - T_1) (1 - T_2)} - \right.$$

$$\left. \frac{c_{54}}{(1 - T_1) (1 - T_2)} + \ll<194>> + \frac{f_{49} T_1 T_2}{(1 - T_1) (1 - T_2)} + \frac{f_{50} T_1 T_2}{(1 - T_1) (1 - T_2)} + \right.$$

$$\left. \frac{f_{51} T_1 T_2}{(1 - T_1) (1 - T_2)} + \frac{f_{58} T_1 T_2}{(1 - T_1) (1 - T_2)} + \frac{f_{59} T_1 T_2}{(1 - T_1) (1 - T_2)} + \frac{f_{60} T_1 T_2}{(1 - T_1) (1 - T_2)} = 0 \right\}$$

Out[*]//Short=

$$\left\{ \left\{ e_1 \rightarrow 0, e_2 \rightarrow 0, e_3 \rightarrow -\frac{b_3}{T_2}, e_4 \rightarrow 0, e_5 \rightarrow -\frac{b_5}{T_1}, e_6 \rightarrow 0, \ll<56>>, f_{55} \rightarrow 0, f_{56} \rightarrow 0, f_{57} \rightarrow 0, \right. \right.$$

$$\left. \left. f_{58} \rightarrow -\frac{-c_{49} - c_{52}}{T_1}, f_{59} \rightarrow -\frac{-c_{50} - c_{53}}{T_2}, f_{60} \rightarrow -\frac{a_2 b_5 + a_2 b_3 T_1 - a_4 b_3 T_1 + \ll<10>>}{T_1^2 T_2 (-1 + T_1 T_2)} \right\} \right\}$$

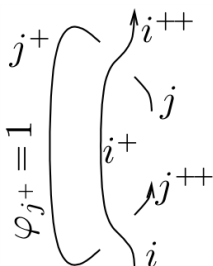
```
In[*]:= sol /. (v_ -> val_) :-> (v = CF[val]);
```

```
In[*]:= r1[-1, i, j]
Short[CF[r42[-1, i, j]], 5]
```

$$\text{Out[*]} = \frac{b_5 p_{1,j} p_{2,i} x_{3,i}}{T_1} - \frac{b_3 p_{1,i} p_{2,j} x_{3,i}}{T_2} + \frac{(b_3 T_1 + b_5 T_2) p_{1,j} p_{2,j} x_{3,i}}{T_1 T_2}$$

$$\begin{aligned} \text{Out[*]//Short} = & -c_{49} p_{1,i} x_{1,i} + \frac{(-c_{49} - c_{52} + c_{49} T_1^2) p_{1,j} x_{1,i}}{T_1^2} + \ll 43 \gg + \\ & \frac{(a_2 b_3 - c_6 T_1 + c_{15} T_1 + a_4 b_3 T_2 - a_4 b_3 T_1 T_2 + c_6 T_1 T_2 - c_{15} T_1^2 T_2) p_{2,i} p_{3,j} x_{2,i} x_{3,j}}{T_1 (-1 + T_2) (-1 + T_1 T_2)} + \\ & \left((a_2 b_3 - a_2 b_3 T_1 + c_{15} T_1 - c_{15} T_1^2 + a_4 b_3 T_2 + a_2 b_5 T_2 - 2 a_4 b_3 T_1 T_2 - a_2 b_5 T_1 T_2 + a_4 b_5 T_1 T_2 + a_4 b_3 T_1^2 T_2 - \right. \\ & \left. c_{15} T_1^2 T_2 + c_{15} T_1^3 T_2 + a_4 b_5 T_2^2 - 2 a_4 b_5 T_1 T_2^2) p_{2,j} p_{3,j} x_{2,i} x_{3,j} \right) / ((-1 + T_1) T_1 T_2 (-1 + T_1 T_2)) \end{aligned}$$

Reidemeister 2c



```
In[*]:= Timing[ Short[ {LeftR2c} = Cases [
    [ F[i, j] x L /@ (Xi+1,j[1] Xi,j+2[-1] Cj+1[1]) d {vs_i, vs_j, vs_i+, vs_j+, vs_j+2}, E[ E_ ] => E ]
]]]
```

$$\text{Out[*]} = \{0.171875, \{\ll 1 \gg\}\}$$

```
In[*]:= Timing[ Short[ {RightR2c} =
    Cases [ [ F[i, j] x L /@ (Ci[0] Ci+1[0] Cj[0] Cj+1[1] Cj+2[0]) d {vs_i, vs_j, vs_i+, vs_j+, vs_j+2},
    E[ E_ ] => E ]
]]]
```

$$\begin{aligned} \text{Out[*]} = & \{0.015625, \\ & \left\{ e\text{Series} \left[p_{1,2+i} \pi_{1,i} + p_{1,3+j} \pi_{1,j} + p_{2,2+i} \pi_{2,i} + p_{2,3+j} \pi_{2,j} + \frac{g_1 p_{3,3+j} \pi_{1,j} \pi_{2,j}}{B} + p_{3,2+i} \pi_{3,i} + p_{3,3+j} \pi_{3,j}, \right. \right. \\ & \left. \left. g_1 g_2 + \ll 12 \gg + (g_1 g_2 + g_8) p_{2,3+j} p_3 \ll 1 \gg \ll 1 \gg \pi_{2,j} \pi_{3,j} \right] \right\} \end{aligned}$$

```
In[*]:= Short[eqn = CF[LeftR2c[[1]] - RightR2c[[1]]]
cvs = Union@Cases[eqn, p__ | π__, ∞]
vars = Union@Cases[γ₀[1, k], g_, ∞]
Short[eqns = CoefficientRules[eqn, cvs] /. (_ -> c_) :-> (c == 0), 3]
{sol} = Solve[eqns, vars]
```

Out[*]//Short=

$$\frac{g_1 (-1 + T_1) (-1 + T_2) p_{3,3+j} \pi_{1,i} \pi_{2,i}}{B T_1 T_2} - \frac{g_1 (-1 + T_2) p_{3, \ll 1 \gg} \pi_{1,j} \pi_{2,i}}{B T_2} - \frac{g_1 (-1 + T_1) p_{3,3+j} \pi_{1,i} \pi_{2,j}}{B T_1}$$

Out[*]=

$$\{p_{3,3+j}, \pi_{1,i}, \pi_{1,j}, \pi_{2,i}, \pi_{2,j}\}$$

Out[*]=

$$\{g_1\}$$

Out[*]//Short=

$$\left\{ \frac{g_1}{B} - \frac{g_1}{B T_1} - \frac{g_1}{B T_2} + \frac{g_1}{B T_1 T_2} == 0, -\frac{g_1}{B} + \frac{g_1}{B T_1} == 0, -\frac{g_1}{B} + \frac{g_1}{B T_2} == 0 \right\}$$

Out[*]=

$$\{\{g_1 \rightarrow 0\}\}$$

```
In[*]:= sol /. (v_ -> val_) :-> (v = CF[val]);
γ₀[1, k]
```

Out[*]=

$$0$$

```

In[*]:= Short[eqn = CF[LeftR2c[[2]] - RightR2c[[2]]]
cvs = Union@Cases[eqn, p__ | π__, ∞]
vars = Union@Cases[γ1[1, k] + γ42[1, k], g_, ∞]
Short[eqns = CoefficientRules[eqn, cvs] /. (_ -> c_) :-> (c == 0), 3]
Short[{sol} = Solve[eqns, vars]]

Out[*]//Short=

$$-\frac{(g_3 + g_6 + g_7)(-1 + T_1) p_{1,3+j} \pi_{1,i}}{T_1} - \frac{(g_4 + g_6 + g_8) \langle\langle 1 \rangle\rangle \langle\langle 1 \rangle\rangle \pi_{2,i}}{T_2} + \langle\langle 18 \rangle\rangle$$


Out[*]=
{p1,3+j, p2,3+j, p3,3+j, π1,i, π1,j, π2,i, π2,j, π3,i, π3,j}

Out[*]=
{g2, g3, g4, g5, g6, g7, g8}

Out[*]//Short=

$$\left\{ g_6 - \frac{g_6}{T_1} - \frac{g_6}{T_2} + \frac{a_2 g_2}{T_1 T_2} + \frac{g_6}{T_1 T_2} = 0, -g_6 + \frac{\langle\langle 1 \rangle\rangle \langle\langle 1 \rangle\rangle}{T_1} + \frac{g_6}{T_1} = 0, \langle\langle 9 \rangle\rangle, \langle\langle 1 \rangle\rangle = 0, \right.$$


$$\frac{2 a_4 b_3}{(1 - T_1)(1 - T_2)} + \frac{a_4 b_3}{(1 - T_1) T_1^2 (1 - T_2)} + \frac{a_2 b_5}{(1 - T_1) T_1^2 (1 - T_2)} - \frac{a_4 b_5}{(1 - T_1) T_1^2 (1 - T_2)} -$$


$$\frac{3 a_4 b_3}{(1 - T_1) T_1 (1 - T_2)} - \frac{a_2 b_5}{(1 - T_1) T_1 (1 - T_2)} + \langle\langle 33 \rangle\rangle + \frac{g_8 T_2}{(1 - T_1)(1 - T_2)} + \frac{a_4 b_5 T_2}{(1 - T_1) T_1^2 (1 - T_2)} -$$


$$\left. \frac{2 a_4 b_5 T_2}{(1 - T_1) T_1 (1 - T_2)} - \frac{g_5 T_1 T_2}{(1 - T_1)(1 - T_2)} - \frac{g_7 T_1 T_2}{(1 - T_1)(1 - T_2)} - \frac{g_8 T_1 T_2}{(1 - T_1)(1 - T_2)} = 0 \right\}$$


Out[*]//Short=

$$\left\{ \left\{ g_2 \rightarrow 0, g_3 \rightarrow 0, g_4 \rightarrow 0, g_5 \rightarrow -\frac{-a_2 b_3 + a_2 b_5 + \langle\langle 22 \rangle\rangle + 2 a_4 b_5 T_1 T_2^2}{(-1 + T_1) T_1 (-1 + T_2) (-1 + T_1 T_2)}, g_6 \rightarrow 0, g_7 \rightarrow 0, g_8 \rightarrow 0 \right\} \right\}$$


In[*]:= sol /. (v_ -> val_) :-> (v = CF[val]);

In[*]:= γ1[1, k]
Short[CF[γ42[1, k]], 5]

Out[*]=
0

Out[*]//Short=

$$\frac{(-b_3 + b_5 + b_3 T_1 - b_5 T_2)(-a_2 + a_2 T_1 - a_4 T_1 - a_4 T_2 + 2 a_4 T_1 T_2) p_{3,k} x_{3,k}}{(-1 + T_1) T_1 (-1 + T_2) (-1 + T_1 T_2)}$$


```

C_k[1] and C_k[-1] are inverses

```

In[*]:= Timing[Short[Cases[{{∫ F[k] × L / @ (Ck[1] Ck+1[-1]) d{vsk, vsk+}}, E[ε_] :-> ε}]]]

Out[*]=
{0., {Series[p1,2+k π1,k + p2,2+k π2,k +  $\frac{h_1 p_{3,2+k} \pi_{1,k} \pi_{2,k}}{B}$  + p3,2+k π3,k, <<1>>]}}

```

```
In[*]:= Timing [ Short [ { RightCC } = Cases [ { {  $\int \mathcal{F}[k] \times \mathcal{L} / @ (C_k[0] C_{k+1}[0]) \, d\{v_{S_k}, v_{S_k^*}\}$  },  $\mathbb{E}[\mathcal{E}_-] \Rightarrow \mathcal{E}$  } ] ] ]
```

```
Out[*]= {0., {Series [ p_{1,2+k} \pi_{1,k} + p_{2,2+k} \pi_{2,k} + p_{3,2+k} \pi_{3,k}, 0 ] } }
```

```
In[*]:= Short [ eqn = CF [ LeftCC [1] - RightCC [1] ] ]
cvs = Union@Cases [ eqn, p_ | \pi_, \infty ]
vars = Union@Cases [ \gamma_0 [-1, k], h_, \infty ]
Short [ eqns = CoefficientRules [ eqn, cvs ] /. ( _ -> c_ ) => ( c == 0 ), 3 ]
{sol} = Solve [ eqns, vars ]
```

```
Out[*]//Short= 
$$\frac{h_1 p_{3,2+k} \pi_{1,k} \pi_{2,k}}{B}$$

```

```
Out[*]= { p_{3,2+k}, \pi_{1,k}, \pi_{2,k} }
```

```
Out[*]= { h_1 }
```

```
Out[*]//Short= 
$$\left\{ \frac{h_1}{B} == 0 \right\}$$

```

```
Out[*]= { { h_1 -> 0 } }
```

```
In[*]:= sol /. ( v_ -> val_ ) => ( v = CF [ val ] );
\gamma_0 [-1, k]
```

```
Out[*]= 0
```

```
In[*]:= Short[eqn = CF[LeftCC[[2]] - RightCC[[2]]]
cvs = Union@Cases[eqn, p_ | π_, ∞]
vars = Union@Cases[γ1[-1, k] + γ42[-1, k], h_, ∞]
Short[eqns = CoefficientRules[eqn, cvs] /. (_ -> c_) :-> (c == 0), 3]
Short[{sol} = Solve[eqns, vars]]
```

Out[*]//Short=

$$\frac{a_2 b_3 - a_2 b_5 - 2 a_2 b_3 T_1 + a_4 b_3 T_1 + \ll 70 \gg + h_6 T_1^3 T_2^2 + h_7 T_1^3 T_2^2 + h_8 T_1^3 T_2^2}{(-1 + T_1) T_1 (-1 + T_2) (-1 + T_1 T_2)} +$$

$$\ll 6 \gg + h_8 p_{2,2+k} p_{3,2+k} \pi_{2,k} \pi_{3,k}$$

Out[*]=

$$\{p_{1,2+k}, p_{2,2+k}, p_{3,2+k}, \pi_{1,k}, \pi_{2,k}, \pi_{3,k}\}$$

Out[*]=

$$\{h_2, h_3, h_4, h_5, h_6, h_7, h_8\}$$

Out[*]//Short=

$$\left\{ h_6 == 0, \ll 6 \gg, \right.$$

$$\frac{2 a_2 b_3}{(1 - T_1) (1 - T_2) (1 - T_1 T_2)} - \frac{a_4 b_3}{(1 - T_1) (1 - T_2) (1 - T_1 T_2)} - \frac{a_2 b_5}{(1 - T_1) (1 - T_2) (1 - T_1 T_2)} +$$

$$\left. \frac{a_4 b_5}{(1 - T_1) (1 - T_2) (1 - T_1 T_2)} + \frac{h_3}{(1 - T_1) (1 - T_2) (1 - T_1 T_2)} + \frac{h_4}{\ll 1 \gg} + \ll 72 \gg == 0 \right\}$$

Out[*]//Short=

$$\left\{ \left\{ h_2 \rightarrow 0, h_3 \rightarrow 0, h_4 \rightarrow 0, \right. \right.$$

$$\left. h_5 \rightarrow - \frac{\ll 1 \gg}{\frac{1}{(1-T_1) (\ll 1 \gg)} - \frac{T_1}{\ll 1 \gg} - \ll 1 \gg + \ll 1 \gg + \frac{\ll 1 \gg}{\ll 1 \gg} - \frac{T_1^2 \ll 1 \gg}{\ll 1 \gg}}, h_6 \rightarrow 0, h_7 \rightarrow 0, h_8 \rightarrow 0 \right\}$$

```
In[*]:= sol /. (v_ -> val_) :-> (v = CF[val]);
```

```
In[*]:= γ1[-1, k]
Short[CF[γ42[-1, k]], 5]
```

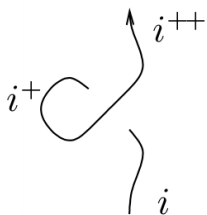
Out[*]=

$$0$$

Out[*]//Short=

$$- \frac{(-b_3 + b_5 + b_3 T_1 - b_5 T_2) (-a_2 + a_2 T_1 - a_4 T_1 - a_4 T_2 + 2 a_4 T_1 T_2) p_{3,k} x_{3,k}}{(-1 + T_1) T_1 (-1 + T_2) (-1 + T_1 T_2)}$$

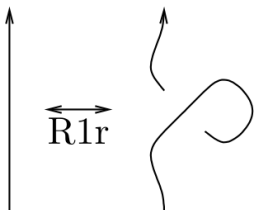
Invariance Under R11



$$In[*]:= \{\text{LeftR1l}\} = \text{Cases} \left[\left\{ \int \mathcal{F}[\mathbf{i}] \times \mathcal{L} / @ (X_{i+2,i}[\mathbf{1}] C_{i+1}[\mathbf{1}]) \, d\{\mathbf{vS}_i, \mathbf{vS}_{i^*}, \mathbf{vS}_{i+2}\} \right\}, \mathbb{E}[\mathcal{E}_-] \Rightarrow \mathcal{E}, \infty \right]$$

$$Out[*]= \{\in \text{Series}[p_{1,3+i} \pi_{1,i} + p_{2,3+i} \pi_{2,i} + p_{3,3+i} \pi_{3,i}, \mathbf{0}]\}$$

Invariance Under R1r



$$In[*]:= \{\text{LeftR1r}\} = \text{Cases} \left[\left\{ \int \mathcal{F}[\mathbf{i}] \times \mathcal{L} / @ (X_{i,i+2}[\mathbf{1}] C_{i+1}[-\mathbf{1}]) \, d\{\mathbf{vS}_i, \mathbf{vS}_{i^*}, \mathbf{vS}_{i+2}\} \right\}, \mathbb{E}[\mathcal{E}_-] \Rightarrow \mathcal{E}, \infty \right]$$

$$Out[*]= \{\in \text{Series}[p_{1,3+i} \pi_{1,i} + p_{2,3+i} \pi_{2,i} + p_{3,3+i} \pi_{3,i}, \mathbf{0}]\}$$

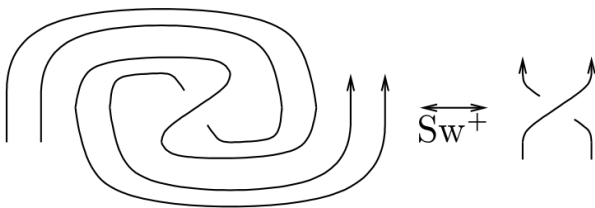
$$In[*]:= \{\text{RightR1r}\} = \text{Cases} \left[\left\{ \int \mathcal{F}[\mathbf{i}] \times \mathcal{L} / @ (C_i[\mathbf{0}] C_{i+1}[\mathbf{0}] C_{i+2}[\mathbf{0}]) \, d\{\mathbf{vS}_i, \mathbf{vS}_{i^*}, \mathbf{vS}_{i+2}\} \right\}, \mathbb{E}[\mathcal{E}_-] \Rightarrow \mathcal{E}, \infty \right]$$

$$Out[*]= \{\in \text{Series}[p_{1,3+i} \pi_{1,i} + p_{2,3+i} \pi_{2,i} + p_{3,3+i} \pi_{3,i}, \mathbf{0}]\}$$

$$In[*]:= \text{LeftR1r} == \text{RightR1r}$$

$$Out[*]= \text{True}$$

Invariance Under Sw



$$In[*]:= \text{Timing} \left[\text{Short} \left[\{\text{LeftSw}\} = \text{Cases} \left[\left\{ \int \mathcal{F}[\mathbf{i}, \mathbf{j}] \times \mathcal{L} / @ (X_{i+1,j+1}[\mathbf{1}] C_i[-\mathbf{1}] C_j[-\mathbf{1}] C_{i+2}[\mathbf{1}] C_{j+2}[\mathbf{1}]) \, d\{\mathbf{vS}_i, \mathbf{vS}_j, \mathbf{vS}_{i^*}, \mathbf{vS}_{j^*}, \mathbf{vS}_{j+2}\} \right\}, \mathbb{E}[\mathcal{E}_-] \Rightarrow \mathcal{E}, \infty \right] \right] \right]$$

$$Out[*]= \{0.03125, \left\{ \in \text{Series} \left[T_1 p_{1,2+i} \pi_{1,i} + \ll 20 \gg + p_{3,3+i} x_{3,2+i}, \frac{\ll 1 \gg}{\ll 1 \gg} + \ll 48 \gg + \frac{\ll 1 \gg}{\ll 1 \gg} \right] \right\}$$

```
In[*]:= Timing [ Short [ { RightSw } = Cases [
    { ∫ ℱ [ i, j ] × ℒ / @ ( Xi+1,j+1 [ 1 ] Ci [ 0 ] Cj [ 0 ] Ci+2 [ 0 ] Cj+2 [ 0 ] ) d { vsi, vsj, vsi+, vsj+, vsj+2 } },
    E [ ℰ- ] ⇒ ℰ, ∞ ]
]]
```

```
Out[*]= { 0.078125, { ∈Series [ T1 p1,2+i π1,i + <<20>> + p3,3+i x3,2+i,  $\frac{\llcorner 1 \gg \llcorner 1 \gg \llcorner 1 \gg}{\llcorner 1 \gg} + \llcorner 48 \gg + \llcorner 1 \gg$  ] ] }
```

```
In[*]:= LeftSw [ 1 ] == RightSw [ 1 ]
```

```
Out[*]= True
```

```
In[*]:= Short [ eqn = CF [ LeftSw [ 2 ] - RightSw [ 2 ] ] ]
cvs = Union @ Cases [ eqn, p- | π-, ∞ ]
vars = Union @ Cases [ eqn, ( c | d | e | f | g | h )-, ∞ ]
Short [ eqns = CoefficientRules [ eqn, cvs ] /. ( _ → c- ) ⇒ ( c == 0 ), 3 ]
Short [ { sol } = Solve [ eqns, vars ] ]
```



```
Out[*]//Short= 
$$-\frac{(-b_3 + b_5 + b_3 \llcorner 1 \gg - b_5 T_2) (\llcorner 1 \gg)}{(-1 + T_1) T_1 (\llcorner 1 \gg) (-1 + T_1 T_2)} - \llcorner 1 \gg + \frac{\llcorner 1 \gg}{\llcorner 1 \gg}$$

```

```
Out[*]= { p3,2+i, π3,i }
```

```
Out[*]= { }
```

```
Out[*]//Short= { <<1>> }
```

 **Set:** Lists {sol} and $\left\{ \left\{ a_2 \rightarrow \frac{a_4 T_1 + a_4 \llcorner 1 \gg - 2 a_4 T_1 T_2}{-1 + T_1} \right\}, \left\{ T_2 \rightarrow \frac{-b_3 + b_5 + b_3 T_1}{b_5} \right\}, \{ b_3 \rightarrow 0, b_5 \rightarrow 0 \} \right\}$ are not the same shape. 

```
Out[*]//Short= 
$$\left\{ \left\{ a_2 \rightarrow \frac{a_4 T_1 + a_4 \llcorner 1 \gg - 2 a_4 T_1 T_2}{-1 + T_1} \right\}, \left\{ T_2 \rightarrow \frac{\llcorner 1 \gg}{\llcorner 1 \gg} \right\}, \{ b_3 \rightarrow 0, b_5 \rightarrow 0 \} \right\}$$

```

```
In[*]:= Factor [ eqn ]
```

```
Out[*]= - ( ( (-b3 + b5 + b3 T1 - b5 T2) (-a2 + a2 T1 - a4 T1 - a4 T2 + 2 a4 T1 T2)
    (1 + T1 T2 p3,2+i π3,i - p3,2+i x3,2+i) ) / ( (-1 + T1) T1 (-1 + T2) (-1 + T1 T2) ) )
```