

Pensieve header: An false implementation of nilpotent integration (still using exp).

Initialization

```
In[*]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\HigherRank"];
Once[<< KnotTheory` ; << Rot.m];
```

Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.

Read more at <http://katlas.org/wiki/KnotTheory>.

Loading Rot.m from <http://drorbn.net/icbs24> to compute rotation numbers.

```
In[*]:= CCF[ε_] := ExpandDenominator@ExpandNumerator@Together[ε];
CCF[ε_] := Factor[ε];
CF[ε_E] := CF /@ ε;
CF[ε_List] := CF /@ ε;
CF[sd_SeriesData] := MapAt[CF, sd, 3];
CF[ε_] := Module[{vs = Cases[ε, (x | p)_, ∞] ∪ {x, p}, ps, c},
  Total[CoefficientRules[Expand[ε], vs] /. (ps_ -> c_) -> CCF[c] (Times @@ vs^ps)]];
```

Integration

```
In[*]:= E /: E[A_] × E[B_] := E[A + B]
```

```
In[*]:= Unprotect[Integrate];
∫ ω_. E[L_] d[vs_List] := Module[{n, Q, G, V, s, t, k, a, b},
  n = Length@vs;
  Q = -Table[(∂vs[[a]], vs[[b]] L) /. Thread[vs -> 0], {a, n}, {b, n}];
  G = Inverse[Q] / 2;
  V = L + vs.Q.vs / 2;
  s = t = V; k = 0;
  While[0 != t,
    s +=
      
$$\frac{1}{(++k)!} (t = \text{CF}@\text{Sum}[G[[a, b]] ((\partial_{vs[[a]], vs[[b]] t) + (\partial_{vs[[a]]} t) (\partial_{vs[[b]]} t)), \{a, n\}, \{b, n\}]]];
    PowerExpand@Factor[\omega (Det[Q] (2 \pi)^n)^{-1/2}] × E[CF@s /. Thread[vs -> 0]]
  ];
Protect[Integrate];$$

```

$$In[*]:= \int \mathbb{E} \left[\frac{i \lambda x_1^2}{2} \right] \mathbb{d} \{x_1\}$$

$$Out[*]= \frac{(-1)^{1/4} \mathbb{E} [0]}{\sqrt{2 \pi} \sqrt{\lambda}}$$

$$In[*]:= \int \mathbb{E} \left[-\frac{i \lambda x_1^2}{2} \right] \mathbb{d} \{x_1\}$$

$$Out[*]= -\frac{(-1)^{3/4} \mathbb{E} [0]}{\sqrt{2 \pi} \sqrt{\lambda}}$$

$$In[*]:= \int \mathbb{E} \left[\frac{i}{2} \{x_1, x_2\} \cdot \begin{pmatrix} a & b \\ b & c \end{pmatrix} \cdot \{x_1, x_2\} \right] \mathbb{d} \{x_1, x_2\}$$

$$Out[*]= \frac{\mathbb{E} [0]}{2 \sqrt{b^2 - a c} \pi}$$

$$In[*]:= \int \mathbb{E} \left[-\frac{\lambda x_1^2}{2} \right] \mathbb{d} \{x_1\}$$

$$Out[*]= \frac{\mathbb{E} [0]}{\sqrt{2 \pi} \sqrt{\lambda}}$$

$$In[*]:= \int \mathbb{E} \left[-\frac{x_1^2}{2} + \xi x_1 \right] \mathbb{d} \{x_1\}$$

$$Out[*]= \frac{\mathbb{E} \left[\frac{\xi^2}{2} \right]}{\sqrt{2 \pi}}$$

$$In[*]:= \int \mathbb{E} \left[-\frac{1}{2} \{x_1, x_2\} \cdot \begin{pmatrix} a & b \\ b & c \end{pmatrix} \cdot \{x_1, x_2\} + \{\xi_1, \xi_2\} \cdot \{x_1, x_2\} \right] \mathbb{d} \{x_1, x_2\}$$

$$Out[*]= \frac{\mathbb{E} \left[\frac{c \xi_1^2 - 2 b \xi_1 \xi_2 + a \xi_2^2}{2 (-b^2 + a c)} \right]}{2 \sqrt{-b^2 + a c} \pi}$$

$$In[*]:= \mathbf{I1} = \int \mathbb{E} \left[-\frac{1}{2} \{x_1, x_2\} \cdot \begin{pmatrix} a & b \\ b & c \end{pmatrix} \cdot \{x_1, x_2\} + \{\xi_1, \xi_2\} \cdot \{x_1, x_2\} \right] \mathbb{d} \{x_1\}$$

$$Out[*]= \frac{\mathbb{E} \left[-\frac{(-b^2 + a c) x_2^2}{2 a} + \frac{\xi_1^2}{2 a} + \frac{x_2 (-b \xi_1 + a \xi_2)}{a} \right]}{\sqrt{a} \sqrt{2 \pi}}$$

$$In[*]:= \int \mathbf{I1} \, d\{\mathbf{x}_2\}$$

$$Out[*]= \frac{\mathbb{E} \left[\frac{c \xi_1^2 - 2b \xi_1 \xi_2 + a \xi_2^2}{2(-b^2 + ac)} \right]}{2 \sqrt{-b^2 + ac} \pi}$$

$$In[*]:= \int \mathbb{E} \left[-\frac{1}{2} \{\mathbf{y}_1, \mathbf{y}_2\} \cdot \begin{pmatrix} a & b \\ b & c \end{pmatrix} \cdot \{\mathbf{y}_1, \mathbf{y}_2\} + \{\eta_1, \eta_2\} \cdot \{\mathbf{y}_1, \mathbf{y}_2\} \right] d\{\mathbf{y}_1, \mathbf{y}_2\}$$

$$Out[*]= \frac{\mathbb{E} \left[\frac{c \eta_1^2 - 2b \eta_1 \eta_2 + a \eta_2^2}{2(-b^2 + ac)} \right]}{2 \sqrt{-b^2 + ac} \pi}$$

$$In[*]:= \mathbf{I1} = \int \mathbb{E} \left[-\frac{1}{2} \{\mathbf{y}_1, \mathbf{y}_2\} \cdot \begin{pmatrix} a & b \\ b & c \end{pmatrix} \cdot \{\mathbf{y}_1, \mathbf{y}_2\} + \{\eta_1, \eta_2\} \cdot \{\mathbf{y}_1, \mathbf{y}_2\} \right] d\{\mathbf{y}_1\}$$

$$Out[*]= \frac{\mathbb{E} \left[\frac{b^2 y_2^2 - ac y_2^2 - 2b y_2 \eta_1 + \eta_1^2 + 2a y_2 \eta_2}{2a} \right]}{\sqrt{a} \sqrt{2} \pi}$$

$$In[*]:= \int \mathbf{I1} \, d\{\mathbf{y}_2\}$$

$$Out[*]= \frac{\mathbb{E} \left[\frac{ac \eta_1^2 - 2ab \eta_1 \eta_2 + a^2 \eta_2^2}{2a(-b^2 + ac)} \right]}{2 \sqrt{-b^2 + ac} \pi}$$

$$In[*]:= \int \mathbb{E} \left[\xi \mathbf{x} + \eta \mathbf{y} + \mathbf{z} (\mathbf{x} - \mathbf{y}) + \mathbf{x}^2 \right] d\{\mathbf{x}, \mathbf{z}\}$$

$$Out[*]= \frac{i \mathbb{E} [\mathbf{y} (\mathbf{y} + \eta + \xi)]}{2 \pi}$$

The ρ_1 Integrand

Adopted from pensieve://Projects/APAI/PerturbedGaussianIntegration.nb.

```

In[*]:=
q[s_, i_, j_] := x_i ((1 - T^s) p_{i+1} + (T^s - 1) p_{j+1});
r1[s_, i_, j_] :=
  s (-1 + 2 p_i x_i - 2 p_j x_i + (T^s - 1) p_i p_j x_i^2 + (1 - T^s) p_j^2 x_i^2 - 2 p_i p_j x_i x_j + 2 p_j^2 x_i x_j) / 2;
γ1[φ_, k_] := ε φ (1 / 2 - x_k p_k);
ρ1i[s_, i_, j_] := T^{-s/2} E[-q[s, i, j] + ε r1[s, i, j] + O[ε]^2];
ρ1i[φ_, k_] := T^{-φ/2} E[-x_k (p_k - p_{k+1}) + γ1[φ, k] + O[ε]^2];
ρ1i[End, k_] := E[-x_k p_k + O[ε]^2];
ρ1i[K_] := Module[{Cs, φ, n, c, k, ε},
  {Cs, φ} = Rot[K]; n = Length[Cs];
  ε = ρ1i[End, 2 n + 1];
  Do[ε *= ρ1i @@ c, {c, Cs}];
  Do[ε *= ρ1i[φ[[k]], k], {k, 2 n}];
  CF@ε
];
ρ1vs[K_] := Union @@ Table[{x_i, p_i}, {i, 2 Crossings[K] + 1}]

```

```

In[*]:= ρ1i[Knot[3, 1]]

```

 KnotTheory: Loading precomputed data in PD4Knots`.

Out[*]=

$$\begin{aligned}
 & T^2 E \left[\left(-((p_1 - p_2) x_1) - (p_2 - p_3) x_2 - \frac{(-1 + T)(p_3 - p_6) x_2}{T} - (p_3 - p_4) x_3 + \frac{(-1 + T)(p_2 - p_5) x_4}{T} - \right. \right. \\
 & \quad \left. \left. (p_4 - p_5) x_4 - (p_5 - p_6) x_5 + \frac{(-1 + T)(p_4 - p_7) x_6}{T} - (p_6 - p_7) x_6 - p_7 x_7 \right) + \right. \\
 & \quad \left(-\frac{1}{2} + p_4 x_4 + \frac{1}{2} \left(1 + 2 p_1 x_4 - 2 p_4 x_4 - 2 p_1^2 x_1 x_4 + 2 p_1 p_4 x_1 x_4 - \left(1 - \frac{1}{T} \right) p_1^2 x_4^2 - \left(-1 + \frac{1}{T} \right) p_1 p_4 x_4^2 \right) + \right. \\
 & \quad \left. \frac{1}{2} \left(1 - 2 p_2 x_2 + 2 p_5 x_2 - \left(-1 + \frac{1}{T} \right) p_2 p_5 x_2^2 - \left(1 - \frac{1}{T} \right) p_5^2 x_2^2 + 2 p_2 p_5 x_2 x_5 - 2 p_5^2 x_2 x_5 \right) + \right. \\
 & \quad \left. \frac{1}{2} \left(1 + 2 p_3 x_6 - 2 p_6 x_6 - 2 p_3^2 x_3 x_6 + 2 p_3 p_6 x_3 x_6 - \left(1 - \frac{1}{T} \right) p_3^2 x_6^2 - \left(-1 + \frac{1}{T} \right) p_3 p_6 x_6^2 \right) \right] \in + O[\epsilon]^2
 \end{aligned}$$

```

In[*]:= ρ1vs[Knot[3, 1]]

```

Out[*]=

{p1, p2, p3, p4, p5, p6, p7, x1, x2, x3, x4, x5, x6, x7}

Integration of ε-Series

Using Picard Iteration!

```

In[*]:= Unprotect[Integrate];
Integrate::sing = "How dare you ask me to integrate a singular Gaussian!";
∫ ω_. E[L_SeriesData] d(vs_List) := Module[{n, m, ε, L0, Q, Δ, G, Z, e, λ, a, b},
  ε = L[[1]]; m = L[[5]];
  n = Length@vs; L0 = Normal@L /. ε → 0;
  Q = Table[(-∂vs[[a], vs[[b]] L0) /. Thread[vs → 0] /. (p | x) → 0, {a, n}, {b, n}];
  If[(Δ = CF@Det[Q]) == 0, Message[Integrate::sing]; Return[]];
  Z = L + vs.Q.vs / 2; G = Inverse[Q];
  While[
    e = Normal@CF[(∂λ Z) - 1/2 ∑_{a=1}^n ∑_{b=1}^n ((∂vs[[a], vs[[b]] Z) + (∂vs[[a]] Z) (∂vs[[b]] Z) + O[ε]^m)];
    θ != e, Z -= ∫_0^λ e dλ
  ];
  PowerExpand@Factor[ω (Δ (2 π)^n)^(-1/2)] × E[CF[Z /. λ → 1 /. Thread[vs → 0]]];
];
Protect[Integrate];

```

In[*]:= $\int \mathbb{E}[-x^2 / 2 + \delta x^3 / 6 + O[\delta]^{20}] d\{x\}$

Out[*]=
$$\frac{\mathbb{E}\left[\frac{5\delta^2}{24} + \frac{5\delta^4}{16} + \frac{1105\delta^6}{1152} + \frac{565\delta^8}{128} + \frac{82825\delta^{10}}{3072} + \frac{19675\delta^{12}}{96} + \frac{1282031525\delta^{14}}{688128} + \frac{80727925\delta^{16}}{4096} + \frac{1683480621875\delta^{18}}{7077888} + O[\delta]^{20}\right]}{\sqrt{2\pi}}$$

In[*]:= $\int \mathbb{E}[-\phi^2 / 2 + \epsilon \phi^4 / 24 + O[\epsilon]^{16}] d\{\phi\}$

Out[*]=
$$\frac{1}{\sqrt{2\pi}} \mathbb{E}\left[\frac{\epsilon}{8} + \frac{\epsilon^2}{12} + \frac{11\epsilon^3}{96} + \frac{17\epsilon^4}{72} + \frac{619\epsilon^5}{960} + \frac{709\epsilon^6}{324} + \frac{858437\epsilon^7}{96768} + \frac{54193\epsilon^8}{1296} + \frac{18639247\epsilon^9}{82944} + \frac{2197187\epsilon^{10}}{1620} + \frac{33152545703\epsilon^{11}}{3649536} + \frac{1169890097\epsilon^{12}}{17496} + \frac{41657327595361\epsilon^{13}}{77635584} + \frac{31722037141\epsilon^{14}}{6804} + \frac{6944870083473751\epsilon^{15}}{159252480} + O[\epsilon]^{16}\right]$$

In[*]:= $\int \mathbb{E}[px + \epsilon ap^2 x + O[\epsilon]^5] d\{p, x\}$

Out[*]=
$$\frac{i \mathbb{E}\left[\frac{15a^2\epsilon^2}{2} + 405a^4\epsilon^4 + O[\epsilon]^5\right]}{2\pi}$$

$$\text{In[*]} := \int \mathbb{E} [\mathbf{p} \mathbf{x} + \epsilon \mathbf{a} \mathbf{p}^2 \mathbf{x} + \epsilon \mathbf{b} \mathbf{p} \mathbf{x}^2 + \mathbf{O}[\epsilon]^5] \, \mathbf{d} \{ \mathbf{p}, \mathbf{x} \}$$

$$\text{Out[*]} = \frac{i \mathbb{E} \left[\frac{15}{2} (a+b)^2 \epsilon^2 + 405 (a+b)^4 \epsilon^4 + \mathbf{O}[\epsilon]^5 \right]}{2 \pi}$$

$$\int \mathbb{E} [\mathbf{p} \mathbf{x} + \mathbf{p}^2 \mathbf{x} + \mathbf{O}[\epsilon]^5] \, \mathbf{d} \{ \mathbf{p}, \mathbf{x} \}$$

$$\text{Out[*]} = \frac{i \mathbb{E} \left[\frac{15 \epsilon^2}{2} + 405 \epsilon^4 + \mathbf{O}[\epsilon]^5 \right]}{2 \pi}$$

$$\text{In[*]} := \int \mathbb{E} [\mathbf{x}_1 \mathbf{p}_1 + \epsilon \mathbf{x}_1^7 \mathbf{p}_1^7 + \mathbf{O}[\epsilon]^2] \, \mathbf{d} \{ \mathbf{x}_1, \mathbf{p}_1 \}$$

$$\text{Out[*]} = \frac{i \mathbb{E} [-5040 \epsilon + \mathbf{O}[\epsilon]^2]}{2 \pi}$$

$$\text{In[*]} := \int \mathbb{E} [\mathbf{x}_1 \mathbf{p}_2 + \epsilon \mathbf{x}_2^7 \mathbf{p}_1^7 + \mathbf{O}[\epsilon]^2] \, \mathbf{d} \{ \mathbf{x}_1, \mathbf{p}_2 \}$$

$$\text{Out[*]} = \frac{i \mathbb{E} [\mathbf{p}_1^7 \mathbf{x}_2^7 \epsilon + \mathbf{O}[\epsilon]^2]}{2 \pi}$$

$$\text{In[*]} := \int \mathbb{E} [\mathbf{x}_1 \mathbf{p}_2 + 3 \mathbf{x}_2 \mathbf{p}_1 + \epsilon \mathbf{p}_2^5 \mathbf{x}_1^5 + \mathbf{O}[\epsilon]^2] \, \mathbf{d} \{ \mathbf{x}_1, \mathbf{x}_2, \mathbf{p}_1, \mathbf{p}_2 \}$$

$$\text{Out[*]} = \frac{\mathbb{E} [-120 \epsilon + \mathbf{O}[\epsilon]^2]}{12 \pi^2}$$

$$\text{In[*]} := \int \mathbb{E} [\mathbf{x}_1 \mathbf{p}_2 + \mathbf{x}_2 \mathbf{p}_3 + \mathbf{x}_3 \mathbf{p}_1 + \epsilon \mathbf{x}_1^5 \mathbf{p}_2^5 + \mathbf{O}[\epsilon]^2] \, \mathbf{d} \{ \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \}$$

$$\text{Out[*]} = \frac{i \mathbb{E} [-120 \epsilon + \mathbf{O}[\epsilon]^2]}{8 \pi^3}$$

$$\text{In[*]} := \text{MatrixForm@Table} [$$

$$\int \mathbb{E} [\mathbf{x}_1 \mathbf{p}_2 + \mathbf{x}_2 \mathbf{p}_3 + \mathbf{x}_3 \mathbf{p}_1 + \xi_i \mathbf{x}_i + \pi_j \mathbf{p}_j + \mathbf{O}[\epsilon]] \, \mathbf{d} \{ \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \},$$

$$\{ \mathbf{i}, 3 \}, \{ \mathbf{j}, 3 \}]$$

Out[*]//MatrixForm=

$$\begin{pmatrix} -\frac{i \mathbb{E} [\mathbf{O}[\epsilon]^1]}{8 \pi^3} & -\frac{i \mathbb{E} [-\pi_2 \xi_1 + \mathbf{O}[\epsilon]^1]}{8 \pi^3} & -\frac{i \mathbb{E} [\mathbf{O}[\epsilon]^1]}{8 \pi^3} \\ -\frac{i \mathbb{E} [\mathbf{O}[\epsilon]^1]}{8 \pi^3} & -\frac{i \mathbb{E} [\mathbf{O}[\epsilon]^1]}{8 \pi^3} & -\frac{i \mathbb{E} [-\pi_3 \xi_2 + \mathbf{O}[\epsilon]^1]}{8 \pi^3} \\ -\frac{i \mathbb{E} [-\pi_1 \xi_3 + \mathbf{O}[\epsilon]^1]}{8 \pi^3} & -\frac{i \mathbb{E} [\mathbf{O}[\epsilon]^1]}{8 \pi^3} & -\frac{i \mathbb{E} [\mathbf{O}[\epsilon]^1]}{8 \pi^3} \end{pmatrix}$$

In[*]:= **K = Knot[5, 2];**

$$\int \rho 1 i [K] \, d(\rho 1 v s @ K)$$

Out[*]=

$$\frac{i T^7 \mathbb{E} \left[\frac{(-1+T)^2 (5-4T+5T^2) \epsilon}{(2-3T+2T^2)^2} + O[\epsilon]^2 \right]}{2048 \pi^{11} (2-3T+2T^2)}$$

In[*]:= **K = Knot[8, 19];**

$$\int \rho 1 i @ K \, d(\rho 1 v s @ K)$$

Out[*]=

$$\frac{i \mathbb{E} \left[-\frac{(-1+T)^2 (1+T^4) (3+4T^3+3T^6) \epsilon}{(1-T+T^2)^2 (1-T^2+T^4)^2} + O[\epsilon]^2 \right]}{131072 \pi^{17} T^5 (1-T+T^3-T^5+T^6)}$$

Invariance Under Reidemeister 3b

In[*]:= **lhsi = $\rho 1 i [1, i, j] \times \rho 1 i [1, i+1, k] \times \rho 1 i [1, j+1, k+1] \times \rho 1 i [0, i] \times$
 $\rho 1 i [0, j] \times \rho 1 i [0, k] \times \rho 1 i [0, i+1] \times \rho 1 i [0, j+1] \times \rho 1 i [0, k+1];$**

lhs = $\int \text{lhsi} \, d\{x_{i+1}, x_{j+1}, x_{k+1}, p_{i+1}, p_{j+1}, p_{k+1}\}$

Out[*]=

$$\frac{1}{8 \pi^3 T^{3/2}} i \mathbb{E} \left[\begin{aligned} & (-p_i x_i + T^2 p_{2+i} x_i - (-1+T) T p_{2+j} x_i + (1-T) p_{2+k} x_i - p_j x_j + T p_{2+j} x_j + (1-T) p_{2+k} x_j - \\ & p_k x_k + p_{2+k} x_k) + \left(-\frac{1}{2} + p_i x_i + T^2 p_{2+i} x_i - p_j x_i + (-2+T) T p_k x_i - \right. \\ & (-1+T) T p_{2+k} x_i + \frac{1}{2} (-1+T) p_i p_j x_i^2 + \frac{1}{2} (1-T) p_j^2 x_i^2 + \frac{1}{2} (-1+T) T^3 p_{2+i} p_k x_i^2 - \\ & \frac{1}{2} (-1+T) T^2 p_k^2 x_i^2 - \frac{1}{2} (-1+T)^2 T (1+T) p_{2+j} p_{2+k} x_i^2 - \frac{1}{2} (-1+T)^2 T^2 p_k p_{2+k} x_i^2 + \\ & \frac{1}{2} (-1+T)^2 T (1+T) p_{2+k}^2 x_i^2 + T p_{2+k} x_j - p_i p_j x_i x_j + p_j^2 x_i x_j + (-1+T) T p_{2+j} p_{2+k} x_i x_j - \\ & (-1+T) T p_{2+k}^2 x_i x_j + \frac{1}{2} (-1+T) T p_{2+j} p_{2+k} x_j^2 - \frac{1}{2} (-1+T) T p_{2+k}^2 x_j^2 - \\ & p_k x_k - p_{2+k} x_k - T^2 p_{2+i} p_k x_i x_k + T p_k^2 x_i x_k + (-1+T) T p_{2+j} p_{2+k} x_i x_k + \\ & \left. (-1+T) T p_k p_{2+k} x_i x_k - (-1+T) T p_{2+k}^2 x_i x_k - T p_{2+j} p_{2+k} x_j x_k + T p_{2+k}^2 x_j x_k \right) \epsilon + O[\epsilon]^2 \end{aligned} \right]$$

In[*]:= **rhsi = $\rho 1 i [1, j, k] \times \rho 1 i [1, i, k+1] \times \rho 1 i [1, i+1, j+1] \times \rho 1 i [0, i] \times$
 $\rho 1 i [0, j] \times \rho 1 i [0, k] \times \rho 1 i [0, i+1] \times \rho 1 i [0, j+1] \times \rho 1 i [0, k+1];$**

rhs = $\int \text{rhsi} \, d\{x_{i+1}, x_{j+1}, x_{k+1}, p_{i+1}, p_{j+1}, p_{k+1}\};$

In[*]:= lhs == rhs

Out[*]=
True

Invariance Under Reidemeister 2b

In[*]:= lhsi = $\rho_{1i}[1, i, j] \times \rho_{1i}[-1, i+1, j+1] \times \rho_{1i}[0, i] \times \rho_{1i}[0, j] \times \rho_{1i}[0, i+1] \times \rho_{1i}[0, j+1]$;
lhs = $\int \text{lhsi} \, d\{x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}\}$

Out[*]=
$$\frac{1}{4\pi^2} \mathbb{E} \left[(-p_i x_i + p_{2+i} x_i - p_j x_j + p_{2+j} x_j) + \left(p_i x_i - p_j x_i - T p_{2+j} x_i + \frac{1}{2} (-1+T) p_i p_j x_i^2 + \frac{1}{2} (1-T) p_j^2 x_i^2 + \frac{1}{2} (1-T) p_{2+i} p_{2+j} x_i^2 + \frac{1}{2} (-1+T) p_{2+j}^2 x_i^2 + p_{2+j} x_j - p_i p_j x_i x_j + p_j^2 x_i x_j + p_{2+i} p_{2+j} x_i x_j - p_{2+j}^2 x_i x_j \right) \epsilon + \mathcal{O}[\epsilon]^2 \right]$$

In[*]:= CF $\left[\left(p_i x_i - p_j x_i - T p_{2+j} x_i + \frac{1}{2} (-1+T) p_i p_j x_i^2 + \frac{1}{2} (1-T) p_j^2 x_i^2 + \frac{1}{2} (1-T) p_{2+i} p_{2+j} x_i^2 + \frac{1}{2} (-1+T) p_{2+j}^2 x_i^2 + p_{2+j} x_j - p_i p_j x_i x_j + p_j^2 x_i x_j + p_{2+i} p_{2+j} x_i x_j - p_{2+j}^2 x_i x_j \right) / \cdot p_{k+2} \rightarrow p_k \right]$

Out[*]=
 $p_i x_i + (-1 - T) p_j x_i + p_j x_j$

In[*]:= lhsi = $\rho_{1i}[1, i, j] \times \rho_{1i}[-1, i+1, j+1] \times \rho_{1i}[0, i] \times \rho_{1i}[0, j] \times \rho_{1i}[0, i+1] \times \rho_{1i}[0, j+1]$;
lhs = $\int \mathbb{E}[\pi_i p_i + \pi_j p_j] \text{lhsi} \, d\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}\}$

Out[*]=
$$\frac{\mathbb{E}[(p_{2+i} \pi_i + p_{2+j} \pi_j) + \mathcal{O}[\epsilon]^2]}{16\pi^4}$$

In[*]:= lhs =
$$\int \mathbb{E}[\pi_i p_i + \pi_j p_j + \mathcal{L}[1, i, j] + \mathcal{L}[-1, i+1, j+1] + \mathcal{O}[\epsilon]^2] \, d\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}\}$$

Out[*]=
$$\frac{\mathbb{E}[(p_{2+i} \pi_i + p_{2+j} \pi_j) + \mathcal{O}[\epsilon]^2]}{16\pi^4}$$

In[*]:= rhs = $\int \mathbb{E}[\pi_i p_i + \pi_j p_j + \mathcal{L}[0, i] + \mathcal{L}[0, i+1] + \mathcal{L}[0, j] + \mathcal{L}[0, j+1] + \mathcal{O}[\epsilon]^2] \, d\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}\}$

Out[*]=
$$\frac{\mathbb{E}[(p_{2+i} \pi_i + p_{2+j} \pi_j) + \mathcal{O}[\epsilon]^2]}{16\pi^4}$$

In[*]:= lhs == rhs

Out[*]= True

Invariance Under R2c

In[*]:= lhs = $\int \mathbb{E} [\pi_i p_i + \pi_j p_j + \mathcal{L}[-1, i, j + 1] + \mathcal{L}[1, i + 1, j] + \Upsilon_1[-1, j + 1] + \mathbf{0}[\epsilon]^2]$
 $\mathfrak{d} \{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}\}$

Out[*]=
$$\frac{\mathbb{E} \left[(p_{2+i} \pi_i + p_{2+j} \pi_j) + \frac{\epsilon}{2} + \mathbf{0}[\epsilon]^2 \right]}{16 \pi^4}$$

In[*]:= rhs = $\int \mathbb{E} [\pi_i p_i + \pi_j p_j + \mathcal{L}[\mathbf{0}, i] + \mathcal{L}[\mathbf{0}, i + 1] + \mathcal{L}[\mathbf{0}, j] + \mathcal{L}[-1, j + 1] + \mathbf{0}[\epsilon]^2]$
 $\mathfrak{d} \{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}\}$

Out[*]=
$$\frac{\mathbb{E} \left[(p_{2+i} \pi_i + p_{2+j} \pi_j) + \frac{\epsilon}{2} + \mathbf{0}[\epsilon]^2 \right]}{16 \pi^4}$$

In[*]:= lhs == rhs

Out[*]= True

Invariance Under R1l

In[*]:= lhs = $\int \mathbb{E} [\pi_i p_i + \mathcal{L}[1, i + 2, i] + \mathcal{L}[1, i + 1] + \mathbf{0}[\epsilon]^2] \mathfrak{d} \{x_i, x_{i+1}, x_{i+2}, p_i, p_{i+1}, p_{i+2}\}$

»
$$\begin{pmatrix} 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 + \Upsilon & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 + \Upsilon & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Out[*]=
$$\frac{i \mathbb{E} [p_{3+i} \pi_i + \mathbf{0}[\epsilon]^2]}{8 \pi^3 \Upsilon}$$

$$\text{In[*]:= rhs} = \int \mathbb{E} [\pi_i p_i + \mathcal{L}[\theta, i] + \mathcal{L}[\theta, i + 1] + \mathcal{L}[\theta, i + 2] + \mathbf{0}[\epsilon]^2] \, d\{x_i, x_{i+1}, x_{i+2}, p_i, p_{i+1}, p_{i+2}\}$$

$$\gg \begin{pmatrix} 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Out[*]=

$$-\frac{i \mathbb{E} [p_{3+i} \pi_i + \mathbf{0}[\epsilon]^2]}{8 \pi^3}$$

$$\text{In[*]:= lhs} == \text{rhs}$$

Out[*]=

$$-\frac{i \mathbb{E} [p_{3+i} \pi_i + \mathbf{0}[\epsilon]^2]}{8 \pi^3 \tau} == -\frac{i \mathbb{E} [p_{3+i} \pi_i + \mathbf{0}[\epsilon]^2]}{8 \pi^3}$$