

Pensieve header: Proof of invariance of ρ_2 using integration techniques.

Initialization

```
In[ ]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\HigherRank"];
Once[<< KnotTheory` ; << Rot.m];
<< FormalGaussianIntegration.m;
```

Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.

Read more at <http://katlas.org/wiki/KnotTheory>.

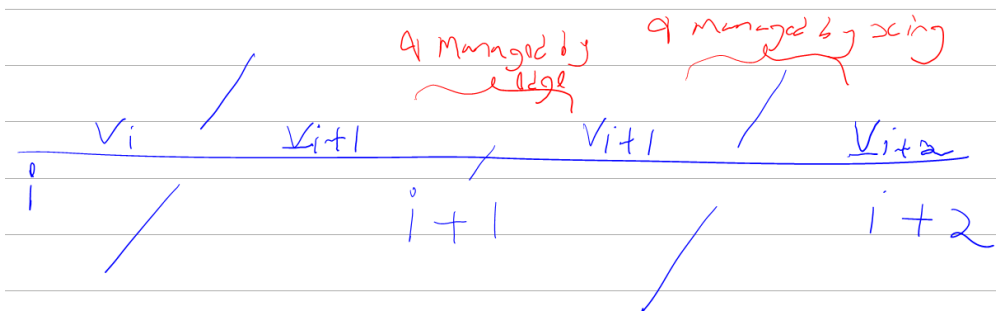
Loading Rot.m from <http://drorbn.net/icbs24> to compute rotation numbers.

```
In[ ]:= T2z[p_] := Module[{q = Expand[p], n, c},
  If[q === 0, 0, c = Coefficient[q, T, n = Exponent[q, T]];
  c z2n + T2z[q - c (T1/2 - T-1/2)2n]];
```

The ρ_2 Integrand

Adopted from pensieve://Talks//Oaxaca-2210/Rho.nb.

Variable convention:



```

In[*]:= S = {x_, p_, x_, p_};
q[s_, i_, j_] := x_i (p_i - p_{i+1}) + x_j (p_j - p_{j+1}) + x_i ((1 - T^s) p_{i+1} + (T^s - 1) p_{j+1});
r1[s_, i_, j_] :=
  s (-1 + 2 p_i x_i - 2 p_j x_i + (T^s - 1) p_i p_j x_i^2 + (1 - T^s) p_j^2 x_i^2 - 2 p_i p_j x_i x_j + 2 p_j^2 x_i x_j) / 2;
r2[1, i_, j_] := (-6 p_i x_i + 6 p_j x_i - 3 (-1 + 3 T) p_i p_j x_i^2 + 3 (-1 + 3 T) p_j^2 x_i^2 + 4 (-1 + T) p_i^2 p_j x_i^3 -
  2 (-1 + T) (5 + T) p_i p_j^2 x_i^3 + 2 (-1 + T) (3 + T) p_j^3 x_i^3 + 18 p_i p_j x_i x_j - 18 p_j^2 x_i x_j -
  6 p_i^2 p_j x_i^2 x_j + 6 (2 + T) p_i p_j^2 x_i^2 x_j - 6 (1 + T) p_j^3 x_i^2 x_j - 6 p_i p_j^2 x_i x_j^2 + 6 p_j^3 x_i x_j^2) / 12;
r2[-1, i_, j_] :=
  (-6 T^2 p_i x_i + 6 T^2 p_j x_i + 3 (-3 + T) T p_i p_j x_i^2 - 3 (-3 + T) T p_j^2 x_i^2 - 4 (-1 + T) T p_i^2 p_j x_i^3 +
  2 (-1 + T) (1 + 5 T) p_i p_j^2 x_i^3 - 2 (-1 + T) (1 + 3 T) p_j^3 x_i^3 + 18 T^2 p_i p_j x_i x_j -
  18 T^2 p_j^2 x_i x_j - 6 T^2 p_i^2 p_j x_i^2 x_j + 6 T (1 + 2 T) p_i p_j^2 x_i^2 x_j -
  6 T (1 + T) p_j^3 x_i^2 x_j - 6 T^2 p_i p_j^2 x_i x_j^2 + 6 T^2 p_j^3 x_i x_j^2) / (12 T^2);
gamma1[phi_, k_] := phi (1 / 2 - x_k p_k);
gamma2[phi_, k_] := -phi^2 p_k x_k / 2;
L[s_, i_, j_] := T^{s/2} E[-q[s, i, j] + e r1[s, i, j] + e^2 r2[s, i, j] + O[e]^3];
L[phi_, k_] := T^{phi/2} E[-x_k (p_k - p_k) + e gamma1[phi, k] + e^2 gamma2[phi, k] + O[e]^3];
L[Dot, i_] := E[-x_i (p_i - p_{i+1}) + O[e]^3];
L[End, k_] := E[-x_k (p_k - p_k) - x_k p_k + O[e]^3];
L[K_] := Module[{Cs, phi, n, c, k, epsilon},
  {Cs, phi} = Rot[K]; n = Length[Cs];
  epsilon = (2 pi)^{4 n + 2} L[End, 2 n + 1];
  Do[epsilon *= L @@ c, {c, Cs}];
  Do[epsilon *= L[phi[[k]], k], {k, 2 n}];
  CF @ epsilon
];
vs_i := Sequence[x_i, p_i, x_i, p_i]
rho2vs[K_] := Union @@ Table[{vs_i}, {i, 2 Crossings[K] + 1}]

```

In[*]:= $\mathcal{L}[\text{Knot}[3, 1]]$

 KnotTheory: Loading precomputed data in PD4Knots`.

Out[*]=

$$\frac{1}{T^2} 16384 \pi^{14}$$

$$\begin{aligned} & \mathbb{E} \left[\in \text{Series} \left[-p_1 x_1 - p_2 x_2 - p_3 x_3 - p_4 x_4 - p_5 x_5 - p_6 x_6 - p_7 x_7 + x_1 p_2 + x_3 p_4 + \frac{x_4 (-p_2 + T p_2 + p_5)}{T} + \right. \right. \\ & x_5 p_6 + \frac{x_2 (p_3 - p_6 + T p_6)}{T} + \frac{x_6 (-p_4 + T p_4 + p_7)}{T} + p_1 x_1 - p_1 x_1 + p_2 x_2 - \\ & p_2 x_2 + p_3 x_3 - p_3 x_3 + p_4 x_4 - p_4 x_4 + p_5 x_5 - p_5 x_5 + p_6 x_6 - p_6 x_6 + p_7 x_7 - p_7 x_7, \\ & 1 - p_2 x_2 + p_5 x_2 + \frac{(-1 + T) p_2 p_5 x_2^2}{2 T} - \frac{(-1 + T) p_5^2 x_2^2}{2 T} + p_1 x_4 - p_4 x_4 - p_1^2 x_1 x_4 + \\ & p_1 p_4 x_1 x_4 - \frac{(-1 + T) p_1^2 x_4^2}{2 T} + \frac{(-1 + T) p_1 p_4 x_4^2}{2 T} + p_2 p_5 x_2 x_5 - p_5^2 x_2 x_5 + p_3 x_6 - \\ & p_6 x_6 - p_3^2 x_3 x_6 + p_3 p_6 x_3 x_6 - \frac{(-1 + T) p_3^2 x_6^2}{2 T} + \frac{(-1 + T) p_3 p_6 x_6^2}{2 T} + p_4 x_4, \\ & - \frac{1}{2} p_2 x_2 + \frac{p_5 x_2}{2} + \frac{(-3 + T) p_2 p_5 x_2^2}{4 T} - \frac{(-3 + T) p_5^2 x_2^2}{4 T} - \frac{(-1 + T) p_2^2 p_5 x_2^3}{3 T} + \\ & \frac{(-1 + T) (1 + 5 T) p_2 p_5^2 x_2^3}{6 T^2} - \frac{(-1 + T) (1 + 3 T) p_5^3 x_2^3}{6 T^2} + \frac{p_1 x_4}{2} - \frac{p_4 x_4}{2} - \frac{3}{2} p_1^2 x_1 x_4 + \\ & \frac{3}{2} p_1 p_4 x_1 x_4 + \frac{1}{2} p_1^3 x_1^2 x_4 - \frac{1}{2} p_1^2 p_4 x_1^2 x_4 - \frac{(-3 + T) p_1^2 x_4^2}{4 T} + \frac{(-3 + T) p_1 p_4 x_4^2}{4 T} - \\ & \frac{(1 + T) p_1^3 x_1 x_4^2}{2 T} + \frac{(1 + 2 T) p_1^2 p_4 x_1 x_4^2}{2 T} - \frac{1}{2} p_1 p_4^2 x_1 x_4^2 - \frac{(-1 + T) (1 + 3 T) p_1^3 x_4^3}{6 T^2} + \\ & \frac{(-1 + T) (1 + 5 T) p_1^2 p_4 x_4^3}{6 T^2} - \frac{(-1 + T) p_1 p_4^2 x_4^3}{3 T} + \frac{3}{2} p_2 p_5 x_2 x_5 - \frac{3}{2} p_5^2 x_2 x_5 - \\ & \frac{1}{2} p_2^2 p_5 x_2^2 x_5 + \frac{(1 + 2 T) p_2 p_5^2 x_2^2 x_5}{2 T} - \frac{(1 + T) p_5^3 x_2^2 x_5}{2 T} - \frac{1}{2} p_2 p_5^2 x_2 x_5^2 + \frac{1}{2} p_5^3 x_2 x_5^2 + \\ & \frac{p_3 x_6}{2} - \frac{p_6 x_6}{2} - \frac{3}{2} p_3^2 x_3 x_6 + \frac{3}{2} p_3 p_6 x_3 x_6 + \frac{1}{2} p_3^3 x_3^2 x_6 - \frac{1}{2} p_3^2 p_6 x_3^2 x_6 - \frac{(-3 + T) p_3^2 x_6^2}{4 T} \\ & \frac{(-3 + T) p_3 p_6 x_6^2}{4 T} - \frac{(1 + T) p_3^3 x_3 x_6^2}{2 T} + \frac{(1 + 2 T) p_3^2 p_6 x_3 x_6^2}{2 T} - \frac{1}{2} p_3 p_6^2 x_3 x_6^2 - \\ & \left. \frac{(-1 + T) (1 + 3 T) p_3^3 x_6^3}{6 T^2} + \frac{(-1 + T) (1 + 5 T) p_3^2 p_6 x_6^3}{6 T^2} - \frac{(-1 + T) p_3 p_6^2 x_6^3}{3 T} - \frac{1}{2} p_4 x_4 \right] \end{aligned}$$

In[*]:= $\rho 2vs[\text{Knot}[3, 1]]$

Out[*]=

$$\{p_1, p_2, p_3, p_4, p_5, p_6, p_7, x_1, x_2, x_3, x_4, x_5, x_6, x_7, p_1, p_2, p_3, p_4, p_5, p_6, p_7, x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$$

In[*]:= **K = Knot [3, 1];** $\int \mathcal{L}[K] \, d(\rho 2vs@K)$

Out[*]=

$$\frac{T \mathbb{E} \left[\in \text{Series} \left[0, \frac{(-1+T)^2 (1+T^2)}{(1-T+T^2)^2}, -\frac{T^2 (1-4T^2+T^4)}{2(1-T+T^2)^4} \right] \right]}{1-T+T^2}$$

In[*]:= **T2z [T^-2 (1 - 4 T^2 + T^4)]**

Out[*]=

$$-2 + 4 z^2 + z^4$$

In[*]:= **Factor @ (2 - 4 z^2 + 3 z^4 + 4 z^6 + z^8)**

Out[*]=

$$2 - 4 z^2 + 3 z^4 + 4 z^6 + z^8$$

In[*]:= **K = Knot [5, 2];** $\int \mathcal{L}[K] \, d(\rho 2vs@K)$

Out[*]=

$$\frac{T \mathbb{E} \left[\in \text{Series} \left[0, \frac{(-1+T)^2 (5-4T+5T^2)}{(2-3T+2T^2)^2}, \frac{1-4T+11T^2-44T^3+76T^4-44T^5+11T^6-4T^7+T^8}{2(2-3T+2T^2)^4} \right] \right]}{2-3T+2T^2}$$

In[*]:= **T2z [(1 - 4 T + 11 T^2 - 44 T^3 + 76 T^4 - 44 T^5 + 11 T^6 - 4 T^7 + T^8) / T^4]**

Out[*]=

$$4 - 20 z^2 + 7 z^4 + 4 z^6 + z^8$$

In[*]:= **K = Knot [8, 19];** $\int \mathcal{L}[K] \, d(\rho 2vs@K)$

Out[*]=

$$\frac{1}{1-T+T^3-T^5+T^6} T^3 \mathbb{E} \left[\in \text{Series} \left[0, -\frac{(-1+T)^2 (1+T^4) (3+4T^3+3T^6)}{(1-T+T^2)^2 (1-T^2+T^4)^2}, \right. \right. \\ \left. \left. - \left((T^3 (4-7T-12T^2+62T^3-100T^4+95T^5-60T^6-42T^7+208T^8-306T^9+208T^{10}-42T^{11}-60T^{12}+95T^{13}-100T^{14}+62T^{15}-12T^{16}-7T^{17}+4T^{18})) / (2(1-T+T^2)^4 (1-T^2+T^4)^4) \right) \right] \right]$$

Concatenating edges

In[*]:= **lhs = 8 π^3** $\int (\mathbb{E} [\underline{\pi}_i \underline{p}_i] \times \mathcal{L}[\varphi 1, i] \times \mathcal{L}[\text{Dot}, i] \times \mathcal{L}[\varphi 2, i+1]) \, d\{\underline{vs}_i, \underline{x}_{i+1}, \underline{p}_{i+1}\}$

rhs = $\left(2 \pi \int \mathbb{E} [\underline{\pi}_i \underline{p}_i] \times \mathcal{L}[\varphi 1 + \varphi 2, i] \, d\{\underline{x}_i, \underline{p}_i\} \right) / \cdot \underline{p}_i \rightarrow \underline{p}_{i+1};$

lhs == rhs

Out[*]=

$$-i T^{\frac{\varphi 1}{2} + \frac{\varphi 2}{2}} \mathbb{E} \left[\in \text{Series} \left[p_{1+i} \underline{\pi}_i, \frac{1}{2} (-\varphi 1 - \varphi 2) - (\varphi 1 + \varphi 2) p_{1+i} \underline{\pi}_i, \frac{1}{2} (\varphi 1 + \varphi 2)^2 p_{1+i} \underline{\pi}_i \right] \right]$$

Out[*]=

True

Invariance Under Reidemeister 3b

$$\begin{aligned}
 \text{In[*]} := & \text{lhs} = \int (\mathbb{E}[\pi_i \mathbf{p}_i + \pi_j \mathbf{p}_j + \pi_k \mathbf{p}_k] \times \mathcal{L}[1, \mathbf{i}, \mathbf{j}] \times \mathcal{L}[1, \mathbf{i} + 1, \mathbf{k}] \times \mathcal{L}[1, \mathbf{j} + 1, \mathbf{k} + 1] \times \\
 & \mathcal{L}[\mathbf{0}, \mathbf{i} + 1] \times \mathcal{L}[\mathbf{0}, \mathbf{j} + 1] \times \mathcal{L}[\mathbf{0}, \mathbf{k} + 1]) \mathfrak{d}\{\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k, \mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k, \mathbf{v}_{S_{i+1}}, \mathbf{v}_{S_{j+1}}, \mathbf{v}_{S_{k+1}}\} \\
 \text{rhs} = & \int (\mathbb{E}[\pi_i \mathbf{p}_i + \pi_j \mathbf{p}_j + \pi_k \mathbf{p}_k] \times \mathcal{L}[1, \mathbf{j}, \mathbf{k}] \times \mathcal{L}[1, \mathbf{i}, \mathbf{k} + 1] \times \mathcal{L}[1, \mathbf{i} + 1, \mathbf{j} + 1] \times \mathcal{L}[\mathbf{0}, \mathbf{i} + 1] \times \\
 & \mathcal{L}[\mathbf{0}, \mathbf{j} + 1] \times \mathcal{L}[\mathbf{0}, \mathbf{k} + 1]) \mathfrak{d}\{\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k, \mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k, \mathbf{v}_{S_{i+1}}, \mathbf{v}_{S_{j+1}}, \mathbf{v}_{S_{k+1}}\}; \\
 \text{lhs} = & \text{rhs}
 \end{aligned}$$

Out[*]=

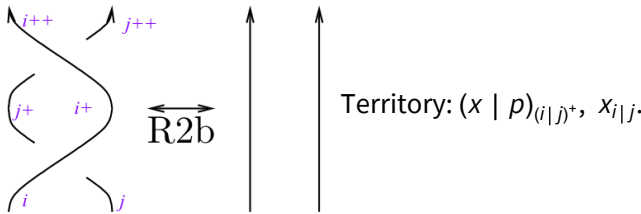
$$\begin{aligned}
 & \frac{1}{512 \pi^9} \\
 & i T^{3/2} \mathbb{E} \left[\in \text{Series} \left[T^2 \pi_i \underline{p}_{2+i} + T \pi_i \underline{p}_{2+j} - T^2 \pi_i \underline{p}_{2+j} + T \pi_j \underline{p}_{2+j} + \pi_i \underline{p}_{2+k} - T \pi_i \underline{p}_{2+k} + \pi_j \underline{p}_{2+k} - T \pi_j \underline{p}_{2+k} + \right. \right. \\
 & \left. \left. \pi_k \underline{p}_{2+k}, \frac{1}{2} \left(-3 + 2 T^2 \pi_i \underline{p}_{2+j} - 2 T \pi_j \underline{p}_{2+j} - T^3 \pi_i^2 \underline{p}_{2+i} \underline{p}_{2+j} + T^4 \pi_i^2 \underline{p}_{2+i} \underline{p}_{2+j} - \right. \right. \right. \\
 & \left. \left. 2 T^3 \pi_i \pi_j \underline{p}_{2+i} \underline{p}_{2+j} + T^3 \pi_i^2 \underline{p}_{2+j}^2 - T^4 \pi_i^2 \underline{p}_{2+j}^2 + 2 T^3 \pi_i \pi_j \underline{p}_{2+j}^2 + 2 T \pi_i \underline{p}_{2+k} - 2 \pi_j \underline{p}_{2+k} + \right. \right. \\
 & \left. \left. 4 T \pi_j \underline{p}_{2+k} - 4 \pi_k \underline{p}_{2+k} - T^2 \pi_i^2 \underline{p}_{2+i} \underline{p}_{2+k} + T^3 \pi_i^2 \underline{p}_{2+i} \underline{p}_{2+k} - 2 T^2 \pi_i \pi_j \underline{p}_{2+i} \underline{p}_{2+k} + \right. \right. \\
 & \left. \left. 2 T^3 \pi_i \pi_j \underline{p}_{2+i} \underline{p}_{2+k} - 2 T^2 \pi_i \pi_k \underline{p}_{2+i} \underline{p}_{2+k} - T \pi_i^2 \underline{p}_{2+j} \underline{p}_{2+k} + 2 T^2 \pi_i^2 \underline{p}_{2+j} \underline{p}_{2+k} - T^3 \pi_i^2 \underline{p}_{2+j} \underline{p}_{2+k} - \right. \right. \\
 & \left. \left. 2 T \pi_i \pi_j \underline{p}_{2+j} \underline{p}_{2+k} + 4 T^2 \pi_i \pi_j \underline{p}_{2+j} \underline{p}_{2+k} - 2 T^3 \pi_i \pi_j \underline{p}_{2+j} \underline{p}_{2+k} - T \pi_j^2 \underline{p}_{2+j} \underline{p}_{2+k} + T^2 \pi_j^2 \underline{p}_{2+j} \underline{p}_{2+k} - \right. \right. \\
 & \left. \left. 2 T \pi_i \pi_k \underline{p}_{2+j} \underline{p}_{2+k} + 2 T^2 \pi_i \pi_k \underline{p}_{2+j} \underline{p}_{2+k} - 2 T \pi_j \pi_k \underline{p}_{2+j} \underline{p}_{2+k} + T \pi_i^2 \underline{p}_{2+k}^2 - T^2 \pi_i^2 \underline{p}_{2+k}^2 + \right. \right. \\
 & \left. \left. 2 T \pi_i \pi_j \underline{p}_{2+k}^2 - 2 T^2 \pi_i \pi_j \underline{p}_{2+k}^2 + T \pi_j^2 \underline{p}_{2+k}^2 - T^2 \pi_j^2 \underline{p}_{2+k}^2 + 2 T \pi_i \pi_k \underline{p}_{2+k}^2 + 2 T \pi_j \pi_k \underline{p}_{2+k}^2 \right) \right), \\
 & \frac{1}{12} \left(-6 T^2 \pi_i \underline{p}_{2+j} + 6 T \pi_j \underline{p}_{2+j} + 3 T^3 \pi_i^2 \underline{p}_{2+i} \underline{p}_{2+j} - 9 T^4 \pi_i^2 \underline{p}_{2+i} \underline{p}_{2+j} + 18 T^3 \pi_i \pi_j \underline{p}_{2+i} \underline{p}_{2+j} + \right. \\
 & \left. 2 T^5 \pi_i^3 \underline{p}_{2+i} \underline{p}_{2+j} - 2 T^6 \pi_i^3 \underline{p}_{2+i} \underline{p}_{2+j} + 6 T^5 \pi_i^2 \pi_j \underline{p}_{2+i} \underline{p}_{2+j} - 9 T^3 \pi_i^2 \underline{p}_{2+j}^2 + 15 T^4 \pi_i^2 \underline{p}_{2+j}^2 - \right. \\
 & \left. 30 T^3 \pi_i \pi_j \underline{p}_{2+j}^2 + 2 T^4 \pi_i^3 \underline{p}_{2+i} \underline{p}_{2+j}^2 - 10 T^5 \pi_i^3 \underline{p}_{2+i} \underline{p}_{2+j}^2 + 8 T^6 \pi_i^3 \underline{p}_{2+i} \underline{p}_{2+j}^2 + 6 T^4 \pi_i^2 \pi_j \underline{p}_{2+i} \underline{p}_{2+j}^2 - \right. \\
 & \left. 24 T^5 \pi_i^2 \pi_j \underline{p}_{2+i} \underline{p}_{2+j}^2 + 6 T^4 \pi_i \pi_j^2 \underline{p}_{2+i} \underline{p}_{2+j}^2 - 2 T^4 \pi_i^3 \underline{p}_{2+j}^3 + 8 T^5 \pi_i^3 \underline{p}_{2+j}^3 - 6 T^6 \pi_i^3 \underline{p}_{2+j}^3 - \right. \\
 & \left. 6 T^4 \pi_i^2 \pi_j \underline{p}_{2+j}^3 + 18 T^5 \pi_i^2 \pi_j \underline{p}_{2+j}^3 - 6 T^4 \pi_i \pi_j^2 \underline{p}_{2+j}^3 - 6 T \pi_i \underline{p}_{2+k} + 6 \pi_j \underline{p}_{2+k} - 24 T \pi_j \underline{p}_{2+k} + \right. \\
 & \left. 24 \pi_k \underline{p}_{2+k} + 3 T^2 \pi_i^2 \underline{p}_{2+i} \underline{p}_{2+k} - 9 T^3 \pi_i^2 \underline{p}_{2+i} \underline{p}_{2+k} + 18 T^2 \pi_i \pi_j \underline{p}_{2+i} \underline{p}_{2+k} - 30 T^3 \pi_i \pi_j \underline{p}_{2+i} \underline{p}_{2+k} + \right. \\
 & \left. 30 T^2 \pi_i \pi_k \underline{p}_{2+i} \underline{p}_{2+k} + 2 T^4 \pi_i^3 \underline{p}_{2+i} \underline{p}_{2+k} - 2 T^5 \pi_i^3 \underline{p}_{2+i} \underline{p}_{2+k} + 6 T^4 \pi_i^2 \pi_j \underline{p}_{2+i} \underline{p}_{2+k} - \right. \\
 & \left. 6 T^5 \pi_i^2 \pi_j \underline{p}_{2+i} \underline{p}_{2+k} + 6 T^4 \pi_i^2 \pi_k \underline{p}_{2+i} \underline{p}_{2+k} + 3 T \pi_i^2 \underline{p}_{2+j} \underline{p}_{2+k} - 18 T^2 \pi_i^2 \underline{p}_{2+j} \underline{p}_{2+k} + \right. \\
 & \left. 15 T^3 \pi_i^2 \underline{p}_{2+j} \underline{p}_{2+k} + 18 T \pi_i \pi_j \underline{p}_{2+j} \underline{p}_{2+k} - 60 T^2 \pi_i \pi_j \underline{p}_{2+j} \underline{p}_{2+k} + 42 T^3 \pi_i \pi_j \underline{p}_{2+j} \underline{p}_{2+k} + \right. \\
 & \left. 15 T \pi_j^2 \underline{p}_{2+j} \underline{p}_{2+k} - 21 T^2 \pi_j^2 \underline{p}_{2+j} \underline{p}_{2+k} + 30 T \pi_i \pi_k \underline{p}_{2+j} \underline{p}_{2+k} - 42 T^2 \pi_i \pi_k \underline{p}_{2+j} \underline{p}_{2+k} + \right. \\
 & \left. 42 T \pi_j \pi_k \underline{p}_{2+j} \underline{p}_{2+k} + 10 T^3 \pi_i^3 \underline{p}_{2+i} \underline{p}_{2+j} \underline{p}_{2+k} - 20 T^4 \pi_i^3 \underline{p}_{2+i} \underline{p}_{2+j} \underline{p}_{2+k} + 10 T^5 \pi_i^3 \underline{p}_{2+i} \underline{p}_{2+j} \underline{p}_{2+k} + \right. \\
 & \left. 30 T^3 \pi_i^2 \pi_j \underline{p}_{2+i} \underline{p}_{2+j} \underline{p}_{2+k} - 54 T^4 \pi_i^2 \pi_j \underline{p}_{2+i} \underline{p}_{2+j} \underline{p}_{2+k} + 24 T^5 \pi_i^2 \pi_j \underline{p}_{2+i} \underline{p}_{2+j} \underline{p}_{2+k} + \right. \\
 & \left. 24 T^3 \pi_i \pi_j^2 \underline{p}_{2+i} \underline{p}_{2+j} \underline{p}_{2+k} - 24 T^4 \pi_i \pi_j^2 \underline{p}_{2+i} \underline{p}_{2+j} \underline{p}_{2+k} + 24 T^3 \pi_i^2 \pi_k \underline{p}_{2+i} \underline{p}_{2+j} \underline{p}_{2+k} - \right. \\
 & \left. 24 T^4 \pi_i^2 \pi_k \underline{p}_{2+i} \underline{p}_{2+j} \underline{p}_{2+k} + 36 T^3 \pi_i \pi_j \pi_k \underline{p}_{2+i} \underline{p}_{2+j} \underline{p}_{2+k} + 2 T^2 \pi_i^3 \underline{p}_{2+j}^2 \underline{p}_{2+k} - 12 T^3 \pi_i^3 \underline{p}_{2+j}^2 \underline{p}_{2+k} + \right. \\
 & \left. 18 T^4 \pi_i^3 \underline{p}_{2+j}^2 \underline{p}_{2+k} - 8 T^5 \pi_i^3 \underline{p}_{2+j}^2 \underline{p}_{2+k} + 6 T^2 \pi_i^2 \pi_j \underline{p}_{2+j}^2 \underline{p}_{2+k} - 36 T^3 \pi_i^2 \pi_j \underline{p}_{2+j}^2 \underline{p}_{2+k} + \right. \\
 & \left. 48 T^4 \pi_i^2 \pi_j \underline{p}_{2+j}^2 \underline{p}_{2+k} - 18 T^5 \pi_i^2 \pi_j \underline{p}_{2+j}^2 \underline{p}_{2+k} + 6 T^2 \pi_i \pi_j^2 \underline{p}_{2+j}^2 \underline{p}_{2+k} - 30 T^3 \pi_i \pi_j^2 \underline{p}_{2+j}^2 \underline{p}_{2+k} + \right. \\
 & \left. 24 T^4 \pi_i \pi_j^2 \underline{p}_{2+j}^2 \underline{p}_{2+k} + 2 T^2 \pi_j^3 \underline{p}_{2+j}^2 \underline{p}_{2+k} - 2 T^3 \pi_j^3 \underline{p}_{2+j}^2 \underline{p}_{2+k} + 6 T^2 \pi_i^2 \pi_k \underline{p}_{2+j}^2 \underline{p}_{2+k} - \right.
 \end{aligned}$$

$$\begin{aligned}
 & 24 T^3 \pi_i^2 \pi_k p_{2+j}^2 p_{2+k} + 18 T^4 \pi_i^2 \pi_k p_{2+j}^2 p_{2+k} + 12 T^2 \pi_i \pi_j \pi_k p_{2+j}^2 p_{2+k} - 36 T^3 \pi_i \pi_j \pi_k p_{2+j}^2 p_{2+k} + \\
 & 6 T^2 \pi_j^2 \pi_k p_{2+j}^2 p_{2+k} - 9 T \pi_i^2 p_{2+k}^2 + 15 T^2 \pi_i^2 p_{2+k}^2 - 30 T \pi_i \pi_j p_{2+k}^2 + 42 T^2 \pi_i \pi_j p_{2+k}^2 - 21 T \pi_j^2 p_{2+k}^2 + \\
 & 27 T^2 \pi_j^2 p_{2+k}^2 - 42 T \pi_i \pi_k p_{2+k}^2 - 54 T \pi_j \pi_k p_{2+k}^2 + 2 T^2 \pi_i^3 p_{2+i}^2 p_{2+k}^2 - 10 T^3 \pi_i^3 p_{2+i}^2 p_{2+k}^2 + \\
 & 8 T^4 \pi_i^3 p_{2+i}^2 p_{2+k}^2 + 6 T^2 \pi_i^2 \pi_j p_{2+i}^2 p_{2+k}^2 - 30 T^3 \pi_i^2 \pi_j p_{2+i}^2 p_{2+k}^2 + 24 T^4 \pi_i^2 \pi_j p_{2+i}^2 p_{2+k}^2 + \\
 & 6 T^2 \pi_i \pi_j^2 p_{2+i}^2 p_{2+k}^2 - 24 T^3 \pi_i \pi_j^2 p_{2+i}^2 p_{2+k}^2 + 18 T^4 \pi_i \pi_j^2 p_{2+i}^2 p_{2+k}^2 + 6 T^2 \pi_i^2 \pi_k p_{2+i}^2 p_{2+k}^2 - \\
 & 24 T^3 \pi_i^2 \pi_k p_{2+i}^2 p_{2+k}^2 + 12 T^2 \pi_i \pi_j \pi_k p_{2+i}^2 p_{2+k}^2 - 36 T^3 \pi_i \pi_j \pi_k p_{2+i}^2 p_{2+k}^2 + 6 T^2 \pi_i \pi_k^2 p_{2+i}^2 p_{2+k}^2 + \\
 & 2 T \pi_i^3 p_{2+j}^2 p_{2+k}^2 - 12 T^2 \pi_i^3 p_{2+j}^2 p_{2+k}^2 + 18 T^3 \pi_i^3 p_{2+j}^2 p_{2+k}^2 - 8 T^4 \pi_i^3 p_{2+j}^2 p_{2+k}^2 + 6 T \pi_i^2 \pi_j p_{2+j}^2 p_{2+k}^2 - \\
 & 36 T^2 \pi_i^2 \pi_j p_{2+j}^2 p_{2+k}^2 + 54 T^3 \pi_i^2 \pi_j p_{2+j}^2 p_{2+k}^2 - 24 T^4 \pi_i^2 \pi_j p_{2+j}^2 p_{2+k}^2 + 6 T \pi_i \pi_j^2 p_{2+j}^2 p_{2+k}^2 - \\
 & 36 T^2 \pi_i \pi_j^2 p_{2+j}^2 p_{2+k}^2 + 48 T^3 \pi_i \pi_j^2 p_{2+j}^2 p_{2+k}^2 - 18 T^4 \pi_i \pi_j^2 p_{2+j}^2 p_{2+k}^2 + 2 T \pi_j^3 p_{2+j}^2 p_{2+k}^2 - \\
 & 10 T^2 \pi_j^3 p_{2+j}^2 p_{2+k}^2 + 8 T^3 \pi_j^3 p_{2+j}^2 p_{2+k}^2 + 6 T \pi_i^2 \pi_k p_{2+j}^2 p_{2+k}^2 - 30 T^2 \pi_i^2 \pi_k p_{2+j}^2 p_{2+k}^2 + \\
 & 24 T^3 \pi_i^2 \pi_k p_{2+j}^2 p_{2+k}^2 + 12 T \pi_i \pi_j \pi_k p_{2+j}^2 p_{2+k}^2 - 60 T^2 \pi_i \pi_j \pi_k p_{2+j}^2 p_{2+k}^2 + 36 T^3 \pi_i \pi_j \pi_k p_{2+j}^2 p_{2+k}^2 + \\
 & 6 T \pi_j^2 \pi_k p_{2+j}^2 p_{2+k}^2 - 24 T^2 \pi_j^2 \pi_k p_{2+j}^2 p_{2+k}^2 + 6 T \pi_i \pi_k^2 p_{2+j}^2 p_{2+k}^2 - 6 T^2 \pi_i \pi_k^2 p_{2+j}^2 p_{2+k}^2 + \\
 & 6 T \pi_j \pi_k^2 p_{2+j}^2 p_{2+k}^2 - 2 T \pi_i^3 p_{2+k}^3 + 8 T^2 \pi_i^3 p_{2+k}^3 - 6 T^3 \pi_i^3 p_{2+k}^3 - 6 T \pi_i^2 \pi_j p_{2+k}^3 + 24 T^2 \pi_i^2 \pi_j p_{2+k}^3 - \\
 & 18 T^3 \pi_i^2 \pi_j p_{2+k}^3 - 6 T \pi_i \pi_j^2 p_{2+k}^3 + 24 T^2 \pi_i \pi_j^2 p_{2+k}^3 - 18 T^3 \pi_i \pi_j^2 p_{2+k}^3 - 2 T \pi_j^3 p_{2+k}^3 + \\
 & 8 T^2 \pi_j^3 p_{2+k}^3 - 6 T^3 \pi_j^3 p_{2+k}^3 - 6 T \pi_i^2 \pi_k p_{2+k}^3 + 18 T^2 \pi_i^2 \pi_k p_{2+k}^3 - 12 T \pi_i \pi_j \pi_k p_{2+k}^3 + \\
 & 36 T^2 \pi_i \pi_j \pi_k p_{2+k}^3 - 6 T \pi_j^2 \pi_k p_{2+k}^3 + 18 T^2 \pi_j^2 \pi_k p_{2+k}^3 - 6 T \pi_i \pi_k^2 p_{2+k}^3 - 6 T \pi_j \pi_k^2 p_{2+k}^3 \Big) \Big]
 \end{aligned}$$

Out[*]=

True

Invariance Under Reidemeister 2b



$$\text{lhs} = \int (\mathbb{E}[\pi_i p_i + \pi_j p_j] \times \mathcal{L}[1, i, j] \times \mathcal{L}[-1, i+1, j+1] \times \mathcal{L}[0, i+1] \times \mathcal{L}[0, j+1])$$

$$\text{d}\{X_i, X_j, p_i, p_j, v_{s_{i+1}}, v_{s_{j+1}}\}$$

Out[*]=

$$\frac{\mathbb{E}[\in \text{Series}[\pi_i p_{2+i} + \pi_j p_{2+j}, 0, 0]]}{64 \pi^6}$$

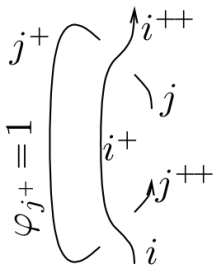
$$\text{In[*]:= rhs} = \int (\mathbb{E}[\pi_i p_i + \pi_j p_j] \times \mathcal{L}[\mathbf{0}, \mathbf{i} + 1] \times \mathcal{L}[\mathbf{0}, \mathbf{j} + 1]) \mathcal{L}[\text{Dot}, \mathbf{i}] \times \mathcal{L}[\text{Dot}, \mathbf{j}] \times \mathcal{L}[\text{Dot}, \mathbf{i} + 1] \times \mathcal{L}[\text{Dot}, \mathbf{j} + 1] \text{d}\{\mathbf{x}_i, \mathbf{x}_j, \mathbf{p}_i, \mathbf{p}_j, \mathbf{v}_{i+1}, \mathbf{v}_{j+1}\}$$

lhs == rhs

$$\text{Out[*]=} \frac{\mathbb{E}[\text{Series}[\pi_i p_{2+i} + \pi_j p_{2+j}, \theta, \theta]]}{64 \pi^6}$$

Out[*]= True

Invariance Under R2c



$$\text{In[*]:= lhs} = \int (\mathbb{E}[\pi_i p_i + \pi_j p_j] \times \mathcal{L}[-1, \mathbf{i}, \mathbf{j} + 1] \times \mathcal{L}[1, \mathbf{i} + 1, \mathbf{j}] \times \mathcal{L}[\mathbf{0}, \mathbf{i} + 1] \times \mathcal{L}[1, \mathbf{j} + 1]) \text{d}\{\mathbf{x}_i, \mathbf{x}_j, \mathbf{p}_i, \mathbf{p}_j, \mathbf{v}_{i+1}, \mathbf{v}_{j+1}\}$$

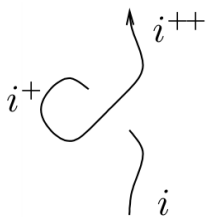
$$\text{Out[*]=} \frac{\sqrt{T} \mathbb{E}[\text{Series}[\pi_i p_{2+i} + \pi_j p_{2+j}, \frac{1}{2}(-1 - 2\pi_j p_{2+j}), \frac{1}{2}\pi_j p_{2+j}]]}{64 \pi^6}$$

$$\text{In[*]:= rhs} = \int (\mathbb{E}[\pi_i p_i + \pi_j p_j] \times \mathcal{L}[\text{Dot}, \mathbf{i}] \times \mathcal{L}[\text{Dot}, \mathbf{j}] \times \mathcal{L}[\text{Dot}, \mathbf{i} + 1] \times \mathcal{L}[\text{Dot}, \mathbf{j} + 1] \times \mathcal{L}[\mathbf{0}, \mathbf{i} + 1] \times \mathcal{L}[1, \mathbf{j} + 1]) \text{d}\{\mathbf{x}_i, \mathbf{x}_j, \mathbf{p}_i, \mathbf{p}_j, \mathbf{v}_{i+1}, \mathbf{v}_{j+1}\};$$

lhs == rhs

Out[*]= True

Invariance Under R1l



$$\text{In[*]:= lhs} = 8 \pi^3 \int (\mathbb{E}[\pi_i \mathbf{p}_i] \times \mathcal{L}[1, \mathbf{i} + 1, \mathbf{i}] \times \mathcal{L}[1, \mathbf{i} + 1]) \, d\{\mathbf{x}_i, \mathbf{p}_i, \mathbf{v}_{\mathbf{i}+1}\}$$

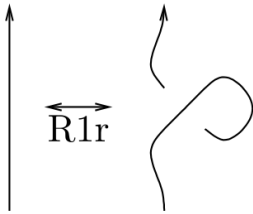
$$\text{rhs} = \left(2 \pi \int (\mathbb{E}[\pi_i \mathbf{p}_i] \times \mathcal{L}[\text{Dot}, \mathbf{i}]) \, d\{\mathbf{x}_i, \mathbf{p}_i\} \right) / \cdot \underline{\mathbf{p}}_{\mathbf{i}+1} \rightarrow \underline{\mathbf{p}}_{\mathbf{i}+2};$$

lhs == rhs

Out[*]=
- i E[Series[$\pi_i \underline{\mathbf{p}}_{2+i}$, 0, 0]]

Out[*]=
True

Invariance Under R1r



$$\text{In[*]:= lhs} = 8 \pi^3 \int (\mathbb{E}[\pi_i \mathbf{p}_i] \times \mathcal{L}[1, \mathbf{i}, \mathbf{i} + 1] \times \mathcal{L}[-1, \mathbf{i} + 1]) \, d\{\mathbf{x}_i, \mathbf{p}_i, \mathbf{v}_{\mathbf{i}+1}\}$$

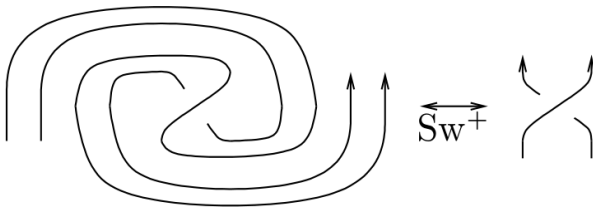
$$\text{rhs} = \left(2 \pi \int (\mathbb{E}[\pi_i \mathbf{p}_i] \times \mathcal{L}[\text{Dot}, \mathbf{i}]) \, d\{\mathbf{x}_i, \mathbf{p}_i\} \right) / \cdot \underline{\mathbf{p}}_{\mathbf{i}+1} \rightarrow \underline{\mathbf{p}}_{\mathbf{i}+2};$$

lhs == rhs

Out[*]=
- i E[Series[$\pi_i \underline{\mathbf{p}}_{2+i}$, 0, 0]]

Out[*]=
True

Invariance Under Sw



CF /@ { $\mathcal{L}[1, \mathbf{j}]$, $\mathcal{L}[1, \mathbf{i}, \mathbf{j}]$ }

$$\begin{aligned}
 \text{In[*]}:= \text{lhs} &= \int \left(\mathbb{E} \left[\underline{x}_i \underline{p}_i + \underline{x}_j \underline{p}_j \right] \times \mathcal{L}[1, i, j] \times \mathcal{L}[-1, i] \times \mathcal{L}[1, i+1] \times \mathcal{L}[-1, j] \times \mathcal{L}[1, j+1] \right) \\
 &\quad \text{d} \left\{ \underline{v}_{s_i}, \underline{v}_{s_j}, \underline{p}_{i+1}, \underline{p}_{j+1}, \underline{x}_{i+1}, \underline{x}_{j+1} \right\} \\
 \text{rhs} &= \int \left(\mathbb{E} \left[\underline{x}_i \underline{p}_i + \underline{x}_j \underline{p}_j \right] \times \mathcal{L}[1, i, j] \times \mathcal{L}[0, i] \times \mathcal{L}[0, i+1] \times \mathcal{L}[0, j] \times \mathcal{L}[0, j+1] \right) \\
 &\quad \text{d} \left\{ \underline{v}_{s_i}, \underline{v}_{s_j}, \underline{p}_{i+1}, \underline{p}_{j+1}, \underline{x}_{i+1}, \underline{x}_{j+1} \right\}; \\
 \text{lhs} &= \text{rhs}
 \end{aligned}$$

Out[*]=

$$\begin{aligned}
 &\frac{1}{64 \pi^6} \sqrt{\mathbb{T}} \mathbb{E} \left[\in \text{Series} \left[\mathbb{T} p_{1+i} \underline{x}_i + p_{1+j} (\underline{x}_i - \mathbb{T} \underline{x}_i + \underline{x}_j), \right. \right. \\
 &\quad - \frac{1}{2} + \frac{1}{2} \mathbb{T} p_{1+i} p_{1+j} \underline{x}_i (-\underline{x}_i + \mathbb{T} \underline{x}_i - 2 \underline{x}_j) - \frac{1}{2} \mathbb{T} p_{1+j}^2 \underline{x}_i (-\underline{x}_i + \mathbb{T} \underline{x}_i - 2 \underline{x}_j) + p_{1+j} (\mathbb{T} \underline{x}_i - \underline{x}_j), \\
 &\quad \frac{1}{4} \mathbb{T} p_{1+j}^2 \underline{x}_i (-3 \underline{x}_i + 5 \mathbb{T} \underline{x}_i - 10 \underline{x}_j) - \frac{1}{4} \mathbb{T} p_{1+i} p_{1+j} \underline{x}_i (-\underline{x}_i + 3 \mathbb{T} \underline{x}_i - 6 \underline{x}_j) - \\
 &\quad \frac{1}{6} \mathbb{T}^2 p_{1+i}^2 p_{1+j} \underline{x}_i^2 (-\underline{x}_i + \mathbb{T} \underline{x}_i - 3 \underline{x}_j) + \frac{1}{2} p_{1+j} (-\mathbb{T} \underline{x}_i + \underline{x}_j) + \\
 &\quad \left. \left. \frac{1}{6} \mathbb{T} p_{1+i} p_{1+j}^2 \underline{x}_i (\underline{x}_i^2 - 5 \mathbb{T} \underline{x}_i^2 + 4 \mathbb{T}^2 \underline{x}_i^2 + 3 \underline{x}_i \underline{x}_j - 12 \mathbb{T} \underline{x}_i \underline{x}_j + 3 \underline{x}_j^2) - \right. \right. \\
 &\quad \left. \left. \frac{1}{6} \mathbb{T} p_{1+j}^3 \underline{x}_i (\underline{x}_i^2 - 4 \mathbb{T} \underline{x}_i^2 + 3 \mathbb{T}^2 \underline{x}_i^2 + 3 \underline{x}_i \underline{x}_j - 9 \mathbb{T} \underline{x}_i \underline{x}_j + 3 \underline{x}_j^2) \right] \right]
 \end{aligned}$$

Out[*]=

True