Pensieve header: Proof of invariance of ρ_1 using integration techniques.

Initialization

```
In[◦]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\HigherRank"];
Once[<< KnotTheory`; << Rot.m];
<< NilpotentIntegration.m;
$π = Normal[# + O[ϵ]^2] &;
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
Loading Rot.m from http://drorbn.net/icbs24 to compute rotation numbers.
```

The ρ_1 Integrand

Adopted from pensieve://Projects/APAI/PerturbedGaussianIntegration.nb.

```
In[◦]:= q[s_, i_, j_] := x_i ((1 - T^s) p_{i+1} + (T^s - 1) p_{j+1});
r1[s_, i_, j_] :=
   s (-1 + 2 p_i x_i - 2 p_j x_i + (T^s - 1) p_i p_j x_i^2 + (1 - T^s) p_j^2 x_i^2 - 2 p_i p_j x_i x_j + 2 p_j^2 x_i x_j) / 2;
γ1[φ_, k_] := ϵ φ (1 / 2 - x_k p_k);
ρ1i[s_, i_, j_] := T^(s/2) 𝔼[-q[s, i, j] + ϵ r1[s, i, j]];
ρ1i[φ_, k_] := T^(φ/2) 𝔼[-x_k (p_k - p_{k+1}) + γ1[φ, k]];
ρ1i[End, k_] := 𝔼[-x_k p_k];
ρ1i[K_] := Module[{Cs, φ, n, c, k, ℰ},
    {Cs, φ} = Rot[K]; n = Length[Cs];
    ℰ = ρ1i[End, 2 n + 1];
    Do[ℰ *= ρ1i @@ c, {c, Cs}];
    Do[ℰ *= ρ1i[φ[[k]], k], {k, 2 n}];
    CF@ℰ
   ];
ρ1vs[K_] := Union @@ Table[{x_i, p_i}, {i, 2 Crossings[K] + 1}]
```

In[•]:= ρ1i[Knot[3, 1]]

KnotTheory: Loading precomputed data in PD4Knots`.

Out[•]=

$$\frac{1}{T^2}\mathbb{E}\Big[\epsilon - p_1 x_1 + p_2 x_1 - p_2 x_2 - \epsilon p_2 x_2 + \frac{p_3 x_2}{T} + \epsilon p_5 x_2 + \frac{(-1+T)\, p_6 x_2}{T} + \frac{(-1+T)\,\epsilon p_2 p_5 x_2^2}{2T} - \frac{(-1+T)\,\epsilon p_5^2 x_2^2}{2T} - p_3 x_3 + p_4 x_3 + \epsilon p_1 x_4 + \frac{(-1+T)\, p_2 x_4}{T} - p_4 x_4 + \frac{p_5 x_4}{T} - \epsilon p_1^2 x_1 x_4 + \epsilon p_1 p_4 x_1 x_4 - \frac{(-1+T)\,\epsilon p_1^2 x_4^2}{2T} + \frac{(-1+T)\,\epsilon p_1 p_4 x_4^2}{2T} - p_5 x_5 + p_6 x_5 + \epsilon p_2 p_5 x_2 x_5 - \epsilon p_5^2 x_2 x_5 + \epsilon p_3 x_6 + \frac{(-1+T)\, p_4 x_6}{T} - p_6 x_6 - \epsilon p_6 x_6 + \frac{p_7 x_6}{T} - \epsilon p_3^2 x_3 x_6 + \epsilon p_3 p_6 x_3 x_6 - \frac{(-1+T)\,\epsilon p_3^2 x_6^2}{2T} + \frac{(-1+T)\,\epsilon p_3 p_6 x_6^2}{2T} - p_7 x_7\Big]$$

In[•]:= ρ1vs[Knot[3, 1]]

Out[•]=

$$\{p_1, p_2, p_3, p_4, p_5, p_6, p_7, x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$$

In[•]:= K = Knot[5, 2]; ρ1i[K]

Out[•]=

$$\frac{1}{T^3}\mathbb{E}\Big[2\epsilon - p_1 x_1 + p_2 x_1 - p_2 x_2 - \epsilon p_2 x_2 + \frac{p_3 x_2}{T} + \epsilon p_7 x_2 + \frac{(-1+T)\, p_8 x_2}{T} + \frac{(-1+T)\,\epsilon p_2 p_7 x_2^2}{2T} - \frac{(-1+T)\,\epsilon p_7^2 x_2^2}{2T} - p_3 x_3 + p_4 x_3 + \epsilon p_1 x_4 + \frac{(-1+T)\, p_2 x_4}{T} - p_4 x_4 + \frac{p_5 x_4}{T} - \epsilon p_1^2 x_1 x_4 + \epsilon p_1 p_4 x_1 x_4 - \frac{(-1+T)\,\epsilon p_1^2 x_4^2}{2T} + \frac{(-1+T)\,\epsilon p_1 p_4 x_4^2}{2T} - p_5 x_5 + p_6 x_5 - p_6 x_6 - \epsilon p_6 x_6 + \frac{p_7 x_6}{T} + \epsilon p_9 x_6 + \frac{(-1+T)\, p_{10} x_6}{T} + \frac{(-1+T)\,\epsilon p_6 p_9 x_6^2}{2T} - \frac{(-1+T)\,\epsilon p_9^2 x_6^2}{2T} - p_7 x_7 + p_8 x_7 + \epsilon p_2 p_7 x_2 x_7 - \epsilon p_7^2 x_2 x_7 + \epsilon p_3 x_8 + \frac{(-1+T)\, p_4 x_8}{T} - p_8 x_8 - \epsilon p_8 x_8 + \frac{p_9 x_8}{T} - \epsilon p_3^2 x_3 x_8 + \epsilon p_3 p_8 x_3 x_8 - \frac{(-1+T)\,\epsilon p_3^2 x_8^2}{2T} + \frac{(-1+T)\,\epsilon p_3 p_8 x_8^2}{2T} - p_9 x_9 - \epsilon p_9 x_9 + p_{10} x_9 + \epsilon p_6 p_9 x_6 x_9 - \epsilon p_9^2 x_6 x_9 + \epsilon p_5 x_{10} + \frac{(-1+T)\, p_6 x_{10}}{T} - p_{10} x_{10} + \frac{p_{11} x_{10}}{T} - \epsilon p_5^2 x_5 x_{10} + \epsilon p_5 p_{10} x_5 x_{10} - \frac{(-1+T)\,\epsilon p_5^2 x_{10}^2}{2T} + \frac{(-1+T)\,\epsilon p_5 p_{10} x_{10}^2}{2T} - p_{11} x_{11}\Big]$$

In[•]:= K = Knot[5, 2]; ∫ρ1i[K] ⅆ(ρ1vs@K)

Out[•]=

$$-\frac{i\, T\, \mathbb{E}\Big[\frac{(-1+T)^2\,(5-4T+5T^2)\,\epsilon}{(2-3T+2T^2)^2}\Big]}{2048\,\pi^{11}\,(2-3T+2T^2)}$$

In[∘]:=
```
K = Knot[8, 19]; ∫ρ1i[K] ⅆ(ρ1vs@K)
```

Out[∘]=

$$-\frac{i\,T^3\,\mathbb{E}\left[-\frac{(-1+T)^2\,(1+T^4)\,(3+4\,T^3+3\,T^6)\,\epsilon}{(1-T+T^2)^2\,(1-T^2+T^4)^2}\right]}{131072\,\pi^{17}\,(1-T+T^3-T^5+T^6)}$$

Concatenating edges

In[∘]:=
```
lhs = ∫(𝔼[π_i p_i] × ρ1i[φ1, i] × ρ1i[φ2, i + 1]) ⅆ{x_i, p_i, x_{i+1}, p_{i+1}}
rhs = ∫(𝔼[π_i p_i] × ρ1i[φ1 + φ2, i]) ⅆ{x_i, p_i}
```

Out[∘]=

$$\frac{T^{\frac{\varphi 1}{2}+\frac{\varphi 2}{2}}\,\mathbb{E}\left[\frac{1}{2}\,\epsilon\,(-\varphi 1-\varphi 2)+p_{2+i}\,\pi_i-\epsilon\,(\varphi 1+\varphi 2)\,p_{2+i}\,\pi_i\right]}{4\,\pi^2}$$

Out[∘]=

$$-\frac{i\,T^{\frac{\varphi 1}{2}+\frac{\varphi 2}{2}}\,\mathbb{E}\left[\frac{1}{2}\,\epsilon\,(-\varphi 1-\varphi 2)+p_{1+i}\,\pi_i-\epsilon\,(\varphi 1+\varphi 2)\,p_{1+i}\,\pi_i\right]}{2\,\pi}$$

Invariance Under Reidemeister 3b

In[*]:=
```
lhs = ∫(𝔼[π_i p_i + π_j p_j + π_k p_k] × ρ1i[1, i, j] × ρ1i[1, i + 1, k] × ρ1i[1, j + 1, k + 1] ×
      ρ1i[0, i] × ρ1i[0, j] × ρ1i[0, k] × ρ1i[0, i + 1] × ρ1i[0, j + 1] × ρ1i[0, k + 1])
    d{x_i, x_j, x_k, p_i, p_j, p_k, x_{i+1}, x_{j+1}, x_{k+1}, p_{i+1}, p_{j+1}, p_{k+1}}
rhs = ∫(𝔼[π_i p_i + π_j p_j + π_k p_k] × ρ1i[1, j, k] × ρ1i[1, i, k + 1] × ρ1i[1, i + 1, j + 1] ×
      ρ1i[0, i] × ρ1i[0, j] × ρ1i[0, k] × ρ1i[0, i + 1] × ρ1i[0, j + 1] × ρ1i[0, k + 1])
    d{x_i, x_j, x_k, p_i, p_j, p_k, x_{i+1}, x_{j+1}, x_{k+1}, p_{i+1}, p_{j+1}, p_{k+1}};
lhs == rhs
```

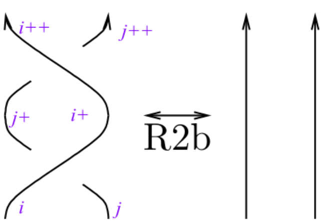
Out[*]=

$$\frac{1}{64\pi^6} T^{3/2}\,\mathbb{E}\Big[-\frac{3\,\epsilon}{2} + T^2 p_{2+i}\pi_i + \frac{1}{2}T^3 \epsilon\, p_{2+i} p_{2+j}\pi_i\,(-\pi_i + T\pi_i - 2\pi_j) - \frac{1}{2}T^3\epsilon\, p_{2+j}^2\pi_i\,(-\pi_i + T\pi_i - 2\pi_j) +$$
$$T\epsilon\, p_{2+j}\,(T\pi_i - \pi_j) - T p_{2+j}\,(-\pi_i + T\pi_i - \pi_j) - \frac{1}{2}T\epsilon\, p_{2+k}^2\,(\pi_i + \pi_j)\,(-\pi_i + T\pi_i - \pi_j + T\pi_j - 2\pi_k) +$$
$$\frac{1}{2}T^2\epsilon\, p_{2+i} p_{2+k}\pi_i\,(-\pi_i + T\pi_i - 2\pi_j + 2T\pi_j - 2\pi_k) +$$
$$\epsilon\, p_{2+k}\,(T\pi_i - \pi_j + 2T\pi_j - 2\pi_k) + p_{2+k}\,(\pi_i - T\pi_i + \pi_j - T\pi_j + \pi_k) - \frac{1}{2}T\epsilon\, p_{2+j} p_{2+k}$$
$$\left(\pi_i^2 - 2T\pi_i^2 + T^2\pi_i^2 + 2\pi_i\pi_j - 4T\pi_i\pi_j + 2T^2\pi_i\pi_j + \pi_j^2 - T\pi_j^2 + 2\pi_i\pi_k - 2T\pi_i\pi_k + 2\pi_j\pi_k\right)\Big]$$

Out[*]=

True

Invariance Under Reidemeister 2b



Territory: $(x \mid p)_{(i|j)^+}$, $x_{i|j}$.

In[∘]:=
```
lhs =
  ∫(𝔼[π_i p_i + π_j p_j] × ρ1i[1, i, j] × ρ1i[-1, i + 1, j + 1] × ρ1i[0, i] × ρ1i[0, j] × ρ1i[0, i + 1] ×
    ρ1i[0, j + 1]) ⅆ{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}}
rhs = ∫(𝔼[π_i p_i + π_j p_j] × ρ1i[0, i] × ρ1i[0, j] × ρ1i[0, i + 1] × ρ1i[0, j + 1])
   ⅆ{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}};
lhs == rhs
```

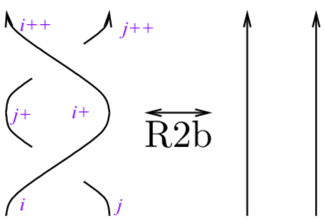
Out[∘]=

$$\frac{\mathbb{E}[p_{2+i}\,\pi_i + p_{2+j}\,\pi_j]}{16\,\pi^4}$$

Out[∘]=

True

Invariance Under Reidemeister 2b (no source terms)



Territory: $(x \mid p)_{(i|j)^+}$, $x_{i|j}$.

In[∘]:=
```
lhs =
  ∫(ρ1i[1, i, j] × ρ1i[-1, i + 1, j + 1] × ρ1i[0, i] × ρ1i[0, j] × ρ1i[0, i + 1] × ρ1i[0, j + 1])
   ⅆ{x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}}
```

Out[∘]=

$$\frac{1}{4\pi^2}\,\mathbb{E}\Big[-p_i x_i + \epsilon\, p_i x_i + p_{2+i} x_i - \epsilon\, p_j x_i - T\,\epsilon\, p_{2+j} x_i + \frac{1}{2}(-1+T)\,\epsilon\, p_i p_j x_i^2 + \frac{1}{2}(1-T)\,\epsilon\, p_j^2 x_i^2 + \frac{1}{2}(1-T)\,\epsilon\, p_{2+i} p_{2+j} x_i^2 + \frac{1}{2}(-1+T)\,\epsilon\, p_{2+j}^2 x_i^2 - p_j x_j + p_{2+j} x_j + \epsilon\, p_{2+j} x_j - \epsilon\, p_i p_j x_i x_j + \epsilon\, p_j^2 x_i x_j + \epsilon\, p_{2+i} p_{2+j} x_i x_j - \epsilon\, p_{2+j}^2 x_i x_j\Big]$$

In[∘]:=
```
CF[lhs /. {p_i → (1 + ϵ) p_{i+2} - ϵ (1 + T) p_{j+2}, p_j → (1 + ϵ) p_{j+2}}] /. ϵ- → 0
```

Out[∘]=

$$\frac{\mathbb{E}[0]}{4\pi^2}$$

In[∘]:=
```
rhs = ∫(ρ1i[0, i] × ρ1i[0, j] × ρ1i[0, i + 1] × ρ1i[0, j + 1]) ⅆ{x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}}
```

Out[∘]=

$$\frac{\mathbb{E}[-p_i x_i + p_{2+i} x_i - p_j x_j + p_{2+j} x_j]}{4\pi^2}$$

In[∘]:= **CF[rhs /. {$p_i \to p_{i+2}$, $p_j \to p_{j+2}$}]**

Out[∘]=

$$\frac{\mathbb{E}[0]}{4\pi^2}$$

In[∘]:= **Coefficient[(4 π² lhs)[[1]], ϵ, 0] == Coefficient[(4 π² rhs)[[1]], ϵ, 0]**

Out[∘]=

True

In[∘]:= **diff = CF[Coefficient[(4 π² lhs)[[1]], ϵ, 0] + ϵ (Coefficient[(4 π² lhs)[[1]], ϵ, 1] - Coefficient[(4 π² rhs)[[1]], ϵ, 1])]**

Out[∘]=

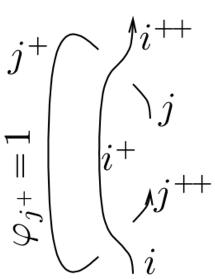
$$-p_i x_i + \epsilon p_i x_i + p_{2+i} x_i - \epsilon p_j x_i - T \epsilon p_{2+j} x_i + \frac{1}{2}(-1+T)\epsilon p_i p_j x_i^2 + \frac{1}{2}(1-T)\epsilon p_j^2 x_i^2 + \frac{1}{2}(1-T)\epsilon p_{2+i} p_{2+j} x_i^2 + \frac{1}{2}(-1+T)\epsilon p_{2+j}^2 x_i^2 - p_j x_j + p_{2+j} x_j + \epsilon p_{2+j} x_j - \epsilon p_i p_j x_i x_j + \epsilon p_j^2 x_i x_j + \epsilon p_{2+i} p_{2+j} x_i x_j - \epsilon p_{2+j}^2 x_i x_j$$

In[∘]:= **∫𝔼[diff + $\pi_i p_i + \pi_j p_j$] 𝕕{x_i, x_j, p_i, p_j}**

Out[∘]=

$$\frac{\mathbb{E}[p_{2+i}\pi_i + p_{2+j}\pi_j]}{4\pi^2}$$

Invariance Under R2c



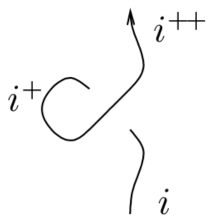
In[∘]:= **lhs = ∫(𝔼[$\pi_i p_i + \pi_j p_j$] × ρ1i[-1, i, j + 1] × ρ1i[1, i + 1, j] × ρ1i[0, i] × ρ1i[0, j] × ρ1i[0, i + 1] × ρ1i[1, j + 1]) 𝕕{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}}**

Out[∘]=

$$\frac{\sqrt{T}\,\mathbb{E}\left[-\frac{\epsilon}{2} + p_{2+i}\pi_i + p_{2+j}\pi_j - \epsilon p_{2+j}\pi_j\right]}{16\pi^4}$$

In[∘]:=
```
rhs = ∫(𝔼[π_i p_i + π_j p_j] × ρ1i[0, i] × ρ1i[0, j] × ρ1i[0, i + 1] × ρ1i[1, j + 1])
    ⅆ{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}};
lhs == rhs
```
Out[∘]=
```
True
```

Invariance Under R1l

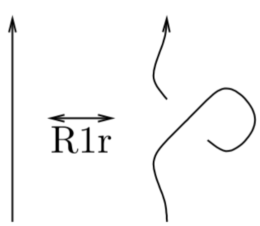


In[∘]:=
```
lhs = ∫(𝔼[π_i p_i] × ρ1i[1, i + 1, i] × ρ1i[0, i] × ρ1i[1, i + 1]) ⅆ{x_i, p_i, x_{i+1}, p_{i+1}}
```
Out[∘]=

$$\frac{\mathbb{E}[p_{2+i}\,\pi_i]}{4\,\pi^2}$$

In[∘]:=
```
rhs = ∫(𝔼[π_i p_i] × ρ1i[0, i] × ρ1i[0, i + 1]) ⅆ{x_i, p_i, x_{i+1}, p_{i+1}};
lhs == rhs
```
Out[∘]=
```
True
```

Invariance Under R1r

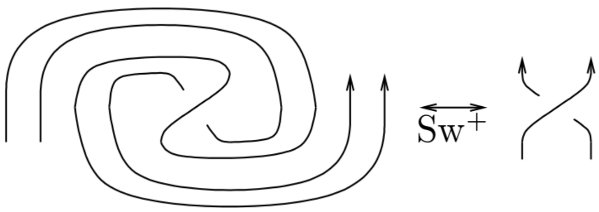


In[∘]:=
```
lhs = ∫(𝔼[π_i p_i] × ρ1i[1, i, i + 1] × ρ1i[0, i] × ρ1i[-1, i + 1]) ⅆ{x_i, p_i, x_{i+1}, p_{i+1}}
```
Out[∘]=

$$\frac{\mathbb{E}[p_{2+i}\,\pi_i]}{4\,\pi^2}$$

In[∘]:=
```
rhs = ∫(𝔼[π_i p_i] × ρ1i[0, i] × ρ1i[0, i + 1]) ⅆ{x_i, p_i, x_{i+1}, p_{i+1}};
lhs == rhs
```
Out[∘]=
```
True
```

Invariance Under Sw



In[∘]:=

$$\text{CF} \,/@\, \{\rho1i[1, j],\ \rho1i[1, i, j]\}$$

Out[∘]=

$$\left\{\sqrt{T}\,\mathbb{E}\left[\frac{\epsilon}{2} - p_j x_j - \epsilon\, p_j x_j + p_{1+j} x_j\right],\ \sqrt{T}\,\mathbb{E}\left[-\frac{\epsilon}{2} + \epsilon\, p_i x_i + (-1+T)\, p_{1+i} x_i - \epsilon\, p_j x_i + (1-T)\, p_{1+j} x_i + \frac{1}{2}(-1+T)\,\epsilon\, p_i p_j x_i^2 + \frac{1}{2}(1-T)\,\epsilon\, p_j^2 x_i^2 - \epsilon\, p_i p_j x_i x_j + \epsilon\, p_j^2 x_i x_j\right]\right\}$$

In[∘]:=

$$\text{lhs} = \int\left(\mathbb{E}[\pi_i p_i + \pi_j p_j + \epsilon\,\pi_{i+1} p_{i+1} + \epsilon\,\pi_{j+1} p_{j+1} + \xi_{i+1} x_{i+1} + \xi_{j+1} x_{j+1}] \times \rho1i[1, i, j] \times \rho1i[-1, i] \times \rho1i[1, i+1] \times \rho1i[-1, j] \times \rho1i[1, j+1]\right) \mathbb{d}\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}\}$$

Out[∘]=

$$\frac{1}{16\pi^4}\sqrt{T}\,\mathbb{E}\Big[T\, p_{2+i}\,\pi_i + \frac{1}{2} T\,\epsilon\, p_{2+i}\, p_{2+j}\,\pi_i\,(-\pi_i + T\,\pi_i - 2\,\pi_j) - \frac{1}{2} T\,\epsilon\, p_{2+j}^2\,\pi_i\,(-\pi_i + T\,\pi_i - 2\,\pi_j) + p_{2+j}\,(\pi_i - T\,\pi_i + \pi_j) + T\,\pi_i\,\xi_{1+i} + \pi_i\,\xi_{1+j} - T\,\pi_i\,\xi_{1+j} + \pi_j\,\xi_{1+j} + \frac{1}{2}\epsilon\, p_{2+i}\left(2\,\pi_{1+i} - T\,\pi_i^2\,\xi_{1+j} + T^2\,\pi_i^2\,\xi_{1+j} - 2\,T\,\pi_i\,\pi_j\,\xi_{1+j}\right) + \frac{1}{2}\epsilon\, p_{2+j}\left(2\,T\,\pi_i - 2\,\pi_j + 2\,\pi_{1+j} - T\,\pi_i^2\,\xi_{1+i} + T^2\,\pi_i^2\,\xi_{1+i} - 2\,T\,\pi_i\,\pi_j\,\xi_{1+i} + 2\,T\,\pi_i^2\,\xi_{1+j} - 2\,T^2\,\pi_i^2\,\xi_{1+j} + 4\,T\,\pi_i\,\pi_j\,\xi_{1+j}\right) + \frac{1}{2}\epsilon\left(-1 + 2\,\pi_{1+i}\,\xi_{1+i} + 2\,T\,\pi_i\,\xi_{1+j} - 2\,\pi_j\,\xi_{1+j} + 2\,\pi_{1+j}\,\xi_{1+j} - T\,\pi_i^2\,\xi_{1+i}\,\xi_{1+j} + T^2\,\pi_i^2\,\xi_{1+i}\,\xi_{1+j} - 2\,T\,\pi_i\,\pi_j\,\xi_{1+i}\,\xi_{1+j} + T\,\pi_i^2\,\xi_{1+j}^2 - T^2\,\pi_i^2\,\xi_{1+j}^2 + 2\,T\,\pi_i\,\pi_j\,\xi_{1+j}^2\right)\Big]$$

In[∘]:=

$$\text{rhs} = \int\left(\mathbb{E}[\pi_i p_i + \pi_j p_j + \epsilon\,\pi_{i+1} p_{i+1} + \epsilon\,\pi_{j+1} p_{j+1} + \xi_{i+1} x_{i+1} + \xi_{j+1} x_{j+1}] \times \rho1i[1, i, j] \times \rho1i[0, i] \times \rho1i[0, i+1] \times \rho1i[0, j] \times \rho1i[0, j+1]\right) \mathbb{d}\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}\};$$

$$\text{lhs} == \text{rhs}$$

Out[∘]=

True