Mathematics
UNIVERSITY OF TORONTO

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Dror Bar-Natan – 2024 Final Report for the Chu Family Foundation

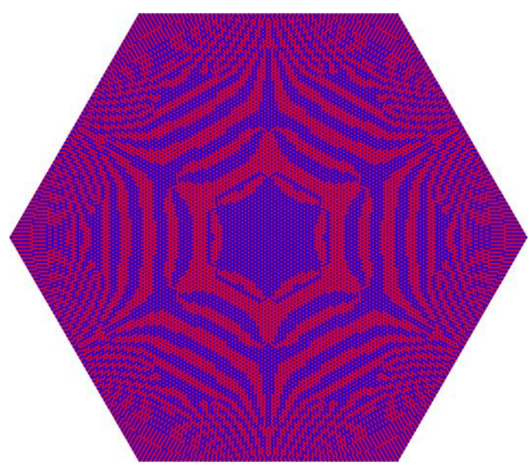
Dear Members of the Chu Family,

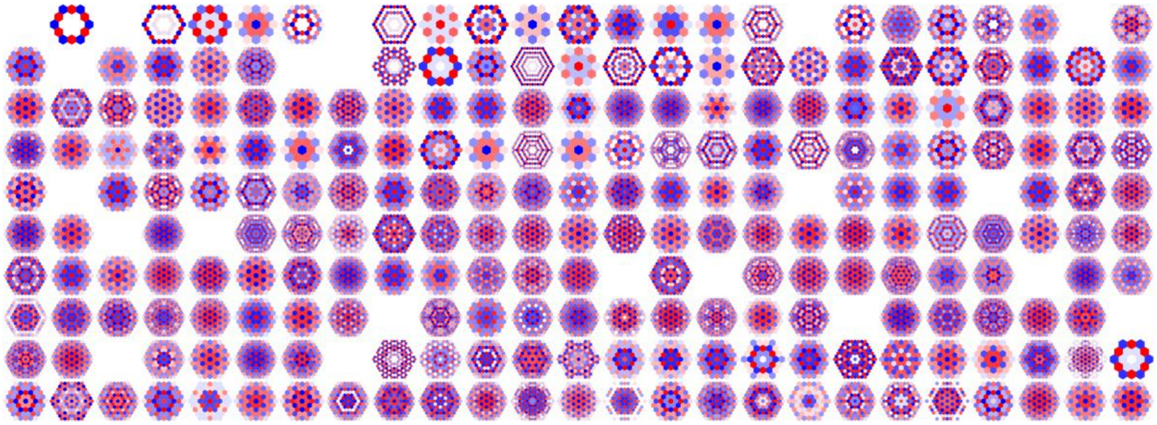
As requested when the grant was offered, this is a report on my research progress over the second and final year of research leave supported by the Chu Family Foundation.

By far, the result I am the most proud of is the knot invariant θ which I developed along with Roland van der Veen during my visit to Groningen in May-June 2024 (following nearly 10 years of work). The invariant θ is the **strongest genuinely computable** knot invariant presently known. By direct computation on tables containing some tens of thousands of knots we know that it has higher separation power than standardly used invariants (HOMFLY-PT and Kh), while these invariants are much harder to compute – they are not polynomial time, while θ is. In fact, we have computed θ on a number of knots with over 300 crossings. That's science fiction, from the perspective of almost anything else in knot theory.

The invariant θ has several other advantages. By theoretical reasoning and by experiment (so without doubt though yet without proof), it provides a "genus bound" and hence it **"sees topology"**, something that most descendants of the Jones polynomial – the so called "quantum invariants" – cannot claim. Our theoretical understanding also suggests that θ may provide a "ribbon obstruction" – one of the highest prizes in low dimensional topology. You can complain that all this is speculative. It is indeed, and it means that we have our work cut out for us for the next few years, and that other will jump in to try to do something too. That would be wonderful!

There is another nice aspect of θ – it is **fun**. It is a 2-variable polynomial, so its coefficients make a 2D array. Coding positive coefficients as red and negative ones as blue, θ becomes a 2D image, and these images are appealing. For example, on the right is the image corresponding to a random 250 crossing knot (supplied by N. Dunfield), and below are the images corresponding to the first 250 prime knots (the "Rolfsen Table" of knots with up to 10 crossings). Clearly there are patterns in these images and viewing them, lots of questions immediately arise. We don't know the answers to practically any of these questions. That's good news! Once we publish, lots of people will be interested in working on the topic.





There is not yet a publication about θ, though hopefully one is coming soon. For now, the only available material is in the form of public lectures that I've given. See http://drorbn.net/icbs24 and http://drorbn.net/ge24.

Unrelated to θ, I spent a significant amount of time over the last few months trying to find the relationship between "w-knots" and "the Goldman-Turaev Lie bialgebra". I will not attempt to explain these notions; I will only say that they both belong to topology, and both lead, via different paths, to the same equations in algebra, the "Kashiwara-Vergne equations". That cannot be a coincidence and there must be a direct relation between the two notions, at the level of topology. I believe that along with Yusuke Kuno of Tokyo, we finally found this relationship! Though a lot of work remains to be done, and the results will not be as pretty as the θ results. I hope I will be able to convince my new post-doctoral fellow, Tamara Hogan (who is supported by a grant from the Chu Family Foundation), to pick up and finish this project. The collaboration may also include my former student Zsuzsanna Dancso and one of her current students.

A paper that I mentioned last year on a very efficient algorithm to compute finite type invariants (joint with my son Itai Bar-Natan, with Iva Halacheva, and with Nancy Scherich) is about one day of editing away from completion and submission. I reported on the work done in the paper in a talk I gave in Tokyo. See http://drorbn.net/tok2309.

Like last year, I did not make progress with what might have been my primary goal for the year – a writeup of my results with Roland van der Veen on the polynomial-time computability of the full portfolio of operations associated with the solvable approximation of quantum groups, and the polynomial-time computability of the associated knot and tangle invariants. I remain stuck in the same pedagogical place as a year ago and as two years ago: the computations work, but they take "two step" procedure which is hard to motivate. And so I find myself having to introduce ugliness into what is otherwise a very beautiful picture, and over the last two years I kept preferring to wait for a better language to arise. It didn't, and perhaps it is time to give up and tolerate a bit of inelegance.

I did make progress, along with several collaborators, on several other aspects of the same project and on several other related and unrelated projects:

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- Along with Roland van der Veen, we've continued our study of the symmetry properties of ρ_1. This is supposed to be easy but turns out to be surprisingly difficult. It seems that we first need to understand the symmetry properties of the "Green function" $g_{\alpha\beta}$, which is an Alexander-level (and hence "classical") quantity. But it turns out that the palindromicity property of the Alexander polynomial is understood mostly from a "Seifert" perspective and not from the edge-edge perspective that is necessary for $g_{\alpha\beta}$, and thus we need to re-understand bits of the Alexander theory. We've continued to make progress but we are not done.

- Along with Zsuzsanna Dancso, Tamara Hogan, Jessica Liu, and Nancy Scherich, we've constructed a homomorphic expansion for the Goldman-Turaev Lie bi-algebra of curves in a punctured plane using the Kontsevich integral for tangles in a "Pole Dancing Studio" – a room with a few vertical lines (the "poles") removed. A paper on the subject moved a bit further and some concepts in it got much clearer. It is near completion.

- Along with Jessica Liu we've found and implemented on a computer extensions of the Tristram-Levine and Kashaev signatures of knots to tangles, using a novel "pushforward" operation for quadratic forms (I'm very proud of it!). This project was detailed in a talk I gave in Providence in May 2023 (available at http://drorbn.net/icerm23) and in Geneva (http://drorbn.net/ge23) It is in initial phases of writeup for publication.

Over the last two years I have acknowledged support from the Chu Family Foundation on two accepted papers ("Over Then Under Tangles", with Zsuzsanna Dancso and Roland van der Veen, arXiv:2007.09828, to appear in the *Journal of Knot Theory and its Ramifications* and "A Perturbed Alexander Invariant", with Roland van der Veen, arXiv:2206.12298, to appear in *Quantum Topology*) and in 26 invited lectures, in Les Diablerets, Oaxaca, UCLA, Toronto, Providence, Ottawa, Kyoto, Tokyo U (x2), Nara, Waseda U (x2), Tokyo U (again), Waterloo, Budapest (x2), Geneva, McMaster U, USC, Bloomington (x2), Groningen, Beijing (BIMSA), Beijing (BJTU), and Geneva (again, x2).

I wish to thank the Chu Family Foundation for their support, which enabled what had been two amazing years!

Sincerely,

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