Dror Bar-Natan: Confessions

- **I've never understood "resolution of singularities".**
- **I don't understand the Koszul condition.**
- **I don't yet appreciate infinity-algebras.**
- **I don't really understand Poisson structures: Why do they automatically arise from action principles? Why do they necessarily emerge in computing path integrals? Why should I care about their deformation quantizations?**
- **I don't understand Tamarkin's work on formality.**
- **Spectral sequences never became me.**
- **I don't understand homotopy theory, loop spaces, spectra, etc.**
- **I don't understand minimal models.** Books on rational homotopy theory: Félix-Helperin-Thomas, Griffiths-Morgan.
- **I don't understand thermal physics - energy, entropy, enthalpy, and all that. Such basic things these are that it is really embarrassing that I don't understand the constraints my air-conditioner is bound by.**

— From Feynman's *Lectures on Physics*: • "equal volumes of gases, at the same pressure and temperature, contain the same number of molecules"; $N_0 = 6.022 \times 10^{23}$ as in (1 mole)=12g of ^{12}C. • $P = F/A$. • $dW = -PdV$. • $PV = \frac{2}{3}N\langle\frac{1}{2}mv^2\rangle = \frac{2}{3}U$ $(\ldots = NkT)$. • With $\gamma - 1 = \frac{2}{3}$, $PV^\gamma = C$. • In gas mixtures, $\frac{1}{2}m_1v_1^2 = \frac{1}{2}m_2v_2^2$ (messy!). • $\frac{1}{2}mv^2 =: \frac{3}{2}kT$, with $k = 1.38 \times 10^{-23}$ J/degree (J = joule = newton metre = watt second).

— From Bamberg-Sternberg: • First law of thermodynamics: $\alpha + \omega = dU$, with α: heat 1-form, ω: work 1-form, U: internal energy. • Second law of thermodynamics: $\alpha = TdS$, with T: temperature, S: entropy.

— From Schroeder: • 1cal = 10^{-3} food calorie := $4.186J$ ~ heat to raise $1g$ of water by $1°C$.

— See also Lieb-Yngvason.

- **I don't understand supersymmetry.**
- **I don't understand renormalization theory.** Minor point: it would be great if I could present the renormalization of associators/vertices as a special case.
- **I don't understand the Mostow rigidity theorem.**
- **I'm not as comfortable with special relativity as I want to be.**
- **I don't really understand general relativity.**
- **I don't know how to put figures in LaTeX efficiently.**
- **I don't fully understand the h-cobordism theorem.** Perhaps follow Milnor's lecture notes?

Def. An h-cobordism is a cobordism in which the boundary inclusions are deformation retracts.

Thm. In *Diff*, *PL*, or *Top*, a simply-connected h-cobordism between simply-connected $(n \geq 5)$-manifolds is trivial.

- **I haven't internalized the distinction between continuous, smooth, and triangulated.**
- **I don't really understand Faddeev-Popov and/or BRST.**
- **I don't understand the Batalin-Vilkovisky formalism.**

— Mnev's example. "Space of fields" $M = R^3_{txy} \times S^1_z$; "classical action" $S_{cl} := \frac{1}{2}t^2$; "Gauge symmetry" $E := \operatorname{span}\left(\partial_y, \partial_x + ty\partial_z\right)$, integrable on $EL = [t = 0]$ surface but not on M, S_{cl} is invariant. M/E is not T_2 and $\int_{M/E} e^{-S}$ makes no sense.

BV space of fields $F = T^*[-1](\mathbb{R}^2[1]\times M)$ with coords $c_{1,2}$ (ghost number 1), t, x, y, z (g.n. 0), $t^\dagger, x^\dagger, y^\dagger, z^\dagger$ (g.n. -1) and $c^\dagger_{1,2}$ (g.n. -2). The BV action is $S = \frac{1}{2}t^2 + c_1y^\dagger + c_2(x^\dagger + tyz^\dagger) + c_1c_2t^\dagger z^\dagger$; satisfies QME & consistent with S_{cl} and E.

Gauge fixing Lagrangian $L = [x = y = t^\dagger = z^\dagger = c^\dagger_{1,2} = 0] \subset F$ gives $$\int_L e^{-S} = \int dtdzdc_1dc_2dx^\dagger dy^\dagger e^{-S_{cl}}c_1c_2x^\dagger y^\dagger = \sqrt{2\pi}T.$$

— Losev: For $\omega \in \Omega^{n-1}(M^n)$, $f\colon M \to \mathbb{R}$, $$\int_{[f=0]} \omega = \int_{TM\oplus\mathbb{R}^{1|1}_{l|\lambda}} \omega e^{-d(f\lambda)}.$$

— Further: old paper by Schwarz; arXiv:0812.0464 by Albert, Bleile, Fröhlich; notes by Kazhdan; thesis by Gwilliam; notes by Ens.

- **I still don't understand the BF TQFT.** From Cattaneo-Rossi's arXiv:math-ph/0210037 *Wilson Surfaces*: $A \in \Omega^1(\mathbb{R}^4, \mathfrak{g})$ a connection, $B \in \Omega^2(\mathbb{R}^4, \mathfrak{g}^*)$, $$S(A, B) := \int_{\mathbb{R}^4} \langle B, F_A\rangle.$$

$\mathcal{G} := \exp\Omega^0(\mathbb{R}^4, \mathfrak{g})$ is (u-)gauge transformations, $(g, \sigma) \in \tilde{\mathcal{G}} := \mathcal{G} \ltimes \Omega^1(\mathbb{R}^4, \mathfrak{g}^*)$ acts by
$$A \mapsto A^g \qquad B \mapsto B^{(g,\sigma)} := \operatorname{Ad}^*_{g^{-1}} B + d_{A^g}\sigma.$$
With $f\colon \mathbb{R}^2 \to \mathbb{R}^4$, $\xi \in \Omega^0(\mathbb{R}^2, \mathfrak{g})$, $\beta \in \Omega^1(\mathbb{R}^2, \mathfrak{g}^*)$, set
$$\mathcal{O}(A, B, f) := \int \mathcal{D}\xi\mathcal{D}\beta \exp\left(\frac{i}{\hbar}\int_{\mathbb{R}^2} \left\langle\xi, d_{f^*A}\beta + f^*B\right\rangle\right).$$

- **I forgot too much of what I used to know about Lie theory.** From Humphreys: Weyl's formula: For $\lambda \in \Lambda^+$,
$$ch_\lambda * \sum_{\sigma\in\mathcal{W}} (-)^\sigma \epsilon_{\sigma\delta} = \sum_{\sigma\in\mathcal{W}} (-)^\sigma \epsilon_{\sigma(\lambda+\delta)}.$$
- **I know nothing about θ functions.**
- **I don't understand Witten's exact solution of Chern-Simons theory (what he understood in 1988).**
- **I'm uncomfortable with quantum groups.** Is there a diagrammatic perspective? On a philosophical level, quantum groups as they appear in topology are "constructions" or "images". I wish I understood them as associated with "kernels". Rotational virtual tangles explain quantum groups as associated with a kernel of an extension, but I don't have an explanation within that context for why clean formulas arise. What is the relationship between quantum groups and expansions?
- **I don't understand the first thing about Heegaard-Floer homology.** Maybe Juhász' arXiv:1310.3418, Manolescu's arXiv:1401.7107, or Lipshitz' arXiv:1411.4540.
- **If it has the word Kähler in it, I shy away.**
- **I don't understand projective and injective resolutions, Ext and Tor, the universal coefficients theorem, etc.**
- **I am yet to internalize "sheafs".**
- **I've never figured "derived".** Perhaps Yekutieli's arXiv:1501.06731?
- **I've never figured "perverse".**
- **I don't understand the Künneth and Eilenberg-Zilber theorems.**
- **I don't understand the relationship between gr and H, as it appears, for example, in braid theory.** — Perhaps Berglund's *Koszul Spaces*?

• **I have no clue what are "motives".**

• **I don't understand Tannakian reconstruction principles, and I wish I did.** — Given an algebra A let $\mathcal{D} := A-\mathrm{Mod}$ (projective (?) left A-modules), let $C := \mathrm{Vect}$ and $G\colon \mathcal{D}\to C$ be the forGetful functor. Then $A \simeq \mathrm{End}(G)$ by

$$a\in A \mapsto (\text{the action of } a \text{ on any } X\in\mathcal{D}),$$
$$\{a_X\colon G(X)\to G(X)\}_{X\in\mathcal{D}} \mapsto a_A(1)\in A.$$

— Given a monoidal $\mathcal{D}$ and an exact $G\colon \mathcal{D}\to C =: \mathrm{Vect}$ with a natural isomorphism $\alpha_{X,Y}\colon G(X)G(Y)\to G(XY)$, there is a Hopf algebra structure on $H := \mathrm{End}(G)$: product is composition, coproduct $\Delta\colon H\to H^2 = \mathrm{End}(G^2\colon \mathcal{D}\times\mathcal{D}\to C)$ by

$$(h_X)_{X\in\mathcal{D}} \mapsto \left((X,Y)\mapsto \alpha_{X,Y}/\!\!/h_{XY}/\!\!/\alpha^{-1}_{X,Y}\in \mathrm{End}(G(X)G(Y))\right).$$

• **I don't understand Pfaffians (though of all my troubles, this is perhaps the least).** — See Wikipedia, Parameswaran, Ledermann. Concisely, if $\lambda_{\{ij\}}=0$, then

$$(\lambda_{ij}dx^i\wedge dx^j)^{n/2} = \sqrt{\det(\lambda_{ij})}\bigwedge_i dx^i$$

(common in symplectic geometry), so $\sqrt{\det(\lambda_{ij})}$ is a polynomial in the λ_{ij}'s. Itai/Yael: with $\omega=\lambda_{ij}dx^i\wedge dx^j$, need

$$\det(\omega(u_i,v_j)) = \omega^{n/2}(u_1,\dots,u_n)\omega^{n/2}(v_1,\dots,v_n).$$

Easy from multi-linearity and anti-symmetry if (u_i) and (v_j) are in a symplectic basis for ω.

• **I don't understand the Goussarov-Polyak-Viro theorem.**

• **I don't understand knot signatures (and signatures in general).**

• **I don't fully understand the Goussarov-Habiro theory of claspers.**

• **I don't understand Gröbner bases.**

• **I still don't know a proof of the Milnor-Moore theorem.** — Maybe "Spencer Bloch's course on Hopf Algebras" ~~or Kreimer's thesis~~. Maybe search inside?

• **I still don't understand Vogel's construction.**

• **I'm missing the key to equivariant cohomology, EG, BG, and all that.** — I need a framework for $X_G := (X\times EG)/G$.

• **I don't understand fusion categories and subfactors.** — Morrison's drorbn.net/dbnvp/Morrison-140220?

• **I don't understand group cohomology.**

— Pensieve: 2013-02: G group; M a G-module; $C^n(G,M) := \{\varphi\colon G^n\to M\}$; "derived from $M\to M^G$"

$$(d\varphi)(g_1,\dots,g_{n+1}) := g_1\varphi(g_2,\dots,g_{n+1}) + \sum_{i=1}^n (-)^i\varphi(\dots,g_ig_{i+1},\dots) + (-)^{n+1}\varphi(g_1,\dots,g_n).$$

$$(\varphi\cup\psi)(g_1,\dots,g_{n+m}) := \sum_{\sigma \text{ monotone on } 1..n \text{ \& on } (n+1)..(n+m)} (-)^\sigma \varphi(g_{\sigma 1},\dots,g_{\sigma n})\psi(g_{\sigma(n+1)},\dots,g_{\sigma(n+m)})$$

At $M=\mathbb{K}$: • $H^* = H^*(K(G,1))$. • $H^1 = \mathrm{Hom}(G,\mathbb{K})$. • $H^2 \leftrightarrow$ central extensions by $\mathbb{K}$. $H^3(G,\mathbb{K}^\times)\leftrightarrow$ categorifications of $\mathbb{Z}G$.

• **I don't understand the basics of three-dimensional topology: the loop and sphere theorems, JSJ decompositions, etc.** Continuing 2013-11: CheatSheet3DTopology.pdf

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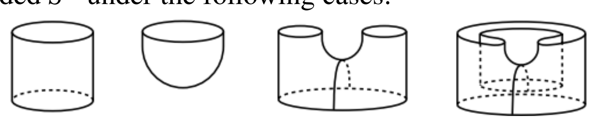
From Hatcher's notes:

Definition. M prime: $M = P\#Q \Rightarrow (P=S^3)\vee(Q=S^3)$. M Irreducible: an embedded 2-sphere in M bounds a 3-ball. (Irreducible $\Rightarrow$ Prime).

Theorem (Alexander, 1920s). S^3 is irreducible.

Proof. Study the change to the "canonical closure" of a cropped embedded S^2 under the following cases:



Theorem. Orientable, prime, not irreducible $\Rightarrow$ $S^2\times S^1$. Nonorientable? Also $S^2\widetilde{\times}S^1$ (Klein 3D).

Theorem. Compact connected orientable 3-manifolds have unique decomposition into primes.

Proof. • Given a system of splitting spheres (sss) and a θ-partition of one member, at least one part will make an sss. • An sss can be simplified relative to a fixed triangulation τ: only disk intersections with simplices; circle and single-edge-arc intersections with faces of τ can be eliminated. • The size of an sss is bounded by $4|\tau|+\mathrm{rank}\,H_1(M;\mathbb{Z}/2)$ and hence prime-decompositions exist. • Uniqueness. □

Nonorientable M? Same but $M\#(S^2\times S^1) = M\#(S^2\widetilde{\times}S^1)$.

Theorem. If a covering is irreducible, so is the base. ([Ha] proof is fishy).

Examples. Lens spaces, surface bundles $F\to M\to S^1$ with $F\neq S^2,\mathbb{R}\mathrm{P}^2$. Yet $S^1\times S^2/(x,y)\sim(\bar{x},-y) = \mathbb{R}\mathrm{P}^3\#\mathbb{R}\mathrm{P}^3$, a prime covers a sum.

Definition. $S\subset M^3$ a 2-sided surface, $S\neq S^2$, $S\neq D^2$. *Compressing disk for S* is a disk $D\subset M$ with $D\cap S=\partial D$. If for every compressing D there's a disk $D'\subset S$ with $\partial D'=\partial D$, S is *incompressible*.

Claims. • $\pi_1(S)\hookrightarrow\pi_1(M)\Rightarrow S$ incompressible. • No incompressibles in $\mathbb{R}^3/S^3$. • In irreducible M^3, T^2 is 2-sided incompressible iff T bounds a $D^2\times S^1$ or T is contained in a B^3. • A T^2 in S^3 bounds a $D^2\times S^1$ on at least one side. • $S\subset M$ incompressible $\Rightarrow$ (M irreducible iff $M|S$ irreducible). • S a collection of disjoint incompressibles or disks or spheres in M, $T\subset M|S$. Then T is incompressible in M iff in $M|S$.

From Hempel's book:

Dehn's Lemma (Dehn 1910 (wrong), Papakyriakopoulos 1950s). M a 3-manifold, $f\colon B^2\to M$ s.t. for some neighborhood A of ∂B^2 in B^2 the restriction $F|_A$ is an embedding and $f^{-1}(f(A))=A$. Then $f|_{\partial B^2}$ extends to an embedding $g\colon B^2\to M$.

The Loop Theorem (Stallings 1960, implies Dehn's lemma). M a 3-manifold, F a connected 2-manifold in ∂M, $\ker(\pi_1(F)\to\pi_1(M)) \not\subset N \triangleleft \pi_1(F)$. Then there is a proper embedding $g\colon(B^2,\partial B^2)\to(M,F)$ s.t. $[g|_{\partial B^2}]\notin N$.

The Sphere Theorem. M orientable 3-manifold, N a $\pi_1(M)$-invariant proper subgroup of $\pi_2(M)$. Then there is an embedding $g\colon S^2\to M$ s.t. $[g]\notin N$.

Redeemed Confessions.

• **I don't understand Galois theory, for real. Abstractness is fun, but Galois surely understood everything in very concrete terms. I wish I did too.** — youtu.be/RhpVSV6iCko and then drorbn.net/dbnvp/AKT-140314.php and `http://www.math.toronto.edu/~drorbn/Talks/CMU-1504/` do the job!