In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\APerturbedAlexanderInvariant"];
<< APAI.m
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
Loading Rot.m from http://drorbn.net/APAI to compute rotation numbers.
```

In[∘]:= $\text{gRules}_{s\_,i\_,j\_} := \{g_{i\beta} == T^s g_{i+1,\beta} + (1 - T^s)\, g_{j+1,\beta},\ g_{j\beta} == g_{j+1,\beta}\}$

Proof of Reidemeister 3:

In[∘]:= $\text{Join}[\text{gRules}_{1,j,k}, \text{gRules}_{1,i,k+1}, \text{gRules}_{1,i+1,j+1}]$ // Column

Out[∘]=
$g_{j,\beta} == T\, g_{1+j,\beta} + (1 - T)\, g_{1+k,\beta}$
$g_{k,\beta} == g_{1+k,\beta}$
$g_{i,\beta} == T\, g_{1+i,\beta} + (1 - T)\, g_{2+k,\beta}$
$g_{1+k,\beta} == g_{2+k,\beta}$
$g_{1+i,\beta} == T\, g_{2+i,\beta} + (1 - T)\, g_{2+j,\beta}$
$g_{1+j,\beta} == g_{2+j,\beta}$

In[∘]:= $\text{Join}[\text{gRules}_{1,20,30}, \text{gRules}_{1,10,31}, \text{gRules}_{1,11,21}]$ // Column

Out[∘]=
$g_{20,\beta} == T\, g_{21,\beta} + (1 - T)\, g_{31,\beta}$
$g_{30,\beta} == g_{31,\beta}$
$g_{10,\beta} == T\, g_{11,\beta} + (1 - T)\, g_{32,\beta}$
$g_{31,\beta} == g_{32,\beta}$
$g_{11,\beta} == T\, g_{12,\beta} + (1 - T)\, g_{22,\beta}$
$g_{21,\beta} == g_{22,\beta}$

In[∘]:= $\text{Eliminate}[\text{Join}[\text{gRules}_{1,j,k}, \text{gRules}_{1,i,k+1}, \text{gRules}_{1,i+1,j+1}], \{g_{i+1,\beta}, g_{j+1,\beta}, g_{k+1,\beta}\}]$

Out[∘]= $g_{i,\beta} == T^2 g_{2+i,\beta} + T\, g_{2+j,\beta} - T^2 g_{2+j,\beta} + g_{2+k,\beta} - T\, g_{2+k,\beta}\ \&\&\ g_{j,\beta} == T\, g_{2+j,\beta} + g_{2+k,\beta} - T\, g_{2+k,\beta}\ \&\&\ g_{k,\beta} == g_{2+k,\beta}$

In[∘]:= $\text{gRules}_{1,10,20} \cup \text{gRules}_{1,11,30} \cup \text{gRules}_{1,21,31}$ // Column

Out[∘]=
$g_{10,\beta} == T\, g_{11,\beta} + (1 - T)\, g_{21,\beta}$
$g_{11,\beta} == T\, g_{12,\beta} + (1 - T)\, g_{31,\beta}$
$g_{20,\beta} == g_{21,\beta}$
$g_{21,\beta} == T\, g_{22,\beta} + (1 - T)\, g_{32,\beta}$
$g_{30,\beta} == g_{31,\beta}$
$g_{31,\beta} == g_{32,\beta}$

In[∘]:= $\text{Eliminate}[\text{gRules}_{1,10,20} \cup \text{gRules}_{1,11,30} \cup \text{gRules}_{1,21,31}, \{g_{11,\beta}, g_{21,\beta}, g_{31,\beta}\}]$

Out[∘]= $g_{10,\beta} == T^2 g_{12,\beta} + T\, g_{22,\beta} - T^2 g_{22,\beta} + g_{32,\beta} - T\, g_{32,\beta}\ \&\&\ g_{20,\beta} == T\, g_{22,\beta} + g_{32,\beta} - T\, g_{32,\beta}\ \&\&\ g_{30,\beta} == g_{32,\beta}$

pdf

In[∘]:=
```
lhs = R₁[1, 20, 30] + R₁[1, 10, 31] + R₁[1, 11, 21] //. gRules_{1,20,30} ⋃ gRules_{1,10,31} ⋃ gRules_{1,11,21};
rhs = R₁[1, 10, 20] + R₁[1, 11, 30] + R₁[1, 21, 31] //. gRules_{1,10,20} ⋃ gRules_{1,11,30} ⋃ gRules_{1,21,31};
Simplify[lhs == rhs]
```

pdf

Out[∘]= True

tex

Next comes Reid1, where we use results from an earlier example:

In[∘]:= $\begin{pmatrix} 1 & T^{-1} & 1 \\ 0 & T^{-1} & 1 \\ 0 & 0 & 1 \end{pmatrix}$ // Inverse // MatrixForm

Out[∘]//MatrixForm=
$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & T & -T \\ 0 & 0 & 1 \end{pmatrix}$$

pdf

In[∘]:= $R_1[1, 2, 1] - 1\,(g_{22} - 1/2)$ /. $g_{\alpha\_,\beta\_} \mapsto \begin{pmatrix} 1 & T^{-1} & 1 \\ 0 & T^{-1} & 1 \\ 0 & 0 & 1 \end{pmatrix}[\![\alpha, \beta]\!]$

pdf

Out[∘]= $\frac{1}{T^2} - \frac{1}{T} - \frac{-1 + \frac{1}{T}}{T}$

tex

Invariance under the other moves is proven similarly.

Alternative R_1's:

In[∘]:= Simplify[
$R_1[s, i, j] == s\,\left((1 - T^s)\, g_{ji}\,(g_{ji} - g_{ii}) + 2\, g_{jj}\, g_{ji} - g_{ji}\, g_{ij} - g_{jj}\, g_{ii} - g_{ji} + g_{ii} - 1/2\right)$ //. $gRules_{s,i,j}$]

Out[∘]= $i \neq j \,||\, s\,(-1 + T^s)\,(g_{1+i,1+i} - g_{1+j,1+i}) == 0$

In[∘]:= Simplify[$R_1[s, i, j] ==$
$s\,\left((g_{j,j+1} - g_{j,j})\,(g_{ji} - g_{ii}) + 2\, g_{jj}\, g_{ji} - g_{ji}\, g_{ij} - g_{jj}\, g_{ii} - g_{ji} + g_{ii} - 1/2\right)$ //. $gRules_{s,i,j}$]

Out[∘]= True

In[∘]:= Simplify[$R_1[s, i, j] == s\,\left(g_{j,i}\,(-1 - g_{i,j} + g_{j,j} + g_{j,1+j}) - g_{i,i}\,(-1 + g_{j,1+j}) - 1/2\right)$ //. $gRules_{s,i,j}$]

Out[∘]= True

In[∘]:= Simplify[$R_1[s, i, j] == s\,\left(g_{ji}\,(g_{jj} + g_{j,j+1} - g_{ij} - 1) - g_{ii}\,(g_{j,j+1} - 1) - 1/2\right)$ //. $gRules_{s,i,j}$]

Out[∘]= True

In[∘]:= Simplify[$R_1[s, i, j] == s\,\left(g_{ji}\,(g_{j+1,j} + g_{j,j+1} - g_{ij}) - g_{ii}\,(g_{j,j+1} - 1) - 1/2\right)$ //. $gRules_{s,i,j}$]

Out[∘]= True

In[∘]:= **Simplify[(g_{ji} ($g_{j+1,j}$ + $g_{j,j+1}$ - g_{ij}) - g_{ii} ($g_{j,j+1}$ - 1) - 1 / 2)]**

Out[∘]= $-\frac{1}{2} - g_{i,i}\,(-1 + g_{j,1+j}) + g_{j,i}\,(-g_{i,j} + g_{j,1+j} + g_{1+j,j})$

In[∘]:= **Simplify[{ g_{jj}, $g_{j,j+1}$, g_{ij}, $g_{j+1,j}$ + $g_{j,j+1}$ } //. $\text{gRules}_{s,i,j}$]**

Out[∘]= $\{-T^{-s}\,(-1 + T^{s})\,(\text{If}[i == j, 1, 0] - g_{1+j,1+i}) + g_{1+j,1+j},$
$g_{1+j,1+j}, \text{If}[i == j, 1, 0] - T^{s}\,\text{If}[i == j, 1, 0] + (-1 + T^{s})\,(-1 + g_{1+i,1+i}) +$
$T^{s}\,g_{1+i,1+j} + (1 - T^{s})\,(-1 - T^{-s}\,(-1 + T^{s})\,(\text{If}[i == j, 1, 0] - g_{1+j,1+i}) + g_{1+j,1+j}),$
$T^{-s}\,(-((-1 + T^{s})\,\text{If}[i == j, 1, 0]) + (-1 + T^{s})\,g_{1+j,1+i} + T^{s}\,(-1 + 2\,g_{1+j,1+j}))\}$