

Pensieve header: The 3-Heisenberg Formalism for Θ . Engine modified from pensieve://Project-s/BabyDoPeGDO/Solving_to_k=3.nb, $\$px\$$ formulas modified from pensieve://Talks/Bonn-2505/. Now in $\$xp\$$ order.

$\mathbb{E}[\omega, Q, P_eSeries]$ represents ωe^{Q+P} , where ω is a scalar, Q is an ϵ -free quadratic, and $P = \sum_{k=0}^{\$k} P[[k]] \epsilon^k$ is a perturbation (it is ill-advised to include ω in P because then it will have log terms).

Scheme: $\mathbb{E}[_]$ // $\mathbb{E}[_]$ calls FZip or Zip, which are functionally the same. Zip works by handling the quadratic part and calling PZip for the perturbation-only part. PZip works by iteratively solving the synthesis equation. FZip works by encapsulating coefficients, calling Zip, and back-substituting.

```
In[*]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\3H"];
Once[<< KnotTheory`];
```

Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at <http://katlas.org/wiki/KnotTheory>.

```
In[*]:= $k = 1;
```

```
In[*]:= HL[ε_] := Style[ε, Background → If[TrueQ@ε, Green, Red]];
```

Minor utilities

```
In[*]:= CCF[ε_] := ExpandDenominator@ExpandNumerator@Together[ε];
(*CoefficientCanonical Form *)
CCF[ε_] := Factor[ε]; (*CoefficientCanonical Form *)
CF[ε_List] := CF /@ ε;
CF[ε_eSeries] := CF /@ Take[ε, $k + 1];
CF[ε_] := Module[{F}, Expand@Collect[ε, (p | x | π | ξ | g) __, F] /. F → CCF];
(*CF[ε_] := Module[
  {vs=Cases[ε, (p | x | π | ξ) __, ∞]},
  Total[(CCF[#[2]] (Times@@vs^#[1])) & /@ CoefficientRules[Expand[ε], vs]]
];*)
CF[ε_E] := CF /@ ε;
CF[ℰsp[[εS___]]] := CF /@ ℰsp[εS];
CF[a_ == b_] := CF[a - b] == 0;
```

```

In[*]:= eSeries /: S1_eSeries ≡ S2_eSeries :=
  Length[S1] == Length[S2] ∧ Inner[CF[#1] == CF[#2] &, S1, S2, And];
eSeries[] := eSeries @@ Table[0, {k, 1}];
eSeries /: Plus[ss___eSeries] := Module[{l = Min[Length /@ {ss}]},
  eSeries @@ Total[Take[List @@ #, l] & /@ {ss}]
];
(*eSeries/: S1_eSeries+S2_eSeries :=
  eSeries@@Table[S1[[k]]+S2[[k]],{k,Min[Length@S1,Length@S2]}};*)
eSeries /: S1_eSeries * S2_eSeries := eSeries @@
  Table[Sum[S1[[j+1]] * S2[[k-j+1]], {j, 0, k}], {k, 0, Min[Length@S1, Length@S2] - 1}];
eSeries /: c_ * S_eSeries := (c #) & /@ S;
eSeries /: ∂vs___S_eSeries := (s ↦ ∂vss) /@ S;

```

```

In[*]:= Options[PolyPlot1] = {Labeled → False};
PolyPlot1[Δ_, OptionsPattern[]] := Module[{Δ1, crs, m, maxc, minc, s, rect},
  Δ1 = PowerExpand@Expand@Δ;
  rect = {{0, 0}, {1, 0}, {1, 1}, {0, 1}};
  If[Δ1 === 0, Graphics[],
    m = Max[-Exponent[Δ1, T, Min], Exponent[Δ1, T, Max]];
    crs = CoefficientRules[Tm Δ1, {T}];
    maxc = N@Log@Max@Abs[Last /@ crs];
    minc = N@Log@Min@Select[Abs[Last /@ crs], # > 0 &];
    If[minc == maxc, s[_] = 0, s[c_] := s[c] = (maxc - Log@c) / (maxc - minc)];
    Graphics[crs /. ({x_} → c_) → {
      Lighter[Which[c == 0, White, c > 0, Red, c < 0, Blue], 0.88 s[Abs@c]],
      Tooltip[Polygon[{(x + m - 1/2, 0) + #} & /@ rect], c Tx-m],
      If[Not@OptionValue[Labeled], {},
        {Black, FontSize → Scaled[ $\frac{1}{(m+1)(6+maxc)}$ ], Text[c Tx-m, {x + m, 0.5}]}]
    }, AspectRatio → Min[1/5, 1/√(m+1)],
    ImagePadding → None, PlotRangePadding → None]
  ];
Options[PolyPlot2] = {Labeled → False, Shear → True};
PolyPlot2[θ_, OptionsPattern[]] :=
  Module[{θ1, crs, m1, m2, maxc, minc, s, stone, mat, p},
    θ1 = PowerExpand@Expand@θ;
    If[θ1 === 0, Graphics[{White, Disk[]}],
      If[OptionValue[Shear],
        stone = Table[{Cos[α], Sin[α]} / Cos[2π/12]/2, {α, 2π/12, 2π, 2π/6}];
        mat =  $\begin{pmatrix} 1 & -1/2 \\ 0 & \sqrt{3}/2 \end{pmatrix}$ ,

```

```

stone = {{1, 1}, {-1, 1}, {-1, -1}, {1, -1}} / 2;
mat = IdentityMatrix[2];
];
m1 = Max[-Exponent[ $\theta_1$ ,  $T_1$ , Min], Exponent[ $\theta_1$ ,  $T_1$ , Max]];
m2 = Max[-Exponent[ $\theta_1$ ,  $T_2$ , Min], Exponent[ $\theta_1$ ,  $T_2$ , Max]];
crs = CoefficientRules[ $T_1^{m1} T_2^{m2} \theta_1$ , { $T_1$ ,  $T_2$ }];
maxc = N@Log@Max@Abs[Last /@ crs];
minc = N@Log@Min@Select[Abs[Last /@ crs], # > 0 &];
If[minc == maxc, s[_] = 0, s[_] := s[_] = (maxc - Log@c) / (maxc - minc)];
Graphics[{{(*{Yellow,Disk[{0,0},1+Cos[2 $\pi$ /12]Norm[{m1,m2}]/ $\sqrt{2}$ ]}},*)
crs /. ({x1_, x2_} -> c_) -> {
  Lighter[Which[c == 0, White, c > 0, Red, c < 0, Blue], 0.88 s[Abs@c]],
  p = mat.{x1 - m1, x2 - m2};
  Tooltip[Polygon[(p + #) & /@ stone],  $c T_1^{x1-m1} T_2^{x2-m2}$ ],
  If[Not@OptionValue[Labeled], {},
    {Black, FontSize -> Scaled[ $\frac{1}{\text{Max}[m1, m2] (10 + \text{maxc})}$ ], Text[ $c T_1^{x1-m1} T_2^{x2-m2}$ , p]}]}
}, ImagePadding -> None, PlotRangePadding -> None]
]];
Options[PolyPlot] = {Labeled -> False, ImageSize -> Automatic};
PolyPlot[{ $\Delta$ ,  $\theta$ }, opts___Rule] := GraphicsColumn[
  {PolyPlot1[ $\Delta$ , FilterRules[Join[{opts}, Options[PolyPlot]], Options[PolyPlot1]]],
  PolyPlot2[ $\theta$ , FilterRules[Join[{opts}, Options[PolyPlot]], Options[PolyPlot2]]]},
  Spacings -> Scaled@0.08, ImagePadding -> None, PlotRangePadding -> None,
  FilterRules[Join[{opts}, Options[PolyPlot]], Options[GraphicsColumn]]
];

```

The Main Program

Variables and their duals:

```

In[*]:= {p*, x*,  $\pi^*$ ,  $\xi^*$ } = { $\pi$ ,  $\xi$ , p, x};
(vs_List)* := (v -> v*) /@ vs;
(u_<math>\nu</math>_)* := (u*)<math>\nu</math>;

```

$\mathbb{E}$ operations:

```

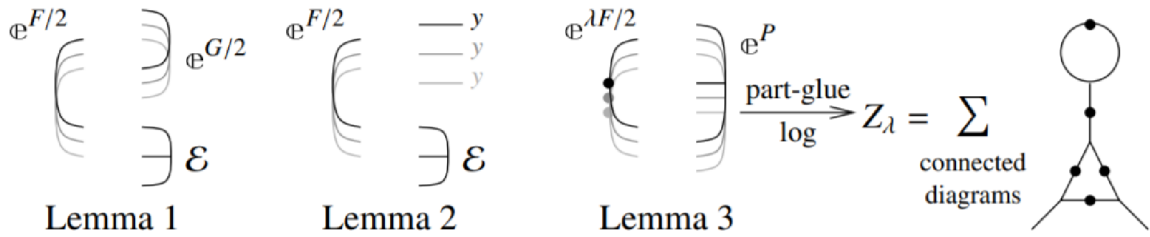
In[*]:= E /: E[ω1_, Q1_, P1_] ≡ E[ω2_, Q2_, P2_] := CF[ω1 == ω2] ∧ CF[Q1 == Q2] ∧ (P1 ≡ P2);
E /: E[ω1_, Q1_, P1_] E[ω2_, Q2_, P2_] := E[ω1 ω2, Q1 + Q2, P1 + P2];
Ed1→r1[ε1s____] ≡ Ed2→r2[ε2s____] ^:= (d1 == d2) ∧ (r1 == r2) ∧ (E[ε1s] ≡ E[ε2s]);
Ed1→r1[ε1s____] Ed2→r2[ε2s____] ^:= E(d1∪d2)→(r1∪r2) @@ (E[ε1s] E[ε2s]);
Edr[εs____]$k := Edr @@ E[εs]$k;

```

```

In[*]:= Ed1→r1[ε1s____] // Ed2→r2[ε2s____] := Module[{is = r1 ∩ d2, lvs},
  lvs = Flatten@Table[{xv, $ei, pv, $ei}, {v, 3}, {i, is}];
  E(d1∪Complement[d2, is])→(r2∪Complement[r1, is]) @@ (Ziplvs∪lvs[lvs*.lvs, Times[
    E[ε1s] /. Table[{u : (x | p)}v,i → uv, $ei, {i, is}],
    E[ε2s] /. Table[{u : (ξ | π)}v,i → uv, $ei, {i, is}]
  ]])
]

```



```

In[*]:= Zipvs[f_, ε_] := (*EchoLabel["EZip called with {vs,f,ε} = "][{vs,f,ε}];*)
  <f, ε> // Zip1vs // EZip23vs;
Zipvs[f_, ε_] := (*EchoLabel["EZip called with {vs,f,ε} = "][{vs,f,ε}];*)
  <f, ε> // Zip1vs // Zip2vs // Zip3vs;

```

Getting rid of the quadratic.

Lemma 1. With convergences left to the reader,

$$\left\langle F : \mathcal{E} \otimes^{\frac{1}{2} \sum_{i,j \in B} G_{ij} z_i z_j} \right\rangle_B = \det(1 - GF)^{-1/2} \left\langle F(1 - GF)^{-1} : \mathcal{E} \right\rangle_B$$

```

In[*]:= Zip1_{\{}} = Identity;
Zip1vs @ <f_, E[ω_, Q_, P_]> := Module[{I, F, G, u, v},
  (*EchoLabel["EZip1 called with {vs,f,ω,Q,P} = "][{vs,f,ω,Q,P}];*)
  I = IdentityMatrix@Length@vs;
  F = Table[∂u,v f, {u, vs*}, {v, vs*}];
  G = Table[∂u,v Q, {u, vs}, {v, vs}];
  <CF[vs*.F.Inverse[I - G.F].vs* / 2],
  E[CF@PowerExpand@Factor[ω Det[I - G.F]^{-1/2}, CF[Q - vs.G.vs / 2], P]>
]

```

Getting rid of linear terms.

Lemma 2. $\left\langle F : \mathcal{E} \otimes^{\sum_{i \in B} y_i z_i} \right\rangle_B = \mathbb{P}^{\frac{1}{2} \sum_{i,j \in B} F_{ij} y_i y_j} \left\langle F : \mathcal{E}|_{z_B \rightarrow z_B + F y_B} \right\rangle_B$.

```

In[*]:= Zip2_{ } = Identity;
Zip2_{vs_} @ <F_, E[ω_, Q_, P_] > := Module[{F, Y, u, v},
  (*EchoLabel["Zip2 called with {vs,F,ω,Q,P} = "] [{vs,F,ω,Q,P}];*)
  F = Table[∂_{u,v} F, {u, vs*}, {v, vs*}];
  Y = Table[∂_v Q, {v, vs}];
  CF /@ <F_, E[ω, Q - Y.v + Y.F.Y / 2, P /. Thread[vs → vs + F.Y]] >
]

```

Dealing with Feynman diagrams.

Lemma 3. With an extra variable λ , $Z_\lambda := \log[\lambda F : \oplus^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2} \sum_{i,j \in B} F_{ij} \left(\partial_{z_i} \partial_{z_j} Z_\lambda + (\partial_{z_i} Z_\lambda)(\partial_{z_j} Z_\lambda) \right).$$

Note that the power m of λ is at most $k - 1 + \frac{2k+2}{2} = 2k$. We write $Z_\lambda = \sum Z[m] \lambda^m$.

```

In[*]:= Zip3_{vs_} @ <F_, E[ω_, Q_, P_] > := Module[{Z, u, v, m, j},
  (*EchoLabel["Zip3 called with {vs,F,ω,Q,P} = "] [{vs,F,ω,Q,P}];*)
  Z[0] = P;
  For[m = 0, m < 2 $k + 5, ++m,
    Z[m + 1] = CF [
      1
      2 (m + 1)
      Sum[∂_{u,v} F (∂_{u,v} Z[m] + Sum[(∂_u Z[j]) (∂_v Z[m - j]), {j, 0, m}]), {u, vs}, {v, vs}]
    ];
  E[ω, Q, CF[Sum[Z[m], {m, 0, 2 $k + 5}] /. Table[v → 0, {v, vs}]]]
]

```

```

In[*]:= EZip23_{vs_} @ <F_, E[ω_, Q_, P_] > := Module[
  {nP, nF, nQ, j = 0, ps, c, t, rr = {(*release rules*)}},
  nP = Total[
    CoefficientRules[#, vs] /.
    (ps_ → c_) => (AppendTo[rr, t[++j] → CF@c]; t[j] (Times @@ vs^{ps}))
  ] & /@ P;
  nQ = Total[CoefficientRules[Q, vs] /.
    (ps_ → c_) => (AppendTo[rr, t[++j] → CF@c]; t[j] (Times @@ vs^{ps}))];
  nF = Total[CoefficientRules[F, vs*] /.
    (ps_ → c_) => (AppendTo[rr, t[++j] → CF@c]; t[j] (Times @@ (vs*)^{ps}))];
  CF[Expand[<nF, E[ω, nQ, nP]> // Zip2_{vs} // Zip3_{vs} /. rr]
]

```

```

In[*]:= m_{i_,j_→k_} := E[{i,j}→{k}] [1, ∑_{v=1}^3 (π_{v,i} ξ_{v,j} + (π_{v,i} + π_{v,j}) p_{v,k} + (ξ_{v,i} + ξ_{v,j}) x_{v,k}), eSeries[]]

```

```

In[*]:= ma,b→c
Out[*]=

$$\mathbb{E}_{\{a,b\} \rightarrow \{c\}} [1, p_{1,c} (\pi_{1,a} + \pi_{1,b}) + p_{2,c} (\pi_{2,a} + \pi_{2,b}) + p_{3,c} (\pi_{3,a} + \pi_{3,b}) + \pi_{1,a} \xi_{1,b} +$$


$$x_{1,c} (\xi_{1,a} + \xi_{1,b}) + \pi_{2,a} \xi_{2,b} + x_{2,c} (\xi_{2,a} + \xi_{2,b}) + \pi_{3,a} \xi_{3,b} + x_{3,c} (\xi_{3,a} + \xi_{3,b}), \in \text{Series}[0, 0]]$$


In[*]:= m1,2→1 // m1,3→1
(m1,2→1 // m1,3→1) ≡ (m2,3→2 // m1,2→1)
Out[*]=

$$\mathbb{E}_{\{1,2,3\} \rightarrow \{1\}} [1,$$


$$p_{1,1} \pi_{1,1} + p_{1,1} \pi_{1,2} + p_{1,1} \pi_{1,3} + p_{2,1} \pi_{2,1} + p_{2,1} \pi_{2,2} + p_{2,1} \pi_{2,3} + p_{3,1} \pi_{3,1} + p_{3,1} \pi_{3,2} + p_{3,1} \pi_{3,3} + x_{1,1} \xi_{1,1} +$$


$$\pi_{1,1} \xi_{1,2} + x_{1,1} \xi_{1,2} + \pi_{1,1} \xi_{1,3} + \pi_{1,2} \xi_{1,3} + x_{1,1} \xi_{1,3} + x_{2,1} \xi_{2,1} + \pi_{2,1} \xi_{2,2} + x_{2,1} \xi_{2,2} + \pi_{2,1} \xi_{2,3} +$$


$$\pi_{2,2} \xi_{2,3} + x_{2,1} \xi_{2,3} + x_{3,1} \xi_{3,1} + \pi_{3,1} \xi_{3,2} + x_{3,1} \xi_{3,2} + \pi_{3,1} \xi_{3,3} + \pi_{3,2} \xi_{3,3} + x_{3,1} \xi_{3,3}, \in \text{Series}[0, 0]]$$

Out[*]=
True

```

Some Knot Theory

```

In[*]:= Kinki := CC3 R1,2 // m2,3→2 // m2,1→i;
Kinki := CC3 R̄1,2 // m1,3→1 // m1,2→i

```

```

In[*]:= RVK[pd_PD] := Module[{n, xs, x, rots, front = {0}, k},
  n = Length[pd]; rots = Table[0, {2 n}];
  xs = Cases[pd, x_X] :> {Xp[x[[4]], x[[1]] PositiveQ@x,
    Xm[x[[2]], x[[1]] True];
  For[k = 0, k < 2 n, ++k, If[k == 0 ∨ FreeQ[front, -k],
    front = Flatten[front /. k -> {xs /. {
      Xp[k + 1, L_] | Xm[L_, k + 1] :> {L, k + 1, 1 - L},
      Xp[L_, k + 1] | Xm[k + 1, L_] :> {++rots[[L]]; {1 - L, k + 1, L}}
    }]],
    Cases[front, k | -k] /. {k, -k} :> --rots[[k + 1]];
  ];
  RVK[xs, rots];
RVK[K_] := RVK[PD[K]];

```

```

In[*]:= rot[i_, 0] := E{i}→{i} [1, 0, eSeries[]];
rot[i_, n_] := Module[{j}, If[n > 0, rot[i, n - 1] CCj, rot[i, n + 1] C̄Cj] // mi,j→i;

```

```

In[*]:= Z[K_] := Z[RVK@K];
Z[rvk_RVK] := (*Z[rvk] =*)
Module[{todo, n, rots, g, done, st, cx, g1, i, j, k, k1, k2, k3},
  {todo, rots} = List@@rvk;
  AppendTo[rots, 0];
  n = Length[todo];
  g = E[{}->{0}][1, 0, eSeries[]];
  done = {0};
  st = Range[0, 2 n + 1];
  While[{ } != ($M = todo),
    {cx} = MaximalBy[todo, Length[done ∩ {#[[1]], #[[2]], #[[1]] - 1, #[[2]] - 1}] &, 1];
    {i, j} = List@@cx;
    g1 = Switch[Head[cx],
      Xp, (Ri,j Kinkk) // mj,k→j,
      Xm, (R̄i,j Kinkk) // mj,k→j,
    ];
    g1 = (rot[k, rots[[i]]] g1) // mk,i→i; rots[[i]] = 0;
    g1 = (g1 rot[k, rots[[i + 1]]]) // mi,k→i; rots[[i + 1]] = 0;
    g1 = (rot[k, rots[[j]]] g1) // mk,j→j; rots[[j]] = 0;
    g1 = (g1 rot[k, rots[[j + 1]]]) // mj,k→j; rots[[j + 1]] = 0;
    g *= g1;
    If[MemberQ[done, i], g = g // mi,i+1→i; st = st /. st[[i + 2]] → st[[i + 1]]];
    If[MemberQ[done, i - 1], g = g // mst[[i],i→st[[i]]; st = st /. st[[i + 1]] → st[[i]]];
    If[MemberQ[done, j], g = g // mj,j+1→j; st = st /. st[[j + 2]] → st[[j + 1]]];
    If[MemberQ[done, j - 1], g = g // mst[[j],j→st[[j]]; st = st /. st[[j + 1]] → st[[j]]];
    done = done ∪ {i - 1, i, j - 1, j};
    todo = DeleteCases[todo, cx];
  ];
  CF /@ (g (* /. {x0→x, y0→y, a0→a} *))
]

```

ρ_1 testing

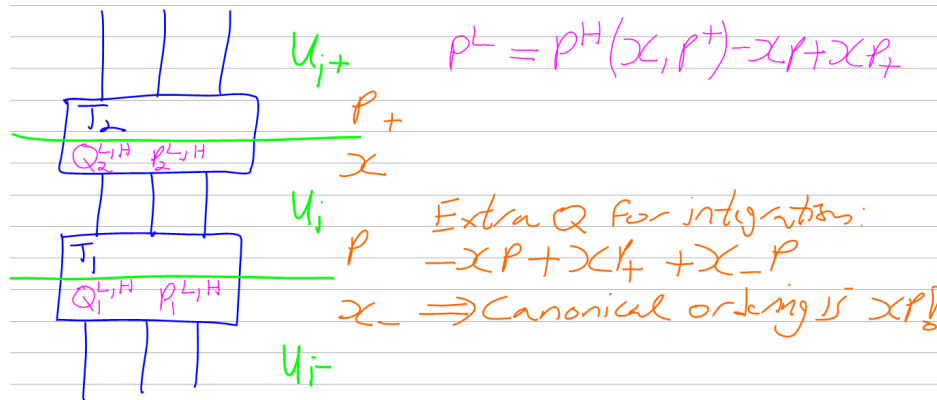
The pPush Lagrangian of Talks/Bonn-2505/Rho11AlternativeIntegrands.nb:

$$In[*]:= \mathcal{L}[X_{i,j}[S_-]] := T^{S/2} \mathbb{E} \left[\left\{ \begin{aligned} & x_i (p_{i+1} - p_i) + x_j (p_{j+1} - p_j) + (T^S - 1) x_i (p_{i+1} - p_{j+1}), \\ & 0, 0, \epsilon \left(-\frac{S}{2} + S T^S p_{1+j} x_i + \frac{1}{2} S T^S (-1 + T^S) p_{1+i} p_{1+j} x_i^2 - \right. \right. \\ & \left. \left. \frac{1}{2} S T^S (-1 + T^S) p_{1+j}^2 x_i^2 - S p_{1+j} x_j - S T^S p_{1+i} p_{1+j} x_i x_j + S T^S p_{1+j}^2 x_i x_j \right) \right\} \right] \end{aligned}$$

$$\mathcal{L}[C_i[\varphi_-]] := T^{\varphi/2} \mathbb{E} \left[\left\{ x_i (p_{i+1} - p_i), \epsilon \left(-\frac{\varphi}{2} - \varphi p_{1+i} x_i \right) \right\} \right]$$

$$\mathcal{L}[K_-] := \text{CF}[\mathcal{L} / \text{@Features}[K][[2]]]$$

$$\text{vs}[K_-] := \text{Join}@\text{Table}[\{p_i, x_i\}, \{i, \text{Features}[K][[1]]\}]$$



$$\begin{aligned} In[*]:= R_{i,j} &= \text{CF}[\mathcal{L}[X_{i,j}[1]] / . \omega_- \mathbb{E}[\{Q_-, P\theta_-, P1a_-, P1b_-\}] \Rightarrow \mathbb{E}_{\{\} \rightarrow \{i,j\}} [\\ & \quad \omega, Q, \epsilon \text{Series}[P\theta, P1a + P1b / . \epsilon \rightarrow 1] \\ & \quad] / . \{p_{i+1} \Rightarrow p_i, p_{j+1} \Rightarrow p_j\} / . \{T \rightarrow T_1, x_{\alpha_-} \Rightarrow x_{1,\alpha}, p_{\alpha_-} \Rightarrow p_{1,\alpha}\}; \\ \bar{R}_{i,j} &= \text{CF}[\mathcal{L}[X_{i,j}[-1]] / . \omega_- \mathbb{E}[\{Q_-, P\theta_-, P1a_-, P1b_-\}] \Rightarrow \mathbb{E}_{\{\} \rightarrow \{i,j\}} [\\ & \quad \omega, Q, \epsilon \text{Series}[P\theta, P1a + P1b / . \epsilon \rightarrow 1] \\ & \quad] / . \{p_{i+1} \Rightarrow p_i, p_{j+1} \Rightarrow p_j\} / . \{T \rightarrow T_1, x_{\alpha_-} \Rightarrow x_{1,\alpha}, p_{\alpha_-} \Rightarrow p_{1,\alpha}\}; \\ CC_{i_-} &= \text{CF}[\mathcal{L}[C_i[1]] / . \omega_- \mathbb{E}[\{Q_-, P_-\}] \Rightarrow \mathbb{E}_{\{\} \rightarrow \{i\}} [\\ & \quad \omega, Q, \epsilon \text{Series}[0, P / . \epsilon \rightarrow 1] \\ & \quad] / . \{p_{i+1} \Rightarrow p_i\} / . \{T \rightarrow T_1, x_{\alpha_-} \Rightarrow x_{1,\alpha}, p_{\alpha_-} \Rightarrow p_{1,\alpha}\}; \\ \overline{CC}_{i_-} &= \text{CF}[\mathcal{L}[C_i[-1]] / . \omega_- \mathbb{E}[\{Q_-, P_-\}] \Rightarrow \mathbb{E}_{\{\} \rightarrow \{i\}} [\\ & \quad \omega, Q, \epsilon \text{Series}[0, P / . \epsilon \rightarrow 1] \\ & \quad] / . \{p_{i+1} \Rightarrow p_i\} / . \{T \rightarrow T_1, x_{\alpha_-} \Rightarrow x_{1,\alpha}, p_{\alpha_-} \Rightarrow p_{1,\alpha}\}; \end{aligned}$$

```

In[*]:= $k = 1;
Ri,j
R̄i,j
CCi
CC̄i

Out[*]=

$$\mathbb{E}_{\{\} \rightarrow \{i,j\}} \left[ \sqrt{T_1}, -p_{1,i} x_{1,i} + T_1 p_{1,i} x_{1,i} + (1 - T_1) p_{1,j} x_{1,i}, \right.$$


$$\in \text{Series} \left[ 0, -\frac{1}{2} + T_1 p_{1,j} x_{1,i} + \frac{1}{2} (-1 + T_1) T_1 p_{1,i} p_{1,j} x_{1,i}^2 - \right.$$


$$\left. \frac{1}{2} (-1 + T_1) T_1 p_{1,j}^2 x_{1,i}^2 - p_{1,j} x_{1,j} - T_1 p_{1,i} p_{1,j} x_{1,i} x_{1,j} + T_1 p_{1,j}^2 x_{1,i} x_{1,j} \right]$$


Out[*]=

$$\mathbb{E}_{\{\} \rightarrow \{i,j\}} \left[ \frac{1}{\sqrt{T_1}}, -p_{1,i} x_{1,i} + \frac{p_{1,i} x_{1,i}}{T_1} + \frac{(-1 + T_1) p_{1,j} x_{1,i}}{T_1}, \right.$$


$$\in \text{Series} \left[ 0, \frac{1}{2} - \frac{p_{1,j} x_{1,i}}{T_1} + \frac{(-1 + T_1) p_{1,i} p_{1,j} x_{1,i}^2}{2 T_1^2} - \right.$$


$$\left. \frac{(-1 + T_1) p_{1,j}^2 x_{1,i}^2}{2 T_1^2} + p_{1,j} x_{1,j} + \frac{p_{1,i} p_{1,j} x_{1,i} x_{1,j}}{T_1} - \frac{p_{1,j}^2 x_{1,i} x_{1,j}}{T_1} \right]$$


Out[*]=

$$\mathbb{E}_{\{\} \rightarrow \{i\}} \left[ \sqrt{T_1}, 0, \in \text{Series} \left[ 0, -\frac{1}{2} - p_{1,i} x_{1,i} \right] \right]$$


Out[*]=

$$\mathbb{E}_{\{\} \rightarrow \{i\}} \left[ \frac{1}{\sqrt{T_1}}, 0, \in \text{Series} \left[ 0, \frac{1}{2} + p_{1,i} x_{1,i} \right] \right]$$


In[*]:= $k = 1; {
  {"CC", (CC1 CC̄2 // m1,2→1) ≡ E{ }→{1} [1, 0, eSeries[]]},
  {"An R1", (CC3 R1,2 // m2,3→2 // m2,1→1) ≡ (CC̄3 R1,2 // m1,3→1 // m1,2→1)},
  {"R2b", (R1,2 R̄3,4 // m1,3→1 // m2,4→2) ≡ E{ }→{1,2} [1, 0, eSeries[]]},
  {"R3", (R1,2 R4,3 R5,6 // m1,4→1 // m2,5→2 // m3,6→3) ≡ (R2,3 R4,5 R1,6 // m1,4→1 // m2,5→2 // m3,6→3) }
}

Out[*]=
{{CC, True}, {An R1, True}, {R2b, True}, {R3, True}}

In[*]:= NewBit[K_] := Module[{Alex = Alexander[K][T1]},
  Alex2 Z[K][[3, 2]] // Factor]

In[*]:= NewBit /@ AllKnots[{3, 5}]

KnotTheory: Loading precomputed data in PD4Knots`.

Out[*]=

$$\left\{ \frac{(-1 + T_1)^2 (1 + T_1^2)}{T_1^2}, 0, \frac{(-1 + T_1)^2 (1 + T_1^2) (2 + T_1^2 + 2 T_1^4)}{T_1^4}, \frac{(-1 + T_1)^2 (5 - 4 T_1 + 5 T_1^2)}{T_1^2} \right\}$$


```

In papers/APAI, page 7, that was:

$$\left\{ \frac{(-1+T_1)^2 (1+T_1^2)}{T_1^2}, 0, \frac{(-1+T_1)^2 (1+T_1^2) (2+T_1^2+2 T_1^4)}{T_1^4}, \frac{(-1+T_1)^2 (5-4 T_1+5 T_1^2)}{T_1^2} \right\}$$

And now θ

The Lagrangian is taken from ThetaAlternativeIntegrands.nb.

In[*]:= $T_3 = T_1 T_2$;

$\mathcal{L}_{\text{pPush}}[X_{i_}, j_ [S_]] :=$

$$\begin{aligned} & (T_1 T_2)^S \mathbb{E} \left[\frac{S \in}{2} - p_{1,i} x_{1,i} + T_1^S p_{1,1+i} x_{1,i} + (1 - T_1^S) p_{1,1+j} x_{1,i} - p_{1,j} x_{1,j} + p_{1,1+j} x_{1,j} - p_{2,i} x_{2,i} + \right. \\ & T_2^S p_{2,1+i} x_{2,i} - S \in T_2^S p_{2,1+i} x_{2,i} + (1 - T_2^S) p_{2,1+j} x_{2,i} + S \in T_1^S T_2^S p_{1,1+i} p_{2,1+j} x_{1,i} x_{2,i} + \\ & \frac{S \in (-1 + T_1^S) T_2^S p_{1,1+j} p_{2,1+j} x_{1,i} x_{2,i}}{-1 + T_2^S} + (-1 + T_1^S) T_2^S p_{3,1+j} x_{1,i} x_{2,i} - p_{2,j} x_{2,j} + p_{2,1+j} x_{2,j} + \\ & S \in p_{2,1+j} x_{2,j} - S \in T_1^S p_{1,1+i} p_{2,1+j} x_{1,i} x_{2,j} - \frac{S \in (-1 + T_1^S) p_{1,1+j} p_{2,1+j} x_{1,i} x_{2,j}}{-1 + T_2^S} + (1 - T_1^S) p_{3,1+j} \\ & x_{1,i} x_{2,j} + \frac{S \in T_2^S (-1 + (T_1 T_2)^S) p_{1,1+j} p_{2,1+i} x_{3,i}}{-1 + T_2^S} - \frac{S \in T_2^S (-1 + (T_1 T_2)^S) p_{1,1+j} p_{2,1+j} x_{3,i}}{-1 + T_2^S} - \\ & p_{3,i} x_{3,i} + (T_1 T_2)^S p_{3,1+i} x_{3,i} + S \in (T_1 T_2)^S p_{3,1+i} x_{3,i} + (1 - (T_1 T_2)^S) p_{3,1+j} x_{3,i} - \\ & \frac{S \in T_2^S (-1 + (T_1 T_2)^S) p_{3,1+j} x_{3,i}}{-1 + T_2^S} - \frac{S \in T_1^S T_2^S (-1 + (T_1 T_2)^S) p_{1,1+i} p_{3,1+j} x_{1,i} x_{3,i}}{-1 + T_2^S} + \\ & \frac{S \in (-1 + T_1^S) T_2^S (-1 + (T_1 T_2)^S) p_{1,1+j} p_{3,1+j} x_{1,i} x_{3,i}}{-1 + T_2^S} - S \in (T_1 T_2)^S (-1 + T_2^S) p_{2,1+j} p_{3,1+i} x_{2,i} x_{3,i} + \\ & S \in T_2^S (-1 + (T_1 T_2)^S) p_{2,1+j} p_{3,1+j} x_{2,i} x_{3,i} + 2 S \in (T_1 T_2)^S p_{2,1+j} p_{3,1+i} x_{2,j} x_{3,i} + \\ & \frac{S \in T_2^S (-1 + (T_1 T_2)^S) p_{2,1+i} p_{3,1+j} x_{2,j} x_{3,i}}{-1 + T_2^S} - \frac{S \in (-1 + 2 T_2^S) (-1 + (T_1 T_2)^S) p_{2,1+j} p_{3,1+j} x_{2,j} x_{3,i}}{-1 + T_2^S} - \\ & p_{3,j} x_{3,j} + p_{3,1+j} x_{3,j} + S \in T_1^S p_{1,1+i} p_{3,1+j} x_{1,i} x_{3,j} + \frac{S \in (-1 + T_1^S) p_{1,1+j} p_{3,1+j} x_{1,i} x_{3,j}}{-1 + T_2^S} - \\ & \left. S \in T_2^S p_{2,1+i} p_{3,1+j} x_{2,i} x_{3,j} - S \in p_{2,1+j} p_{3,1+j} x_{2,i} x_{3,j} \right] \end{aligned}$$

In[*]:= $\mathcal{L}_{\text{pPush}}[C_{i_} [\varphi_]] :=$

$$(T_1 T_2)^\varphi \mathbb{E} \left[(-p_{1,i} + p_{1,1+i}) x_{1,i} + (-p_{2,i} + p_{2,1+i}) x_{2,i} + (-p_{3,i} + p_{3,1+i}) x_{3,i} + \varphi \left(\frac{1}{2} + p_{3,1+i} x_{3,i} \right) \right]$$

```

In[*]:= $k = 1;

Ri,j = CF[ $\mathcal{L}_{\text{pPush}}[X_{i,j}[1]]$  /.  $\omega_{-} \mathbb{E}[L_{-}] \Rightarrow \text{Module}[\{B, L1, Q, P0, P1a, P1b\},$ 
  L1 = CoefficientRules[B L /. { $p_{\alpha\beta} \Rightarrow B p_{\alpha\beta}, x_{\alpha\beta} \Rightarrow B^{-1} x_{\alpha\beta}$ }, { $\epsilon, B$ }];
  { $\omega, Q = \{0, 1\}$  /. L1, P0 = {0, 0} /. L1, P1a = {1, 2} /. L1, P1b = {1, 1} /. L1};
   $\mathbb{E}_{\{i \rightarrow \{i,j\}\}}[$ 
     $\omega, \sum_{v=1}^3 ((T_v - 1) x_{v,i} (p_{v,i} - p_{v,j}))$ ,
    eSeries[P0, P1a + P1b]
  ] /. { $p_{v,i+1} \Rightarrow p_{v,i}, p_{v,j+1} \Rightarrow p_{v,j}$ }
];

R̄i,j = CF[ $\mathcal{L}_{\text{pPush}}[X_{i,j}[-1]]$  /.  $\omega_{-} \mathbb{E}[L_{-}] \Rightarrow \text{Module}[\{B, L1, Q, P0, P1a, P1b\},$ 
  L1 = CoefficientRules[B L /. { $p_{\alpha\beta} \Rightarrow B p_{\alpha\beta}, x_{\alpha\beta} \Rightarrow B^{-1} x_{\alpha\beta}$ }, { $\epsilon, B$ }];
  { $\omega, Q = \{0, 1\}$  /. L1, P0 = {0, 0} /. L1, P1a = {1, 2} /. L1, P1b = {1, 1} /. L1};
   $\mathbb{E}_{\{i \rightarrow \{i,j\}\}}[$ 
     $\omega, \sum_{v=1}^3 ((T_v^{-1} - 1) x_{v,i} (p_{v,i} - p_{v,j}))$ ,
    eSeries[P0, P1a + P1b]
  ] /. { $p_{v,i+1} \Rightarrow p_{v,i}, p_{v,j+1} \Rightarrow p_{v,j}$ }
];

CCi = CF[ $\mathcal{L}_{\text{pPush}}[C_i[1]]$  /.  $\omega_{-} \mathbb{E}[L_{-}] \Rightarrow \text{Module}[\{B, L1, Q, P0, P1a, P1b\},$ 
  L1 = CoefficientRules[B L /. { $p_{\alpha\beta} \Rightarrow B p_{\alpha\beta}, x_{\alpha\beta} \Rightarrow B^{-1} x_{\alpha\beta}$ }, { $\epsilon, B$ }];
  { $\omega, Q = \{0, 1\}$  /. L1, P0 = 0, P1a = 0, P1b = {1, 1} /. L1};
   $\mathbb{E}_{\{i \rightarrow \{i\}\}}[$ 
     $\omega, 0$ ,
    eSeries[P0, P1a + P1b]
  ] /. { $p_{v,i+1} \Rightarrow p_{v,i}$ }
];

CC̄i = CF[ $\mathcal{L}_{\text{pPush}}[C_i[-1]]$  /.  $\omega_{-} \mathbb{E}[L_{-}] \Rightarrow \text{Module}[\{B, L1, Q, P0, P1a, P1b\},$ 
  L1 = CoefficientRules[B L /. { $p_{\alpha\beta} \Rightarrow B p_{\alpha\beta}, x_{\alpha\beta} \Rightarrow B^{-1} x_{\alpha\beta}$ }, { $\epsilon, B$ }];
  { $\omega, Q = \{0, 1\}$  /. L1, P0 = 0, P1a = 0, P1b = {1, 1} /. L1};
   $\mathbb{E}_{\{i \rightarrow \{i\}\}}[$ 
     $\omega, 0$ ,
    eSeries[P0, P1a + P1b]
  ] /. { $p_{v,i+1} \Rightarrow p_{v,i}$ }
];

```

In[*]:= **R_{i,j}**

Out[*]=

$$\begin{aligned}
& \mathbb{E}_{\{\} \rightarrow \{i,j\}} \left[T_1 T_2, (-1 + T_1) p_{1,i} x_{1,i} + (1 - T_1) p_{1,j} x_{1,i} + \right. \\
& \quad (-1 + T_2) p_{2,i} x_{2,i} + (1 - T_2) p_{2,j} x_{2,i} + (-1 + T_1 T_2) p_{3,i} x_{3,i} + (1 - T_1 T_2) p_{3,j} x_{3,i}, \\
& \quad \left. \in \text{Series} \left[(-1 + T_1) T_2 p_{3,j} x_{1,i} x_{2,i} + (1 - T_1) p_{3,j} x_{1,i} x_{2,j}, \frac{1}{2} - T_2 p_{2,i} x_{2,i} + T_1 T_2 p_{1,i} p_{2,j} x_{1,i} x_{2,i} + \right. \right. \\
& \quad \frac{(-1 + T_1) T_2 p_{1,j} p_{2,j} x_{1,i} x_{2,i}}{-1 + T_2} + p_{2,j} x_{2,j} - T_1 p_{1,i} p_{2,j} x_{1,i} x_{2,j} - \frac{(-1 + T_1) p_{1,j} p_{2,j} x_{1,i} x_{2,j}}{-1 + T_2} + \\
& \quad \frac{T_2 (-1 + T_1 T_2) p_{1,j} p_{2,i} x_{3,i}}{-1 + T_2} - \frac{T_2 (-1 + T_1 T_2) p_{1,j} p_{2,j} x_{3,i}}{-1 + T_2} + T_1 T_2 p_{3,i} x_{3,i} - \frac{T_2 (-1 + T_1 T_2) p_{3,j} x_{3,i}}{-1 + T_2} - \\
& \quad \frac{T_1 T_2 (-1 + T_1 T_2) p_{1,i} p_{3,j} x_{1,i} x_{3,i}}{-1 + T_2} + \frac{(-1 + T_1) T_2 (-1 + T_1 T_2) p_{1,j} p_{3,j} x_{1,i} x_{3,i}}{-1 + T_2} - \\
& \quad \frac{T_1 (-1 + T_2) T_2 p_{2,j} p_{3,i} x_{2,i} x_{3,i} + T_2 (-1 + T_1 T_2) p_{2,j} p_{3,j} x_{2,i} x_{3,i} + 2 T_1 T_2 p_{2,j} p_{3,i} x_{2,j} x_{3,i} +}{-1 + T_2} \\
& \quad \frac{T_2 (-1 + T_1 T_2) p_{2,i} p_{3,j} x_{2,j} x_{3,i}}{-1 + T_2} - \frac{(-1 + 2 T_2) (-1 + T_1 T_2) p_{2,j} p_{3,j} x_{2,j} x_{3,i}}{-1 + T_2} + \\
& \quad \left. \left. T_1 p_{1,i} p_{3,j} x_{1,i} x_{3,j} + \frac{(-1 + T_1) p_{1,j} p_{3,j} x_{1,i} x_{3,j}}{-1 + T_2} - T_2 p_{2,i} p_{3,j} x_{2,i} x_{3,j} - p_{2,j} p_{3,j} x_{2,i} x_{3,j} \right] \right]
\end{aligned}$$

In[*]:= **$\bar{R}_{i,j}$**

Out[*]=

$$\begin{aligned}
& \mathbb{E}_{\{\} \rightarrow \{i,j\}} \left[\frac{1}{T_1 T_2}, -\frac{(-1 + T_1) p_{1,i} x_{1,i}}{T_1} + \frac{(-1 + T_1) p_{1,j} x_{1,i}}{T_1} - \right. \\
& \quad \frac{(-1 + T_2) p_{2,i} x_{2,i}}{T_2} + \frac{(-1 + T_2) p_{2,j} x_{2,i}}{T_2} - \frac{(-1 + T_1 T_2) p_{3,i} x_{3,i}}{T_1 T_2} + \frac{(-1 + T_1 T_2) p_{3,j} x_{3,i}}{T_1 T_2}, \\
& \quad \left. \in \text{Series} \left[-\frac{(-1 + T_1) p_{3,j} x_{1,i} x_{2,i}}{T_1 T_2} + \frac{(-1 + T_1) p_{3,j} x_{1,i} x_{2,j}}{T_1}, \right. \right. \\
& \quad -\frac{1}{2} + \frac{p_{2,i} x_{2,i}}{T_2} - \frac{p_{1,i} p_{2,j} x_{1,i} x_{2,i}}{T_1 T_2} - \frac{(-1 + T_1) p_{1,j} p_{2,j} x_{1,i} x_{2,i}}{T_1 (-1 + T_2)} - p_{2,j} x_{2,j} + \frac{p_{1,i} p_{2,j} x_{1,i} x_{2,j}}{T_1} + \\
& \quad \frac{(-1 + T_1) T_2 p_{1,j} p_{2,j} x_{1,i} x_{2,j}}{T_1 (-1 + T_2)} - \frac{(-1 + T_1 T_2) p_{1,j} p_{2,i} x_{3,i}}{T_1 (-1 + T_2) T_2} + \frac{(-1 + T_1 T_2) p_{1,j} p_{2,j} x_{3,i}}{T_1 (-1 + T_2) T_2} - \frac{p_{3,i} x_{3,i}}{T_1 T_2} + \\
& \quad \frac{(-1 + T_1 T_2) p_{3,j} x_{3,i}}{T_1 (-1 + T_2) T_2} + \frac{(-1 + T_1 T_2) p_{1,i} p_{3,j} x_{1,i} x_{3,i}}{T_1^2 (-1 + T_2) T_2} + \frac{(-1 + T_1) (-1 + T_1 T_2) p_{1,j} p_{3,j} x_{1,i} x_{3,i}}{T_1^2 (-1 + T_2) T_2} - \\
& \quad \frac{(-1 + T_2) p_{2,j} p_{3,i} x_{2,i} x_{3,i}}{T_1 T_2^2} + \frac{(-1 + T_1 T_2) p_{2,j} p_{3,j} x_{2,i} x_{3,i}}{T_1 T_2^2} - \frac{2 p_{2,j} p_{3,i} x_{2,j} x_{3,i}}{T_1 T_2} - \\
& \quad \frac{(-1 + T_1 T_2) p_{2,i} p_{3,j} x_{2,j} x_{3,i}}{T_1 (-1 + T_2) T_2} - \frac{(-2 + T_2) (-1 + T_1 T_2) p_{2,j} p_{3,j} x_{2,j} x_{3,i}}{T_1 (-1 + T_2) T_2} - \\
& \quad \left. \left. \frac{p_{1,i} p_{3,j} x_{1,i} x_{3,j}}{T_1} - \frac{(-1 + T_1) T_2 p_{1,j} p_{3,j} x_{1,i} x_{3,j}}{T_1 (-1 + T_2)} + \frac{p_{2,i} p_{3,j} x_{2,i} x_{3,j}}{T_2} + p_{2,j} p_{3,j} x_{2,i} x_{3,j} \right] \right]
\end{aligned}$$

In[*]:= **$\mathcal{L}_{\text{pPush}}[\mathbf{C}_i[1]]$**

Out[*]=

$$T_1 T_2 \mathbb{E} \left[(-p_{1,i} + p_{1,1+i}) x_{1,i} + (-p_{2,i} + p_{2,1+i}) x_{2,i} + (-p_{3,i} + p_{3,1+i}) x_{3,i} + \left(\frac{1}{2} + p_{3,1+i} x_{3,i} \right) \right]$$

In[*]:= **CC**₁ // **CF**

Out[*]=

$$\mathbb{E}_{\{\} \rightarrow \{1\}} \left[T_1 T_2, 0, \text{eSeries} \left[0, \frac{1}{2} + p_{3,1} x_{3,1} \right] \right]$$

In[*]:= **CC**₁ // **CF**

Out[*]=

$$\mathbb{E}_{\{\} \rightarrow \{1\}} \left[\frac{1}{T_1 T_2}, 0, \text{eSeries} \left[0, -\frac{1}{2} - p_{3,1} x_{3,1} \right] \right]$$

In[*]:= **\$k = 0**; {
 {"CC", (CC₁ CC₂ // m_{1,2→1}) ≡ $\mathbb{E}_{\{\} \rightarrow \{1\}} [1, 0, \text{eSeries}[]]$ },
 {"An R1", (CC₃ R_{1,2} // m_{2,3→2} // m_{2,1→1}) ≡ (CC₃ R_{1,2} // m_{1,3→1} // m_{1,2→1})},
 {"R2b", (R_{1,2} R_{3,4} // m_{1,3→1} // m_{2,4→2}) ≡ $\mathbb{E}_{\{\} \rightarrow \{1,2\}} [1, 0, \text{eSeries}[]]$ },
 {"R3", (R_{1,2} R_{4,3} R_{5,6} // m_{1,4→1} // m_{2,5→2} // m_{3,6→3}) ≡ (R_{2,3} R_{4,5} R_{1,6} // m_{1,4→1} // m_{2,5→2} // m_{3,6→3}) }
 }

Out[*]=

{ {CC, True}, {An R1, True}, {R2b, True}, {R3, True} }

In[*]:= **\$k = 1**; **HL**[(CC₁ CC₂ // m_{1,2→1}) ≡ $\mathbb{E}_{\{\} \rightarrow \{1\}} [1, 0, \text{eSeries}[]]$]

Out[*]=

True

In[*]:= **\$π**[$\mathcal{E}_-$] := (CF[$\mathcal{E}_-$ / . {p_{αβ} → B⁻¹ p_{αβ}, x_{αβ} → B x_{αβ}}] / . B → 0)

In[*]:= **\$k = 1**; **HL**@**\$π**[(CC₃ R_{1,2} // m_{2,3→2} // m_{2,1→1}) ≡ (CC₃ R_{1,2} // m_{1,3→1} // m_{1,2→1})]

Out[*]=

True

In[*]:= R_{1,2} R_{3,4} // m_{1,3→1} // m_{2,4→2}

Out[*]=

$$\begin{aligned} & \mathbb{E}_{\{\} \rightarrow \{1,2\}} \left[1, 0, \right. \\ & \quad \text{eSeries} \left[0, \frac{(-1 + T_1)(1 + T_2)p_{3,2}x_{1,1}x_{2,1} + (-1 + T_1)T_2p_{2,1}p_{3,2}x_{1,1}x_{2,1}^2 - (-1 + T_1)^2T_2^2p_{3,2}^2x_{1,1}^2x_{2,1}^2 + (1 - T_1)p_{2,1}p_{3,2}x_{1,1}x_{2,1}x_{2,2} + (-1 + T_1)T_2p_{2,2}p_{3,2}x_{1,1}x_{2,1}x_{2,2} + 2(-1 + T_1)^2T_2p_{3,2}^2x_{1,1}^2x_{2,1}x_{2,2} + (1 - T_1)p_{2,2}p_{3,2}x_{1,1}x_{2,2}^2 - (-1 + T_1)^2p_{3,2}^2x_{1,1}^2x_{2,2}^2 - (-1 + T_1)(-1 + T_2)p_{3,1}p_{3,2}x_{1,1}x_{2,1}x_{3,1} + (-1 + T_1)(-1 + T_1T_2)(-1 + T_2 + T_2^2)p_{3,2}^2x_{1,1}x_{2,1}x_{3,1}}{-1 + T_2} + \right. \\ & \quad \left. 2(-1 + T_1)p_{3,1}p_{3,2}x_{1,1}x_{2,2}x_{3,1} - \frac{(-1 + T_1)(2 + T_2)(-1 + T_1T_2)p_{3,2}^2x_{1,1}x_{2,2}x_{3,1}}{-1 + T_2} - \right. \\ & \quad \left. (-1 + T_1)T_2p_{3,2}^2x_{1,1}x_{2,1}x_{3,2} + (-1 + T_1)p_{3,2}^2x_{1,1}x_{2,2}x_{3,2} \right] \end{aligned}$$

In[*]:= **\$k = 1**; **HL**@**\$π**[(R_{1,2} R_{3,4} // m_{1,3→1} // m_{2,4→2}) ≡ $\mathbb{E}_{\{\} \rightarrow \{1,2\}} [1, 0, \text{eSeries}[]]$]

Out[*]=

True

```
In[ ]:= $k = 1;
HL@$π[ (R1,2 R4,3 R5,6 // m1,4→1 // m2,5→2 // m3,6→3) ≡ (R2,3 R4,5 R1,6 // m1,4→1 // m2,5→2 // m3,6→3) ]
Out[ ]:=
True
```

```
In[ ]:= Z[Knot[3, 1]]
```

KnotTheory: Loading precomputed data in PD4Knots`.

```
Out[ ]:=
```

$$\mathbb{E}_{\{\} \rightarrow \{\emptyset\}} \left[\frac{T_1^2 T_2^2}{(1 - T_1 + T_1^2) (1 - T_2 + T_2^2) (1 - T_1 T_2 + T_1^2 T_2^2)}, \emptyset, \right. \\ \left. \in \text{Series} \left[\emptyset, - \frac{1 - T_1 + T_1^2 - T_2 - T_1^3 T_2 + T_2^2 + T_1^4 T_2^2 - T_1 T_2^3 - T_1^4 T_2^3 + T_1^2 T_2^4 - T_1^3 T_2^4 + T_1^4 T_2^4}{(1 - T_1 + T_1^2) (1 - T_2 + T_2^2) (1 - T_1 T_2 + T_1^2 T_2^2)} - \right. \right. \\ \left. \left. \left((-1 + T_1) (3 - 11 T_2 + 2 T_1 T_2 + 5 T_2^2 + 14 T_1 T_2^2 - 3 T_1^2 T_2^2 - T_2^3 - 8 T_1 T_2^3 - 13 T_1^2 T_2^3 + \right. \right. \right. \\ \left. \left. \left. 6 T_1^3 T_2^3 - 5 T_2^4 + 9 T_1 T_2^4 + 5 T_1^2 T_2^4 + 5 T_1 T_2^5 - 7 T_1^2 T_2^5 - 7 T_1^2 T_2^6 + 6 T_1^3 T_2^6 \right) p_{3,\emptyset} x_{1,\emptyset} x_{2,\emptyset} \right) / \right. \\ \left. \left. \left((-1 + T_2) (1 - T_2 + T_2^2) (1 - T_1 T_2 + T_1^2 T_2^2) \right) - \frac{3 (-1 + T_1)^2 T_2 (1 + T_2) p_{3,\emptyset}^2 x_{1,\emptyset} x_{2,\emptyset} x_{3,\emptyset}}{-1 + T_2} \right] \right]$$

```
In[ ]:= PolyPlot2[1 - T1 + T12 - T2 - T13 T2 + T22 + T14 T22 - T1 T23 - T14 T23 + T12 T24 - T13 T24 + T14 T24]
```

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Out[ ]:=
```

