Pensieve header: The 3-Heisenberg Formalism for Θ. Engine modified from pensieve://Projects/BabyDoPeGDO/Solving_to_k=3.nb, px formulas modified from pensieve://Talks/Bonn-2505/. Last version before the transition to the xp order.

$\mathbb{E}$[ω, Q, P_ϵSeries] represents $\omega\, e^{Q+P}$, where ω is a scalar, Q is an ϵ-free quadratic, and $P = \sum_{k=0}^{\$k} P[\![k]\!]\,\epsilon^k$ is a perturbation (it is ill-advised to include ω in P because then it will have log terms).

Scheme: $\mathbb{E}$_[_] // $\mathbb{E}$_[_] calls FZip or Zip, which are functionally the same. Zip works by handling the quadratic part and calling PZip for the perturbation-only part. PZip works by iteratively solving the synthesis equation. FZip works by encapsulating coefficients, calling Zip, and back-substituting.

```
In[•]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\Banff-2607"];
Once[<< KnotTheory`];
```

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

```
In[•]:= $k = 1;
```

Minor utilities

```
In[•]:= CCF[ℰ_] := ExpandDenominator@ExpandNumerator@Together[ℰ];
(*CoefficientCanonical Form *)
CCF[ℰ_] := Factor[ℰ]; (*CoefficientCanonical Form *)
CF[ℰ_List] := CF /@ ℰ;
CF[ℰ_ϵSeries] := CF /@ Take[ℰ, $k + 1];
CF[ℰ_] := Module[{F}, Expand@Collect[ℰ, (p | x | π | ξ | g)__, F] /. F → CCF];
(*CF[ℰ_] := Module[
    {vs=Cases[ℰ, (p|x|π|ξ)__,∞]},
    Total[(CCF[#[[2]]] (Times@@vs^#[[1]]))& /@ CoefficientRules[Expand[ℰ], vs]]
  ];*)
CF[ℰ_𝔼] := CF /@ ℰ;
CF[𝔼sp___[ℰs___]] := CF /@ 𝔼sp[ℰs];
```

```
In[∘]:= ϵSeries /: S1_ϵSeries ≡ S2_ϵSeries :=
          Length[S1] == Length[S2] ∧ Inner[CF[#1] == CF[#2] &, S1, S2, And];
        ϵSeries[] := ϵSeries @@ Table[0, $k + 1];
        ϵSeries /: Plus[ss___ϵSeries] := Module[{l = Min[Length /@ {ss}]},
            ϵSeries @@ Total[Take[List @@ #, l] & /@ {ss}]
          ];
        (*ϵSeries/: S1_ϵSeries+S2_ϵSeries :=
          ϵSeries@@Table[S1〚k〛+S2〚k〛,{k,Min[Length@S1,Length@S2]}];*)
        ϵSeries /: S1_ϵSeries * S2_ϵSeries := ϵSeries @@
            Table[Sum[S1〚j + 1〛 * S2〚k - j + 1〛, {j, 0, k}], {k, 0, Min[Length@S1, Length@S2] - 1}];
        ϵSeries /: c_ * S_ϵSeries := (c #) & /@ S;
        ϵSeries /: ∂_{vs___} S_ϵSeries := (s ↦ ∂_{vs} s) /@ S;
```

The Main Program

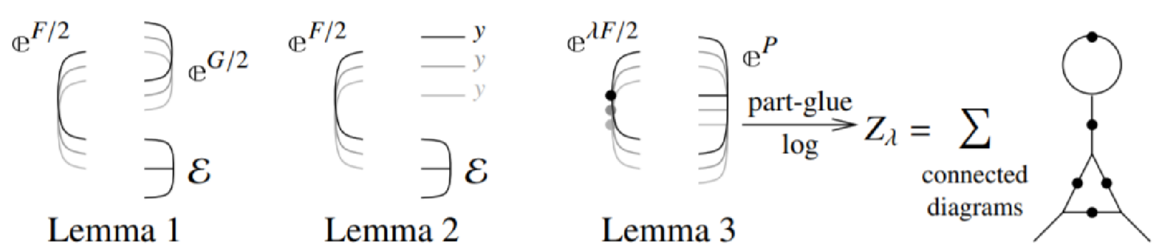
Variables and their duals:

```
In[∘]:= {p*, x*, π*, ξ*} = {π, ξ, p, x};
        (vs_List)* := (v ↦ v*) /@ vs;
        (u_{vi__})* := (u*)_{vi};
```

𝔼 operations:

```
In[∘]:= 𝔼 /: 𝔼[ω1_, Q1_, P1_] ≡ 𝔼[ω2_, Q2_, P2_] := CF[ω1 == ω2] ∧ CF[Q1 == Q2] ∧ (P1 ≡ P2);
        𝔼 /: 𝔼[ω1_, Q1_, P1_] 𝔼[ω2_, Q2_, P2_] := 𝔼[ω1 ω2, Q1 + Q2, P1 + P2];
        𝔼_{d1_→r1_}[ℰ1s___] ≡ 𝔼_{d2_→r2_}[ℰ2s___] ^:= (d1 == d2) ∧ (r1 == r2) ∧ (𝔼[ℰ1s] ≡ 𝔼[ℰ2s]);
        𝔼_{d1_→r1_}[ℰ1s___] 𝔼_{d2_→r2_}[ℰ2s___] ^:= 𝔼_{(d1⋃d2)→(r1⋃r2)} @@ (𝔼[ℰ1s] 𝔼[ℰ2s]);
        𝔼_{dr_}[ℰs___]_{$k_} := 𝔼_{dr} @@ 𝔼[ℰs]_{$k};
```

```
In[∘]:= 𝔼_{d1_→r1_}[ℰ1s___] // 𝔼_{d2_→r2_}[ℰ2s___] := Module[{is = r1 ⋂ d2, lvs},
          lvs = Flatten@Table[{x_{v,$@i}, p_{v,$@i}}, {v, 3}, {i, is}];
          𝔼_{(d1⋃Complement[d2,is])→(r2⋃Complement[r1,is])} @@ (Zip_{lvs⋃lvs*}[lvs*.lvs, Times[
              𝔼[ℰ1s] /. Table[(u : (x | p))_{v_,i} → u_{v,$@i}, {i, is}],
              𝔼[ℰ2s] /. Table[(u : (ξ | π))_{v_,i} → u_{v,$@i}, {i, is}]
            ]])
        ]
```



```
In[∘]:= Zip_vs_[ℱ_, ℰ_] := ((*EchoLabel["EZip called with {vs,ℱ,ℰ} = "][{vs,ℱ,ℰ}];*)
          ⟨ℱ, ℰ⟩ // Zip1_vs // EZip23_vs);
        Zip_vs_[ℱ_, ℰ_] := ((*EchoLabel["EZip called with {vs,ℱ,ℰ} = "][{vs,ℱ,ℰ}];*)
          ⟨ℱ, ℰ⟩ // Zip1_vs // Zip2_vs // Zip3_vs);
```

Getting rid of the quadratic.

Lemma 1. With convergences left to the reader,

$$\left\langle F:\mathcal{E}\,\mathbb{e}^{\frac{1}{2}\sum_{i,j\in B}G_{ij}z_iz_j}\right\rangle_B=\det(1-GF)^{-1/2}\left\langle F(1-GF)^{-1}:\mathcal{E}\right\rangle_B$$

```
In[∘]:= Zip1_{} = Identity;
        Zip1_vs_@⟨ℱ_, 𝔼[ω_, Q_, P_]⟩ := Module[{ℐ, F, G, u, v},
          (*EchoLabel["EZip1 called with {vs,ℱ,ω,Q,P} = "][{vs,ℱ,ω,Q,P}];*)
          ℐ = IdentityMatrix@Length@vs;
          F = Table[∂_{u,v}ℱ, {u, vs*}, {v, vs*}];
          G = Table[∂_{u,v}Q, {u, vs}, {v, vs}];
          ⟨CF[vs*.F.Inverse[ℐ - G.F].vs* / 2],
           𝔼[CF@PowerExpand@Factor[ω Det[ℐ - G.F]^(-1/2)], CF[Q - vs.G.vs / 2], P]⟩
         ]
```

Getting rid of linear terms.

Lemma 2. $\left\langle F:\mathcal{E}\,\mathbb{e}^{\sum_{i\in B}y_iz_i}\right\rangle_B=\mathbb{e}^{\frac{1}{2}\sum_{i,j\in B}F_{ij}y_iy_j}\left\langle F:\ \mathcal{E}|_{z_B\to z_B+Fy_B}\right\rangle_B.$

```
In[∘]:= Zip2_{} = Identity;
        Zip2_vs_@⟨ℱ_, 𝔼[ω_, Q_, P_]⟩ := Module[{F, Y, u, v},
          (*EchoLabel["EZip2 called with {vs,ℱ,ω,Q,P} = "][{vs,ℱ,ω,Q,P}];*)
          F = Table[∂_{u,v}ℱ, {u, vs*}, {v, vs*}];
          Y = Table[∂_v Q, {v, vs}];
          CF /@ ⟨ℱ, 𝔼[ω, Q - Y.vs + Y.F.Y / 2, P /. Thread[vs → vs + F.Y]]⟩
         ]
```

Dealing with Feynman diagrams.

Lemma 3. With an extra variable λ, $Z_\lambda:=\log[\lambda F:\mathbb{e}^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0=P\quad\text{and}\quad\partial_\lambda Z_\lambda=\frac{1}{2}\sum_{i,j\in B}F_{ij}\left(\partial_{z_i}\partial_{z_j}Z_\lambda+(\partial_{z_i}Z_\lambda)(\partial_{z_j}Z_\lambda)\right).$$

Note that the power m of λ is at most $k-1+\frac{2k+2}{2}=2k$. We write $Z_\lambda=\sum Z[m]\,\lambda^m$.

In[∘]:=
```
Zip3vs_@⟨ℱ_, 𝔼[ω_, Q_, P_]⟩ := Module[{Z, u, v, m, j},
  (*EchoLabel["Zip3 called with {vs,ℱ,ω,Q,P} = "][{vs,ℱ,ω,Q,P}];*)
  Z[0] = P;
  For[m = 0, m < 2 $k + 3, ++m,
   Z[m + 1] = CF[1/(2 (m + 1))
      Sum[∂u*,v* ℱ (∂u,v Z[m] + Sum[(∂u Z[j]) (∂v Z[m - j]), {j, 0, m}]), {u, vs}, {v, vs}]]
   ];
  𝔼[ω, Q, CF[Sum[Z[m], {m, 0, 2 $k + 3}] /. Table[v → 0, {v, vs}]]]
 ]
```

In[∘]:=
```
EZip23vs_@⟨ℱ_, 𝔼[ω_, Q_, P_]⟩ := Module[
  {nP, nℱ, nQ, j = 0, ps, c, t, rr = {(*release rules*)}},
  nP = Total[
      CoefficientRules[#, vs] /.
       (ps_ → c_) :→ (AppendTo[rr, t[++j] → CF@c]; t[j] (Times @@ vs^ps))
     ] & /@ P;
  nQ = Total[CoefficientRules[Q, vs] /.
     (ps_ → c_) :→ (AppendTo[rr, t[++j] → CF@c]; t[j] (Times @@ vs^ps))];
  nℱ = Total[CoefficientRules[ℱ, vs*] /.
     (ps_ → c_) :→ (AppendTo[rr, t[++j] → CF@c]; t[j] (Times @@ (vs*)^ps))];
  CF[Expand[⟨nℱ, 𝔼[ω, nQ, nP]⟩ // Zip2vs // Zip3vs] /. rr]
 ]
```

In[∘]:=
```
mi_,j_→k_ := 𝔼{i,j}→{k}[1, ∑_{ν=1}^{3} (-ξν,i πν,j + (πν,i + πν,j) pν,k + (ξν,i + ξν,j) xν,k), ϵSeries[]]
```

In[∘]:=
```
ma,b→c
```

Out[∘]=

$\mathbb{E}_{\{a,b\}\to\{c\}}[1,\ p_{1,c}\ (\pi_{1,a}+\pi_{1,b})+p_{2,c}\ (\pi_{2,a}+\pi_{2,b})+p_{3,c}\ (\pi_{3,a}+\pi_{3,b})-\pi_{1,b}\ \xi_{1,a}+$
$x_{1,c}\ (\xi_{1,a}+\xi_{1,b})-\pi_{2,b}\ \xi_{2,a}+x_{2,c}\ (\xi_{2,a}+\xi_{2,b})-\pi_{3,b}\ \xi_{3,a}+x_{3,c}\ (\xi_{3,a}+\xi_{3,b}),\ \epsilon\text{Series}[0,\ 0]]$

g2px and px2g

From pensieve://Talks/Geneva-2408/DataConversions.nb

In[∘]:=
```
g2px[ℰ_] := CF@Module[{λ}, Expand[ℰ /. gν_,i_,j_ :→ λν pν,i xν,j] /. λ_^k_. :→ 1 / k!]
```

In[∘]:=
```
Zip{}[ℰ_] := ℰ;
Zip{ξ_,ξs___}[ℰ_] := (Collect[ℰ // Zip{ξs}, ξ] /. f_. ξ^d_. :→ (D[f, {ξ*, d}])) /. ξ* → 0
```

```
In[∘]:= px2g[ℰ_] := CF@Module[{ps, xs, Q, α, β},
    ps = Union[Cases[ℰ, p__, ∞]]; xs = Union[Cases[ℰ, x__, ∞]];
    Q = Sum[p0* x0* g_{p0[[2]],x0[[2]],p0[[3]],x0[[3]]}, {p0, ps}, {x0, xs}];
    Expand[Zip_{ps⋃xs}[ℰ e^Q] /. g_{α_,β_,i_,j_} ⧴ If[α === β, g_{α,i,j}, 0]]
  ]
```

gRules

From pensieve://Talks/MonteVerita-2604/Theta.nb

```
In[∘]:= gR_{s_,i_,j_} := {
   g_{ν_ jβ_} ⧴ g_{ν j⁺ β} + δ_{jβ}, g_{ν_ iβ_} ⧴ T_ν^s g_{ν i⁺ β} + (1 - T_ν^s) g_{ν j⁺ β} + δ_{iβ},
   g_{ν_ α_ i⁺} ⧴ T_ν^s g_{ν α i} + δ_{α i⁺}, g_{ν_ α_ j⁺} ⧴ g_{ν α j} + (1 - T_ν^s) g_{ν α i} + δ_{α j⁺}
  }
```

Inverse gRules written locally:

```
In[∘]:= igR_{s_,i_,j_} := {
   g_{ν_,j⁺,β_} ⧴ g_{ν,j,β} - δ_{j,β}, g_{ν_,i⁺,β_} ⧴ T_ν^{-s} g_{ν i β} - (T_ν^{-s} - 1) g_{ν j⁺ β} - T_ν^{-s} δ_{iβ},
   g_{ν_,α_,i} ⧴ T_ν^{-s} (g_{ν,α,i⁺} - δ_{α,i⁺}), g_{ν_,α_,j} ⧴ g_{ν,α,j⁺} - (1 - T_ν^s) g_{ν,α,i} - δ_{α,j⁺}
  }
```

Some Knot Theory

```
In[∘]:= Kink_{i_} := CC_3 R_{1,2} // m_{2,3→2} // m_{2,1→i};
        Kink‾_{i_} := CC_3 R‾_{1,2} // m_{1,3→1} // m_{1,2→i}
```

```
In[∘]:= RVK[pd_PD] := Module[{n, xs, x, rots, front = {0}, k},
    n = Length@pd; rots = Table[0, {2 n}];
    xs = Cases[pd, x_X ⧴ { Xp[x[[4]], x[[1]]]   PositiveQ@x
                           Xm[x[[2]], x[[1]]]   True       ];
    For[k = 0, k < 2 n, ++k, If[k == 0 ∨ FreeQ[front, -k],
      front = Flatten[front /. k → (xs /. {
              Xp[k + 1, l_] | Xm[l_, k + 1] ⧴ {l, k + 1, 1 - l},
              Xp[l_, k + 1] | Xm[k + 1, l_] ⧴ (++rots[[l]]; {1 - l, k + 1, l})
            })],
      Cases[front, k | -k] /. {k, -k} ⧴ --rots[[k + 1]];
     ]];
    RVK[xs, rots] ];
RVK[K_] := RVK[PD[K]];
```

```
In[•]:= rot[i_, 0] := 𝔼{}→{i}[1, 0, ϵSeries[]];
        rot[i_, n_] := Module[{j}, If[n > 0, rot[i, n - 1] CCj, rot[i, n + 1] C̄C̄j] // mi,j→i];
```

```
In[•]:= Z[K_] := Z[RVK@K];
        Z[rvk_RVK] := (*Z[rvk] =*)
         Module[{todo, n, rots, ζ, done, st, cx, ζ1, i, j, k, k1, k2, k3},
          {todo, rots} = List @@ rvk;
          AppendTo[rots, 0];
          n = Length[todo];
          ζ = 𝔼{}→{0}[1, 0, ϵSeries[]];
          done = {0};
          st = Range[0, 2 n + 1];
          While[{} =!= ($M = todo),
           {cx} = MaximalBy[todo, Length[done ⋂ {#[[1]], #[[2]], #[[1]] - 1, #[[2]] - 1}] &, 1];
           {i, j} = List @@ cx;
           ζ1 = Switch[Head[cx],
              Xp, (Ri,j K̄inkk) // mj,k→j,
              Xm, (R̄i,j Kinkk) // mj,k→j
             ];
           ζ1 = (rot[k, rots[[i]]] ζ1) // mk,i→i; rots[[i]] = 0;
           ζ1 = (ζ1 rot[k, rots[[i + 1]]]) // mi,k→i; rots[[i + 1]] = 0;
           ζ1 = (rot[k, rots[[j]]] ζ1) // mk,j→j; rots[[j]] = 0;
           ζ1 = (ζ1 rot[k, rots[[j + 1]]]) // mj,k→j; rots[[j + 1]] = 0;
           ζ *= ζ1;
           If[MemberQ[done, i], ζ = ζ // mi,i+1→i; st = st /. st[[i + 2]] → st[[i + 1]]];
           If[MemberQ[done, i - 1], ζ = ζ // mst[[i]],i→st[[i]]; st = st /. st[[i + 1]] → st[[i]]];
           If[MemberQ[done, j], ζ = ζ // mj,j+1→j; st = st /. st[[j + 2]] → st[[j + 1]]];
           If[MemberQ[done, j - 1], ζ = ζ // mst[[j]],j→st[[j]]; st = st /. st[[j + 1]] → st[[j]]];
           done = done ⋃ {i - 1, i, j - 1, j};
           todo = DeleteCases[todo, cx]
          ];
          CF /@ (ζ (*/. {x0→x,y0→y,a0→a}*))
         ]
```

ρ_1 testing

From Talks/Bonn-2505/Itype.nb:

In[∘]:=
```
ℒ[X_{i_,j_}[s_]] := T^{s/2} 𝔼[{
    x_i (p_{i+1} - p_i) + x_j (p_{j+1} - p_j) + (T^s - 1) x_i (p_{i+1} - p_{j+1}),
    0, 0, (ϵ s / 2) × (x_i (p_i - p_j) ((T^s - 1) x_i p_j + 2 (1 - x_j p_j)) - 1)
  }]
ℒ[C_{i_}[φ_]] := T^{φ/2} 𝔼[x_i (p_{i+1} - p_i) + ϵ φ (1/2 - x_i p_i)]
ℒ[K_] := CF[ℒ /@ Features[K]〚2〛]
vs[K_] := Join @@ Table[{p_i, x_i}, {i, Features[K]〚1〛}]
```

From Projects/BabyDoPeGDO/Solving to $k=3.nb:

In[∘]:=

$$R_{i\_,j\_} := \mathbb{E}_{\{\}\to\{i,j\}}\left[1,\ (T_1-1)\,x_{1,j}\,(p_{1,i}-p_{1,j}),\ \epsilon\text{Series}\left[0,\ \frac{1}{2}(-1+T_1)\,x_{1,j}^2\,p_{1,i}^2 + x_{1,i}\,x_{1,j}\,p_{1,i}\,p_{1,j} + \frac{1}{2}(1-3T_1)\,x_{1,j}^2\,p_{1,i}\,p_{1,j}\right]\right];$$

$$\overline{R}_{i\_,j\_} := \mathbb{E}_{\{\}\to\{i,j\}}\left[1,\ (T_1^{-1}-1)\,x_{1,j}\,(p_{1,i}-p_{1,j}),\ \epsilon\text{Series}\left[0,\ -\frac{(-1+T_1)\,x_{1,i}\,x_{1,j}\,p_{1,i}^2}{T_1^2} - \frac{(1-T_1)\,x_{1,j}^2\,p_{1,i}^2}{2\,T_1^3} - \frac{x_{1,i}\,x_{1,j}\,p_{1,i}\,p_{1,j}}{T_1^2} - \frac{(-1-T_1)\,x_{1,j}^2\,p_{1,i}\,p_{1,j}}{2\,T_1^3}\right]\right];$$

$$CC_{i\_} := \mathbb{E}_{\{\}\to\{i\}}\left[\sqrt{T_1},\ 0,\ \epsilon\text{Series}\left[0,\ -\frac{x_{1,i}\,p_{1,i}}{T_1}\right]\right];$$

$$\overline{CC}_{i\_} := \mathbb{E}_{\{\}\to\{i\}}\left[\frac{1}{\sqrt{T_1}},\ 0,\ \epsilon\text{Series}\left[0,\ \frac{x_{1,i}\,p_{1,i}}{T_1}\right]\right];$$

In[∘]:=
```
$k = 1; {
  {"CC̄", (CC_1 CC̄_2 // m_{1,2→1}) ≡ 𝔼_{{}→{1}}[1, 0, ϵSeries[]]},
  {"An R1", (CC_3 R_{1,2} // m_{2,3→2} // m_{2,1→1}) ≡ (CC̄_3 R_{1,2} // m_{1,3→1} // m_{1,2→1})},
  {"R2b", (R_{1,2} R̄_{3,4} // m_{1,3→1} // m_{2,4→2}) ≡ 𝔼_{{}→{1,2}}[1, 0, ϵSeries[]]},
  {"R3", (R_{1,2} R_{4,3} R_{5,6} // m_{1,4→1} // m_{2,5→2} // m_{3,6→3}) ≡ (R_{2,3} R_{4,5} R_{1,6} // m_{1,4→1} // m_{2,5→2} // m_{3,6→3})}
}
```

Out[∘]=
```
{{CC̄, True}, {An R1, True}, {R2b, True}, {R3, True}}
```

In[∘]:=

$$\text{NewBit}[K\_] := \text{Module}\left[\{\text{Alex} = \text{Alexander}[K][T_1]\},\ T_1^3\,\frac{\text{Alex}^2}{T_1-1}\,Z[K]〚3,2〛 \text{ // Factor}\right]$$

In[∘]:=
```
NewBit /@ AllKnots[{3, 5}]
```

KnotTheory: Loading precomputed data in PD4Knots`.

Out[∘]=

$$\left\{2-T_1+T_1^2,\ (1+T_1)\,(1-3\,T_1+T_1^2),\ \frac{4-3\,T_1+5\,T_1^2-3\,T_1^3+3\,T_1^4-T_1^5+T_1^6}{T_1^2},\ 9-11\,T_1+7\,T_1^2-T_1^3\right\}$$

In pensieve://Projects/BabyDoPeGDO/Solving to $k=3.nb that was

$$\left\{2-T+T^2,\ (1+T)\ \left(1-3\,T+T^2\right),\ \frac{4-3\,T+5\,T^2-3\,T^3+3\,T^4-T^5+T^6}{T^2},\ 9-11\,T+7\,T^2-T^3\right\}$$

And now θ

In[ ]:=
```
T₃ = T₁ T₂;
ℒ[X_{i_,j_}[s_]] := T₃^s 𝔼[CF@{
    Sum_{ν=1}^{3} (x_{νi} (p_{νi⁺} - p_{νi}) + x_{νj} (p_{νj⁺} - p_{νj}) + (T_ν^s - 1) x_{νi} (p_{νi⁺} - p_{νj⁺})),
    (T₁^s - 1) p_{3j} x_{1i} (T₂^s x_{2i} - x_{2j}),
    ϵ s (T₃^s - 1) p_{1j} (p_{2i} - p_{2j}) x_{3i} / (T₂^s - 1),
    ϵ s (1/2 + T₂^s p_{1i} p_{2j} x_{1i} x_{2i} - p_{1i} p_{2j} x_{1i} x_{2j} - p_{3i} x_{3i} - (T₂^s - 1) p_{2j} p_{3i} x_{2i} x_{3i} +
      (T₃^s - 1) p_{2j} p_{3j} x_{2i} x_{3i} + 2 p_{2j} p_{3i} x_{2j} x_{3i} + p_{1i} p_{3j} x_{1i} x_{3j} - p_{2i} p_{3j} x_{2i} x_{3j} - T₂^s p_{2j} p_{3j} x_{2i} x_{3j} +
      ((T₁^s - 1) p_{1j} x_{1i} (T₂^{2s} p_{2j} x_{2i} - T₂^s p_{2j} x_{2j} - (T₂^s + 1) (T₃^s - 1) p_{3j} x_{3i} + T₂^s p_{3j} x_{3j}) +
        (T₃^s - 1) p_{3j} x_{3i} (1 - T₂^s p_{1i} x_{1i} + p_{2i} x_{2j} + (T₂^s - 2) p_{2j} x_{2j})) / (T₂^s - 1))}]
```

In[ ]:=
```
ℒ[C_{i_}[φ_]] := T₃^φ 𝔼[{Sum_{ν=1}^{3} x_{νi} (p_{νi⁺} - p_{νi}), ϵ φ (p_{3i} x_{3i} - 1/2)}]
```

In[ ]:=
```
R_{i_,j_} := CF[ℒ[X_{i,j}[1]] /. ω_ 𝔼[{Q_, P0_, P1a_, P1b_}] ⧴ 𝔼_{{}→{i,j}}[
    ω, Sum_{ν=1}^{3} ((T_ν - 1) x_{ν,j} (p_{ν,i} - p_{ν,j})),
    ϵSeries[P0, P1a + P1b /. ϵ → 1]
  ]];
R̄_{i_,j_} := CF[ℒ[X_{i,j}[-1]] /. ω_ 𝔼[{Q_, P0_, P1a_, P1b_}] ⧴ 𝔼_{{}→{i,j}}[
    ω, Sum_{ν=1}^{3} ((T_ν^{-1} - 1) x_{ν,j} (p_{ν,i} - p_{ν,j})),
    ϵSeries[P0, P1a + P1b /. ϵ → 1]
  ]];
CC_{i_} := CF[ℒ[C_i[1]] /. ω_ 𝔼[{Q_, P_}] ⧴ 𝔼_{{}→{i}}[
    ω, 0, ϵSeries[0, P /. ϵ → 1]
  ]];
C̄C̄_{i_} := CF[ℒ[C_i[-1]] /. ω_ 𝔼[{Q_, P_}] ⧴ 𝔼_{{}→{i}}[
    ω, 0, ϵSeries[0, P /. ϵ → 1]
  ]];
```

In[ ]:= $\overline{\mathrm{R}}_{i,j}$ // CF

Out[ ]=

$$\mathbb{E}_{\{\}\to\{i,j\}}\left[\frac{1}{T_1 T_2}, -\frac{(-1+T_1)\,p_{1,i}\,x_{1,j}}{T_1} + \frac{(-1+T_1)\,p_{1,j}\,x_{1,j}}{T_1} - \frac{(-1+T_2)\,p_{2,i}\,x_{2,j}}{T_2} + \frac{(-1+T_2)\,p_{2,j}\,x_{2,j}}{T_2} - \frac{(-1+T_1 T_2)\,p_{3,i}\,x_{3,j}}{T_1 T_2} + \frac{(-1+T_1 T_2)\,p_{3,j}\,x_{3,j}}{T_1 T_2},\right.$$
$$\in\mathrm{Series}\left[-\frac{(-1+T_1)\,p_{3,j}\,x_{1,i}\,x_{2,i}}{T_1 T_2} + \frac{(-1+T_1)\,p_{3,j}\,x_{1,i}\,x_{2,j}}{T_1},\right.$$
$$-\frac{1}{2} - \frac{p_{1,i}\,p_{2,j}\,x_{1,i}\,x_{2,i}}{T_2} - \frac{(-1+T_1)\,p_{1,j}\,p_{2,j}\,x_{1,i}\,x_{2,i}}{T_1\,(-1+T_2)\,T_2} + p_{1,i}\,p_{2,j}\,x_{1,i}\,x_{2,j} + \frac{(-1+T_1)\,p_{1,j}\,p_{2,j}\,x_{1,i}\,x_{2,j}}{T_1\,(-1+T_2)} -$$
$$\frac{(-1+T_1 T_2)\,p_{1,j}\,p_{2,i}\,x_{3,i}}{T_1\,(-1+T_2)} + \frac{(-1+T_1 T_2)\,p_{1,j}\,p_{2,j}\,x_{3,i}}{T_1\,(-1+T_2)} + p_{3,i}\,x_{3,i} - \frac{(-1+T_1 T_2)\,p_{3,j}\,x_{3,i}}{T_1\,(-1+T_2)} +$$
$$\frac{(-1+T_1 T_2)\,p_{1,i}\,p_{3,j}\,x_{1,i}\,x_{3,i}}{T_1\,(-1+T_2)\,T_2} - \frac{(-1+T_1)\,(1+T_2)\,(-1+T_1 T_2)\,p_{1,j}\,p_{3,j}\,x_{1,i}\,x_{3,i}}{T_1^2\,(-1+T_2)\,T_2} -$$
$$\frac{(-1+T_2)\,p_{2,j}\,p_{3,i}\,x_{2,i}\,x_{3,i}}{T_2} + \frac{(-1+T_1 T_2)\,p_{2,j}\,p_{3,j}\,x_{2,i}\,x_{3,i}}{T_1 T_2} - 2\,p_{2,j}\,p_{3,i}\,x_{2,j}\,x_{3,i} -$$
$$\frac{(-1+T_1 T_2)\,p_{2,i}\,p_{3,j}\,x_{2,j}\,x_{3,i}}{T_1\,(-1+T_2)} + \frac{(-1+2T_2)\,(-1+T_1 T_2)\,p_{2,j}\,p_{3,j}\,x_{2,j}\,x_{3,i}}{T_1\,(-1+T_2)\,T_2} -$$
$$\left.\left. p_{1,i}\,p_{3,j}\,x_{1,i}\,x_{3,j} - \frac{(-1+T_1)\,p_{1,j}\,p_{3,j}\,x_{1,i}\,x_{3,j}}{T_1\,(-1+T_2)} + p_{2,i}\,p_{3,j}\,x_{2,i}\,x_{3,j} + \frac{p_{2,j}\,p_{3,j}\,x_{2,i}\,x_{3,j}}{T_2}\right]\right]$$

In[ ]:= CC_1 // CF

Out[ ]=

$$\mathbb{E}_{\{\}\to\{1\}}\left[T_1 T_2, 0, \in\mathrm{Series}\left[0, -\frac{1}{2} + p_{3,1}\,x_{3,1}\right]\right]$$

In[ ]:= $\overline{\mathrm{CC}}_1$ // CF

Out[ ]=

$$\mathbb{E}_{\{\}\to\{1\}}\left[\frac{1}{T_1 T_2}, 0, \in\mathrm{Series}\left[0, \frac{1}{2} - p_{3,1}\,x_{3,1}\right]\right]$$

In[∘]:=
```
$k = 0; {
  {"CC̄", (CC₁ CC̄₂ // m_{1,2→1}) ≡ 𝔼_{{}→{1}}[1, 0, ϵSeries[]]},
  {"An R1", (CC₃ R_{1,2} // m_{2,3→2} // m_{2,1→1}) ≡ (CC̄₃ R_{1,2} // m_{1,3→1} // m_{1,2→1})},
  {"R2b", (R_{1,2} R̄_{3,4} // m_{1,3→1} // m_{2,4→2}) ≡ 𝔼_{{}→{1,2}}[1, 0, ϵSeries[]]},
  {"R3", (R_{1,2} R_{4,3} R_{5,6} // m_{1,4→1} // m_{2,5→2} // m_{3,6→3}) ≡ (R_{2,3} R_{4,5} R_{1,6} // m_{1,4→1} // m_{2,5→2} // m_{3,6→3})}
}
```

Out[∘]=

{{CC̄, True},

{An R1, $\frac{(-1+T_1)(-1+T_2)(1+T_2)\,p_{3,1}\,x_{1,1}\,x_{2,1}}{T_2} == (-1+T_1)\,T_1\,(-1+T_2)(1+T_2)\,p_{3,1}\,x_{1,1}\,x_{2,1}$},

{R2b, $\frac{(-1+T_1)(-1+T_1 T_2)\,p_{3,1}\,x_{1,1}\,x_{2,1}}{T_1 T_2} + (-1+T_1)(-1+T_2)\,p_{3,2}\,x_{1,1}\,x_{2,1} + \frac{(-1+T_1)^2\,T_2\,p_{3,2}\,x_{1,2}\,x_{2,1}}{T_1} - \frac{(-1+T_1)(-1+T_1 T_2)\,p_{3,1}\,x_{1,1}\,x_{2,2}}{T_1} + \frac{(-1+T_1)(-1+T_2)(1+2T_2)\,p_{3,2}\,x_{1,1}\,x_{2,2}}{T_2} + \frac{(-1+T_1)^2\left(-1-T_2+T_2^2\right)p_{3,2}\,x_{1,2}\,x_{2,2}}{T_1 T_2} == 0$},

{R3, $(-1+T_1)\,T_2\,p_{3,2}\,x_{1,1}\,x_{2,1} + (-1+T_1)\,T_2\,p_{3,3}\,x_{1,1}\,x_{2,1} - (-1+T_1)^2\,T_1\,T_2\,p_{3,2}\,x_{1,3}\,x_{2,1} + (1-T_1)\,p_{3,2}\,x_{1,1}\,x_{2,2} - (-1+T_1)\,T_2\,(-1+T_1 T_2)\,p_{3,1}\,x_{1,2}\,x_{2,2} + (-1+T_1)\,T_1\,T_2^2\,p_{3,3}\,x_{1,2}\,x_{2,2} + (-1+T_1)^2\,T_1\,p_{3,2}\,x_{1,3}\,x_{2,2} - (-1+T_1)(-1+T_2)^2(1+T_2)\,p_{3,2}\,x_{1,1}\,x_{2,3} - (-1+T_1)\,T_2\,p_{3,3}\,x_{1,1}\,x_{2,3} + (-1+T_1)(-1+T_1 T_2)\,p_{3,1}\,x_{1,2}\,x_{2,3} - (-1+T_1)\,T_1\,T_2\,p_{3,3}\,x_{1,2}\,x_{2,3} + (-1+T_1)^2\,T_1\,(-1+T_2)^2(1+T_2)\,p_{3,2}\,x_{1,3}\,x_{2,3} == -((-1+T_1)\,T_2\,(-2+T_1 T_2)\,p_{3,2}\,x_{1,1}\,x_{2,1}) + (-1+T_1)\,T_1\,T_2^2\,p_{3,3}\,x_{1,1}\,x_{2,1} + (-1+T_1)^2\,T_2\,(-1+T_1 T_2)\,p_{3,2}\,x_{1,2}\,x_{2,1} - (-1+T_1)^2\,T_1\,T_2^2\,p_{3,3}\,x_{1,2}\,x_{2,1} + (-1+T_1)\left(-1+T_2-T_2^2-T_1 T_2^2+T_1 T_2^3\right)p_{3,2}\,x_{1,1}\,x_{2,2} - (-1+T_1)\,T_1\,(-1+T_2)\,T_2^2\,p_{3,3}\,x_{1,1}\,x_{2,2} - (-1+T_1)^2(-1+T_2)\,T_2\,(-1+T_1 T_2)\,p_{3,2}\,x_{1,2}\,x_{2,2} + (-1+T_1)\,T_1\,T_2^2\,(2-T_1-T_2+T_1 T_2)\,p_{3,3}\,x_{1,2}\,x_{2,2} + (-1+T_1)(-1+T_1 T_2)\,p_{3,2}\,x_{1,1}\,x_{2,3} - (-1+T_1)\,T_1\,T_2\,p_{3,3}\,x_{1,1}\,x_{2,3} - (-1+T_1)^2(-1+T_1 T_2)\,p_{3,2}\,x_{1,2}\,x_{2,3} + (-2+T_1)(-1+T_1)\,T_1\,T_2\,p_{3,3}\,x_{1,2}\,x_{2,3}$}}

In[◦]:=
```
$k = 0;
R_{1,2} // CF
R̄_{3,4} // CF
R2LHS = (R_{1,2} R̄_{3,4} // m_{1,3→1} // m_{2,4→2})
```

Out[◦]=

$$\mathbb{E}_{\{\}\to\{1,2\}}[T_1 T_2,\ (-1+T_1)\,p_{1,1}\,x_{1,2} + (1-T_1)\,p_{1,2}\,x_{1,2} + (-1+T_2)\,p_{2,1}\,x_{2,2} + (1-T_2)\,p_{2,2}\,x_{2,2} + (-1+T_1 T_2)\,p_{3,1}\,x_{3,2} + (1-T_1 T_2)\,p_{3,2}\,x_{3,2},\ \epsilon\text{Series}[(-1+T_1)\,T_2\,p_{3,2}\,x_{1,1}\,x_{2,1} + (1-T_1)\,p_{3,2}\,x_{1,1}\,x_{2,2}]]$$

Out[◦]=

$$\mathbb{E}_{\{\}\to\{3,4\}}\left[\frac{1}{T_1 T_2},\ -\frac{(-1+T_1)\,p_{1,3}\,x_{1,4}}{T_1} + \frac{(-1+T_1)\,p_{1,4}\,x_{1,4}}{T_1} - \frac{(-1+T_2)\,p_{2,3}\,x_{2,4}}{T_2} + \frac{(-1+T_2)\,p_{2,4}\,x_{2,4}}{T_2} - \frac{(-1+T_1 T_2)\,p_{3,3}\,x_{3,4}}{T_1 T_2} + \frac{(-1+T_1 T_2)\,p_{3,4}\,x_{3,4}}{T_1 T_2},\ \epsilon\text{Series}\left[-\frac{(-1+T_1)\,p_{3,4}\,x_{1,3}\,x_{2,3}}{T_1 T_2} + \frac{(-1+T_1)\,p_{3,4}\,x_{1,3}\,x_{2,4}}{T_1}\right]\right]$$

Out[◦]=

$$\mathbb{E}_{\{\}\to\{1,2\}}\left[1,\ 0,\ \epsilon\text{Series}\left[\frac{(-1+T_1)\,(-1+T_1 T_2)\,p_{3,1}\,x_{1,1}\,x_{2,1}}{T_1 T_2} + (-1+T_1)\,(-1+T_2)\,p_{3,2}\,x_{1,1}\,x_{2,1} + \frac{(-1+T_1)^2\,T_2\,p_{3,2}\,x_{1,2}\,x_{2,1}}{T_1} - \frac{(-1+T_1)\,(-1+T_1 T_2)\,p_{3,1}\,x_{1,1}\,x_{2,2}}{T_1} + \frac{(-1+T_1)\,(-1+T_2)\,(1+2T_2)\,p_{3,2}\,x_{1,1}\,x_{2,2}}{T_2} + \frac{(-1+T_1)^2\left(-1-T_2+T_2^2\right)p_{3,2}\,x_{1,2}\,x_{2,2}}{T_1 T_2}\right]\right]$$

In[◦]:=
```
R2LHS0 = R_{1,2} R̄_{3,4} // m_{1,3→1}
```

Out[◦]=

$$\mathbb{E}_{\{\}\to\{1,2,4\}}\left[1,\ (-1+T_1)\,p_{1,1}\,x_{1,2} + (1-T_1)\,p_{1,2}\,x_{1,2} - \frac{(-1+T_1)\,p_{1,1}\,x_{1,4}}{T_1} + \frac{(-1+T_1)\,p_{1,4}\,x_{1,4}}{T_1} + (-1+T_2)\,p_{2,1}\,x_{2,2} + (1-T_2)\,p_{2,2}\,x_{2,2} - \frac{(-1+T_2)\,p_{2,1}\,x_{2,4}}{T_2} + \frac{(-1+T_2)\,p_{2,4}\,x_{2,4}}{T_2} + (-1+T_1 T_2)\,p_{3,1}\,x_{3,2} + (1-T_1 T_2)\,p_{3,2}\,x_{3,2} - \frac{(-1+T_1 T_2)\,p_{3,1}\,x_{3,4}}{T_1 T_2} + \frac{(-1+T_1 T_2)\,p_{3,4}\,x_{3,4}}{T_1 T_2},\ \epsilon\text{Series}\left[(-1+T_1)\,T_2\,p_{3,2}\,x_{1,1}\,x_{2,1} - \frac{(-1+T_1)\,p_{3,4}\,x_{1,1}\,x_{2,1}}{T_1 T_2} + \frac{(-1+T_1)^2\,T_2\,p_{3,2}\,x_{1,4}\,x_{2,1}}{T_1} + (1-T_1)\,p_{3,2}\,x_{1,1}\,x_{2,2} - \frac{(-1+T_1)^2\,p_{3,2}\,x_{1,4}\,x_{2,2}}{T_1} + (-1+T_1)\,(-1+T_2)\,p_{3,2}\,x_{1,1}\,x_{2,4} + \frac{(-1+T_1)\,p_{3,4}\,x_{1,1}\,x_{2,4}}{T_1} + \frac{(-1+T_1)^2\,(-1+T_2)\,p_{3,2}\,x_{1,4}\,x_{2,4}}{T_1}\right]\right]$$

In[#]:= **CF[R2LHS0[[2]] + R2LHS0[[3, 1]]]**

Out[#]=

$$(-1+T_1)\,p_{1,1}\,x_{1,2}+(1-T_1)\,p_{1,2}\,x_{1,2}-\frac{(-1+T_1)\,p_{1,1}\,x_{1,4}}{T_1}+\frac{(-1+T_1)\,p_{1,4}\,x_{1,4}}{T_1}+$$
$$(-1+T_1)\,T_2\,p_{3,2}\,x_{1,1}\,x_{2,1}-\frac{(-1+T_1)\,p_{3,4}\,x_{1,1}\,x_{2,1}}{T_1\,T_2}+\frac{(-1+T_1)^2\,T_2\,p_{3,2}\,x_{1,4}\,x_{2,1}}{T_1}+$$
$$(-1+T_2)\,p_{2,1}\,x_{2,2}+(1-T_2)\,p_{2,2}\,x_{2,2}+(1-T_1)\,p_{3,2}\,x_{1,1}\,x_{2,2}-\frac{(-1+T_1)^2\,p_{3,2}\,x_{1,4}\,x_{2,2}}{T_1}-$$
$$\frac{(-1+T_2)\,p_{2,1}\,x_{2,4}}{T_2}+\frac{(-1+T_2)\,p_{2,4}\,x_{2,4}}{T_2}+(-1+T_1)\,(-1+T_2)\,p_{3,2}\,x_{1,1}\,x_{2,4}+$$
$$\frac{(-1+T_1)\,p_{3,4}\,x_{1,1}\,x_{2,4}}{T_1}+\frac{(-1+T_1)^2\,(-1+T_2)\,p_{3,2}\,x_{1,4}\,x_{2,4}}{T_1}+(-1+T_1\,T_2)\,p_{3,1}\,x_{3,2}+$$
$$(1-T_1\,T_2)\,p_{3,2}\,x_{3,2}-\frac{(-1+T_1\,T_2)\,p_{3,1}\,x_{3,4}}{T_1\,T_2}+\frac{(-1+T_1\,T_2)\,p_{3,4}\,x_{3,4}}{T_1\,T_2}$$