Pensieve header: The 3-Heisenberg Formalism for Θ. Engine modified from pensieve://Projects/BabyDoPeGDO/Solving_to_k=3.nb, px formulas modified from pensieve://Talks/Bonn-2505/. Last version before the transition to the xp order.

𝔼[ω, Q, P_ϵSeries] represents $\omega\, e^{Q+P}$, where ω is a scalar, Q is an ϵ-free quadratic, and $P = \sum_{k=0}^{\$k} P[\![k]\!]\, \epsilon^k$ is a perturbation (it is ill-advised to include ω in P because then it will have log terms).

Scheme: 𝔼_[_] // 𝔼_[_] calls FZip or Zip, which are functionally the same. Zip works by handling the quadratic part and calling PZip for the perturbation-only part. PZip works by iteratively solving the synthesis equation. FZip works by encapsulating coefficients, calling Zip, and back-substituting.

(Alt) In[ ]:=

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\Banff-2607"];
Once[<< KnotTheory`];
```

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

(Alt) In[ ]:=

```
$k = 1;
```

Minor utilities

(Alt) In[ ]:=

```
CCF[ℰ_] := ExpandDenominator@ExpandNumerator@Together[ℰ];
(*CoefficientCanonical Form *)
CCF[ℰ_] := Factor[ℰ]; (*CoefficientCanonical Form *)
CF[ℰ_List] := CF /@ ℰ;
CF[ℰ_ϵSeries] := CF /@ Take[ℰ, $k + 1];
CF[ℰ_] := Module[{F}, Expand@Collect[ℰ, (p | x | π | ξ | g)__, F] /. F → CCF];
(*CF[ℰ_] := Module[
    {vs=Cases[ℰ,(p|x|π|ξ)__,∞]},
    Total[(CCF[#[[2]]] (Times@@vs^#[[1]]))& /@ CoefficientRules[Expand[ℰ], vs]]
  ];*)
CF[ℰ_𝔼] := CF /@ ℰ;
CF[𝔼_sp___[ℰs___]] := CF /@ 𝔼_sp[ℰs];
```

(Alt) In[ ]:=

```
ϵSeries /: S1_ϵSeries ≡ S2_ϵSeries :=
  Length[S1] == Length[S2] ∧ Inner[CF[#1] == CF[#2] &, S1, S2, And];
ϵSeries[] := ϵSeries @@ Table[0, $k + 1];
ϵSeries /: Plus[ss___ϵSeries] := Module[{l = Min[Length /@ {ss}]},
   ϵSeries @@ Total[Take[List @@ #, l] & /@ {ss}]
  ];
(*ϵSeries/: S1_ϵSeries+S2_ϵSeries :=
  ϵSeries@@Table[S1⟦k⟧+S2⟦k⟧,{k,Min[Length@S1,Length@S2]}];*)
ϵSeries /: S1_ϵSeries * S2_ϵSeries := ϵSeries @@
   Table[Sum[S1⟦j + 1⟧ * S2⟦k - j + 1⟧, {j, 0, k}], {k, 0, Min[Length@S1, Length@S2] - 1}];
ϵSeries /: c_ * S_ϵSeries := (c #) & /@ S;
ϵSeries /: ∂_vs___ S_ϵSeries := (s ↦ ∂_vs s) /@ S;
```

The Main Program

Variables and their duals:

(Alt) In[ ]:=

```
{p*, x*, π*, ξ*} = {π, ξ, p, x};
(vs_List)* := (v ↦ v*) /@ vs;
(u_vi__)* := (u*)_vi;
```

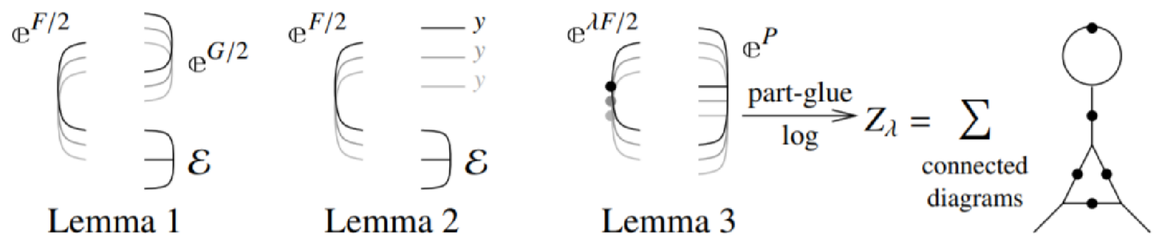
𝔼 operations:

(Alt) In[ ]:=

```
𝔼 /: 𝔼[ω1_, Q1_, P1_] ≡ 𝔼[ω2_, Q2_, P2_] := CF[ω1 == ω2] ∧ CF[Q1 == Q2] ∧ (P1 ≡ P2);
𝔼 /: 𝔼[ω1_, Q1_, P1_] 𝔼[ω2_, Q2_, P2_] := 𝔼[ω1 ω2, Q1 + Q2, P1 + P2];
𝔼_{d1_→r1_}[ℰ1s___] ≡ 𝔼_{d2_→r2_}[ℰ2s___] ^:= (d1 == d2) ∧ (r1 == r2) ∧ (𝔼[ℰ1s] ≡ 𝔼[ℰ2s]);
𝔼_{d1_→r1_}[ℰ1s___] 𝔼_{d2_→r2_}[ℰ2s___] ^:= 𝔼_{(d1⋃d2)→(r1⋃r2)} @@ (𝔼[ℰ1s] 𝔼[ℰ2s]);
𝔼_dr_[ℰs___]_$k_ := 𝔼_dr @@ 𝔼[ℰs]_$k;
```

(Alt) In[ ]:=

```
𝔼_{d1_→r1_}[ℰ1s___] // 𝔼_{d2_→r2_}[ℰ2s___] := Module[{is = r1 ⋂ d2, lvs},
  lvs = Flatten@Table[{x_{v,$@i}, p_{v,$@i}}, {v, 3}, {i, is}];
  𝔼_{(d1⋃Complement[d2,is])→(r2⋃Complement[r1,is])} @@ (Zip_{lvs⋃lvs*}[lvs*.lvs, Times[
      𝔼[ℰ1s] /. Table[(u : (x | p))_{v_,i} → u_{v,$@i}, {i, is}],
      𝔼[ℰ2s] /. Table[(u : (ξ | π))_{v_,i} → u_{v,$@i}, {i, is}]
     ]])
 ]
```



(Alt) In[ ]:=

```
Zip_vs_[𝓕_, ℰ_] := ((*EchoLabel["EZip called with {vs,𝓕,ℰ} = "][{vs,𝓕,ℰ}];*)
   ⟨𝓕, ℰ⟩ // Zip1_vs // EZip23_vs);
Zip_vs_[𝓕_, ℰ_] := ((*EchoLabel["EZip called with {vs,𝓕,ℰ} = "][{vs,𝓕,ℰ}];*)
   ⟨𝓕, ℰ⟩ // Zip1_vs // Zip2_vs // Zip3_vs);
```

Getting rid of the quadratic.

Lemma 1. With convergences left to the reader,

$$\left\langle F : \mathcal{E}\,\mathbb{e}^{\frac{1}{2}\sum_{i,j\in B} G_{ij}z_i z_j} \right\rangle_B = \det(1-GF)^{-1/2} \left\langle F(1-GF)^{-1} : \mathcal{E} \right\rangle_B$$

(Alt) In[ ]:=

```
Zip1_{} = Identity;
Zip1_vs_@⟨𝓕_, 𝔼[ω_, Q_, P_]⟩ := Module[{𝓘, F, G, u, v},
  (*EchoLabel["EZip1 called with {vs,𝓕,ω,Q,P} = "][{vs,𝓕,ω,Q,P}];*)
  𝓘 = IdentityMatrix@Length@vs;
  F = Table[∂_{u,v}𝓕, {u, vs*}, {v, vs*}];
  G = Table[∂_{u,v}Q, {u, vs}, {v, vs}];
  ⟨CF[vs*.F.Inverse[𝓘 - G.F].vs* / 2],
   𝔼[CF@PowerExpand@Factor[ω Det[𝓘 - G.F]^(-1/2)], CF[Q - vs.G.vs / 2], P]⟩
 ]
```

Getting rid of linear terms.

Lemma 2. $\left\langle F : \mathcal{E}\,\mathbb{e}^{\sum_{i\in B} y_i z_i} \right\rangle_B = \mathbb{e}^{\frac{1}{2}\sum_{i,j\in B} F_{ij}y_i y_j} \left\langle F :\ \mathcal{E}|_{z_B\to z_B+F y_B} \right\rangle_B$.

(Alt) In[ ]:=

```
Zip2_{} = Identity;
Zip2_vs_@⟨𝓕_, 𝔼[ω_, Q_, P_]⟩ := Module[{F, Y, u, v},
  (*EchoLabel["EZip2 called with {vs,𝓕,ω,Q,P} = "][{vs,𝓕,ω,Q,P}];*)
  F = Table[∂_{u,v}𝓕, {u, vs*}, {v, vs*}];
  Y = Table[∂_v Q, {v, vs}];
  CF /@ ⟨𝓕, 𝔼[ω, Q - Y.vs + Y.F.Y / 2, P /. Thread[vs → vs + F.Y]]⟩
 ]
```

Dealing with Feynman diagrams.
Lemma 3. With an extra variable λ, $Z_\lambda := \log[\lambda F : \mathbb{e}^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2}\sum_{i,j\in B} F_{ij}\left(\partial_{z_i}\partial_{z_j} Z_\lambda + (\partial_{z_i} Z_\lambda)(\partial_{z_j} Z_\lambda)\right).$$

Note that the power m of λ is at most $k-1+\frac{2k+2}{2}=2k$. We write $Z_\lambda = \sum Z[m]\,\lambda^m$.

(Alt) In[ ]:=

```
Zip3_vs_@⟨𝓕_, 𝔼[ω_, Q_, P_]⟩ := Module[{Z, u, v, m, j},
  (*EchoLabel["Zip3 called with {vs,𝓕,ω,Q,P} = "][{vs,𝓕,ω,Q,P}];*)
  Z[0] = P;
  For[m = 0, m < 2 $k + 3, ++m,
   Z[m + 1] = CF[1/(2 (m + 1))
      Sum[∂_{u*,v*} 𝓕 (∂_{u,v} Z[m] + Sum[(∂_u Z[j]) (∂_v Z[m - j]), {j, 0, m}]), {u, vs}, {v, vs}]]
  ];
  𝔼[ω, Q, CF[Sum[Z[m], {m, 0, 2 $k + 3}] /. Table[v → 0, {v, vs}]]]
 ]
```

(Alt) In[ ]:=

```
EZip23_vs_@⟨𝓕_, 𝔼[ω_, Q_, P_]⟩ := Module[
  {nP, n𝓕, nQ, j = 0, ps, c, t, rr = {(*release rules*)}},
  nP = Total[
      CoefficientRules[#, vs] /.
       (ps_ → c_) :→ (AppendTo[rr, t[++j] → CF@c]; t[j] (Times @@ vs^ps))
     ] & /@ P;
  nQ = Total[CoefficientRules[Q, vs] /.
     (ps_ → c_) :→ (AppendTo[rr, t[++j] → CF@c]; t[j] (Times @@ vs^ps))];
  n𝓕 = Total[CoefficientRules[𝓕, vs*] /.
     (ps_ → c_) :→ (AppendTo[rr, t[++j] → CF@c]; t[j] (Times @@ (vs*)^ps))];
  CF[Expand[⟨n𝓕, 𝔼[ω, nQ, nP]⟩ // Zip2_vs // Zip3_vs] /. rr]
 ]
```

(Alt) In[ ]:=

```
m_{i_,j_→k_} := 𝔼_{{i,j}→{k}}[1, Sum_{ν=1}^{3} (-ξ_{ν,i} π_{ν,j} + (π_{ν,i} + π_{ν,j}) p_{ν,k} + (ξ_{ν,i} + ξ_{ν,j}) x_{ν,k}), ϵSeries[]]
```

In[ ]:= $m_{a,b\to c}$

Out[ ]=

$\mathbb{E}_{\{a,b\}\to\{c\}}[1,\ p_{1,c}\,(\pi_{1,a}+\pi_{1,b}) + p_{2,c}\,(\pi_{2,a}+\pi_{2,b}) + p_{3,c}\,(\pi_{3,a}+\pi_{3,b}) - \pi_{1,b}\,\xi_{1,a} + x_{1,c}\,(\xi_{1,a}+\xi_{1,b}) - \pi_{2,b}\,\xi_{2,a} + x_{2,c}\,(\xi_{2,a}+\xi_{2,b}) - \pi_{3,b}\,\xi_{3,a} + x_{3,c}\,(\xi_{3,a}+\xi_{3,b}),\ \epsilon\mathrm{Series}[0, 0]]$

g2px and px2g

From pensieve://Talks/Geneva-2408/DataConversions.nb

(Alt) In[ ]:=

```
g2px[ℰ_] := CF@Module[{λ}, Expand[ℰ /. g_{ν_,i_,j_} :> λ_ν p_{ν,i} x_{ν,j}] /. λ_^{k_.} :> 1/k!]
```

(Alt) In[ ]:=

```
Zip_{}[ℰ_] := ℰ;
Zip_{ζ_,ζs___}[ℰ_] := (Collect[ℰ // Zip_{ζs}, ζ] /. f_. ζ^{d_.} :> (D[f, {ζ*, d}])) /. ζ* → 0
```

(Alt) In[ ]:=

```
px2g[ℰ_] := CF@Module[{ps, xs, Q, α, β},
   ps = Union[Cases[ℰ, p__, ∞]]; xs = Union[Cases[ℰ, x__, ∞]];
   Q = Sum[p0* x0* g_{p0[[2]],x0[[2]],p0[[3]],x0[[3]]}, {p0, ps}, {x0, xs}];
   Expand[Zip_{ps⋃xs}[ℰ 𝕖^Q] /. g_{α_,β_,i_,j_} :> If[α === β, g_{α,i,j}, 0]]
  ]
```

gRules

From pensieve://Talks/MonteVerita-2604/Theta.nb

(Alt) In[ ]:=

```
gR_{s_,i_,j_} := {
   g_{ν_ jβ_} :> g_{ν j⁺ β} + δ_{jβ}, g_{ν_ iβ_} :> T_ν^s g_{ν i⁺ β} + (1 - T_ν^s) g_{ν j⁺ β} + δ_{iβ},
   g_{ν_ α_ i⁺} :> T_ν^s g_{ν α i} + δ_{α i⁺}, g_{ν_ α_ j⁺} :> g_{ν α j} + (1 - T_ν^s) g_{ν α i} + δ_{α j⁺}
  }
```

Inverse gRules written locally:

(Alt) In[ ]:=

```
igR_{s_,i_,j_} := {
   g_{ν_,j⁺,β_} :> g_{ν,j,β} - δ_{j,β}, g_{ν_,i⁺,β_} :> T_ν^{-s} g_{ν i β} - (T_ν^{-s} - 1) g_{ν j⁺ β} - T_ν^{-s} δ_{iβ},
   g_{ν_,α_,i} :> T_ν^{-s} (g_{ν,α,i⁺} - δ_{α,i⁺}), g_{ν_,α_,j} :> g_{ν,α,j⁺} - (1 - T_ν^s) g_{ν,α,i} - δ_{α,j⁺}
  }
```

Some Knot Theory

(Alt) In[ ]:=

```
Kink_{i_} := CC_3 R_{1,2} // m_{2,3→2} // m_{2,1→i};
K̅i̅n̅k̅_{i_} := CC_3 R̅_{1,2} // m_{1,3→1} // m_{1,2→i}
```

(Alt) In[ ]:=

```
RVK[pd_PD] := Module[{n, xs, x, rots, front = {0}, k},
   n = Length@pd; rots = Table[0, {2 n}];
   xs = Cases[pd, x_X :> { Xp[x[[4]], x[[1]]]  PositiveQ@x
                           Xm[x[[2]], x[[1]]]  True      ];
   For[k = 0, k < 2 n, ++k, If[k == 0 ∨ FreeQ[front, -k],
     front = Flatten[front /. k → (xs /. {
            Xp[k + 1, l_] | Xm[l_, k + 1] :> {l, k + 1, 1 - l},
            Xp[l_, k + 1] | Xm[k + 1, l_] :> (++rots[[l]]; {1 - l, k + 1, l})
          })],
     Cases[front, k | -k] /. {k, -k} :> --rots[[k + 1]];
    ]];
   RVK[xs, rots] ];
RVK[K_] := RVK[PD[K]];
```

(Alt) In[ ]:=

```
rot[i_, 0] := 𝔼_{{}→{i}}[1, 0, ϵSeries[]];
rot[i_, n_] := Module[{j}, If[n > 0, rot[i, n - 1] CC_j, rot[i, n + 1] \overline{CC}_j] // m_{i,j→i}];
```

(Alt) In[ ]:=

```
Z[K_] := Z[RVK@K];
Z[rvk_RVK] := (*Z[rvk] =*)
 Module[{todo, n, rots, ξ, done, st, cx, ξ1, i, j, k, k1, k2, k3},
  {todo, rots} = List @@ rvk;
  AppendTo[rots, 0];
  n = Length[todo];
  ξ = 𝔼_{}→{0}[1, 0, ϵSeries[]];
  done = {0};
  st = Range[0, 2 n + 1];
  While[{} =!= ($M = todo),
   {cx} = MaximalBy[todo, Length[done ⋂ {#[[1]], #[[2]], #[[1]] - 1, #[[2]] - 1}] &, 1];
   {i, j} = List @@ cx;
   ξ1 = Switch[Head[cx],
     Xp, (R_{i,j} OverBar[Kink_k]) // m_{j,k→j},
     Xm, (OverBar[R]_{i,j} Kink_k) // m_{j,k→j}
    ];
   ξ1 = (rot[k, rots[[i]]] ξ1) // m_{k,i→i}; rots[[i]] = 0;
   ξ1 = (ξ1 rot[k, rots[[i + 1]]]) // m_{i,k→i}; rots[[i + 1]] = 0;
   ξ1 = (rot[k, rots[[j]]] ξ1) // m_{k,j→j}; rots[[j]] = 0;
   ξ1 = (ξ1 rot[k, rots[[j + 1]]]) // m_{j,k→j}; rots[[j + 1]] = 0;
   ξ *= ξ1;
   If[MemberQ[done, i], ξ = ξ // m_{i,i+1→i}; st = st /. st[[i + 2]] → st[[i + 1]]];
   If[MemberQ[done, i - 1], ξ = ξ // m_{st[[i]],i→st[[i]]}; st = st /. st[[i + 1]] → st[[i]]];
   If[MemberQ[done, j], ξ = ξ // m_{j,j+1→j}; st = st /. st[[j + 2]] → st[[j + 1]]];
   If[MemberQ[done, j - 1], ξ = ξ // m_{st[[j]],j→st[[j]]}; st = st /. st[[j + 1]] → st[[j]]];
   done = done ⋃ {i - 1, i, j - 1, j};
   todo = DeleteCases[todo, cx]
  ];
  CF /@ (ξ (*/. {x_0→x,y_0→y,a_0→a}*))
 ]
```

ρ_1 testing

From Talks/Bonn-2505/Itype.nb:

(Alt) In[ ]:=

$$\mathcal{L}[X_{i\_,j\_}[s\_]] := T^{s/2}\,\mathbb{E}\Big[\Big\{ x_i\,(p_{i+1}-p_i) + x_j\,(p_{j+1}-p_j) + \left(T^s-1\right) x_i\,(p_{i+1}-p_{j+1}), 0, 0, (\epsilon\, s/2)\times\left(x_i\,(p_i-p_j)\,\left(\left(T^s-1\right) x_i\,p_j + 2\,(1-x_j\,p_j)\right)-1\right) \Big\}\Big]$$

$$\mathcal{L}[C_{i\_}[\varphi\_]] := T^{\varphi/2}\,\mathbb{E}\left[x_i\,(p_{i+1}-p_i) + \epsilon\,\varphi\left(\frac{1}{2}-x_i\,p_i\right)\right]$$

$$\mathcal{L}[K\_] := \text{CF}[\mathcal{L}\ /@\ \text{Features}[K][[2]]]$$

$$\text{vs}[K\_] := \text{Join}\ @@\ \text{Table}[\{p_i,\ x_i\},\ \{i,\ \text{Features}[K][[1]]\}]$$

From Projects/BabyDoPeGDO/Solving to $k=3.nb:

(Alt) In[ ]:=

$$R_{i\_,j\_} := \mathbb{E}_{\{\}\to\{i,j\}}[1,\ (T_1-1)\,x_{1,j}\,(p_{1,i}-p_{1,j}),\ \epsilon\text{Series}[0]];$$
$$\overline{R}_{i\_,j\_} := \mathbb{E}_{\{\}\to\{i,j\}}\left[1,\ \left(T_1^{-1}-1\right)x_{1,j}\,(p_{1,i}-p_{1,j}),\ \epsilon\text{Series}[0]\right];$$
$$CC_{i\_} := \mathbb{E}_{\{\}\to\{i\}}\left[\sqrt{T_1},\ 0,\ \epsilon\text{Series}[0,\ 0]\right];$$
$$\overline{CC}_{i\_} := \mathbb{E}_{\{\}\to\{i\}}\left[\frac{1}{\sqrt{T_1}},\ 0,\ \epsilon\text{Series}[0,\ 0]\right];$$

(Alt) In[ ]:=

```
$k = 0;
```
$R_{i,j}$

$\overline{R}_{i,j}$

CC_i

$\overline{CC}_i$

(Alt) Out[ ]=

$$\mathbb{E}_{\{\}\to\{i,j\}}[1,\ (-1+T_1)\,(p_{1,i}-p_{1,j})\,x_{1,j},\ \epsilon\text{Series}[0]]$$

(Alt) Out[ ]=

$$\mathbb{E}_{\{\}\to\{i,j\}}\left[1,\ \left(-1+\frac{1}{T_1}\right)(p_{1,i}-p_{1,j})\,x_{1,j},\ \epsilon\text{Series}[0]\right]$$

(Alt) Out[ ]=

$$\mathbb{E}_{\{\}\to\{i\}}\left[\sqrt{T_1},\ 0,\ \epsilon\text{Series}[0,\ 0]\right]$$

(Alt) Out[ ]=

$$\mathbb{E}_{\{\}\to\{i\}}\left[\frac{1}{\sqrt{T_1}},\ 0,\ \epsilon\text{Series}[0,\ 0]\right]$$

(Alt) In[ ]:=

$$CC_3\,R_{1,2}\ //\ m_{2,3\to 2}\ //\ m_{2,1\to 1}$$

(Alt) Out[ ]=

$$\mathbb{E}_{\{\}\to\{1\}}\left[\frac{1}{\sqrt{T_1}},\ 0,\ \epsilon\text{Series}[0]\right]$$

(Alt) In[ ]:=

$$\text{CC}_3\,R_{1,2}$$
$$\text{CC}_3\,R_{1,2}\;//\;m_{2,3\to 2}$$

(Alt) Out[ ]=

$$\mathbb{E}_{\{\}\to\{1,2,3\}}\left[\sqrt{T_1},\ (-1+T_1)\ (p_{1,1}-p_{1,2})\ x_{1,2},\ \epsilon\text{Series}[0]\right]$$

(Alt) Out[ ]=

$$\mathbb{E}_{\{\}\to\{1,2\}}\left[\sqrt{T_1},\ (-1+T_1)\ p_{1,1}\,x_{1,2}+(1-T_1)\ p_{1,2}\,x_{1,2},\ \epsilon\text{Series}[0]\right]$$

(Alt) In[ ]:=

$$\begin{aligned}
&\$k=0;\ \{\\
&\ \{\text{"C}\overline{\text{C}}\text{"},\ \left(\text{CC}_1\,\overline{\text{CC}}_2\;//\;m_{1,2\to 1}\right)\equiv\mathbb{E}_{\{\}\to\{1\}}[1,\,0,\,\epsilon\text{Series}[\,]]\},\\
&\ \{\text{"An R1"},\ (\text{CC}_3\,R_{1,2}\;//\;m_{2,3\to 2}\;//\;m_{2,1\to 1})\equiv\left(\overline{\text{CC}}_3\,R_{1,2}\;//\;m_{1,3\to 1}\;//\;m_{1,2\to 1}\right)\},\\
&\ \{\text{"R2b"},\ \left(R_{1,2}\,\overline{R}_{3,4}\;//\;m_{1,3\to 1}\;//\;m_{2,4\to 2}\right)\equiv\mathbb{E}_{\{\}\to\{1,2\}}[1,\,0,\,\epsilon\text{Series}[\,]]\},\\
&\ \{\text{"R3"},\ (R_{1,2}\,R_{4,3}\,R_{5,6}\;//\;m_{1,4\to 1}\;//\;m_{2,5\to 2}\;//\;m_{3,6\to 3})\equiv(R_{2,3}\,R_{4,5}\,R_{1,6}\;//\;m_{1,4\to 1}\;//\;m_{2,5\to 2}\;//\;m_{3,6\to 3})\}\\
&\}
\end{aligned}$$

(Alt) Out[ ]=

$$\left\{\left\{\text{C}\overline{\text{C}},\ \text{True}\right\},\ \{\text{An R1},\ \text{True}\},\ \{\text{R2b},\ \text{True}\},\ \{\text{R3},\ \text{True}\}\right\}$$

In[ ]:=

$$\text{NewBit}[K\_]:=\text{Module}\left[\{\text{Alex}=\text{Alexander}[K][T_1]\},\ T_1^3\,\frac{\text{Alex}^2}{T_1-1}\,Z[K][\![3,\,2]\!]\;//\;\text{Factor}\right]$$

In[ ]:=

$$\text{NewBit}\;/@\;\text{AllKnots}[\{3,\,5\}]$$

KnotTheory: Loading precomputed data in PD4Knots\`.

Out[ ]=

$$\left\{2-T_1+T_1^2,\ (1+T_1)\ \left(1-3\,T_1+T_1^2\right),\ \frac{4-3\,T_1+5\,T_1^2-3\,T_1^3+3\,T_1^4-T_1^5+T_1^6}{T_1^2},\ 9-11\,T_1+7\,T_1^2-T_1^3\right\}$$

In pensieve://Projects/BabyDoPeGDO/Solving to $k=3.nb that was

$$\left\{2-T+T^2,\ (1+T)\ \left(1-3\,T+T^2\right),\ \frac{4-3\,T+5\,T^2-3\,T^3+3\,T^4-T^5+T^6}{T^2},\ 9-11\,T+7\,T^2-T^3\right\}$$

And now θ

(Alt) In[ ]:=

$$\begin{aligned}
&T_3=T_1\,T_2;\\
&\mathcal{L}[X_{i_,j_}[s_]]:=T_3^s\,\mathbb{E}\Big[\text{CF}@\Big\{\\
&\qquad \sum_{\nu=1}^{3}\left(x_{\nu i}\,(p_{\nu i^+}-p_{\nu i})+x_{\nu j}\,(p_{\nu j^+}-p_{\nu j})+\left(T_\nu^s-1\right)x_{\nu i}\,(p_{\nu i^+}-p_{\nu j^+})\right),\\
&\qquad \left(T_1^s-1\right)p_{3j}\,x_{1i}\left(T_2^s\,x_{2i}-x_{2j}\right),\\
&\qquad \epsilon\,s\left(T_3^s-1\right)p_{1j}\,(p_{2i}-p_{2j})\,x_{3i}\Big/\left(T_2^s-1\right),\\
&\qquad \epsilon\,s\Big(1/2+T_2^s\,p_{1i}\,p_{2j}\,x_{1i}\,x_{2i}-p_{1i}\,p_{2j}\,x_{1i}\,x_{2j}-p_{3i}\,x_{3i}-\left(T_2^s-1\right)p_{2j}\,p_{3i}\,x_{2i}\,x_{3i}+\\
&\qquad\quad \left(T_3^s-1\right)p_{2j}\,p_{3j}\,x_{2i}\,x_{3i}+2\,p_{2j}\,p_{3i}\,x_{2j}\,x_{3i}+p_{1i}\,p_{3j}\,x_{1i}\,x_{3j}-p_{2i}\,p_{3j}\,x_{2i}\,x_{3j}-T_2^s\,p_{2j}\,p_{3j}\,x_{2i}\,x_{3j}+\\
&\qquad\quad \Big(\left(T_1^s-1\right)p_{1j}\,x_{1i}\left(T_2^{2\,s}\,p_{2j}\,x_{2i}-T_2^s\,p_{2j}\,x_{2j}-\left(T_2^s+1\right)\left(T_3^s-1\right)p_{3j}\,x_{3i}+T_2^s\,p_{3j}\,x_{3j}\right)+\\
&\qquad\qquad \left(T_3^s-1\right)p_{3j}\,x_{3i}\left(1-T_2^s\,p_{1i}\,x_{1i}+p_{2i}\,x_{2j}+\left(T_2^s-2\right)p_{2j}\,x_{2j}\right)\Big)\Big/\left(T_2^s-1\right)\Big)\Big\}\Big]
\end{aligned}$$

(Alt) In[∘]:=

```
ℒ[C_i_[φ_]] := T_3^φ 𝔼[{∑_{ν=1}^3 x_{νi} (p_{νi⁺} − p_{νi}), ∈ φ (p_{3i} x_{3i} − 1 / 2)}]
```

In[∘]:=

```
R_{i_,j_} := CF[ℒ[X_{i,j}[1]] /. ω_ 𝔼[{Q_, P0_, P1a_, P1b_}] ⧴ 𝔼_{{}→{i,j}}[
    ω, ∑_{ν=1}^3 ((T_ν − 1) x_{ν,j} (p_{ν,i} − p_{ν,j})),
    ∈Series[P0, P1a + P1b /. ∈ → 1]
  ]];
R̄_{i_,j_} := CF[ℒ[X_{i,j}[−1]] /. ω_ 𝔼[{Q_, P0_, P1a_, P1b_}] ⧴ 𝔼_{{}→{i,j}}[
    ω, ∑_{ν=1}^3 ((T_ν^{−1} − 1) x_{ν,j} (p_{ν,i} − p_{ν,j})),
    ∈Series[P0, P1a + P1b /. ∈ → 1]
  ]];
CC_i_ := CF[ℒ[C_i[1]] /. ω_ 𝔼[{Q_, P_}] ⧴ 𝔼_{{}→{i}}[
    ω, 0, ∈Series[0, P /. ∈ → 1]
  ]];
C̄C̄_i_ := CF[ℒ[C_i[−1]] /. ω_ 𝔼[{Q_, P_}] ⧴ 𝔼_{{}→{i}}[
    ω, 0, ∈Series[0, P /. ∈ → 1]
  ]];
```

In[∘]:=

```
R̄_{i,j} // CF
```

Out[∘]=

$$\mathbb{E}_{\{\}\to\{i,j\}}\Big[\frac{1}{T_1 T_2},\ -\frac{(-1+T_1)\,p_{1,i}\,x_{1,j}}{T_1}+\frac{(-1+T_1)\,p_{1,j}\,x_{1,j}}{T_1}-\frac{(-1+T_2)\,p_{2,i}\,x_{2,j}}{T_2}+\frac{(-1+T_2)\,p_{2,j}\,x_{2,j}}{T_2}-\frac{(-1+T_1T_2)\,p_{3,i}\,x_{3,j}}{T_1T_2}+\frac{(-1+T_1T_2)\,p_{3,j}\,x_{3,j}}{T_1T_2},$$

$$\epsilon\text{Series}\Big[-\frac{(-1+T_1)\,p_{3,j}\,x_{1,i}\,x_{2,i}}{T_1T_2}+\frac{(-1+T_1)\,p_{3,j}\,x_{1,i}\,x_{2,j}}{T_1},$$

$$-\frac{1}{2}-\frac{p_{1,i}\,p_{2,j}\,x_{1,i}\,x_{2,i}}{T_2}-\frac{(-1+T_1)\,p_{1,j}\,p_{2,j}\,x_{1,i}\,x_{2,i}}{T_1\,(-1+T_2)\,T_2}+p_{1,i}\,p_{2,j}\,x_{1,i}\,x_{2,j}+\frac{(-1+T_1)\,p_{1,j}\,p_{2,j}\,x_{1,i}\,x_{2,j}}{T_1\,(-1+T_2)}-$$

$$\frac{(-1+T_1T_2)\,p_{1,j}\,p_{2,i}\,x_{3,i}}{T_1\,(-1+T_2)}+\frac{(-1+T_1T_2)\,p_{1,j}\,p_{2,j}\,x_{3,i}}{T_1\,(-1+T_2)}+p_{3,i}\,x_{3,i}-\frac{(-1+T_1T_2)\,p_{3,j}\,x_{3,i}}{T_1\,(-1+T_2)}+$$

$$\frac{(-1+T_1T_2)\,p_{1,i}\,p_{3,j}\,x_{1,i}\,x_{3,i}}{T_1\,(-1+T_2)\,T_2}-\frac{(-1+T_1)\,(1+T_2)\,(-1+T_1T_2)\,p_{1,j}\,p_{3,j}\,x_{1,i}\,x_{3,i}}{T_1^2\,(-1+T_2)\,T_2}-$$

$$\frac{(-1+T_2)\,p_{2,j}\,p_{3,i}\,x_{2,i}\,x_{3,i}}{T_2}+\frac{(-1+T_1T_2)\,p_{2,j}\,p_{3,j}\,x_{2,i}\,x_{3,i}}{T_1T_2}-2\,p_{2,j}\,p_{3,i}\,x_{2,j}\,x_{3,i}-$$

$$\frac{(-1+T_1T_2)\,p_{2,i}\,p_{3,j}\,x_{2,j}\,x_{3,i}}{T_1\,(-1+T_2)}+\frac{(-1+2T_2)\,(-1+T_1T_2)\,p_{2,j}\,p_{3,j}\,x_{2,j}\,x_{3,i}}{T_1\,(-1+T_2)\,T_2}-$$

$$p_{1,i}\,p_{3,j}\,x_{1,i}\,x_{3,j}-\frac{(-1+T_1)\,p_{1,j}\,p_{3,j}\,x_{1,i}\,x_{3,j}}{T_1\,(-1+T_2)}+p_{2,i}\,p_{3,j}\,x_{2,i}\,x_{3,j}+\frac{p_{2,j}\,p_{3,j}\,x_{2,i}\,x_{3,j}}{T_2}\Big]\Big]$$

In[◦]:= CC_1 // CF

Out[◦]=

$$\mathbb{E}_{\{\}\to\{1\}}\left[T_1 T_2,\ 0,\ \epsilon\text{Series}\left[0,\ -\frac{1}{2}+p_{3,1}\,x_{3,1}\right]\right]$$

In[◦]:= $\overline{CC}_1$ // CF

Out[◦]=

$$\mathbb{E}_{\{\}\to\{1\}}\left[\frac{1}{T_1 T_2},\ 0,\ \epsilon\text{Series}\left[0,\ \frac{1}{2}-p_{3,1}\,x_{3,1}\right]\right]$$

In[◦]:= $k = 0; {

{"C$\overline{\text{C}}$", ($CC_1\,\overline{CC}_2$ // $m_{1,2\to1}$) ≡ $\mathbb{E}_{\{\}\to\{1\}}$[1, 0, ϵSeries[]]},

{"An R1", ($CC_3\,R_{1,2}$ // $m_{2,3\to2}$ // $m_{2,1\to1}$) ≡ ($\overline{CC}_3\,R_{1,2}$ // $m_{1,3\to1}$ // $m_{1,2\to1}$)},

{"R2b", ($R_{1,2}\,\overline{R}_{3,4}$ // $m_{1,3\to1}$ // $m_{2,4\to2}$) ≡ $\mathbb{E}_{\{\}\to\{1,2\}}$[1, 0, ϵSeries[]]},

{"R3", ($R_{1,2}\,R_{4,3}\,R_{5,6}$ // $m_{1,4\to1}$ // $m_{2,5\to2}$ // $m_{3,6\to3}$) ≡ ($R_{2,3}\,R_{4,5}\,R_{1,6}$ // $m_{1,4\to1}$ // $m_{2,5\to2}$ // $m_{3,6\to3}$)}

}

Out[◦]=

$$\Big\{\{C\overline{C},\ \text{True}\},$$

$$\Big\{\text{An R1},\ \frac{(-1+T_1)(-1+T_2)(1+T_2)\,p_{3,1}\,x_{1,1}\,x_{2,1}}{T_2} == (-1+T_1)\,T_1\,(-1+T_2)(1+T_2)\,p_{3,1}\,x_{1,1}\,x_{2,1}\Big\},$$

$$\Big\{\text{R2b},\ \frac{(-1+T_1)(-1+T_1T_2)\,p_{3,1}\,x_{1,1}\,x_{2,1}}{T_1 T_2}+(-1+T_1)(-1+T_2)\,p_{3,2}\,x_{1,1}\,x_{2,1}+\frac{(-1+T_1)^2\,T_2\,p_{3,2}\,x_{1,2}\,x_{2,1}}{T_1}-\frac{(-1+T_1)(-1+T_1T_2)\,p_{3,1}\,x_{1,1}\,x_{2,2}}{T_1}+\frac{(-1+T_1)(-1+T_2)(1+2T_2)\,p_{3,2}\,x_{1,1}\,x_{2,2}}{T_2}+\frac{(-1+T_1)^2\left(-1-T_2+T_2^2\right)p_{3,2}\,x_{1,2}\,x_{2,2}}{T_1T_2} == 0\Big\},$$

$$\Big\{\text{R3},\ (-1+T_1)\,T_2\,p_{3,2}\,x_{1,1}\,x_{2,1}+(-1+T_1)\,T_2\,p_{3,3}\,x_{1,1}\,x_{2,1}-(-1+T_1)^2\,T_1\,T_2\,p_{3,2}\,x_{1,3}\,x_{2,1}+(1-T_1)\,p_{3,2}\,x_{1,1}\,x_{2,2}-(-1+T_1)\,T_2\,(-1+T_1T_2)\,p_{3,1}\,x_{1,2}\,x_{2,2}+(-1+T_1)\,T_1\,T_2^2\,p_{3,3}\,x_{1,2}\,x_{2,2}+(-1+T_1)^2\,T_1\,p_{3,2}\,x_{1,3}\,x_{2,2}-(-1+T_1)(-1+T_2)^2(1+T_2)\,p_{3,2}\,x_{1,1}\,x_{2,3}-(-1+T_1)\,T_2\,p_{3,3}\,x_{1,1}\,x_{2,3}+(-1+T_1)(-1+T_1T_2)\,p_{3,1}\,x_{1,2}\,x_{2,3}-(-1+T_1)\,T_1\,T_2\,p_{3,3}\,x_{1,2}\,x_{2,3}+(-1+T_1)^2\,T_1\,(-1+T_2)^2(1+T_2)\,p_{3,2}\,x_{1,3}\,x_{2,3} == -\big((-1+T_1)\,T_2\,(-2+T_1T_2)\,p_{3,2}\,x_{1,1}\,x_{2,1}\big)+(-1+T_1)\,T_1\,T_2^2\,p_{3,3}\,x_{1,1}\,x_{2,1}+(-1+T_1)^2\,T_2\,(-1+T_1T_2)\,p_{3,2}\,x_{1,2}\,x_{2,1}-(-1+T_1)^2\,T_1\,T_2^2\,p_{3,3}\,x_{1,2}\,x_{2,1}+(-1+T_1)\left(-1+T_2-T_2^2-T_1T_2^2+T_1T_2^3\right)p_{3,2}\,x_{1,1}\,x_{2,2}-(-1+T_1)\,T_1\,(-1+T_2)\,T_2^2\,p_{3,3}\,x_{1,1}\,x_{2,2}-(-1+T_1)^2(-1+T_2)\,T_2\,(-1+T_1T_2)\,p_{3,2}\,x_{1,2}\,x_{2,2}+(-1+T_1)\,T_1\,T_2^2\,(2-T_1-T_2+T_1T_2)\,p_{3,3}\,x_{1,2}\,x_{2,2}+(-1+T_1)(-1+T_1T_2)\,p_{3,2}\,x_{1,1}\,x_{2,3}-(-1+T_1)\,T_1\,T_2\,p_{3,3}\,x_{1,1}\,x_{2,3}-(-1+T_1)^2(-1+T_1T_2)\,p_{3,2}\,x_{1,2}\,x_{2,3}+(-2+T_1)(-1+T_1)\,T_1\,T_2\,p_{3,3}\,x_{1,2}\,x_{2,3}\Big\}\Big\}$$

In[∘]:=
```
$k = 0;
R1,2 // CF
R̄3,4 // CF
R2LHS = (R1,2 R̄3,4 // m1,3→1 // m2,4→2)
```

Out[∘]=

$$\mathbb{E}_{\{\}\to\{1,2\}}[T_1 T_2,\ (-1+T_1)\, p_{1,1} x_{1,2} + (1-T_1)\, p_{1,2} x_{1,2} + (-1+T_2)\, p_{2,1} x_{2,2} + (1-T_2)\, p_{2,2} x_{2,2} + (-1+T_1 T_2)\, p_{3,1} x_{3,2} + (1-T_1 T_2)\, p_{3,2} x_{3,2},\ \epsilon\text{Series}[(-1+T_1)\, T_2\, p_{3,2} x_{1,1} x_{2,1} + (1-T_1)\, p_{3,2} x_{1,1} x_{2,2}]]$$

Out[∘]=

$$\mathbb{E}_{\{\}\to\{3,4\}}\left[\frac{1}{T_1 T_2},\ -\frac{(-1+T_1)\, p_{1,3} x_{1,4}}{T_1} + \frac{(-1+T_1)\, p_{1,4} x_{1,4}}{T_1} - \frac{(-1+T_2)\, p_{2,3} x_{2,4}}{T_2} + \frac{(-1+T_2)\, p_{2,4} x_{2,4}}{T_2} - \frac{(-1+T_1 T_2)\, p_{3,3} x_{3,4}}{T_1 T_2} + \frac{(-1+T_1 T_2)\, p_{3,4} x_{3,4}}{T_1 T_2},\ \epsilon\text{Series}\left[-\frac{(-1+T_1)\, p_{3,4} x_{1,3} x_{2,3}}{T_1 T_2} + \frac{(-1+T_1)\, p_{3,4} x_{1,3} x_{2,4}}{T_1}\right]\right]$$

Out[∘]=

$$\mathbb{E}_{\{\}\to\{1,2\}}\left[1,\ 0,\ \epsilon\text{Series}\left[\frac{(-1+T_1)\,(-1+T_1 T_2)\, p_{3,1} x_{1,1} x_{2,1}}{T_1 T_2} + (-1+T_1)\,(-1+T_2)\, p_{3,2} x_{1,1} x_{2,1} + \frac{(-1+T_1)^2\, T_2\, p_{3,2} x_{1,2} x_{2,1}}{T_1} - \frac{(-1+T_1)\,(-1+T_1 T_2)\, p_{3,1} x_{1,1} x_{2,2}}{T_1} + \frac{(-1+T_1)\,(-1+T_2)\,(1+2 T_2)\, p_{3,2} x_{1,1} x_{2,2}}{T_2} + \frac{(-1+T_1)^2 \left(-1-T_2+T_2^2\right) p_{3,2} x_{1,2} x_{2,2}}{T_1 T_2}\right]\right]$$

In[∘]:=
```
R2LHS0 = R1,2 R̄3,4 // m1,3→1
```

Out[∘]=

$$\mathbb{E}_{\{\}\to\{1,2,4\}}\left[1,\ (-1+T_1)\, p_{1,1} x_{1,2} + (1-T_1)\, p_{1,2} x_{1,2} - \frac{(-1+T_1)\, p_{1,1} x_{1,4}}{T_1} + \frac{(-1+T_1)\, p_{1,4} x_{1,4}}{T_1} + (-1+T_2)\, p_{2,1} x_{2,2} + (1-T_2)\, p_{2,2} x_{2,2} - \frac{(-1+T_2)\, p_{2,1} x_{2,4}}{T_2} + \frac{(-1+T_2)\, p_{2,4} x_{2,4}}{T_2} + (-1+T_1 T_2)\, p_{3,1} x_{3,2} + (1-T_1 T_2)\, p_{3,2} x_{3,2} - \frac{(-1+T_1 T_2)\, p_{3,1} x_{3,4}}{T_1 T_2} + \frac{(-1+T_1 T_2)\, p_{3,4} x_{3,4}}{T_1 T_2},\ \epsilon\text{Series}\left[(-1+T_1)\, T_2\, p_{3,2} x_{1,1} x_{2,1} - \frac{(-1+T_1)\, p_{3,4} x_{1,1} x_{2,1}}{T_1 T_2} + \frac{(-1+T_1)^2\, T_2\, p_{3,2} x_{1,4} x_{2,1}}{T_1} + (1-T_1)\, p_{3,2} x_{1,1} x_{2,2} - \frac{(-1+T_1)^2\, p_{3,2} x_{1,4} x_{2,2}}{T_1} + (-1+T_1)\,(-1+T_2)\, p_{3,2} x_{1,1} x_{2,4} + \frac{(-1+T_1)\, p_{3,4} x_{1,1} x_{2,4}}{T_1} + \frac{(-1+T_1)^2\,(-1+T_2)\, p_{3,2} x_{1,4} x_{2,4}}{T_1}\right]\right]$$

In[•]:= **CF[R2LHS0[[2]] + R2LHS0[[3, 1]]]**

Out[•]=

$$(-1+T_1)\,p_{1,1}\,x_{1,2}+(1-T_1)\,p_{1,2}\,x_{1,2}-\frac{(-1+T_1)\,p_{1,1}\,x_{1,4}}{T_1}+\frac{(-1+T_1)\,p_{1,4}\,x_{1,4}}{T_1}+$$
$$(-1+T_1)\,T_2\,p_{3,2}\,x_{1,1}\,x_{2,1}-\frac{(-1+T_1)\,p_{3,4}\,x_{1,1}\,x_{2,1}}{T_1\,T_2}+\frac{(-1+T_1)^2\,T_2\,p_{3,2}\,x_{1,4}\,x_{2,1}}{T_1}+$$
$$(-1+T_2)\,p_{2,1}\,x_{2,2}+(1-T_2)\,p_{2,2}\,x_{2,2}+(1-T_1)\,p_{3,2}\,x_{1,1}\,x_{2,2}-\frac{(-1+T_1)^2\,p_{3,2}\,x_{1,4}\,x_{2,2}}{T_1}-$$
$$\frac{(-1+T_2)\,p_{2,1}\,x_{2,4}}{T_2}+\frac{(-1+T_2)\,p_{2,4}\,x_{2,4}}{T_2}+(-1+T_1)\,(-1+T_2)\,p_{3,2}\,x_{1,1}\,x_{2,4}+$$
$$\frac{(-1+T_1)\,p_{3,4}\,x_{1,1}\,x_{2,4}}{T_1}+\frac{(-1+T_1)^2\,(-1+T_2)\,p_{3,2}\,x_{1,4}\,x_{2,4}}{T_1}+(-1+T_1\,T_2)\,p_{3,1}\,x_{3,2}+$$
$$(1-T_1\,T_2)\,p_{3,2}\,x_{3,2}-\frac{(-1+T_1\,T_2)\,p_{3,1}\,x_{3,4}}{T_1\,T_2}+\frac{(-1+T_1\,T_2)\,p_{3,4}\,x_{3,4}}{T_1\,T_2}$$