Pensieve header: The 3-Heisenberg Formalism for Θ. Engine modified from pensieve://Projects/BabyDoPeGDO/Solving_to_k=3.nb, px formulas modified from pensieve://Talks/Bonn-2505/.

$\mathbb{E}$[ω, Q, P_ϵSeries] represents ωe^{Q+P}, where ω is a scalar, Q is an ϵ-free quadratic, and $P=\sum_{k=0}^{\$k} P[[k]]\,\epsilon^k$ is a perturbation (it is ill-advised to include ω in P because then it will have log terms).

Scheme: $\mathbb{E}$_[_] // $\mathbb{E}$_[_] calls FZip or Zip, which are functionally the same. Zip works by handling the quadratic part and calling PZip for the perturbation-only part. PZip works by iteratively solving the synthesis equation. FZip works by encapsulating coefficients, calling Zip, and back-substituting.

In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\Banff-2607"];
Once[<< KnotTheory`];
```

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

In[∘]:=
```
$k = 1;
```

Minor utilities

In[∘]:=
```
CCF[ℰ_] := ExpandDenominator@ExpandNumerator@Together[ℰ];
(*CoefficientCanonical Form *)
CF[ℰ_List] := CF /@ ℰ;
CF[ℰ_ϵSeries] := CF /@ Take[ℰ, $k + 1];
CF[ℰ_] := Module[{F}, Expand@Collect[ℰ, (p | x | π | ξ)__, F] /. F → CCF];
(*CF[ℰ_] := Module[
   {vs=Cases[ℰ,(p|x|π|ξ)__,∞]},
   Total[(CCF[#[[2]]] (Times@@vs^#[[1]]))& /@ CoefficientRules[Expand[ℰ], vs]]
  ];*)
CF[ℰ_𝔼] := CF /@ ℰ;
CF[𝔼sp___[ℰs___]] := CF /@ 𝔼sp[ℰs];
```

In[∘]:=
```
ϵSeries /: S1_ϵSeries ≡ S2_ϵSeries :=
  Length[S1] == Length[S2] ∧ Inner[CF[#1] == CF[#2] &, S1, S2, And];
ϵSeries[] := ϵSeries @@ Table[0, $k + 1];
ϵSeries /: Plus[ss___ϵSeries] := Module[{l = Min[Length /@ {ss}]},
   ϵSeries @@ Total[Take[List @@ #, l] & /@ {ss}]
  ];
(*ϵSeries/: S1_ϵSeries+S2_ϵSeries :=
  ϵSeries@@Table[S1[[k]]+S2[[k]],{k,Min[Length@S1,Length@S2]}];*)
ϵSeries /: S1_ϵSeries * S2_ϵSeries := ϵSeries @@
   Table[Sum[S1[[j + 1]] * S2[[k - j + 1]], {j, 0, k}], {k, 0, Min[Length@S1, Length@S2] - 1}];
ϵSeries /: c_ * S_ϵSeries := (c #) & /@ S;
ϵSeries /: ∂vs___ S_ϵSeries := (s ↦ ∂vs s) /@ S;
```

The Main Program

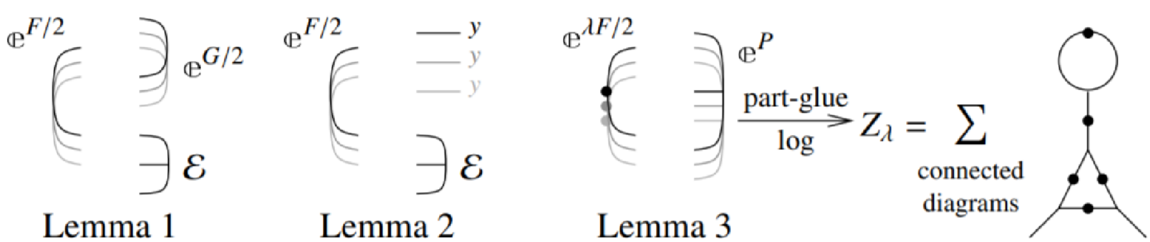
Variables and their duals:

```
In[∘]:= {p*, x*, π*, ξ*} = {π, ξ, p, x};
(vs_List)* := (v ↦ v*) /@ vs;
(u_ vi__)* := (u*) vi;
```

𝔼 operations:

```
In[∘]:= 𝔼 /: 𝔼[ω1_, Q1_, P1_] ≡ 𝔼[ω2_, Q2_, P2_] := CF[ω1 == ω2] ∧ CF[Q1 == Q2] ∧ (P1 ≡ P2);
𝔼 /: 𝔼[ω1_, Q1_, P1_] 𝔼[ω2_, Q2_, P2_] := 𝔼[ω1 ω2, Q1 + Q2, P1 + P2];
𝔼d1_→r1_[ℰ1s___] ≡ 𝔼d2_→r2_[ℰ2s___] ^:= (d1 == d2) ∧ (r1 == r2) ∧ (𝔼[ℰ1s] ≡ 𝔼[ℰ2s]);
𝔼d1_→r1_[ℰ1s___] 𝔼d2_→r2_[ℰ2s___] ^:= 𝔼(d1⋃d2)→(r1⋃r2) @@ (𝔼[ℰ1s] 𝔼[ℰ2s]);
𝔼dr_[ℰs___]$k_ := 𝔼dr @@ 𝔼[ℰs]$k;
```

```
In[∘]:= 𝔼d1_→r1_[ℰ1s___] // 𝔼d2_→r2_[ℰ2s___] := Module[{is = r1⋂d2, lvs},
  lvs = Flatten@Table[{xv,$@i, pv,$@i}, {v, 3}, {i, is}];
  𝔼(d1⋃Complement[d2,is])→(r2⋃Complement[r1,is]) @@ (Ziplvs⋃lvs*[lvs*.lvs, Times[
      𝔼[ℰ1s] /. Table[(u : x | p)v_,i → uv,$@i, {i, is}],
      𝔼[ℰ2s] /. Table[(u : ξ | π)v_,i → uv,$@i, {i, is}]
    ]])
 ]
```




```
In[∘]:= Zipvs_[ℱ_, ℰ_] := ((*EchoLabel["EZip called with {vs,ℱ,ℰ} = "][{vs,ℱ,ℰ}];*)
   ⟨ℱ, ℰ⟩ // Zip1vs // EZip23vs);
Zipvs_[ℱ_, ℰ_] := ((*EchoLabel["EZip called with {vs,ℱ,ℰ} = "][{vs,ℱ,ℰ}];*)
   ⟨ℱ, ℰ⟩ // Zip1vs // Zip2vs // Zip3vs);
```

Getting rid of the quadratic.

Lemma 1. With convergences left to the reader,

$$\left\langle F : \mathcal{E} e^{\frac{1}{2}\sum_{i,j\in B} G_{ij} z_i z_j} \right\rangle_B = \det(1-GF)^{-1/2} \left\langle F(1-GF)^{-1} : \mathcal{E} \right\rangle_B$$

In[◦]:=
```
Zip1{} = Identity;
Zip1vs_ @⟨ℱ_, 𝔼[ω_, Q_, P_]⟩ := Module[{ℐ, F, G, u, v},
  (*EchoLabel["EZip1 called with {vs,ℱ,ω,Q,P} = "][{vs,ℱ,ω,Q,P}];*)
  ℐ = IdentityMatrix@Length@vs;
  F = Table[∂u,v ℱ, {u, vs*}, {v, vs*}];
  G = Table[∂u,v Q, {u, vs}, {v, vs}];
  ⟨CF[vs*.F.Inverse[ℐ - G.F].vs* / 2],
   𝔼[CF@PowerExpand@Factor[ω Det[ℐ - G.F]^(-1/2)], CF[Q - vs.G.vs / 2], P]⟩
 ]
```

Getting rid of linear terms.

Lemma 2. $\left\langle F:\mathcal{E}\,\mathbb{e}^{\sum_{i\in B}y_iz_i}\right\rangle_B=\mathbb{e}^{\frac{1}{2}\sum_{i,j\in B}F_{ij}y_iy_j}\left\langle F:\ \mathcal{E}|_{z_B\to z_B+Fy_B}\right\rangle_B.$

In[◦]:=
```
Zip2{} = Identity;
Zip2vs_ @⟨ℱ_, 𝔼[ω_, Q_, P_]⟩ := Module[{F, Y, u, v},
  (*EchoLabel["EZip2 called with {vs,ℱ,ω,Q,P} = "][{vs,ℱ,ω,Q,P}];*)
  F = Table[∂u,v ℱ, {u, vs*}, {v, vs*}];
  Y = Table[∂v Q, {v, vs}];
  CF /@ ⟨ℱ, 𝔼[ω, Q - Y.vs + Y.F.Y / 2, P /. Thread[vs → vs + F.Y]]⟩
 ]
```

Dealing with Feynman diagrams.

Lemma 3. With an extra variable λ, $Z_\lambda\coloneqq\log[\lambda F:\mathbb{e}^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0=P\quad\text{and}\quad\partial_\lambda Z_\lambda=\frac{1}{2}\sum_{i,j\in B}F_{ij}\left(\partial_{z_i}\partial_{z_j}Z_\lambda+(\partial_{z_i}Z_\lambda)(\partial_{z_j}Z_\lambda)\right).$$

Note that the power m of λ is at most $k-1+\frac{2k+2}{2}=2k$. We write $Z_\lambda=\sum Z[m]\,\lambda^m$.

In[◦]:=
```
Zip3vs_ @⟨ℱ_, 𝔼[ω_, Q_, P_]⟩ := Module[{Z, u, v, m, j},
  (*EchoLabel["EZip3 called with {vs,ℱ,ω,Q,P} = "][{vs,ℱ,ω,Q,P}];*)
  Z[0] = P;
  For[m = 0, m < 2 $k, ++m,
   Z[m + 1] = CF[1/(2 (m + 1))
      Sum[∂u*,v* ℱ (∂u,v Z[m] + Sum[(∂u Z[j]) (∂v Z[m - j]), {j, 0, m}]), {u, vs}, {v, vs}]]
  ];
  𝔼[ω, Q, CF[Sum[Z[m], {m, 0, 2 $k}] /. Table[v → 0, {v, vs}]]]
 ]
```

In[∘]:=
```
EZip23vs_@⟨ℱ_, 𝔼[ω_, Q_, P_]⟩ := Module[
  {nP, nℱ, nQ, j = 0, ps, c, t, rr = {(*release rules*)}},
  nP = Total[
      CoefficientRules[#, vs] /.
        (ps_ → c_) :→ (AppendTo[rr, t[++j] → CF@c]; t[j] (Times @@ vs^ps))
    ] & /@ P;
  nQ = Total[CoefficientRules[Q, vs] /.
      (ps_ → c_) :→ (AppendTo[rr, t[++j] → CF@c]; t[j] (Times @@ vs^ps))];
  nℱ = Total[CoefficientRules[ℱ, vs*] /.
      (ps_ → c_) :→ (AppendTo[rr, t[++j] → CF@c]; t[j] (Times @@ (vs*)^ps))];
  CF[Expand[⟨nℱ, 𝔼[ω, nQ, nP]⟩ // Zip2vs // Zip3vs] /. rr]
 ]
```

In[∘]:=
$$m_{i\_,j\_\to k\_} := \mathbb{E}_{\{i,j\}\to\{k\}}\left[1, \sum_{\nu=1}^{3} \left(-\xi_{\nu,i}\,\pi_{\nu,j} + (\pi_{\nu,i}+\pi_{\nu,j})\,p_{\nu,k} + (\xi_{\nu,i}+\xi_{\nu,j})\,x_{\nu,k}\right),\ \epsilon\text{Series}[]\right]$$

In[∘]:= $m_{a,b\to c}$

Out[∘]=
$$\mathbb{E}_{\{a,b\}\to\{c\}}[1,\ p_{1,c}\,(\pi_{1,a}+\pi_{1,b}) + p_{2,c}\,(\pi_{2,a}+\pi_{2,b}) + p_{3,c}\,(\pi_{3,a}+\pi_{3,b}) - \pi_{1,b}\,\xi_{1,a} + x_{1,c}\,(\xi_{1,a}+\xi_{1,b}) - \pi_{2,b}\,\xi_{2,a} + x_{2,c}\,(\xi_{2,a}+\xi_{2,b}) - \pi_{3,b}\,\xi_{3,a} + x_{3,c}\,(\xi_{3,a}+\xi_{3,b}),\ \epsilon\text{Series}[0, 0]]$$

Some Knot Theory

In[∘]:=
$$\text{Kink}_{i\_} := CC_3\, R_{1,2} \,//\, m_{2,3\to 2} \,//\, m_{2,1\to i};$$
$$\overline{\text{Kink}}_{i\_} := CC_3\, \overline{R}_{1,2} \,//\, m_{1,3\to 1} \,//\, m_{1,2\to i}$$

In[∘]:=
```
RVK[pd_PD] := Module[{n, xs, x, rots, front = {0}, k},
   n = Length@pd; rots = Table[0, {2 n}];
   xs = Cases[pd, x_X :→ { Xp[x[[4]], x[[1]]]  PositiveQ@x
                           Xm[x[[2]], x[[1]]]  True       ];
   For[k = 0, k < 2 n, ++k, If[k == 0 ∨ FreeQ[front, -k],
     front = Flatten[front /. k → (xs /. {
            Xp[k + 1, l_] | Xm[l_, k + 1] :→ {l, k + 1, 1 - l},
            Xp[l_, k + 1] | Xm[k + 1, l_] :→ (++rots[[l]]; {1 - l, k + 1, l})
          })],
     Cases[front, k | -k] /. {k, -k} :→ --rots[[k + 1]];
    ]];
   RVK[xs, rots] ];
RVK[K_] := RVK[PD[K]];
```

In[∘]:=
```
rot[i_, 0] := 𝔼{}→{i}[1, 0, ϵSeries[]];
rot[i_, n_] := Module[{j}, If[n > 0, rot[i, n - 1] CCj, rot[i, n + 1] CC̄j] // mi,j→i];
```

In[∘]:=
```
Z[K_] := Z[RVK@K];
Z[rvk_RVK] := (*Z[rvk] =*)
 Module[{todo, n, rots, ξ, done, st, cx, ξ1, i, j, k, k1, k2, k3},
  {todo, rots} = List @@ rvk;
  AppendTo[rots, 0];
  n = Length[todo];
  ξ = 𝔼_{{}→{0}}[1, 0, ϵSeries[]];
  done = {0};
  st = Range[0, 2 n + 1];
  While[{} =!= ($M = todo),
   {cx} = MaximalBy[todo, Length[done ⋂ {#[[1]], #[[2]], #[[1]] - 1, #[[2]] - 1}] &, 1];
   {i, j} = List @@ cx;
   ξ1 = Switch[Head[cx],
      Xp, (R_{i,j} OverBar[Kink]_k) // m_{j,k→j},
      Xm, (OverBar[R]_{i,j} Kink_k) // m_{j,k→j}
     ];
   ξ1 = (rot[k, rots[[i]]] ξ1) // m_{k,i→i}; rots[[i]] = 0;
   ξ1 = (ξ1 rot[k, rots[[i + 1]]]) // m_{i,k→i}; rots[[i + 1]] = 0;
   ξ1 = (rot[k, rots[[j]]] ξ1) // m_{k,j→j}; rots[[j]] = 0;
   ξ1 = (ξ1 rot[k, rots[[j + 1]]]) // m_{j,k→j}; rots[[j + 1]] = 0;
   ξ *= ξ1;
   If[MemberQ[done, i], ξ = ξ // m_{i,i+1→i}; st = st /. st[[i + 2]] → st[[i + 1]]];
   If[MemberQ[done, i - 1], ξ = ξ // m_{st[[i]],i→st[[i]]}; st = st /. st[[i + 1]] → st[[i]]];
   If[MemberQ[done, j], ξ = ξ // m_{j,j+1→j}; st = st /. st[[j + 2]] → st[[j + 1]]];
   If[MemberQ[done, j - 1], ξ = ξ // m_{st[[j]],j→st[[j]]}; st = st /. st[[j + 1]] → st[[j]]];
   done = done ⋃ {i - 1, i, j - 1, j};
   todo = DeleteCases[todo, cx]
  ];
  CF /@ (ξ (*/. {x_0→x,y_0→y,a_0→a}*))
 ]
```

ρ_1 testing

In[∘]:=

$$R_{i\_,j\_} := \mathbb{E}_{\{\}\to\{i,j\}}\left[1,\ (T_1-1)\, x_{1,j}\,(p_{1,i}-p_{1,j}),\ \epsilon\text{Series}\left[0,\ \frac{1}{2}(-1+T_1)\,x_{1,j}^2\,p_{1,i}^2 + x_{1,i}\,x_{1,j}\,p_{1,i}\,p_{1,j} + \frac{1}{2}(1-3T_1)\,x_{1,j}^2\,p_{1,i}\,p_{1,j}\right]\right];$$

$$\overline{R}_{i\_,j\_} := \mathbb{E}_{\{\}\to\{i,j\}}\left[1,\ \left(T_1^{-1}-1\right) x_{1,j}\,(p_{1,i}-p_{1,j}),\ \epsilon\text{Series}\left[0,\ -\frac{(-1+T_1)\,x_{1,i}\,x_{1,j}\,p_{1,i}^2}{T_1^2} - \frac{(1-T_1)\,x_{1,j}^2\,p_{1,i}^2}{2\,T_1^3} - \frac{x_{1,i}\,x_{1,j}\,p_{1,i}\,p_{1,j}}{T_1^2} - \frac{(-1-T_1)\,x_{1,j}^2\,p_{1,i}\,p_{1,j}}{2\,T_1^3}\right]\right];$$

$$CC_{i\_} := \mathbb{E}_{\{\}\to\{i\}}\left[\sqrt{T_1},\ 0,\ \epsilon\text{Series}\left[0,\ -\frac{x_{1,i}\,p_{1,i}}{T_1}\right]\right];$$

$$\overline{CC}_{i\_} := \mathbb{E}_{\{\}\to\{i\}}\left[\frac{1}{\sqrt{T_1}},\ 0,\ \epsilon\text{Series}\left[0,\ \frac{x_{1,i}\,p_{1,i}}{T_1}\right]\right];$$

In[∘]:=

$$\$k = 1;\ \{$$
$$\{\text{"C}\overline{\text{C}}\text{"},\ \left(CC_1\,\overline{CC}_2\ //\ m_{1,2\to1}\right) \equiv \mathbb{E}_{\{\}\to\{1\}}[1,\ 0,\ \epsilon\text{Series}[\,]]\},$$
$$\{\text{"An R1"},\ (CC_3\,R_{1,2}\ //\ m_{2,3\to2}\ //\ m_{2,1\to1}) \equiv \left(\overline{CC}_3\,R_{1,2}\ //\ m_{1,3\to1}\ //\ m_{1,2\to1}\right)\},$$
$$\{\text{"R2b"},\ \left(R_{1,2}\,\overline{R}_{3,4}\ //\ m_{1,3\to1}\ //\ m_{2,4\to2}\right) \equiv \mathbb{E}_{\{\}\to\{1,2\}}[1,\ 0,\ \epsilon\text{Series}[\,]]\},$$
$$\{\text{"R3"},\ (R_{1,2}\,R_{4,3}\,R_{5,6}\ //\ m_{1,4\to1}\ //\ m_{2,5\to2}\ //\ m_{3,6\to3}) \equiv (R_{2,3}\,R_{4,5}\,R_{1,6}\ //\ m_{1,4\to1}\ //\ m_{2,5\to2}\ //\ m_{3,6\to3})\}$$
$$\}$$

Out[∘]=

$$\{\{\text{C}\overline{\text{C}},\ \text{True}\},\ \{\text{An R1},\ \text{True}\},\ \{\text{R2b},\ \text{True}\},\ \{\text{R3},\ \text{True}\}\}$$

In[∘]:=

$$\text{NewBit}[K\_] := \text{Module}\left[\{\text{Alex} = \text{Alexander}[K][T_1]\},\ T_1^3\,\frac{\text{Alex}^2}{T_1-1}\,Z[K][\![3,2]\!]\ //\ \text{Factor}\right]$$

In[∘]:=

```
NewBit /@ AllKnots[{3, 5}]
```

KnotTheory: Loading precomputed data in PD4Knots`.

Out[∘]=

$$\left\{2-T_1+T_1^2,\ (1+T_1)\left(1-3T_1+T_1^2\right),\ \frac{4-3T_1+5T_1^2-3T_1^3+3T_1^4-T_1^5+T_1^6}{T_1^2},\ 9-11T_1+7T_1^2-T_1^3\right\}$$

In pensieve://Projects/BabyDoPeGDO/Solving to $k=3.nb that was

$$\left\{2-T+T^2,\ (1+T)\left(1-3T+T^2\right),\ \frac{4-3T+5T^2-3T^3+3T^4-T^5+T^6}{T^2},\ 9-11T+7T^2-T^3\right\}$$

And now θ

In[∘]:=
```
T3 = T1 T2;
ℒ[X_{i_,j_}[s_]] := T3^s 𝔼[CF@{
    Σ_{ν=1}^{3} (x_{νi} (p_{νi⁺} - p_{νi}) + x_{νj} (p_{νj⁺} - p_{νj}) + (T_ν^s - 1) x_{νi} (p_{νi⁺} - p_{νj⁺})),
    (T1^s - 1) p_{3j} x_{1i} (T2^s x_{2i} - x_{2j}),
    ϵ s (T3^s - 1) p_{1j} (p_{2i} - p_{2j}) x_{3i} / (T2^s - 1),
    ϵ s (1/2 + T2^s p_{1i} p_{2j} x_{1i} x_{2i} - p_{1i} p_{2j} x_{1i} x_{2j} - p_{3i} x_{3i} - (T2^s - 1) p_{2j} p_{3i} x_{2i} x_{3i} +
        (T3^s - 1) p_{2j} p_{3j} x_{2i} x_{3i} + 2 p_{2j} p_{3i} x_{2j} x_{3i} + p_{1i} p_{3j} x_{1i} x_{3j} - p_{2i} p_{3j} x_{2i} x_{3j} - T2^s p_{2j} p_{3j} x_{2i} x_{3j} +
        ((T1^s - 1) p_{1j} x_{1i} (T2^{2s} p_{2j} x_{2i} - T2^s p_{2j} x_{2j} - (T2^s + 1) (T3^s - 1) p_{3j} x_{3i} + T2^s p_{3j} x_{3j}) +
            (T3^s - 1) p_{3j} x_{3i} (1 - T2^s p_{1i} x_{1i} + p_{2i} x_{2j} + (T2^s - 2) p_{2j} x_{2j})) / (T2^s - 1))}]
```

In[∘]:=
```
ℒ[C_{i_}[φ_]] := T3^φ 𝔼[{Σ_{ν=1}^{3} x_{νi} (p_{νi⁺} - p_{νi}), ϵ φ (p_{3i} x_{3i} - 1/2)}]
```

In[∘]:=
```
R_{i_,j_} := CF[ℒ[X_{i,j}[1]] /. ω_ 𝔼[{Q_, P0_, P1a_, P1b_}] ⧴ 𝔼_{{}→{i,j}}[
      ω, Σ_{ν=1}^{3} ((T_ν - 1) x_{ν,j} (p_{ν,i} - p_{ν,j})),
      ϵSeries[P0, P1a + P1b /. ϵ → 1]
    ]];
R̄_{i_,j_} := CF[ℒ[X_{i,j}[-1]] /. ω_ 𝔼[{Q_, P0_, P1a_, P1b_}] ⧴ 𝔼_{{}→{i,j}}[
      ω, Σ_{ν=1}^{3} ((T_ν^{-1} - 1) x_{ν,j} (p_{ν,i} - p_{ν,j})),
      ϵSeries[P0, P1a + P1b /. ϵ → 1]
    ]];
CC_{i_} := CF[ℒ[C_i[1]] /. ω_ 𝔼[{Q_, P_}] ⧴ 𝔼_{{}→{i}}[
      ω, 0, ϵSeries[0, P /. ϵ → 1]
    ]];
C̄C̄_{i_} := CF[ℒ[C_i[-1]] /. ω_ 𝔼[{Q_, P_}] ⧴ 𝔼_{{}→{i}}[
      ω, 0, ϵSeries[0, P /. ϵ → 1]
    ]];
```

In[∘]:= $\overline{R}_{i,j}$ // CF

Out[∘]=

$$\mathbb{E}_{\{\}\to\{i,j\}}\Big[\frac{1}{T_1 T_2},\ \frac{(1-T_1)\,p_{1,i}\,x_{1,j}}{T_1}+\frac{(-1+T_1)\,p_{1,j}\,x_{1,j}}{T_1}+\frac{(1-T_2)\,p_{2,i}\,x_{2,j}}{T_2}+\frac{(-1+T_2)\,p_{2,j}\,x_{2,j}}{T_2}+\frac{(1-T_1 T_2)\,p_{3,i}\,x_{3,j}}{T_1 T_2}+\frac{(-1+T_1 T_2)\,p_{3,j}\,x_{3,j}}{T_1 T_2},\ \epsilon\text{Series}\Big[\frac{(1-T_1)\,p_{3,j}\,x_{1,i}\,x_{2,i}}{T_1 T_2}+\frac{(-1+T_1)\,p_{3,j}\,x_{1,i}\,x_{2,j}}{T_1}\Big]\Big]$$

In[∘]:= CC_1 // CF

Out[∘]=

$$\mathbb{E}_{\{\}\to\{1\}}[T_1 T_2,\ 0,\ \epsilon\text{Series}[0]]$$

In[∘]:= $\overline{\mathrm{CC}}_1$ // CF

Out[∘]=

$$\mathbb{E}_{\{\}\to\{1\}}\left[\frac{1}{T_1\,T_2},\,0,\,\epsilon\mathrm{Series}[0]\right]$$

In[∘]:=
```
$k = 0; {
 {"CC̄", (CC1 CC̄2 // m1,2→1) ≡ E{}→{1}[1, 0, ϵSeries[]]},
 {"An R1", (CC3 R1,2 // m2,3→2 // m2,1→1) ≡ (CC̄3 R1,2 // m1,3→1 // m1,2→1)},
 {"R2b", (R1,2 R̄3,4 // m1,3→1 // m2,4→2) ≡ E{}→{1,2}[1, 0, ϵSeries[]]},
 {"R3", (R1,2 R4,3 R5,6 // m1,4→1 // m2,5→2 // m3,6→3) ≡ (R2,3 R4,5 R1,6 // m1,4→1 // m2,5→2 // m3,6→3)}
}
```

Out[∘]=

$$\Big\{\{C\overline{C},\,\mathrm{True}\},\,\Big\{\mathrm{An\ R1},\,\frac{\left(1-T_1-T_2^2+T_1T_2^2\right)p_{3,1}x_{1,1}x_{2,1}}{T_2}==\left(T_1-T_1^2-T_1T_2^2+T_1^2T_2^2\right)p_{3,1}x_{1,1}x_{2,1}\Big\},$$

$$\Big\{\mathrm{R2b},\,\frac{\left(1-T_1-T_1T_2+T_1^2T_2\right)p_{3,1}x_{1,1}x_{2,1}}{T_1T_2}+\left(1-T_1-T_2+T_1T_2\right)p_{3,2}x_{1,1}x_{2,1}+\frac{\left(T_2-2T_1T_2+T_1^2T_2\right)p_{3,2}x_{1,2}x_{2,1}}{T_1}+\frac{\left(-1+T_1+T_1T_2-T_1^2T_2\right)p_{3,1}x_{1,1}x_{2,2}}{T_1}+\frac{\left(1-T_1+T_2-T_1T_2-2T_2^2+2T_1T_2^2\right)p_{3,2}x_{1,1}x_{2,2}}{T_2}+\frac{\left(-1+2T_1-T_1^2-T_2+2T_1T_2-T_1^2T_2+T_2^2-2T_1T_2^2+T_1^2T_2^2\right)p_{3,2}x_{1,2}x_{2,2}}{T_1T_2}==0\Big\},$$

$$\Big\{\mathrm{R3},\,\left(-T_2+T_1T_2\right)p_{3,2}x_{1,1}x_{2,1}+\left(-T_2+T_1T_2\right)p_{3,3}x_{1,1}x_{2,1}+\left(-T_1T_2+2T_1^2T_2-T_1^3T_2\right)p_{3,2}x_{1,3}x_{2,1}+\left(1-T_1\right)p_{3,2}x_{1,1}x_{2,2}+\left(-T_2+T_1T_2+T_1T_2^2-T_1^2T_2^2\right)p_{3,1}x_{1,2}x_{2,2}+\left(-T_1T_2^2+T_1^2T_2^2\right)p_{3,3}x_{1,2}x_{2,2}+\left(T_1-2T_1^2+T_1^3\right)p_{3,2}x_{1,3}x_{2,2}+\left(1-T_1-T_2+T_1T_2-T_2^2+T_1T_2^2+T_2^3-T_1T_2^3\right)p_{3,2}x_{1,1}x_{2,3}+\left(T_2-T_1T_2\right)p_{3,3}x_{1,1}x_{2,3}+\left(1-T_1-T_1T_2+T_1^2T_2\right)p_{3,1}x_{1,2}x_{2,3}+\left(T_1T_2-T_1^2T_2\right)p_{3,3}x_{1,2}x_{2,3}+\left(T_1-2T_1^2+T_1^3-T_1T_2+2T_1^2T_2-T_1^3T_2-T_1T_2^2+2T_1^2T_2^2-T_1^3T_2^2+T_1T_2^3-2T_1^2T_2^3+T_1^3T_2^3\right)p_{3,2}x_{1,3}x_{2,3}==$$

$$\left(-2T_2+2T_1T_2+T_1T_2^2-T_1^2T_2^2\right)p_{3,2}x_{1,1}x_{2,1}+\left(-T_1T_2^2+T_1^2T_2^2\right)p_{3,3}x_{1,1}x_{2,1}+\left(-T_2+2T_1T_2-T_1^2T_2+T_1T_2^2-2T_1^2T_2^2+T_1^3T_2^2\right)p_{3,2}x_{1,2}x_{2,1}+\left(-T_1T_2^2+2T_1^2T_2^2-T_1^3T_2^2\right)p_{3,3}x_{1,2}x_{2,1}+\left(1-T_1-T_2+T_1T_2+T_2^2-T_1^2T_2^2-T_1T_2^3+T_1^2T_2^3\right)p_{3,2}x_{1,1}x_{2,2}+\left(-T_1T_2^2+T_1^2T_2^2+T_1T_2^3-T_1^2T_2^3\right)p_{3,3}x_{1,1}x_{2,2}+\left(-T_2+2T_1T_2-T_1^2T_2+T_2^2-T_1T_2^2-T_1^2T_2^2+T_1^3T_2^2-T_1T_2^3+2T_1^2T_2^3-T_1^3T_2^3\right)p_{3,2}x_{1,2}x_{2,2}+\left(-2T_1T_2^2+3T_1^2T_2^2-T_1^3T_2^2+T_1T_2^3-2T_1^2T_2^3+T_1^3T_2^3\right)p_{3,3}x_{1,2}x_{2,2}+\left(1-T_1-T_1T_2+T_1^2T_2\right)p_{3,2}x_{1,1}x_{2,3}+\left(T_1T_2-T_1^2T_2\right)p_{3,3}x_{1,1}x_{2,3}+\left(1-2T_1+T_1^2-T_1T_2+2T_1^2T_2-T_1^3T_2\right)p_{3,2}x_{1,2}x_{2,3}+\left(2T_1T_2-3T_1^2T_2+T_1^3T_2\right)p_{3,3}x_{1,2}x_{2,3}\Big\}\Big\}$$

```
In[∘]:= $k = 0;
        R1,2 // CF
        R̄3,4 // CF
        (R1,2 R̄3,4 // m1,3→1 // m2,4→2)
```

Out[∘]=

$$\mathbb{E}_{\{\}\to\{1,2\}}[T_1 T_2,\ (-1+T_1)\,p_{1,1}\,x_{1,2} + (1-T_1)\,p_{1,2}\,x_{1,2} + (-1+T_2)\,p_{2,1}\,x_{2,2} + (1-T_2)\,p_{2,2}\,x_{2,2} + (-1+T_1 T_2)\,p_{3,1}\,x_{3,2} + (1-T_1 T_2)\,p_{3,2}\,x_{3,2},\ \epsilon\text{Series}[(-T_2+T_1 T_2)\,p_{3,2}\,x_{1,1}\,x_{2,1} + (1-T_1)\,p_{3,2}\,x_{1,1}\,x_{2,2}]]$$

Out[∘]=

$$\mathbb{E}_{\{\}\to\{3,4\}}\Big[\frac{1}{T_1 T_2},\ \frac{(1-T_1)\,p_{1,3}\,x_{1,4}}{T_1} + \frac{(-1+T_1)\,p_{1,4}\,x_{1,4}}{T_1} + \frac{(1-T_2)\,p_{2,3}\,x_{2,4}}{T_2} + \frac{(-1+T_2)\,p_{2,4}\,x_{2,4}}{T_2} + \frac{(1-T_1 T_2)\,p_{3,3}\,x_{3,4}}{T_1 T_2} + \frac{(-1+T_1 T_2)\,p_{3,4}\,x_{3,4}}{T_1 T_2},\ \epsilon\text{Series}\Big[\frac{(1-T_1)\,p_{3,4}\,x_{1,3}\,x_{2,3}}{T_1 T_2} + \frac{(-1+T_1)\,p_{3,4}\,x_{1,3}\,x_{2,4}}{T_1}\Big]\Big]$$

Out[∘]=

$$\mathbb{E}_{\{\}\to\{1,2\}}\Big[1,\ 0,\ \epsilon\text{Series}\Big[\frac{\left(1-T_1-T_1 T_2+T_1^2 T_2\right)\,p_{3,1}\,x_{1,1}\,x_{2,1}}{T_1 T_2} + (1-T_1-T_2+T_1 T_2)\,p_{3,2}\,x_{1,1}\,x_{2,1} + \frac{\left(T_2-2\,T_1 T_2+T_1^2 T_2\right)\,p_{3,2}\,x_{1,2}\,x_{2,1}}{T_1} + \frac{\left(-1+T_1+T_1 T_2-T_1^2 T_2\right)\,p_{3,1}\,x_{1,1}\,x_{2,2}}{T_1} + \frac{\left(1-T_1+T_2-T_1 T_2-2\,T_2^2+2\,T_1 T_2^2\right)\,p_{3,2}\,x_{1,1}\,x_{2,2}}{T_2} + \frac{\left(-1+2\,T_1-T_1^2-T_2+2\,T_1 T_2-T_1^2 T_2+T_2^2-2\,T_1 T_2^2+T_1^2 T_2^2\right)\,p_{3,2}\,x_{1,2}\,x_{2,2}}{T_1 T_2}\Big]\Big]$$