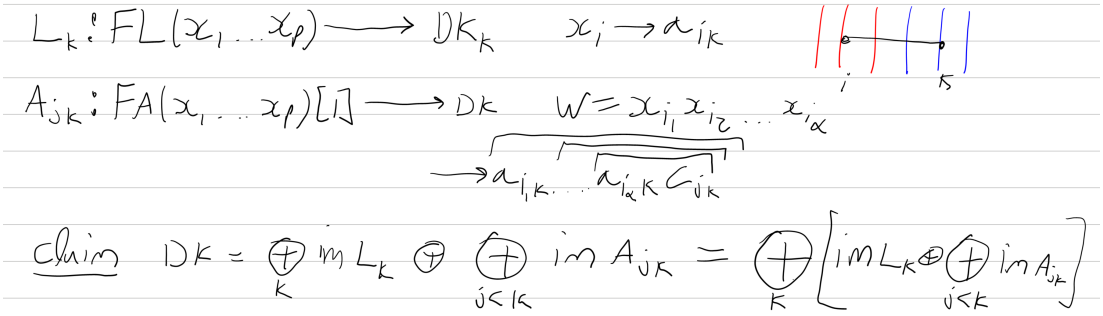
```
In[•]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\People\\Kuno"];
        << FreeLie.m
```

```
FreeLie` implements / extends
 {*, +, **, $SeriesShowDegree, ⟨⟩, ∫, ≡, ad, Ad, adSeries, AllCyclicWords, AllLyndonWords,
  AllWords, Arbitrator, AS, ASeries, AW, b, BCH, BooleanSequence, BracketForm, BS, CC, Crop,
  cw, CW, CWS, CWSeries, D, Deg, DegreeScale, DerivationSeries, div, DK, DKS, DKSeries, EulerE,
  Exp, FreeLieFormatting, Inverse, j, J, JA, LieDerivation, LieMorphism, LieSeries, LS, LW,
  LyndonFactorization, Morphism, New, RandomCWSeries, Randomizer, RandomLieSeries, RC, SeriesSolve,
  Support, t, tb, TopBracketForm, tr, UndeterminedCoefficients, αMap, Γ, ι, Λ, σ, τ, ħ, ⟌, ⌢}.

FreeLie` is in the public domain. Dror Bar-Natan is committed
  to support it within reason until July 15, 2022. This is version 240218.
```

```
In[•]:= Antipode[w_AW] := Antipode[w] = (-1)^Length[w] Reverse[w];
        Antipode[expr_] := Expand[expr /. w_AW :> Antipode[w]]
```

Goal: Implement the "Emergent Drinfel'd-Kohno Algebra" EDK following



EDK[k_Integer, x_FL]: an element in the image of L_k.
EDK[j_Integer,k_Integer, w_AW]: an element in the image of A_{jk}.

EDKBasis[ps, s, d] produces the basis of the EDK algebra with poles ps, strands 1,...,s, and at degree d.

```
In[•]:= EDKBasis[ps_List, s_Integer, d_Integer] := Flatten@{
          Table[EDK[j, k, #] & /@ AllWords[d - 1, ps], {j, 1, s - 1}, {k, j + 1, s}],
          Table[EDK[k, #] & /@ AllLyndonWords[d, ps], {k, s}]
        }
```

```
In[•]:= EDKBasis[{x, y}, 3, 3]
```

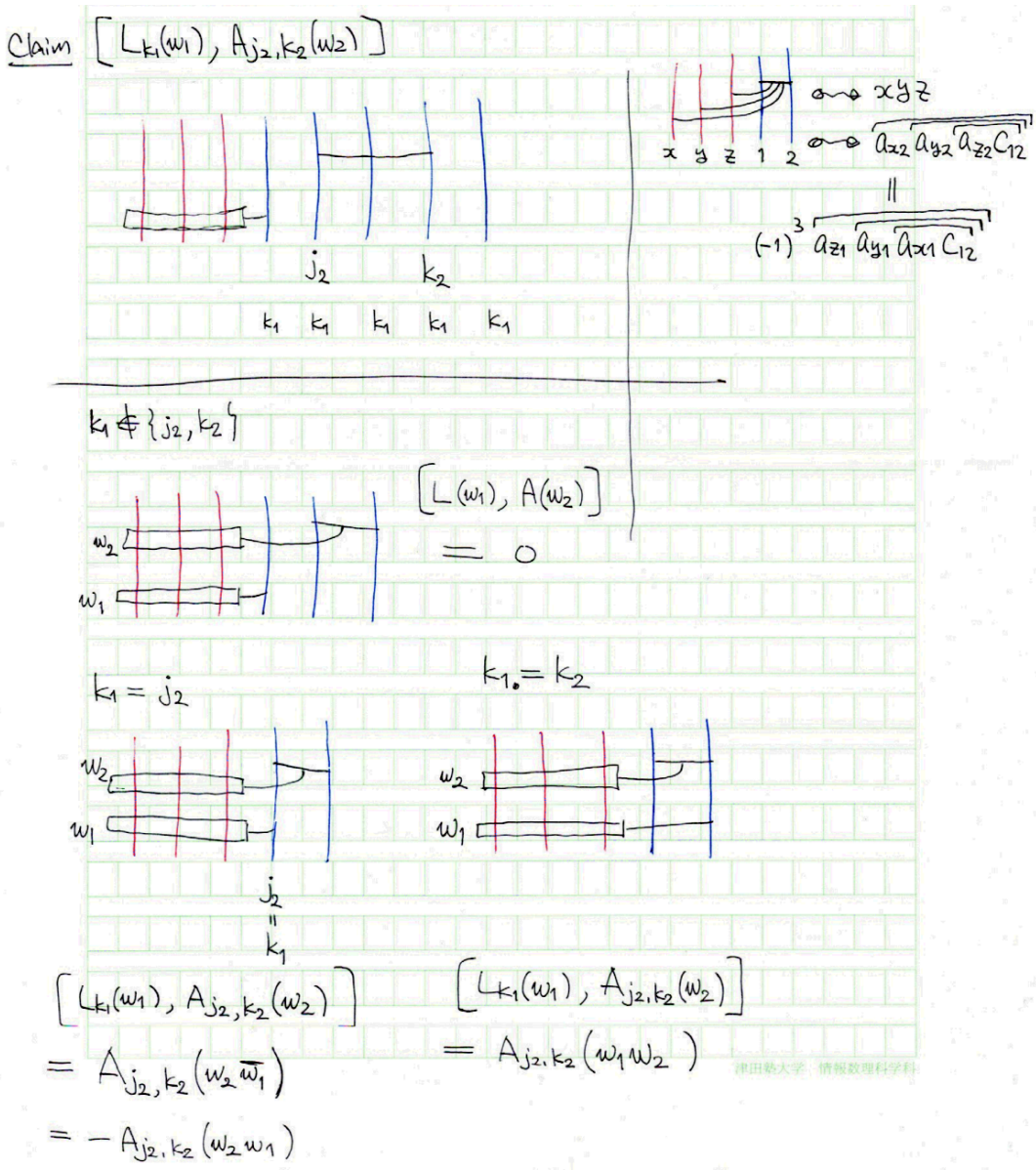
```
Out[•]= {EDK[1, 2, AW[x, x]], EDK[1, 2, AW[x, y]], EDK[1, 2, AW[y, x]], EDK[1, 2, AW[y, y]],
 EDK[1, 3, AW[x, x]], EDK[1, 3, AW[x, y]], EDK[1, 3, AW[y, x]], EDK[1, 3, AW[y, y]],
 EDK[2, 3, AW[x, x]], EDK[2, 3, AW[x, y]], EDK[2, 3, AW[y, x]], EDK[2, 3, AW[y, y]],
 EDK[1, x xy], EDK[1, xyy], EDK[2, x xy], EDK[2, xyy], EDK[3, x xy], EDK[3, xyy]}
```

In[•]:=
```
EDK[j_, k_, w_AW] /; j > k := EDK[k, j, Antipode[w]];
```

In[•]:=
```
EDK[__, 0] = 0;
EDK /: EDK[jk__, x1_] + EDK[jk__, x2_] := EDK[jk, x1 + x2];
EDK /: c_ * EDK[jk__, x_] := EDK[jk, Expand[c x]];
EDK /: b[EDK[jk1__, c_ * (x1_LW | x1_AW)], EDK[jk2__, x2_]] :=
  Expand[c b[EDK[jk1, x1], EDK[jk2, x2]]];
EDK /: b[EDK[jk1__, x1_], EDK[jk2__, c_ * (x2_LW | x2_AW)]] :=
  Expand[c b[EDK[jk1, x1], EDK[jk2, x2]]];
EDK /: b[EDK[jk1__, x1_], EDK[jk2__, x2_Plus]] :=
  b[EDK[jk1, x1], EDK[jk2, #]] & /@ x2;
EDK /: b[EDK[jk1__, x1_Plus], EDK[jk2__, x2_]] := b[EDK[jk1, #], EDK[jk2, x2]] & /@ x1;
```

$$\left[A_{0,k_1}(w_1),\ A_{j_2 k_2}(w_2)\right] = 0 \quad \text{obvious}$$

In[•]:=
```
EDK /: b[EDK[_, _, _], EDK[_, _, _]] := 0
```



In[•]:=
```
EDK /: b[EDK[k1_, w1_LW], EDK[j2_, k2_, w2_AW]] := Which[
   k1 == j2, -EDK[j2, k2, w2 ** ι[w1]],
   k1 == k2, EDK[j2, k2, ι[w1] ** w2],
   True, 0
  ];
EDK /: b[EDK[j2_, k2_, w2_AW], EDK[k1_, w1_LW]] :=
 Expand[-b[EDK[k1, w1], EDK[j2, k2, w2]]]
```

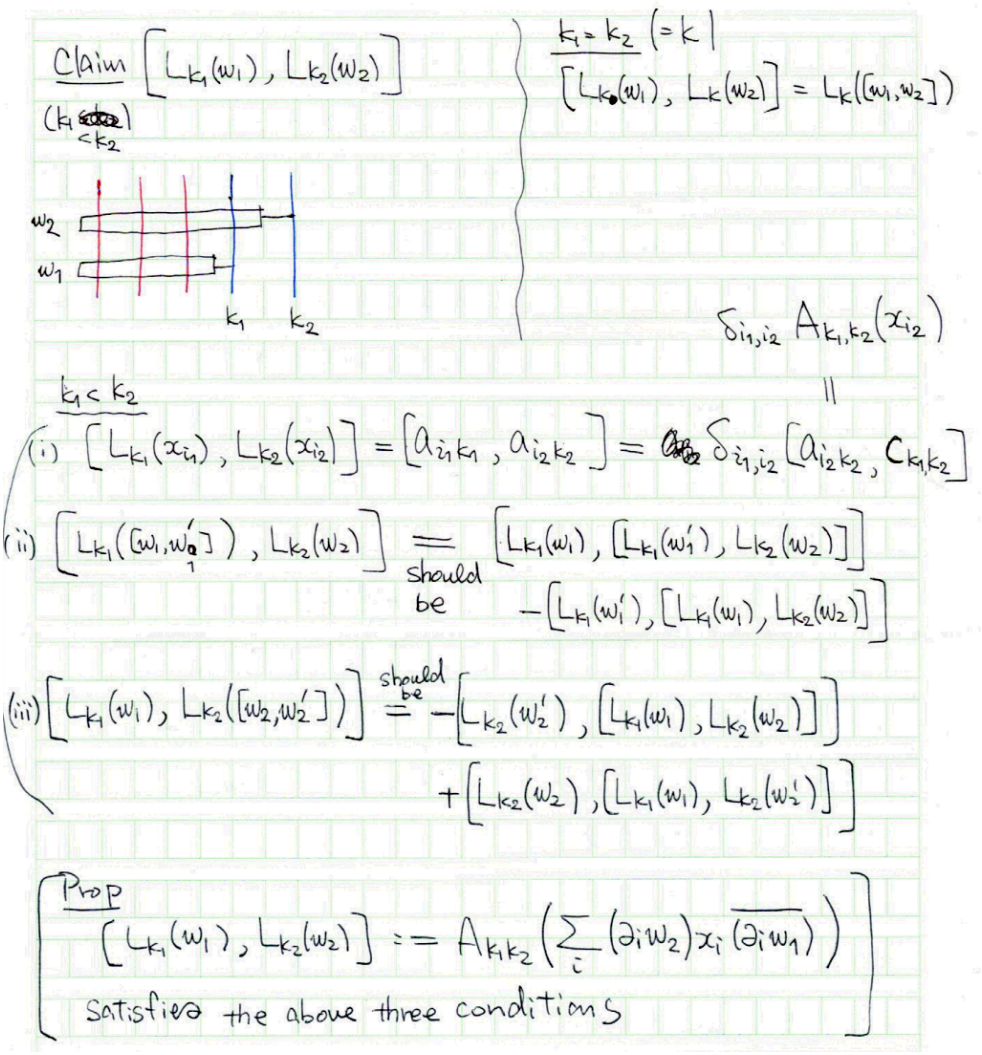
In[•]:=
```
b[EDK[1, LW[x, y]], EDK[1, 2, AW[z, z, x, y]]]
```

Out[•]=
```
EDK[1, 2, -AW[z, z, x, y, x, y] + AW[z, z, x, y, y, x]]
```

In[•]:=
```
Cases[{AW[1], LW[2], Yu[3]}, x_LW | x_AW]
```

Out[•]=

$\{\text{AW}[1], \overline{2}\}$



In[•]:=
```
AllLyndonWords[4, {x, y, z}] /. w_LW :> Union[List @@ w]
```

Out[•]=

```
{{x, y}, {x, z}, {x, y}, {x, y, z}, {x, y, z}, {x, z}, {x, y, z}, {x, y}, {x, y, z},
 {x, y, z}, {x, y, z}, {x, y, z}, {x, y, z}, {x, y, z}, {x, z}, {y, z}, {y, z}, {y, z}}
```

In[•]:=
```
EDK /: b[EDK[k1_, w1_LW], EDK[k2_, w2_LW]] := Which[
  k1 == k2, EDK[k1, b[w1, w2]],
  k1 < k2, EDK[k1, k2, Sum[τ[LW@p, w2] ** AW[p] ** Antipode[τ[LW@p, w1]],
    {p, Union[List @@ w1] ⋂ Union[List @@ w2]}]],
  k2 < k1, EDK[k2, k1, Sum[-τ[LW@p, w1] ** AW[p] ** Antipode[τ[LW@p, w2]],
    {p, Union[List @@ w1] ⋂ Union[List @@ w2]}]]
 ]
```

Testing anti-symmetry and Jacobi for EDK:

In[•]:=
```
bas = EDKBasis[{x, y, z}, 3, 4]
Union@Table[{α, β} = αβ; b[α, β] + b[β, α], {αβ, Subsets[bas, {2}]}]
```

Out[•]=

$$\{\mathrm{EDK}[1,2,\mathrm{AW}[x,x,x]],\ \mathrm{EDK}[1,2,\mathrm{AW}[x,x,y]],\ \mathrm{EDK}[1,2,\mathrm{AW}[x,x,z]],$$
$$\mathrm{EDK}[1,2,\mathrm{AW}[x,y,x]],\ \mathrm{EDK}[1,2,\mathrm{AW}[x,y,y]],\ \mathrm{EDK}[1,2,\mathrm{AW}[x,y,z]],$$
$$\mathrm{EDK}[1,2,\mathrm{AW}[x,z,x]],\ \mathrm{EDK}[1,2,\mathrm{AW}[x,z,y]],\ \mathrm{EDK}[1,2,\mathrm{AW}[x,z,z]],$$
$$\mathrm{EDK}[1,2,\mathrm{AW}[y,x,x]],\ \mathrm{EDK}[1,2,\mathrm{AW}[y,x,y]],\ \mathrm{EDK}[1,2,\mathrm{AW}[y,x,z]],$$
$$\mathrm{EDK}[1,2,\mathrm{AW}[y,y,x]],\ \mathrm{EDK}[1,2,\mathrm{AW}[y,y,y]],\ \mathrm{EDK}[1,2,\mathrm{AW}[y,y,z]],$$
$$\mathrm{EDK}[1,2,\mathrm{AW}[y,z,x]],\ \mathrm{EDK}[1,2,\mathrm{AW}[y,z,y]],\ \mathrm{EDK}[1,2,\mathrm{AW}[y,z,z]],$$
$$\mathrm{EDK}[1,2,\mathrm{AW}[z,x,x]],\ \mathrm{EDK}[1,2,\mathrm{AW}[z,x,y]],\ \mathrm{EDK}[1,2,\mathrm{AW}[z,x,z]],$$
$$\mathrm{EDK}[1,2,\mathrm{AW}[z,y,x]],\ \mathrm{EDK}[1,2,\mathrm{AW}[z,y,y]],\ \mathrm{EDK}[1,2,\mathrm{AW}[z,y,z]],$$
$$\mathrm{EDK}[1,2,\mathrm{AW}[z,z,x]],\ \mathrm{EDK}[1,2,\mathrm{AW}[z,z,y]],\ \mathrm{EDK}[1,2,\mathrm{AW}[z,z,z]],$$
$$\mathrm{EDK}[1,3,\mathrm{AW}[x,x,x]],\ \mathrm{EDK}[1,3,\mathrm{AW}[x,x,y]],\ \mathrm{EDK}[1,3,\mathrm{AW}[x,x,z]],$$
$$\mathrm{EDK}[1,3,\mathrm{AW}[x,y,x]],\ \mathrm{EDK}[1,3,\mathrm{AW}[x,y,y]],\ \mathrm{EDK}[1,3,\mathrm{AW}[x,y,z]],$$
$$\mathrm{EDK}[1,3,\mathrm{AW}[x,z,x]],\ \mathrm{EDK}[1,3,\mathrm{AW}[x,z,y]],\ \mathrm{EDK}[1,3,\mathrm{AW}[x,z,z]],$$
$$\mathrm{EDK}[1,3,\mathrm{AW}[y,x,x]],\ \mathrm{EDK}[1,3,\mathrm{AW}[y,x,y]],\ \mathrm{EDK}[1,3,\mathrm{AW}[y,x,z]],$$
$$\mathrm{EDK}[1,3,\mathrm{AW}[y,y,x]],\ \mathrm{EDK}[1,3,\mathrm{AW}[y,y,y]],\ \mathrm{EDK}[1,3,\mathrm{AW}[y,y,z]],$$
$$\mathrm{EDK}[1,3,\mathrm{AW}[y,z,x]],\ \mathrm{EDK}[1,3,\mathrm{AW}[y,z,y]],\ \mathrm{EDK}[1,3,\mathrm{AW}[y,z,z]],$$
$$\mathrm{EDK}[1,3,\mathrm{AW}[z,x,x]],\ \mathrm{EDK}[1,3,\mathrm{AW}[z,x,y]],\ \mathrm{EDK}[1,3,\mathrm{AW}[z,x,z]],$$
$$\mathrm{EDK}[1,3,\mathrm{AW}[z,y,x]],\ \mathrm{EDK}[1,3,\mathrm{AW}[z,y,y]],\ \mathrm{EDK}[1,3,\mathrm{AW}[z,y,z]],$$
$$\mathrm{EDK}[1,3,\mathrm{AW}[z,z,x]],\ \mathrm{EDK}[1,3,\mathrm{AW}[z,z,y]],\ \mathrm{EDK}[1,3,\mathrm{AW}[z,z,z]],$$
$$\mathrm{EDK}[2,3,\mathrm{AW}[x,x,x]],\ \mathrm{EDK}[2,3,\mathrm{AW}[x,x,y]],\ \mathrm{EDK}[2,3,\mathrm{AW}[x,x,z]],$$
$$\mathrm{EDK}[2,3,\mathrm{AW}[x,y,x]],\ \mathrm{EDK}[2,3,\mathrm{AW}[x,y,y]],\ \mathrm{EDK}[2,3,\mathrm{AW}[x,y,z]],$$
$$\mathrm{EDK}[2,3,\mathrm{AW}[x,z,x]],\ \mathrm{EDK}[2,3,\mathrm{AW}[x,z,y]],\ \mathrm{EDK}[2,3,\mathrm{AW}[x,z,z]],$$
$$\mathrm{EDK}[2,3,\mathrm{AW}[y,x,x]],\ \mathrm{EDK}[2,3,\mathrm{AW}[y,x,y]],\ \mathrm{EDK}[2,3,\mathrm{AW}[y,x,z]],$$
$$\mathrm{EDK}[2,3,\mathrm{AW}[y,y,x]],\ \mathrm{EDK}[2,3,\mathrm{AW}[y,y,y]],\ \mathrm{EDK}[2,3,\mathrm{AW}[y,y,z]],$$
$$\mathrm{EDK}[2,3,\mathrm{AW}[y,z,x]],\ \mathrm{EDK}[2,3,\mathrm{AW}[y,z,y]],\ \mathrm{EDK}[2,3,\mathrm{AW}[y,z,z]],$$
$$\mathrm{EDK}[2,3,\mathrm{AW}[z,x,x]],\ \mathrm{EDK}[2,3,\mathrm{AW}[z,x,y]],\ \mathrm{EDK}[2,3,\mathrm{AW}[z,x,z]],$$
$$\mathrm{EDK}[2,3,\mathrm{AW}[z,y,x]],\ \mathrm{EDK}[2,3,\mathrm{AW}[z,y,y]],\ \mathrm{EDK}[2,3,\mathrm{AW}[z,y,z]],$$
$$\mathrm{EDK}[2,3,\mathrm{AW}[z,z,x]],\ \mathrm{EDK}[2,3,\mathrm{AW}[z,z,y]],\ \mathrm{EDK}[2,3,\mathrm{AW}[z,z,z]],$$
$$\mathrm{EDK}\left[1,\overline{x\ \overline{x\ \overline{xy}}}\right],\ \mathrm{EDK}\left[1,\overline{x\ \overline{x\ \overline{xz}}}\right],\ \mathrm{EDK}\left[1,\overline{x\ \overline{xy\,y}}\right],\ \mathrm{EDK}\left[1,\overline{x\ \overline{x\ \overline{yz}}}\right],$$
$$\mathrm{EDK}\left[1,\overline{x\ \overline{xz\,y}}\right],\ \mathrm{EDK}\left[1,\overline{x\ \overline{xz\,z}}\right],\ \mathrm{EDK}\left[1,\overline{\overline{xy}\ \overline{xz}}\right],\ \mathrm{EDK}\left[1,\overline{\overline{xy\,y}\ y}\right],\ \mathrm{EDK}\left[1,\overline{x\ \overline{y\ \overline{yz}}}\right],$$
$$\mathrm{EDK}\left[1,\overline{x\ \overline{yz}\ y}\right],\ \mathrm{EDK}\left[1,\overline{x\ \overline{yz\,z}}\right],\ \mathrm{EDK}\left[1,\overline{\overline{xz\,y}\ y}\right],\ \mathrm{EDK}\left[1,\overline{\overline{xz}\ \overline{yz}}\right],\ \mathrm{EDK}\left[1,\overline{\overline{xz\,z}\ y}\right],$$
$$\mathrm{EDK}\left[1,\overline{\overline{xz\,z}\ z}\right],\ \mathrm{EDK}\left[1,\overline{y\ \overline{y\ \overline{yz}}}\right],\ \mathrm{EDK}\left[1,\overline{y\ \overline{yz\,z}}\right],\ \mathrm{EDK}\left[1,\overline{yz\,z\,z}\right],\ \mathrm{EDK}\left[2,\overline{x\ \overline{x\ \overline{xy}}}\right],$$
$$\mathrm{EDK}\left[2,\overline{x\ \overline{x\ \overline{xz}}}\right],\ \mathrm{EDK}\left[2,\overline{x\ \overline{xy\,y}}\right],\ \mathrm{EDK}\left[2,\overline{x\ \overline{x\ \overline{yz}}}\right],\ \mathrm{EDK}\left[2,\overline{x\ \overline{xz\,y}}\right],\ \mathrm{EDK}\left[2,\overline{x\ \overline{xz\,z}}\right],$$
$$\mathrm{EDK}\left[2,\overline{\overline{xy}\ \overline{xz}}\right],\ \mathrm{EDK}\left[2,\overline{\overline{xy\,y}\ y}\right],\ \mathrm{EDK}\left[2,\overline{x\ \overline{y\ \overline{yz}}}\right],\ \mathrm{EDK}\left[2,\overline{x\ \overline{yz}\ y}\right],\ \mathrm{EDK}\left[2,\overline{x\ \overline{yz\,z}}\right],$$
$$\mathrm{EDK}\left[2,\overline{\overline{xz\,y}\ y}\right],\ \mathrm{EDK}\left[2,\overline{\overline{xz}\ \overline{yz}}\right],\ \mathrm{EDK}\left[2,\overline{\overline{xz\,z}\ y}\right],\ \mathrm{EDK}\left[2,\overline{\overline{xz\,z}\ z}\right],\ \mathrm{EDK}\left[2,\overline{y\ \overline{y\ \overline{yz}}}\right],$$
$$\mathrm{EDK}\left[2,\overline{y\ \overline{yz\,z}}\right],\ \mathrm{EDK}\left[2,\overline{yz\,z\,z}\right],\ \mathrm{EDK}\left[3,\overline{x\ \overline{x\ \overline{xy}}}\right],\ \mathrm{EDK}\left[3,\overline{x\ \overline{x\ \overline{xz}}}\right],\ \mathrm{EDK}\left[3,\overline{x\ \overline{xy\,y}}\right],$$

EDK[3, x x yz], EDK[3, x xzy], EDK[3, x xzz], EDK[3, xy xz], EDK[3, xyy y],
EDK[3, x y yz], EDK[3, x yz y], EDK[3, x yzz], EDK[3, xzy y], EDK[3, xz yz],
EDK[3, xzz y], EDK[3, xzz z], EDK[3, y y yz], EDK[3, y yz z], EDK[3, yz z z]}

Out[•]=

{0}

In[•]:=
```
bas = EDKBasis[{x, y, z}, 3, 4]
Union@Table[{α, β, γ} = αβγ;
  b[α, b[β, γ]] + b[β, b[γ, α]] + b[γ, b[α, β]], {αβγ, Subsets[bas, {3}]}
 ]
```

Out[•]=

{EDK[1, 2, AW[x, x, x]], EDK[1, 2, AW[x, x, y]], EDK[1, 2, AW[x, x, z]],
EDK[1, 2, AW[x, y, x]], EDK[1, 2, AW[x, y, y]], EDK[1, 2, AW[x, y, z]],
EDK[1, 2, AW[x, z, x]], EDK[1, 2, AW[x, z, y]], EDK[1, 2, AW[x, z, z]],
EDK[1, 2, AW[y, x, x]], EDK[1, 2, AW[y, x, y]], EDK[1, 2, AW[y, x, z]],
EDK[1, 2, AW[y, y, x]], EDK[1, 2, AW[y, y, y]], EDK[1, 2, AW[y, y, z]],
EDK[1, 2, AW[y, z, x]], EDK[1, 2, AW[y, z, y]], EDK[1, 2, AW[y, z, z]],
EDK[1, 2, AW[z, x, x]], EDK[1, 2, AW[z, x, y]], EDK[1, 2, AW[z, x, z]],
EDK[1, 2, AW[z, y, x]], EDK[1, 2, AW[z, y, y]], EDK[1, 2, AW[z, y, z]],
EDK[1, 2, AW[z, z, x]], EDK[1, 2, AW[z, z, y]], EDK[1, 2, AW[z, z, z]],
EDK[1, 3, AW[x, x, x]], EDK[1, 3, AW[x, x, y]], EDK[1, 3, AW[x, x, z]],
EDK[1, 3, AW[x, y, x]], EDK[1, 3, AW[x, y, y]], EDK[1, 3, AW[x, y, z]],
EDK[1, 3, AW[x, z, x]], EDK[1, 3, AW[x, z, y]], EDK[1, 3, AW[x, z, z]],
EDK[1, 3, AW[y, x, x]], EDK[1, 3, AW[y, x, y]], EDK[1, 3, AW[y, x, z]],
EDK[1, 3, AW[y, y, x]], EDK[1, 3, AW[y, y, y]], EDK[1, 3, AW[y, y, z]],
EDK[1, 3, AW[y, z, x]], EDK[1, 3, AW[y, z, y]], EDK[1, 3, AW[y, z, z]],
EDK[1, 3, AW[z, x, x]], EDK[1, 3, AW[z, x, y]], EDK[1, 3, AW[z, x, z]],
EDK[1, 3, AW[z, y, x]], EDK[1, 3, AW[z, y, y]], EDK[1, 3, AW[z, y, z]],
EDK[1, 3, AW[z, z, x]], EDK[1, 3, AW[z, z, y]], EDK[1, 3, AW[z, z, z]],
EDK[2, 3, AW[x, x, x]], EDK[2, 3, AW[x, x, y]], EDK[2, 3, AW[x, x, z]],
EDK[2, 3, AW[x, y, x]], EDK[2, 3, AW[x, y, y]], EDK[2, 3, AW[x, y, z]],
EDK[2, 3, AW[x, z, x]], EDK[2, 3, AW[x, z, y]], EDK[2, 3, AW[x, z, z]],
EDK[2, 3, AW[y, x, x]], EDK[2, 3, AW[y, x, y]], EDK[2, 3, AW[y, x, z]],
EDK[2, 3, AW[y, y, x]], EDK[2, 3, AW[y, y, y]], EDK[2, 3, AW[y, y, z]],
EDK[2, 3, AW[y, z, x]], EDK[2, 3, AW[y, z, y]], EDK[2, 3, AW[y, z, z]],
EDK[2, 3, AW[z, x, x]], EDK[2, 3, AW[z, x, y]], EDK[2, 3, AW[z, x, z]],
EDK[2, 3, AW[z, y, x]], EDK[2, 3, AW[z, y, y]], EDK[2, 3, AW[z, y, z]],
EDK[2, 3, AW[z, z, x]], EDK[2, 3, AW[z, z, y]], EDK[2, 3, AW[z, z, z]],
EDK[1, x x xy], EDK[1, x x xz], EDK[1, x xyy], EDK[1, x x yz],
EDK[1, x xzy], EDK[1, x xzz], EDK[1, xy xz], EDK[1, xyy y], EDK[1, x y yz],
EDK[1, x yz y], EDK[1, x yzz], EDK[1, xzy y], EDK[1, xz yz], EDK[1, xzz y],

$$\text{EDK}\left[1, \overline{\text{x z z z}}\right], \text{EDK}\left[1, \text{y } \overline{\text{y y z}}\right], \text{EDK}\left[1, \text{y } \overline{\text{y z z}}\right], \text{EDK}\left[1, \overline{\text{y z z z}}\right], \text{EDK}\left[2, \text{x } \overline{\text{x x y}}\right],$$
$$\text{EDK}\left[2, \text{x } \overline{\text{x x z}}\right], \text{EDK}\left[2, \text{x } \overline{\text{x y y}}\right], \text{EDK}\left[2, \text{x } \overline{\text{x y z}}\right], \text{EDK}\left[2, \text{x } \overline{\text{x z y}}\right], \text{EDK}\left[2, \text{x } \overline{\text{x z z}}\right],$$
$$\text{EDK}\left[2, \overline{\text{x y}\ \text{x z}}\right], \text{EDK}\left[2, \overline{\text{x y y y}}\right], \text{EDK}\left[2, \text{x } \overline{\text{y y z}}\right], \text{EDK}\left[2, \text{x } \overline{\text{y z}\ \text{y}}\right], \text{EDK}\left[2, \text{x } \overline{\text{y z z}}\right],$$
$$\text{EDK}\left[2, \overline{\text{x z y y}}\right], \text{EDK}\left[2, \overline{\text{x z}\ \text{y z}}\right], \text{EDK}\left[2, \overline{\text{x z z y}}\right], \text{EDK}\left[2, \overline{\text{x z z z}}\right], \text{EDK}\left[2, \text{y } \overline{\text{y y z}}\right],$$
$$\text{EDK}\left[2, \text{y } \overline{\text{y z z}}\right], \text{EDK}\left[2, \overline{\text{y z z z}}\right], \text{EDK}\left[3, \text{x } \overline{\text{x x y}}\right], \text{EDK}\left[3, \text{x } \overline{\text{x x z}}\right], \text{EDK}\left[3, \text{x } \overline{\text{x y y}}\right],$$
$$\text{EDK}\left[3, \text{x } \overline{\text{x y z}}\right], \text{EDK}\left[3, \text{x } \overline{\text{x z y}}\right], \text{EDK}\left[3, \text{x } \overline{\text{x z z}}\right], \text{EDK}\left[3, \overline{\text{x y}\ \text{x z}}\right], \text{EDK}\left[3, \overline{\text{x y y y}}\right],$$
$$\text{EDK}\left[3, \text{x } \overline{\text{y z}\ \text{y z}}\right], \text{EDK}\left[3, \text{x } \overline{\text{y z}\ \text{y}}\right], \text{EDK}\left[3, \text{x } \overline{\text{y z z}}\right], \text{EDK}\left[3, \overline{\text{x z y y}}\right], \text{EDK}\left[3, \overline{\text{x z}\ \text{y z}}\right],$$
$$\text{EDK}\left[3, \overline{\text{x z z y}}\right], \text{EDK}\left[3, \overline{\text{x z z z}}\right], \text{EDK}\left[3, \text{y } \overline{\text{y y z}}\right], \text{EDK}\left[3, \text{y } \overline{\text{y z z}}\right], \text{EDK}\left[3, \overline{\text{y z z z}}\right]\}$$

Out[•]=

$$\{0\}$$

σ

$$\Phi^{\sigma[1,23,4]} \to \Phi^{\sigma[\{1\},\{2,3\},\{4\}]} \to \sigma[\{1\}, \{2, 3\}, \{4\}][\Phi]$$
$$\Phi^{\sigma[2,3,1]}$$
$$\Phi^{\sigma[1,x23,4]}$$
$$\triangle[\text{i} \to \text{jk}]$$
$$\text{EDK}[\text{j1, sum of Lie words}] + \text{EDK}[\text{j2, sum of Lie words}] +$$
$$\text{EDK}[\text{j, k, sum of Associative words}] +$$
$$\text{p2s}$$

In[•]:=
```
σ /: ℰ_s_σ := s[ℰ];
σ[lft___, i_Integer, rgt___] := σ[lft, IntegerDigits[i], rgt];
σ[___][0] = 0;
x_Plus // s_σ := s[#] & /@ x;
EDK[jk__, x_Plus] // s_σ := s[EDK[jk, #]] & /@ x;
EDK[jk__, c_ * (w_LW | w_AW)] // s_σ := Expand[c * s[EDK[jk, w]]];
EDK[k_, LW[x_]] // s_σ := Sum[EDK[α, LW[x]], {α, s[[k]]}];
EDK[k_, w_LW] // s_σ := b @@ (s[EDK[k, #]] & /@ LyndonFactorization[w]);
EDK[j_, k_, AW[]] // s_σ := Sum[EDK[α, β, AW[]], {α, s[[j]]}, {β, s[[k]]}];
EDK[j_, k_, AW[x_, w___]] // s_σ := b[s@EDK[k, LW@x], s@EDK[j, k, AW[w]]]
```

In[•]:=
```
bas = EDKBasis[{x, y}, 1, 1];
Column@Table[α → α^σ[12], {α, bas}]
```

Out[•]=

$$\text{EDK}\left[1, \overline{\text{x}}\right] \to \text{EDK}\left[1, \overline{\text{x}}\right] + \text{EDK}\left[2, \overline{\text{x}}\right]$$
$$\text{EDK}\left[1, \overline{\text{y}}\right] \to \text{EDK}\left[1, \overline{\text{y}}\right] + \text{EDK}\left[2, \overline{\text{y}}\right]$$

In[•]:=
```
bas = EDKBasis[{x, y}, 2, 1];
Column@Table[α → α^σ[12,3], {α, bas}]
```

Out[•]=

$$\text{EDK}[1, 2, \text{AW}[\,]] \to \text{EDK}[1, 3, \text{AW}[\,]] + \text{EDK}[2, 3, \text{AW}[\,]]$$
$$\text{EDK}[1, \overline{x}] \to \text{EDK}[1, \overline{x}] + \text{EDK}[2, \overline{x}]$$
$$\text{EDK}[1, \overline{y}] \to \text{EDK}[1, \overline{y}] + \text{EDK}[2, \overline{y}]$$
$$\text{EDK}[2, \overline{x}] \to \text{EDK}[3, \overline{x}]$$
$$\text{EDK}[2, \overline{y}] \to \text{EDK}[3, \overline{y}]$$

In[•]:=
```
bas = EDKBasis[{x, y}, 2, 2];
Column@Table[α → α^σ[12,3], {α, bas}]
```

Out[•]=

$$\text{EDK}[1, 2, \text{AW}[x]] \to \text{EDK}[1, 3, \text{AW}[x]] + \text{EDK}[2, 3, \text{AW}[x]]$$
$$\text{EDK}[1, 2, \text{AW}[y]] \to \text{EDK}[1, 3, \text{AW}[y]] + \text{EDK}[2, 3, \text{AW}[y]]$$
$$\text{EDK}[1, \overline{xy}] \to \text{EDK}[1, \overline{xy}] + \text{EDK}[2, \overline{xy}]$$
$$\text{EDK}[2, \overline{xy}] \to \text{EDK}[3, \overline{xy}]$$

In[•]:=
```
bas = EDKBasis[{x, y}, 2, 3];
Column@Table[α → α^σ[12,3], {α, bas}]
```

Out[•]=

$$\text{EDK}[1, 2, \text{AW}[x, x]] \to \text{EDK}[1, 3, \text{AW}[x, x]] + \text{EDK}[2, 3, \text{AW}[x, x]]$$
$$\text{EDK}[1, 2, \text{AW}[x, y]] \to \text{EDK}[1, 3, \text{AW}[x, y]] + \text{EDK}[2, 3, \text{AW}[x, y]]$$
$$\text{EDK}[1, 2, \text{AW}[y, x]] \to \text{EDK}[1, 3, \text{AW}[y, x]] + \text{EDK}[2, 3, \text{AW}[y, x]]$$
$$\text{EDK}[1, 2, \text{AW}[y, y]] \to \text{EDK}[1, 3, \text{AW}[y, y]] + \text{EDK}[2, 3, \text{AW}[y, y]]$$
$$\text{EDK}[1, \overline{x\ \overline{xy}}] \to \text{EDK}[1, \overline{x\ \overline{xy}}] + \text{EDK}[2, \overline{x\ \overline{xy}}] + \text{EDK}[1, 2, -\text{AW}[x, y] - \text{AW}[y, x]]$$
$$\text{EDK}[1, \overline{\overline{xy}\,y}] \to \text{EDK}[1, \overline{\overline{xy}\,y}] + \text{EDK}[2, \overline{\overline{xy}\,y}] + \text{EDK}[1, 2, -\text{AW}[x, y] - \text{AW}[y, x]]$$
$$\text{EDK}[2, \overline{x\ \overline{xy}}] \to \text{EDK}[3, \overline{x\ \overline{xy}}]$$
$$\text{EDK}[2, \overline{\overline{xy}\,y}] \to \text{EDK}[3, \overline{\overline{xy}\,y}]$$

In[•]:=
```
bas = EDKBasis[{x, y}, 2, 4];
Column@Table[α → α^σ[12,3], {α, bas}]
```

Out[•]=

$$\text{EDK}[1,2,\text{AW}[x,x,x]] \to \text{EDK}[1,3,\text{AW}[x,x,x]] + \text{EDK}[2,3,\text{AW}[x,x,x]]$$
$$\text{EDK}[1,2,\text{AW}[x,x,y]] \to \text{EDK}[1,3,\text{AW}[x,x,y]] + \text{EDK}[2,3,\text{AW}[x,x,y]]$$
$$\text{EDK}[1,2,\text{AW}[x,y,x]] \to \text{EDK}[1,3,\text{AW}[x,y,x]] + \text{EDK}[2,3,\text{AW}[x,y,x]]$$
$$\text{EDK}[1,2,\text{AW}[x,y,y]] \to \text{EDK}[1,3,\text{AW}[x,y,y]] + \text{EDK}[2,3,\text{AW}[x,y,y]]$$
$$\text{EDK}[1,2,\text{AW}[y,x,x]] \to \text{EDK}[1,3,\text{AW}[y,x,x]] + \text{EDK}[2,3,\text{AW}[y,x,x]]$$
$$\text{EDK}[1,2,\text{AW}[y,x,y]] \to \text{EDK}[1,3,\text{AW}[y,x,y]] + \text{EDK}[2,3,\text{AW}[y,x,y]]$$
$$\text{EDK}[1,2,\text{AW}[y,y,x]] \to \text{EDK}[1,3,\text{AW}[y,y,x]] + \text{EDK}[2,3,\text{AW}[y,y,x]]$$
$$\text{EDK}[1,2,\text{AW}[y,y,y]] \to \text{EDK}[1,3,\text{AW}[y,y,y]] + \text{EDK}[2,3,\text{AW}[y,y,y]]$$
$$\text{EDK}\left[1,\overline{x\ \overline{x\ xy}}\right] \to \text{EDK}\left[1,\overline{x\ \overline{x\ xy}}\right] + \text{EDK}\left[2,\overline{x\ \overline{x\ xy}}\right] + \text{EDK}[1,2,-2\,\text{AW}[x,x,y] + 2\,\text{AW}[y,x,x]]$$
$$\text{EDK}\left[1,\overline{x\ \overline{xy\ y}}\right] \to \text{EDK}\left[1,\overline{x\ \overline{xy\ y}}\right] + \text{EDK}\left[2,\overline{x\ \overline{xy\ y}}\right] + \text{EDK}[1,2,-\text{AW}[x,x,y] - \text{AW}[x,y,y] + \text{AW}[y,x,x] + \text{AW}[y,y,x]]$$
$$\text{EDK}\left[1,\overline{\overline{xy\ y}\ y}\right] \to \text{EDK}\left[1,\overline{\overline{xy\ y}\ y}\right] + \text{EDK}\left[2,\overline{\overline{xy\ y}\ y}\right] + \text{EDK}[1,2,-2\,\text{AW}[x,y,y] + 2\,\text{AW}[y,y,x]]$$
$$\text{EDK}\left[2,\overline{x\ \overline{x\ xy}}\right] \to \text{EDK}\left[3,\overline{x\ \overline{x\ xy}}\right]$$
$$\text{EDK}\left[2,\overline{x\ \overline{xy\ y}}\right] \to \text{EDK}\left[3,\overline{x\ \overline{xy\ y}}\right]$$
$$\text{EDK}\left[2,\overline{\overline{xy\ y}\ y}\right] \to \text{EDK}\left[3,\overline{\overline{xy\ y}\ y}\right]$$

In[•]:=
```
bas = EDKBasis[{x, y}, 2, 4];
Column@Table[α → α^σ[1,23], {α, bas}]
```

Out[•]=

$$\text{EDK}[1,2,\text{AW}[x,x,x]] \to \text{EDK}[1,2,\text{AW}[x,x,x]] + \text{EDK}[1,3,\text{AW}[x,x,x]]$$
$$\text{EDK}[1,2,\text{AW}[x,x,y]] \to \text{EDK}[1,2,\text{AW}[x,x,y]] + \text{EDK}[1,3,\text{AW}[x,x,y]]$$
$$\text{EDK}[1,2,\text{AW}[x,y,x]] \to \text{EDK}[1,2,\text{AW}[x,y,x]] + \text{EDK}[1,3,\text{AW}[x,y,x]]$$
$$\text{EDK}[1,2,\text{AW}[x,y,y]] \to \text{EDK}[1,2,\text{AW}[x,y,y]] + \text{EDK}[1,3,\text{AW}[x,y,y]]$$
$$\text{EDK}[1,2,\text{AW}[y,x,x]] \to \text{EDK}[1,2,\text{AW}[y,x,x]] + \text{EDK}[1,3,\text{AW}[y,x,x]]$$
$$\text{EDK}[1,2,\text{AW}[y,x,y]] \to \text{EDK}[1,2,\text{AW}[y,x,y]] + \text{EDK}[1,3,\text{AW}[y,x,y]]$$
$$\text{EDK}[1,2,\text{AW}[y,y,x]] \to \text{EDK}[1,2,\text{AW}[y,y,x]] + \text{EDK}[1,3,\text{AW}[y,y,x]]$$
$$\text{EDK}[1,2,\text{AW}[y,y,y]] \to \text{EDK}[1,2,\text{AW}[y,y,y]] + \text{EDK}[1,3,\text{AW}[y,y,y]]$$
$$\text{EDK}\left[1,\overline{x\ \overline{x\ xy}}\right] \to \text{EDK}\left[1,\overline{x\ \overline{x\ xy}}\right]$$
$$\text{EDK}\left[1,\overline{x\ \overline{xy\ y}}\right] \to \text{EDK}\left[1,\overline{x\ \overline{xy\ y}}\right]$$
$$\text{EDK}\left[1,\overline{\overline{xy\ y}\ y}\right] \to \text{EDK}\left[1,\overline{\overline{xy\ y}\ y}\right]$$
$$\text{EDK}\left[2,\overline{x\ \overline{x\ xy}}\right] \to \text{EDK}\left[2,\overline{x\ \overline{x\ xy}}\right] + \text{EDK}\left[3,\overline{x\ \overline{x\ xy}}\right] + \text{EDK}[2,3,-2\,\text{AW}[x,x,y] + 2\,\text{AW}[y,x,x]]$$
$$\text{EDK}\left[2,\overline{x\ \overline{xy\ y}}\right] \to \text{EDK}\left[2,\overline{x\ \overline{xy\ y}}\right] + \text{EDK}\left[3,\overline{x\ \overline{xy\ y}}\right] + \text{EDK}[2,3,-\text{AW}[x,x,y] - \text{AW}[x,y,y] + \text{AW}[y,x,x] + \text{AW}[y,y,x]]$$
$$\text{EDK}\left[2,\overline{\overline{xy\ y}\ y}\right] \to \text{EDK}\left[2,\overline{\overline{xy\ y}\ y}\right] + \text{EDK}\left[3,\overline{\overline{xy\ y}\ y}\right] + \text{EDK}[2,3,-2\,\text{AW}[x,y,y] + 2\,\text{AW}[y,y,x]]$$

```
In[•]:= bas = EDKBasis[{x, y}, 2, 2];
        Column@Table[α → α^σ[3,12], {α, bas}]
```

```
Out[•]=
EDK[1, 2, AW[x]] → EDK[1, 3, -AW[x]] + EDK[2, 3, -AW[x]]
EDK[1, 2, AW[y]] → EDK[1, 3, -AW[y]] + EDK[2, 3, -AW[y]]
EDK[1, 2, xy] → EDK[3, xy]
EDK[2, xy] → EDK[1, xy] + EDK[2, xy]
```

```
In[•]:= bas = EDKBasis[{x, y}, 2, 3];
        Column@Table[α → α^σ[3,12], {α, bas}]
```

```
Out[•]=
EDK[1, 2, AW[x, x]] → EDK[1, 3, AW[x, x]] + EDK[2, 3, AW[x, x]]
EDK[1, 2, AW[x, y]] → EDK[1, 3, AW[y, x]] + EDK[2, 3, AW[y, x]]
EDK[1, 2, AW[y, x]] → EDK[1, 3, AW[x, y]] + EDK[2, 3, AW[x, y]]
EDK[1, 2, AW[y, y]] → EDK[1, 3, AW[y, y]] + EDK[2, 3, AW[y, y]]
EDK[1, x xy] → EDK[3, x xy]
EDK[1, xyy] → EDK[3, xyy]
EDK[2, x xy] → EDK[1, x xy] + EDK[2, x xy] + EDK[1, 2, -AW[x, y] - AW[y, x]]
EDK[2, xyy] → EDK[1, xyy] + EDK[2, xyy] + EDK[1, 2, -AW[x, y] - AW[y, x]]
```

```
In[•]:= bas1 = EDKBasis[{x, y}, 2, 4];
        bas2 = EDKBasis[{x, y}, 2, 4];
        s = σ[12, 3];
        Column@
         DeleteCases[_ → 0][Flatten@Table[{α, β} → b[α, β]^s - b[α^s, β^s], {α, bas1}, {β, bas2}]]
```

```
Out[•]=
```

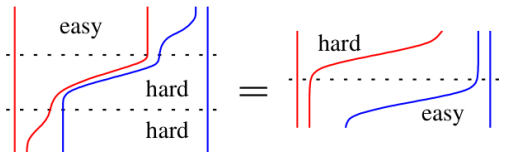
```
In[•]:= bas1 = EDKBasis[{x, y}, 2, 4];
        bas2 = EDKBasis[{x, y}, 2, 4];
        s = σ[1, 23];
        Column@
         DeleteCases[_ → 0][Flatten@Table[{α, β} → b[α, β]^s - b[α^s, β^s], {α, bas1}, {β, bas2}]]
```

```
Out[•]=
```

```
In[•]:= bas1 = EDKBasis[{x, y}, 2, 4];
        bas2 = EDKBasis[{x, y}, 2, 4];
        s = σ[23, 1];
        Column@
         DeleteCases[_ → 0][Flatten@Table[{α, β} → b[α, β]^s - b[α^s, β^s], {α, bas1}, {β, bas2}]]
```

```
Out[•]=
```

The Full Pole-Strand “Operad”.



Goal: Implement the pentagon,

$\Phi \star\star \Phi^{\sigma[y\to y+1,,2]} \star\star \Phi^{\sigma[x\to y,y\to 1,,2]} == \Phi^{\sigma[x\to x+y,y\to 1,,2]} \star\star \Phi^{\sigma[12]}$.

A more modest goal: Implement $\text{ps}\Delta_s[y]$, which shifts all the strands $(1\,..\,s)$ one unit to the right and creates and new strand 1 by doubling pole y.

The pentagon will now be:

$\Phi \star\star \text{ps}\Delta[y][\Phi] \star\star \ldots$

In[•]:=
```
psΔ[_][0] = 0;
x_Plus // psΔ[y_] := psΔ[y] /@ x;
EDK[k_, x_] // psΔ[y_] := EDK[k + 1, x] + EDK[1, k + 1, τ[LW@y, x]];
EDK[j_, k_, x_] // psΔ[_] := EDK[j + 1, k + 1, x];
```

In[•]:=
```
bas = EDKBasis[{x, y}, 2, 3];
Table[α → psΔ[y][α], {α, bas}] // Column
```

Out[•]=

$$\text{EDK}[1,2,\text{AW}[x,x]] \to \text{EDK}[2,3,\text{AW}[x,x]]$$
$$\text{EDK}[1,2,\text{AW}[x,y]] \to \text{EDK}[2,3,\text{AW}[x,y]]$$
$$\text{EDK}[1,2,\text{AW}[y,x]] \to \text{EDK}[2,3,\text{AW}[y,x]]$$
$$\text{EDK}[1,2,\text{AW}[y,y]] \to \text{EDK}[2,3,\text{AW}[y,y]]$$
$$\text{EDK}\left[1,\overline{x\ \overline{xy}}\right] \to \text{EDK}\left[2,\overline{x\ \overline{xy}}\right] + \text{EDK}[1,2,\text{AW}[x,x]]$$
$$\text{EDK}\left[1,\overline{\overline{xy}\,y}\right] \to \text{EDK}\left[2,\overline{\overline{xy}\,y}\right] + \text{EDK}[1,2,\text{AW}[x,y]] - 2\,\text{AW}[y,x]]$$
$$\text{EDK}\left[2,\overline{x\ \overline{xy}}\right] \to \text{EDK}\left[3,\overline{x\ \overline{xy}}\right] + \text{EDK}[1,3,\text{AW}[x,x]]$$
$$\text{EDK}\left[2,\overline{\overline{xy}\,y}\right] \to \text{EDK}\left[3,\overline{\overline{xy}\,y}\right] + \text{EDK}[1,3,\text{AW}[x,y]] - 2\,\text{AW}[y,x]]$$

In[•]:=
```
bas1 = EDKBasis[{x, y}, 2, 3];
bas2 = EDKBasis[{x, y}, 2, 4];
Column@DeleteCases[_ → 0][Flatten@
   Table[{α, β} → psΔ[y][b[α, β]] - b[psΔ[y][α], psΔ[y][β]], {α, bas1}, {β, bas2}]]
```

Out[•]=

```
In[•]:= LM[___][0] = 0;
        x_Plus // lm_LM := lm /@ x;
        lm_LM[c_ w_LW] := Expand[c lm[w]];
        lm_LM[c_ w_AW] := Expand[c lm[w]];
        lm_LM[EDK[jk__, x_]] := EDK[jk, Expand[lm[x]]];
        LM[rules___][LW[w_]] := Distribute[LW[w] /. {rules}] /. LW[0] → 0;
        LM[rules___][w_LW] :=
          (LM[rules][w] = b @@ (LM[rules] /@ LyndonFactorization[w]));
        LM[rules___][AW[]] = AW[];
        LM[rules___][AW[w_]] := Distribute[AW[w] /. {rules}] /. AW[0] → 0;
        LM[rules___][w_AW] := LM[rules][w] = Module[{w1, w2},
            w1 = Take[w, Floor[Length[w] / 2]];
            w2 = Drop[w, Floor[Length[w] / 2]];
            LM[rules][w1] ** LM[rules][w2]
           ];
```

```
In[•]:= LM[x → 0][LW[x, y]]
Out[•]=
        0
```

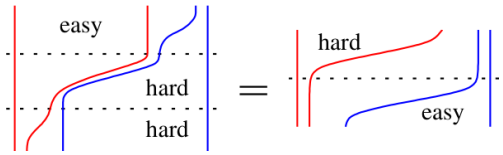
```
In[•]:= LM[x → x + y, y → z][LW[x]]
Out[•]=
        x̄ + ȳ
```

```
In[•]:= LM[x → x + y, y → z][LW[y]]
Out[•]=
        z̄
```

```
In[•]:= LM[x → x + y, y → z][LW[x, y]]
Out[•]=
        x̄ z̄ + ȳ z̄
```

```
In[•]:= LM[x → x + y, y → z][AW[x, x, y, x]]
Out[•]=
        AW[x, x, z, x] + AW[x, x, z, y] + AW[x, y, z, x] + AW[x, y, z, y] +
         AW[y, x, z, x] + AW[y, x, z, y] + AW[y, y, z, x] + AW[y, y, z, y]
```

```
In[•]:= AW[x, y, x] // LM[x → x + y, y → z] // LM[z → z1 + z2]
Out[•]=
        AW[x, z1, x] + AW[x, z1, y] + AW[x, z2, x] + AW[x, z2, y] +
         AW[y, z1, x] + AW[y, z1, y] + AW[y, z2, x] + AW[y, z2, y]
```



$\Phi^{**}\Phi^{\sigma[y\to y+1,,2]}{}^{**}\Phi^{\sigma[x\to y,y\to 1,,2]} == \Phi^{\sigma[x\to x+y,y\to 1,,2]}{}^{**}\Phi^{\sigma[12]}$.

In[•]:=
```
bas = EDKBasis[{x, y}, 1, 4]
```

Out[•]=

$$\left\{\text{EDK}\left[1, \overline{x\ \overline{x\ \overline{xy}}}\right], \text{EDK}\left[1, \overline{x\ \overline{\overline{xy}\,y}}\right], \text{EDK}\left[1, \overline{\overline{\overline{xy}\,y}\,y}\right]\right\}$$

```
In[•]:= Table[α → Column@{α, α // psΔ[y], α // psΔ[y] // LM[x → y, y → 0],
          α // psΔ[y] // LM[x → x + y, y → 0], α^σ[12]}, {α, bas}] // Column
```

Out[•]=

$$\text{EDK}\left[1, \overline{x\ \overline{x\ \overline{xy}}}\right] \to$$

$$\text{EDK}\left[1, \overline{x\ \overline{x\ \overline{xy}}}\right]$$
$$\text{EDK}\left[2, \overline{x\ \overline{x\ \overline{xy}}}\right] + \text{EDK}[1, 2, \text{AW}[x, x, x]]$$
$$\text{EDK}[1, 2, \text{AW}[y, y, y]]$$
$$\text{EDK}[1, 2, \text{AW}[x, x, x] + \text{AW}[x, x, y] + \text{AW}[x, y, x] + \text{AW}[x, y, y] + \text{AW}[y, x, x] + \text{AW}[y, x, y] + \text{AW}[y, y, x] + \text{AW}[y, y, y]]$$
$$\text{EDK}\left[1, \overline{x\ \overline{x\ \overline{xy}}}\right] + \text{EDK}\left[2, \overline{x\ \overline{x\ \overline{xy}}}\right] + \text{EDK}[1, 2, -2\,\text{AW}[x, x, y] + 2\,\text{AW}[y, x, x]]$$

$$\text{EDK}\left[1, \overline{x\ \overline{\overline{xy}\,y}}\right] \to$$

$$\text{EDK}\left[1, \overline{x\ \overline{\overline{xy}\,y}}\right]$$
$$\text{EDK}\left[2, \overline{x\ \overline{\overline{xy}\,y}}\right] + \text{EDK}[1, 2, \text{AW}[x, x, y] - 2\,\text{AW}[x, y, x]]$$
$$0$$
$$0$$
$$\text{EDK}\left[1, \overline{x\ \overline{\overline{xy}\,y}}\right] + \text{EDK}\left[2, \overline{x\ \overline{\overline{xy}\,y}}\right] + \text{EDK}[1, 2, -\text{AW}[x, x, y] - \text{AW}[x, y, y] + \text{AW}[y, x, x] + \text{AW}[y, y, x]]$$

$$\text{EDK}\left[1, \overline{\overline{xy}\,y\ y}\right] \to \begin{array}{l} \text{EDK}\left[1, \overline{\overline{\overline{xy}\,y}\ y}\right] \\ \text{EDK}\left[2, \overline{\overline{\overline{xy}\,y}\ y}\right] + \text{EDK}[1, 2, \text{AW}[x, y, y] - 3\,\text{AW}[y, x, y] + 3\,\text{AW}[y, y, x]] \\ 0 \\ 0 \\ \text{EDK}\left[1, \overline{\overline{\overline{xy}\,y}\ y}\right] + \text{EDK}\left[2, \overline{\overline{\overline{xy}\,y}\ y}\right] + \text{EDK}[1, 2, -2\,\text{AW}[x, y, y] + 2\,\text{AW}[y, y, x]] \end{array}$$

```
In[•]:= Pentagon[α_] := α + (α // psΔ[y]) +
          (α // psΔ[y] // LM[x → y, y → 0]) - (α // psΔ[y] // LM[x → x + y, y → 0]) - α^σ[12]
```

In[•]:=
```
Pentagon /@ EDKBasis[{x, y}, 1, 5]
```

Out[•]=
```
{EDK[1, 2, 2 AW[x, x, x, y] - 3 AW[x, x, y, x] - AW[x, x, y, y] - 3 AW[x, y, x, x] -
    AW[x, y, x, y] - AW[x, y, y, x] - AW[x, y, y, y] + 2 AW[y, x, x, x] - AW[y, x, x, y] -
    AW[y, x, y, x] - AW[y, x, y, y] - AW[y, y, x, x] - AW[y, y, x, y] - AW[y, y, y, x]],
 EDK[1, 2, 2 AW[x, x, x, y] - 3 AW[x, x, y, x] + 2 AW[x, x, y, y] - AW[x, y, x, x] -
    2 AW[x, y, x, y] - 2 AW[x, y, y, x] + AW[y, x, x, x] - 2 AW[y, x, y, x] + 2 AW[y, y, x, x]],
 EDK[1, 2, -4 AW[x, y, x, x] - 2 AW[x, y, x, y] - 2 AW[x, y, y, x] +
    2 AW[y, x, x, x] + 4 AW[y, x, x, y] - 2 AW[y, x, y, x]],
 EDK[1, 2, 3 AW[x, x, y, y] - 3 AW[x, y, x, y] - AW[x, y, y, x] +
    AW[x, y, y, y] + 2 AW[y, y, x, x] + AW[y, y, y, x]],
 EDK[1, 2, -AW[x, y, x, y] + AW[x, y, y, x] - 3 AW[y, x, x, y] +
    2 AW[y, x, y, x] - AW[y, x, y, y] - AW[y, y, x, x] - AW[y, y, x, y]],
 EDK[1, 2, 4 AW[x, y, y, y] - 6 AW[y, x, y, y] + 4 AW[y, y, x, y] - AW[y, y, y, x]]}
```

In[•]:=
```
EDKBasis[{x, y}, 2, 5]
```

Out[•]=
```
{EDK[1, 2, AW[x, x, x, x]], EDK[1, 2, AW[x, x, x, y]],
 EDK[1, 2, AW[x, x, y, x]], EDK[1, 2, AW[x, x, y, y]], EDK[1, 2, AW[x, y, x, x]],
 EDK[1, 2, AW[x, y, x, y]], EDK[1, 2, AW[x, y, y, x]], EDK[1, 2, AW[x, y, y, y]],
 EDK[1, 2, AW[y, x, x, x]], EDK[1, 2, AW[y, x, x, y]], EDK[1, 2, AW[y, x, y, x]],
 EDK[1, 2, AW[y, x, y, y]], EDK[1, 2, AW[y, y, x, x]], EDK[1, 2, AW[y, y, x, y]],
 EDK[1, 2, AW[y, y, y, x]], EDK[1, 2, AW[y, y, y, y]], EDK[1, x x x x y],
 EDK[1, x x x y y], EDK[1, x x y x y], EDK[1, x x y y y],
 EDK[1, x y x y], EDK[1, x y y x y], EDK[2, x x x x y], EDK[2, x x x y y],
 EDK[2, x x y x y], EDK[2, x x y y y]}
```

In[•]:=
```
AllWords[4, {x, y}]
```

Out[•]=
```
{AW[x, x, x, x], AW[x, x, x, y], AW[x, x, y, x], AW[x, x, y, y], AW[x, y, x, x], AW[x, y, x, y],
 AW[x, y, y, x], AW[x, y, y, y], AW[y, x, x, x], AW[y, x, x, y], AW[y, x, y, x],
 AW[y, x, y, y], AW[y, y, x, x], AW[y, y, x, y], AW[y, y, y, x], AW[y, y, y, y]}
```

In[•]:=
```
mat[d_] := Module[{p}, Table[
   p = Pentagon[α][[3]];
   Table[Coefficient[p, β], {β, AllWords[d - 1, {x, y}]}],
   {α, EDKBasis[{x, y}, 1, d]}
  ]]
```

```
In[•]:= mat[5] // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 0 & 2 & -3 & -1 & -3 & -1 & -1 & -1 & 2 & -1 & -1 & -1 & -1 & -1 & 0 \\ 0 & 2 & -3 & 2 & -1 & -2 & -2 & 0 & 1 & 0 & -2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & -2 & -2 & 0 & 2 & 4 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & -3 & -1 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & -3 & 2 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & -6 & 0 & 4 & -1 & 0 \end{pmatrix}$$

```
In[•]:= NullSpace[mat[5]^T]
```

Out[•]=

```
{{-2, 2, 1, -2, 2, 0}}
```

```
In[•]:= Table[Echo[d → Timing@Length@NullSpace[mat[d]^T]], {d, 3, 20}]
```

Part: Part specification 0[[3]] is longer than depth of object.

» 3 → {0.015625, 1}

» 4 → {0., 0}

» 5 → {0., 1}

» 6 → {0.015625, 0}

» 7 → {0.03125, 1}

» 8 → {0.078125, 1}

» 9 → {0.234375, 1}

» 10 → {0.71875, 1}

» 11 → {4.42188, 2}

» 12 → {27.0313, 2}

» 13 → {113.891, 3}

» 14 → {407.453, 3}

» 15 → {9948.27, 4}