Pensieve header: Computing $H^4$ of the Drinfel'd-Kohno cohomology, which rules the degree by degree extension of horizontal associators.

(Alt) In[•]:=

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\People\\Kuno"];
<< FreeLie.m
```

```
FreeLie` implements / extends
 {*, +, **, $SeriesShowDegree, ⟨⟩, ∫, ≡, ad, Ad, adSeries, AllCyclicWords, AllLyndonWords,
  AllWords, Arbitrator, AS, ASeries, AW, b, BCH, BooleanSequence, BracketForm, BS, CC, Crop,
  cw, CW, CWS, CWSeries, D, Deg, DegreeScale, DerivationSeries, div, DK, DKS, DKSeries, EulerE,
  Exp, FreeLieFormatting, Inverse, j, J, JA, LieDerivation, LieMorphism, LieSeries, LS, LW,
  LyndonFactorization, Morphism, New, RandomCWSeries, Randomizer, RandomLieSeries, RC, SeriesSolve,
  Support, t, tb, TopBracketForm, tr, UndeterminedCoefficients, αMap, Γ, ι, Λ, σ, τ, ħ, ⟞, ⟝}.

FreeLie` is in the public domain. Dror Bar-Natan is committed
  to support it within reason until July 15, 2022. This is version 240218.
```

(Alt) In[•]:=

```
Basis[n_, m_] := Union @@ Table[DK[j, #] & /@ AllLyndonWords[m, Range[j - 1]], {j, 2, n}]
```

In[•]:=
```
Basis[4, 3]
```

Out[•]=

$$\left\{\mathrm{DK}\left[3,\overline{1\ \overline{12}}\right],\mathrm{DK}\left[3,\overline{\overline{12}2}\right],\mathrm{DK}\left[4,\overline{1\ \overline{12}}\right],\mathrm{DK}\left[4,\overline{1\ \overline{13}}\right],\mathrm{DK}\left[4,\overline{\overline{12}2}\right],\right.$$
$$\left.\mathrm{DK}\left[4,\overline{1\ \overline{23}}\right],\mathrm{DK}\left[4,\overline{\overline{13}2}\right],\mathrm{DK}\left[4,\overline{\overline{13}3}\right],\mathrm{DK}\left[4,\overline{2\ \overline{23}}\right],\mathrm{DK}\left[4,\overline{\overline{23}3}\right]\right\}$$

(Alt) In[•]:=

```
d_n_[ℰ_] := Expand[ℰ /. dk_DK :> Plus[
      dk^σ@@Range[2,n+1],
      Sum[(-1)^i dk^Join[σ@@Range[i-1],σ[{i,i+1}],σ@@Range[i+2,n+1]], {i, n}],
      (-1)^(n+1) dk
     ]]
```

In[•]:=
```
Basis[3, 2] // d_3
```

Out[•]=

$$\{0\}$$

In[•]:=
```
Basis[4, 2] // d_4
```

Out[•]=

$$\left\{\mathrm{DK}\left[3,-\overline{12}\right],\mathrm{DK}\left[5,\overline{12}\right],\mathrm{DK}\left[5,-\overline{14}\right],\mathrm{DK}\left[5,\overline{34}\right]\right\}$$

(Alt) In[•]:=
```
r[n_, m_] := r[n, m] = Module[{dk, B0, B1, b0, b1, db0, mat},
   B0 = Basis[n, m];
   If[B0 == {}, Return[0]];
   B1 = Basis[n + 1, m] /. DK → dk;
   mat = Table[
     db0 = dn[b0] /. DK → dk /. dk[k_, x_Plus] :> (dk[k, #] & /@ x) /.
       dk[k_, c_?NumberQ * x_] :> c * dk[k, x];
     Table[Coefficient[db0, b1], {b1, B1}],
     {b0, B0}
    ];
   MatrixRank[mat]
  ]
```

In[•]:=
```
r[4, 2]
```
Out[•]=
```
4
```

In[•]:=
```
r[3, 2]
```
Out[•]=
```
0
```

(Alt) In[•]:=
```
Betti[n_, m_] := Length[Basis[n, m]] - r[n - 1, m] - r[n, m]
```

In[•]:=
```
Betti[4, 2]
```
Out[•]=
```
0
```

(Alt) In[•]:=
```
Betti[4, 3]
```
(Alt) Out[•]=
```
1
```

In[•]:=
```
Betti[4, 4]
```
Out[•]=
```
0
```

In[•]:=
```
Betti[4, 5]
```
Out[•]=
```
0
```

In[•]:=
```
Betti[4, 6]
```
Out[•]=
```
0
```

(Alt) In[•]:=
```
AbsoluteTiming@Table[Echo@Betti[3, m], {m, 10}]
```
(Alt) Out[•]=
```
{3620.03, {0, 1, 1, 0, 1, 0, 1, 1, 1, 1}}
```

In[•]:=
```
Table[Echo@Betti[3, m], {m, 10}]
```
Out[•]=
```
{0, 1, 1, 0, 1, 0, 1, 1, 1, 1}
```

(Alt) In[•]:=
```
AbsoluteTiming[Betti[3, 11]]
```
(Alt) Out[•]=
```
{41889.2, 2}
```

(Alt) In[•]:=
```
Table[Betti[3, m], {m, 11}]
```
(Alt) Out[•]=
```
{0, 1, 1, 0, 1, 0, 1, 1, 1, 1, 2}
```

In[•]:=
```
Table[Echo@Betti[4, m], {m, 8}]
```
Out[•]=
```
{0, 0, 1, 0, 0, 0, 0, 0}
```

In[•]:=
```
Table[Echo@Betti[4, m], {m, 9}]
```
Out[•]=
```
{0, 0, 1, 0, 0, 0, 0, 0, 0}
```