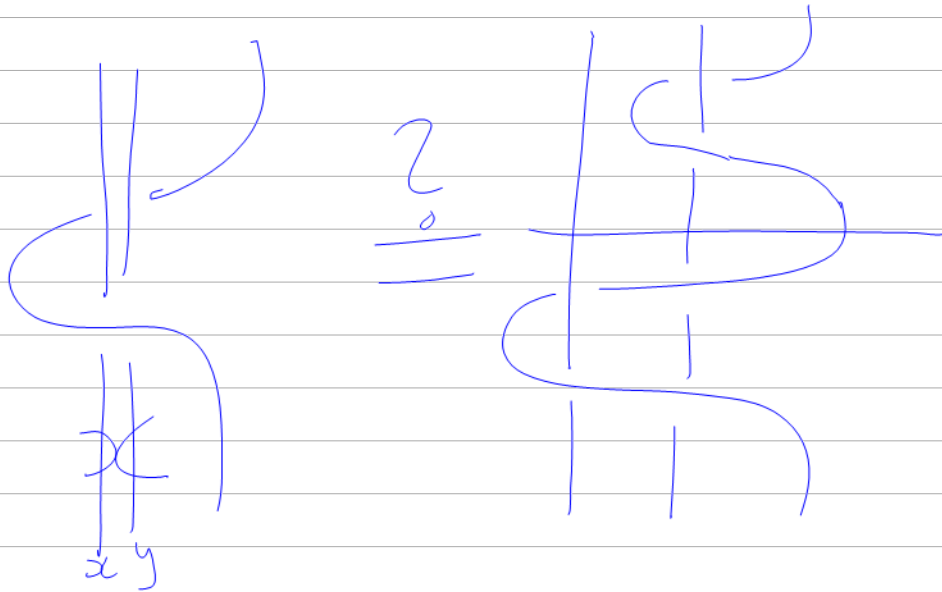
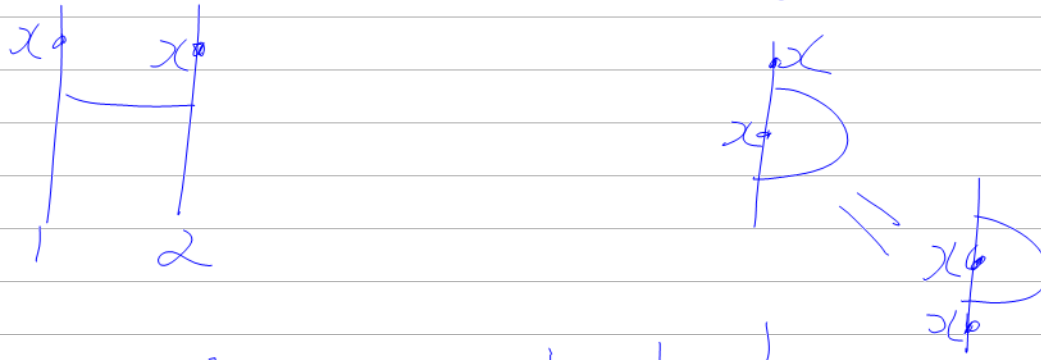
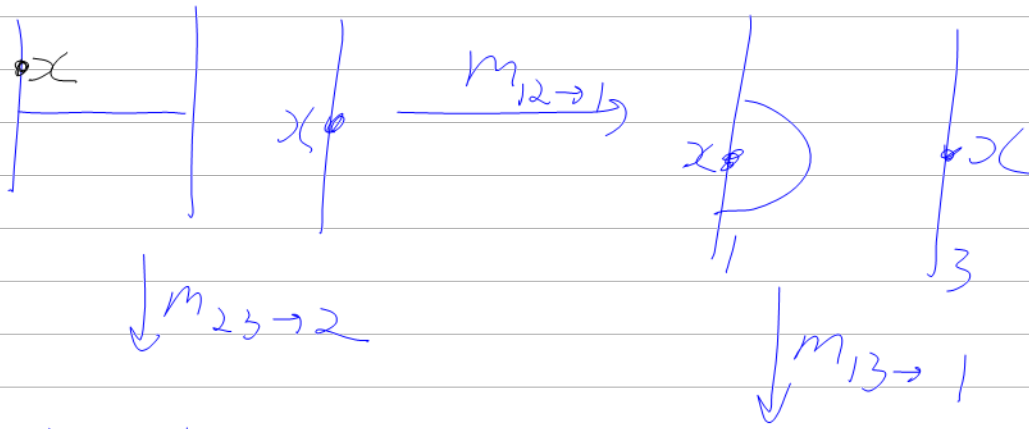


$OAR, \{x, y\}, \{1, 2, 3\} [Ac[1, 2] [AW_1[] AW_2[] AW_3[x] AW_1[x] AW_2[]]]$



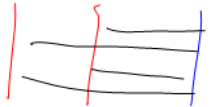
$$FA[x_1, \dots, x_n] \xrightarrow[\Phi_{x_1, \dots, x_n}]{x_i \mapsto \lambda_i} FA[y_1, \dots, y_m]$$

FA: Vect \rightarrow FreeAlg

$$FA[x_1 \rightarrow \sum a_{1j} y_j, x_2 \rightarrow \sum a_{2j} y_j, \dots]$$

$\Phi_{1, 2, 3, 4}$

$\Phi_{\dots\star} = Z \left(\begin{array}{c} | \curvearrowright | \\ \text{PP} \quad \text{S} \end{array} \right), \quad \Phi_{\dots\star\star} = Z \left(\begin{array}{c} | \curvearrowright | \\ \text{PS} \quad \text{S} \end{array} \right), \quad \Phi_{\star\star\star} = Z \left(\begin{array}{c} | \curvearrowright | \\ \text{SS} \quad \text{S} \end{array} \right)$


solved
solved (=1)

Q. Does $\Phi_{\dots\star\star}$ make sense?
 If so, is it related to $\Phi_{\dots\star}$?

Φ^{213}

$\alpha = \begin{array}{c} | \curvearrowright | \\ \text{PP} \quad \text{S} \\ (\bullet \bullet) \star \end{array}, \quad \beta = \begin{array}{c} | \curvearrowright | \\ \text{PS} \quad \text{S} \\ (\bullet \bullet) \star \end{array}, \quad \alpha\beta = \begin{array}{c} | \curvearrowright | \\ \text{SS} \quad \text{S} \\ (\bullet \bullet) \star \end{array} = \Delta \cdot \begin{array}{c} | \curvearrowright | \\ \text{PS} \quad \text{S} \\ \bullet \star \end{array}$

$f^{-1} \exp(x) f \cdot g^{-1} \exp(y) g = \exp(x+y)$

\xleftarrow{Z}

{solutions} $\rightarrow \exp(L)$
 $(f, g) \mapsto g$

Q. How f and g are related?

 $\alpha = \gamma \cdot \beta^{-1}$

$\alpha = \begin{array}{c} | \curvearrowright | \\ \text{PP} \quad \text{S} \\ (\bullet \bullet) \star \end{array} = \begin{array}{c} \text{---} \text{---} \text{---} \\ | \quad | \quad | \\ \bullet \quad \bullet \quad \star \\ \text{---} \text{---} \text{---} \\ | \quad | \quad | \\ \bullet \quad \bullet \quad \star \\ \text{---} \text{---} \text{---} \\ | \quad | \quad | \\ \bullet \quad \bullet \quad \star \\ \text{---} \text{---} \text{---} \\ | \quad | \quad | \\ \bullet \quad \bullet \quad \star \\ \text{---} \text{---} \text{---} \\ | \quad | \quad | \\ \bullet \quad \bullet \quad \star \\ \text{---} \text{---} \text{---} \\ | \quad | \quad | \\ \bullet \quad \bullet \quad \star \end{array}$

$\beta = \begin{array}{c} | \curvearrowright | \\ \text{PS} \quad \text{S} \\ (\bullet \bullet) \star \end{array} = \begin{array}{c} \text{---} \text{---} \text{---} \\ | \quad | \quad | \\ \bullet \quad \bullet \quad \star \\ \text{---} \text{---} \text{---} \\ | \quad | \quad | \\ \bullet \quad \bullet \quad \star \\ \text{---} \text{---} \text{---} \\ | \quad | \quad | \\ \bullet \quad \bullet \quad \star \\ \text{---} \text{---} \text{---} \\ | \quad | \quad | \\ \bullet \quad \bullet \quad \star \\ \text{---} \text{---} \text{---} \\ | \quad | \quad | \\ \bullet \quad \bullet \quad \star \end{array}$

$\exp(x)$

$\Phi_{(\bullet \star) \bullet}^{-1}$

$\exp(\frac{1}{2}y)$

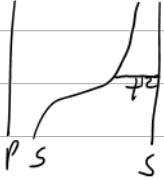
$\Phi_{(\bullet \bullet) \star}$

$\exp(y)$

$\Phi_{(\bullet \bullet) \star}$

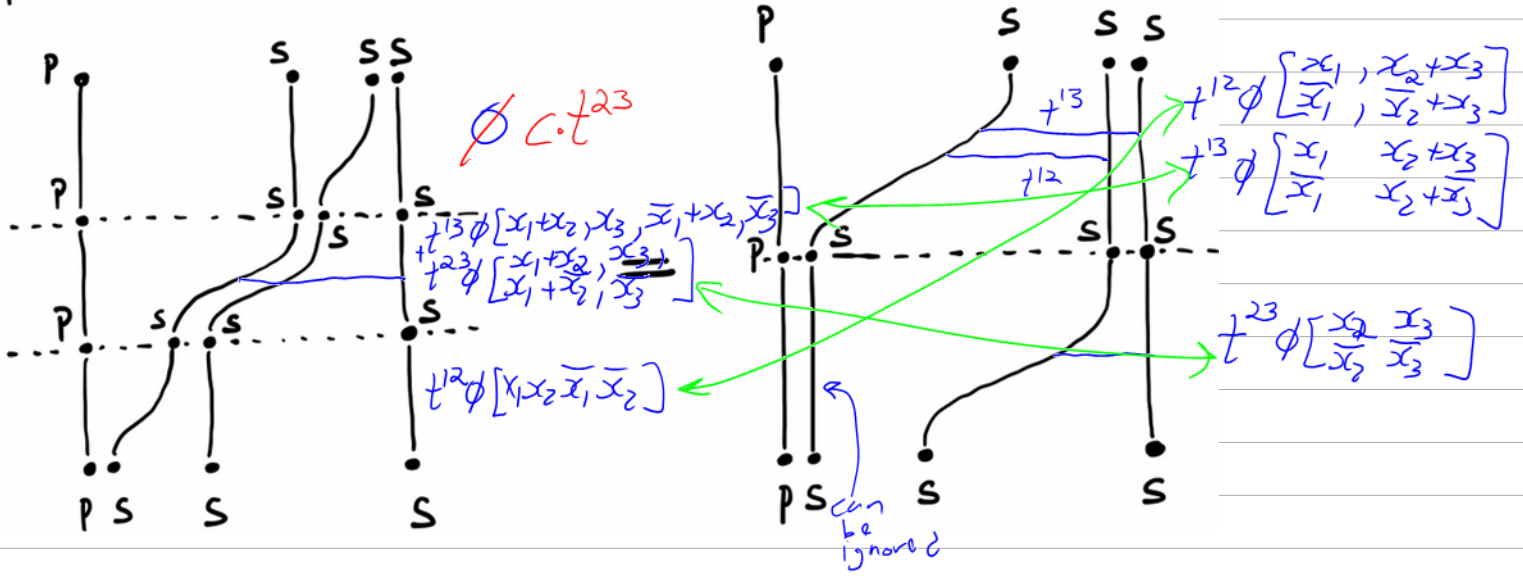
$\xrightarrow{Z} f^{-1}$

$\xrightarrow{Z} g^{-1}$

$$\Phi = \Phi[1, t^{12}\phi]$$


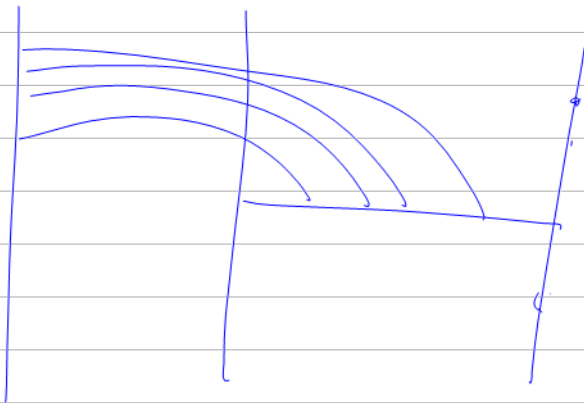
$$t^{12}\phi[x_1, x_2, \bar{x}_1, \bar{x}_2]$$

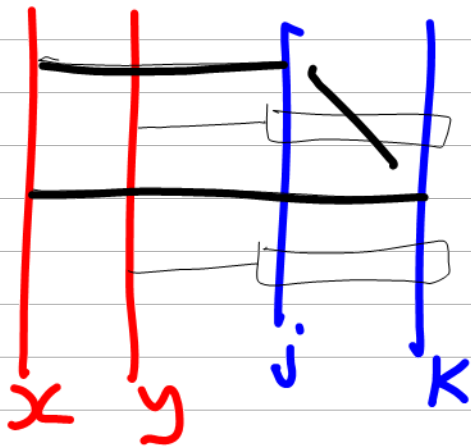
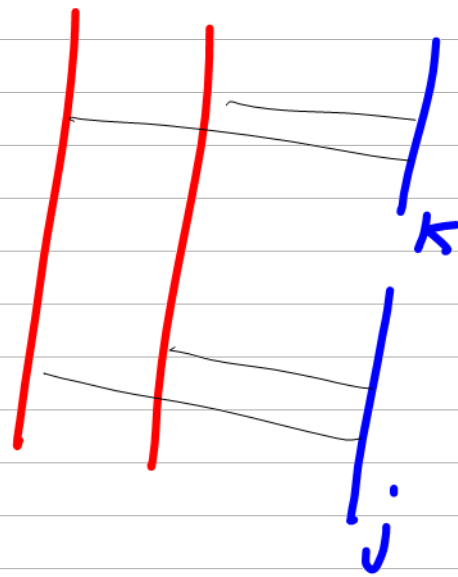
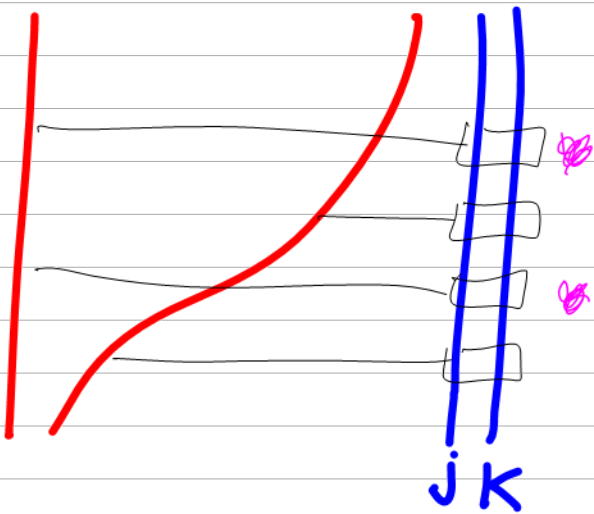
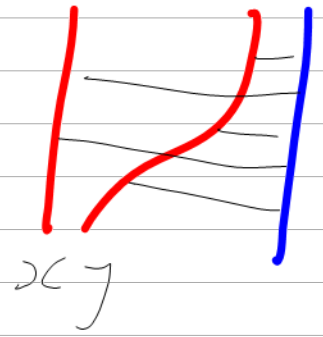
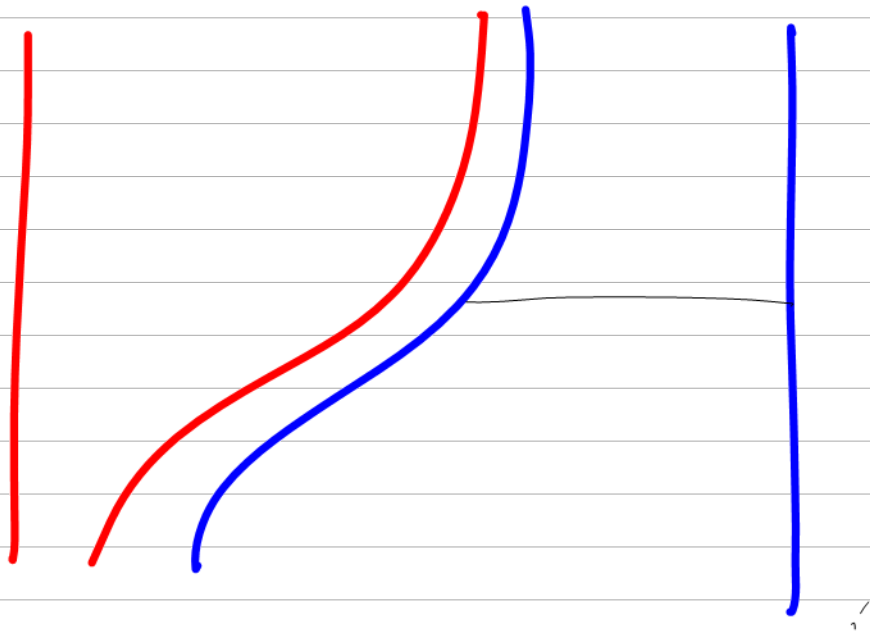
pentagon eq. for pss :



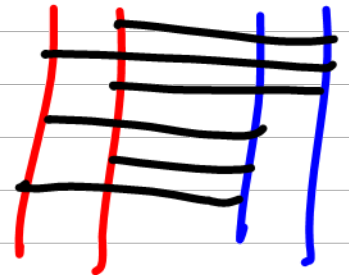
$$t^{12} : \phi[x_1, x_2, \bar{x}_1, \bar{x}_2] = \phi[x_1, x_2+x_3, \bar{x}_1, \bar{x}_2+x_3]$$

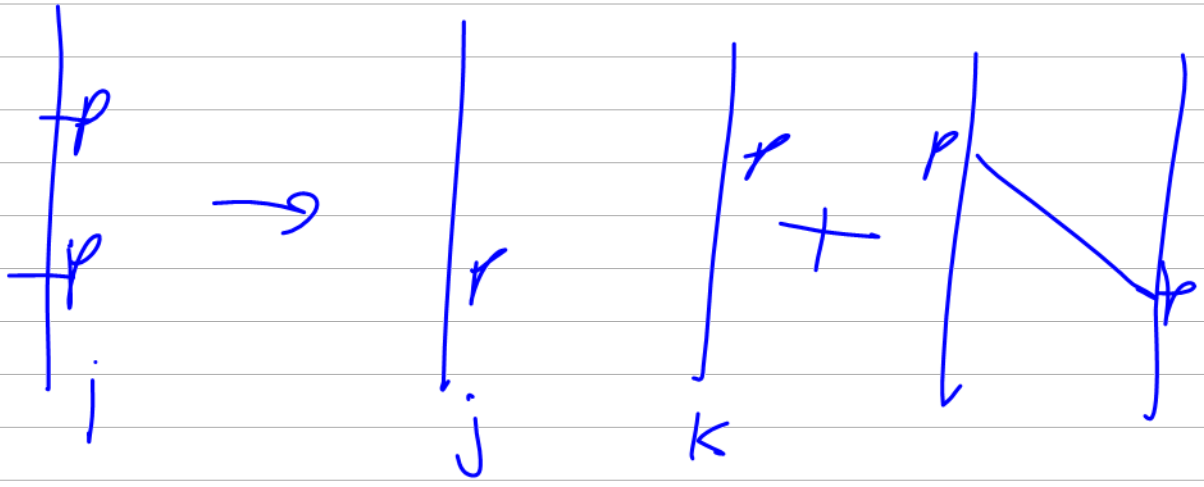
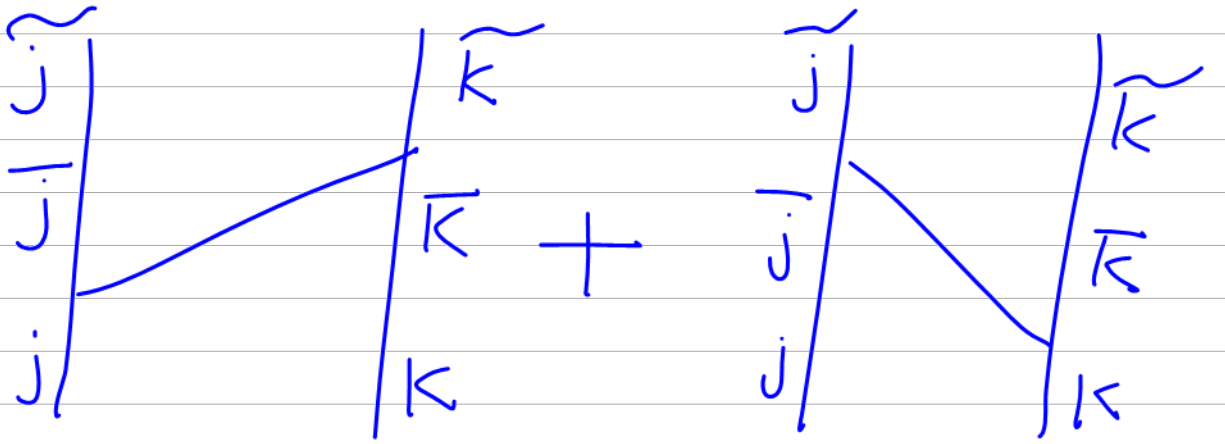
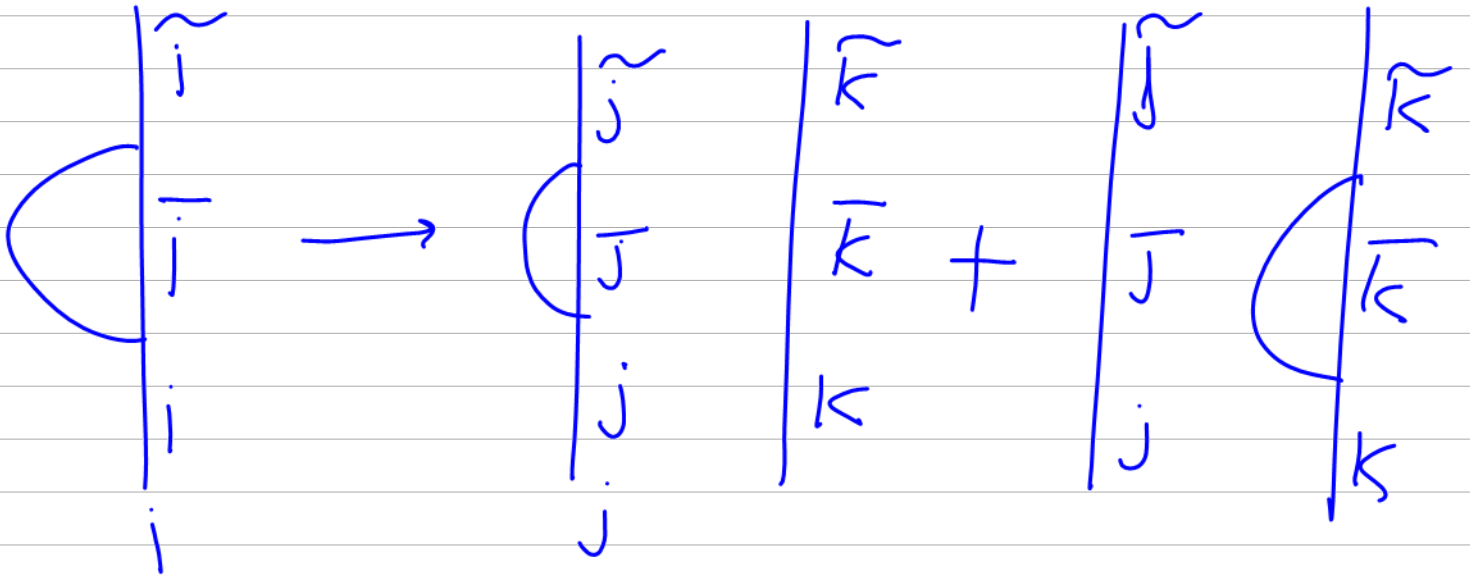
$$\psi = \frac{1}{2}\eta(\bar{x}_2 - x_2) + \beta(\bar{x}_2 - x_2)$$



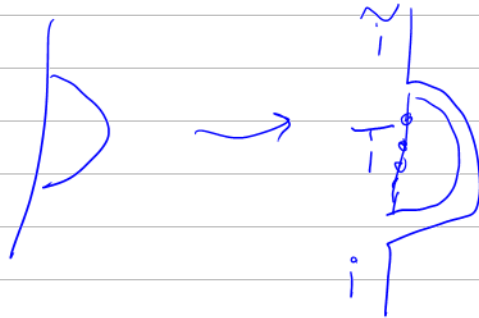


$j: xyx$
 $k: yxy$





$$\begin{array}{c|c}
 y_2 & z_2 \\
 \hline
 y_1 & z_1
 \end{array}
 \quad
 \begin{array}{l}
 y_1 + z_1 = y_2 + z_2 \\
 z_2 = z_1 + y_1 - y_2
 \end{array}$$



$\text{Pert}(\Phi)$
 ~~Φ~~

Φ_{m-1} s.t. $\text{Pert}(\Phi_{m-1}) = 0 + E_m + \dots$
↑
deg m

$\Phi_m = \Phi_{m-1} + \Psi_m$
deg m

$\text{Pert}(\Phi_m) = \text{Pert}(\Phi_{m-1} + \Psi_m)$

$(\Phi_{m-1} + \Psi_m)^{123} (\Phi_{m-1} + \Psi_m)^{1(23)4} (+) (+) (+) - 1$ at deg m

$= E_m + \Psi^{234} - \Psi^{(12)34} + \Psi^{1(23)4} - \Psi^{12(34)} + \Psi^{123} = E_m + d^3 \Psi = 0$

$d^3 \Psi = \sum_{i=0}^{3+1} (-1)^i d_i^3 \Psi$

$d_i^3 \Psi$: add a strand/pole in position i



Need * $d^4 E_m = 0$...

Follow from "associhedra" see my paper on Non-Associative Tangles

* $H^4(\mathcal{G}) = 0$

sometimes works sometimes doesn't.



$$e(x) = 1 + x + \dots \quad (\text{hom}) \quad e(x+y) = e(x)e(y) \quad \text{should not exist!}$$

Suppose e_{m-1} satisfies hom_{m-1} meaning $e_{m-1}(x+y) - e_{m-1}(x)e_{m-1}(y) = 0 + M_m + \dots$

\swarrow a polynomial of deg m

$$e_m = e_{m-1} + C_m$$

to deg $m-1$
 mistake in deg m .

$$\begin{aligned} \text{hom}(e_m) &= (e_{m-1} + C_m)(x+y) - (e_{m-1} + C_m)(x) \cdot (e_{m-1} + C_m)(y) \\ &= M_m + C_m(x+y) - C_m(x) - C_m(y) = M_m - dC_m \end{aligned}$$

Need $* \mathcal{E} \quad * dM_m = 0 \quad * H(\mathcal{E})$

Need an eq'n that M satisfies?

$$\begin{array}{ccc} M_m(x+y, z) & \swarrow e_{m-1}(x+y)e_{m-1}(z) & \searrow M_m(x, y) \\ e_{m-1}(x+y+z) & & e_{m-1}(x)e_{m-1}(y)e_{m-1}(z) \quad \text{in deg } m \\ M_m(x, y+z) & \swarrow e_{m-1}(x)e_{m-1}(y+z) & \searrow M_m(y, z) \end{array}$$

$$d^2 M_m = M_m(y, z) - M_m(x+y, z) + M_m(x, y+z) - M_m(x, y) = 0$$

$$C^k = \mathbb{Q}[x_1, \dots, x_k] \quad d^k: C^k \rightarrow C^{k+1}$$

$$d^k = \sum_{j=0}^{k+1} (-1)^j d_j^k$$

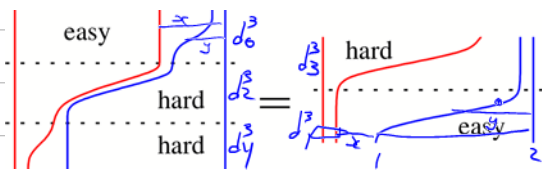
d_j^k : "insert variable in position j "

claim $d^k // d^{k+1} = 0$

Pentagon_d [$\bar{\alpha}$] :=

$$\text{IM}_d [\bar{\alpha} // s\eta_2, \bar{\alpha} // s\sigma_{1 \rightarrow 2} // p\Delta_{y \rightarrow y, z} // p2s_{z \rightarrow 1}, \bar{\alpha} // s\sigma_{1 \rightarrow 2} // p2s_{y \rightarrow 1} // \frac{p\sigma_{x \rightarrow y}}{d_0} // p\eta_x]$$

$$\text{IM}_d [\bar{\alpha} // s\sigma_{1 \rightarrow 2} // p2s_{y \rightarrow 1} // p\Delta_{x \rightarrow x, y}, \bar{\alpha} // s\Delta_{1 \rightarrow 1, 2}]$$



$$d^3 : \mathcal{A}_{PPS} \rightarrow \mathcal{A}_{PPSS} \quad d^3 = \sum_{j=0}^4 (-1)^j d_j^3$$

$$\mathcal{A}_{PPS} = \text{FA} \langle x, y \rangle = \{ \psi \} \quad \text{H}$$

$$d_0^3 \psi = \otimes \psi(y, 0) + t(\partial_y' \psi)(y, 0) \otimes (\partial_y' \psi)(y, 0)$$

$$\mathcal{A}_{PPSS} = \text{FA} \langle x, y \rangle^{\otimes 2} \oplus \text{FA} \langle x, y \rangle^{\otimes 2} [1]$$

$$d_0^3 x^k = \otimes y^k \quad d_0^3 x^k y x^l = t y^l \otimes y^k \quad d_0^3 (\text{otherwise}) = 0$$

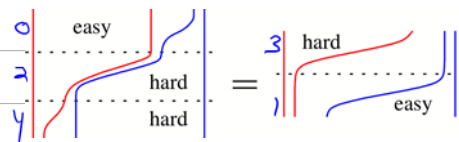
$$d_1^3 \psi = \otimes \psi(x+y, 0) + t(\partial_y' \psi)(x+y, 0) \otimes (\partial_y' \psi)(x+y, 0)$$

$$d_1^3 x^k = \otimes (x+y)^k \quad d_1^3 x^k y x^l = t (x+y)^l \otimes (x+y)^k \quad d_1^3 (\text{otherwise}) = 0$$

$$d_2^3 \psi = \otimes \psi + t(\partial_y'' \psi) \otimes (\partial_y' \psi)$$

$$d_3^3 \psi = \Delta \psi // \emptyset$$

$$d_4^3 \psi = \psi \otimes 1$$



$$d^n : \mathcal{A}_{PPS^{n-2}} \rightarrow \mathcal{A}_{PPS^{n-1}}$$

$$d^n = \sum_{j=0}^{n+1} (-1)^j d_j^n$$

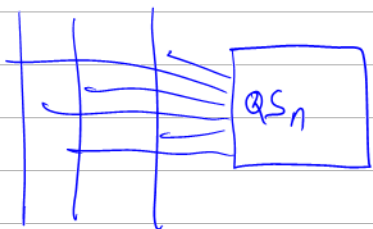
d_0^n : "Add pole on left" & convert

d_1^n : Double pole #1 & convert

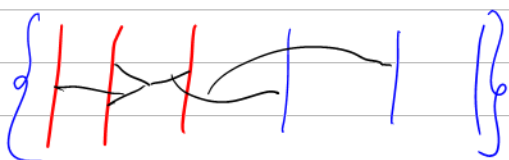
d_2^n : Double pole #2 & convert

d_j^n for $3 \leq j \leq n$ double strand $j-2$

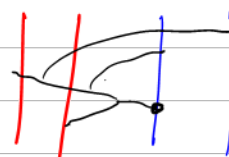
d_{n+1}^n : add strand on right



$$d^p_{\{x_1, x_2, x_3\}, \{1, 2, 3\}} =$$



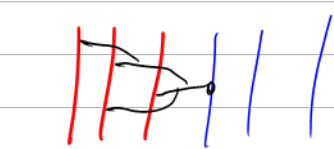
can touch blue at most twice.



Doesn't touch blue

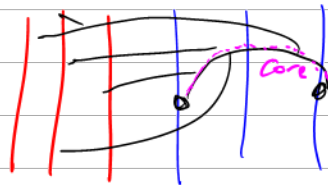


Touches blue once

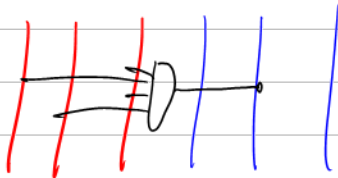


$$\oplus_{SS} FL\langle P_S \rangle$$

Touches blue twice

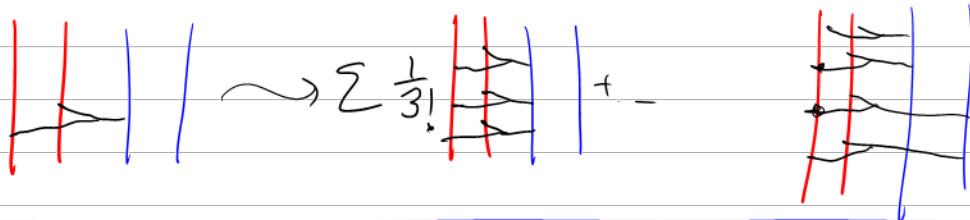


$$\oplus_{P_1 < P_2} FA\langle P_S \rangle [1]$$

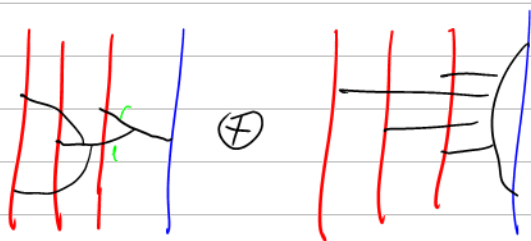


$$\oplus_{SS} FA\langle P_S \rangle [1]$$

Div dV

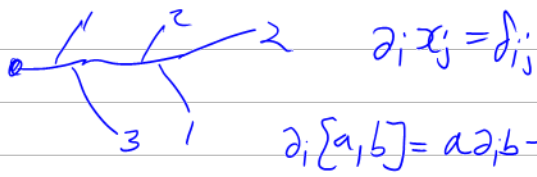


claims there is a funny Lie algebra structure on $FL\langle P_S \rangle \oplus FA\langle P_S \rangle [1]$:
 Funny := not canonical.



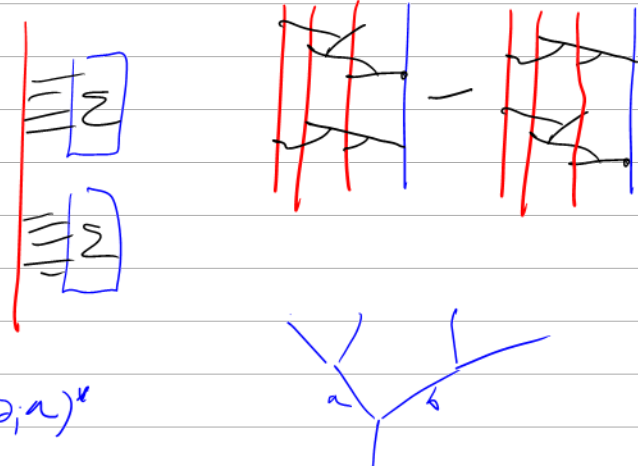
$$0 \rightarrow FA[1] \xrightarrow{\text{Abelian}} \text{Funny} \xrightarrow{\text{Lie}} FL \rightarrow 0$$

$$\partial_i: FL(x_1 \dots x_n) \xrightarrow{\text{deg}} FA(x_1 \dots x_n)$$

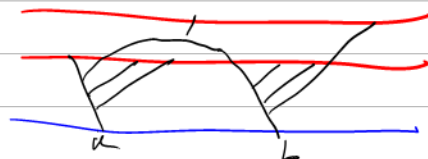
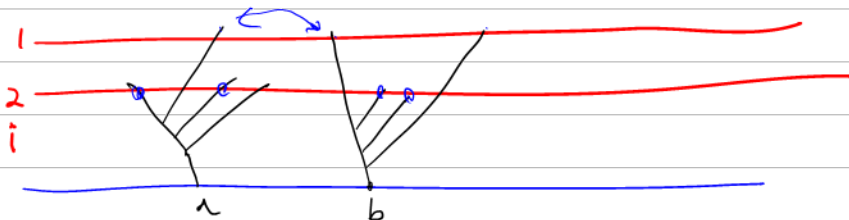


$$\partial_i x_j = \delta_{ij}$$

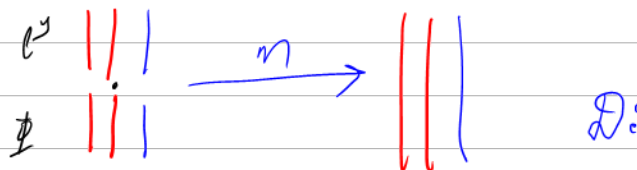
$$\partial_i [a, b] = a \partial_i b - b \partial_i a$$



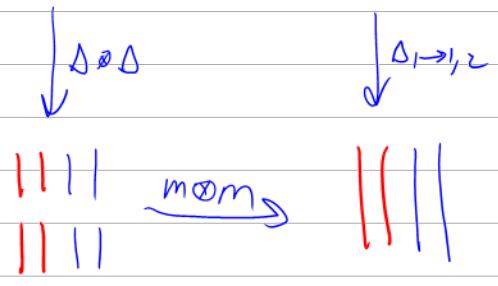
$$[a, b]_F := [a, b]_L + \sum_i (\partial_i a) x_i (\partial_i b)^* - (\partial_i b) x_i (\partial_i a)^*$$



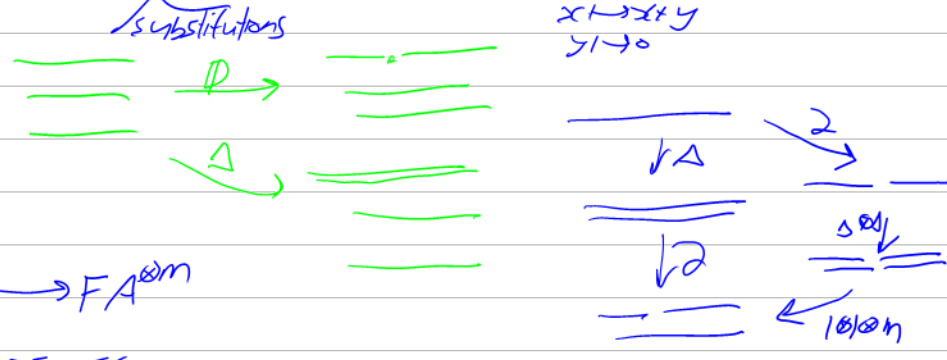




Sum [Expand [A (AW_{u1}[P] AW_{u2}[] - AW_{u1}[] AW_{u2}[P])] // D [P] i→i, ī // D [P] ī→i, i // Δ_{i→j, k} // Δ_{ī→j̄, k̄} // Δ_{i→j̄, k̄} // Δ_{ī→j, k} // m_{j, j̄→j} // σ_{j̄→j} // m_{k̄, k̄→k} // m_{j, u1→j} // m_{u2, j̄→j̄}, {p, ps}]



230406 The AKKN operational envelope: Cuts, then doubles (and reversals), then infusion of constants, then merges.



$D/D = D/D/D$
 $20m/20m = 2000mm = 2000mm$

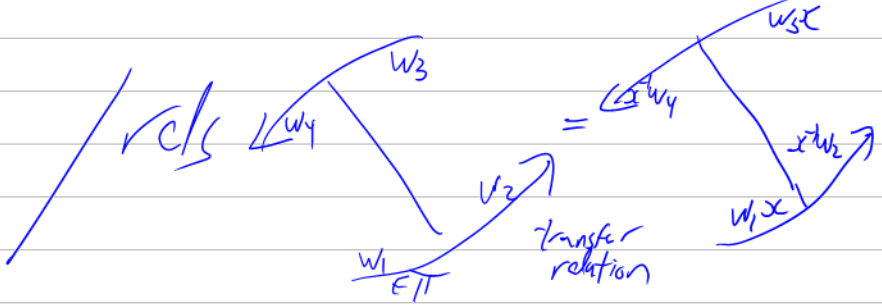
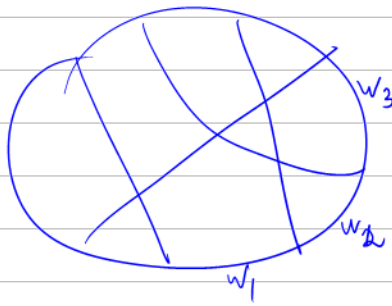
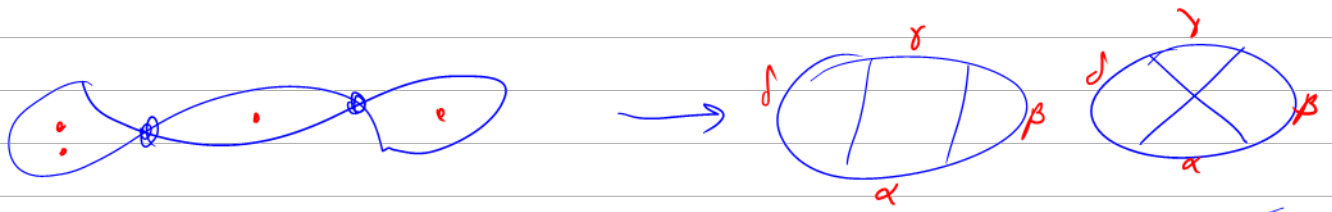
$FA^{\otimes n} \rightarrow FA^{\otimes m}$
 $22.. \Delta 055.. CC mmm$

Theorem (FA-level) Any composition of the operations $\partial_x, \Delta, S, FA_{\text{substitutions}}, C, m$ can be written uniquely in this order:

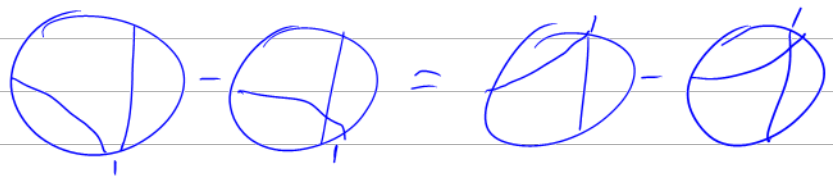
$(\partial_x \partial_{y|_{i \rightarrow j, k}} \dots) (\Delta \Delta \dots S S) (\text{substitutions}) C (m m m m)$

$FA \xrightarrow{SUS} FA \xrightarrow{\partial} FA \otimes FA$
 $\underline{xyyx} \rightarrow \underline{abba} \rightarrow$

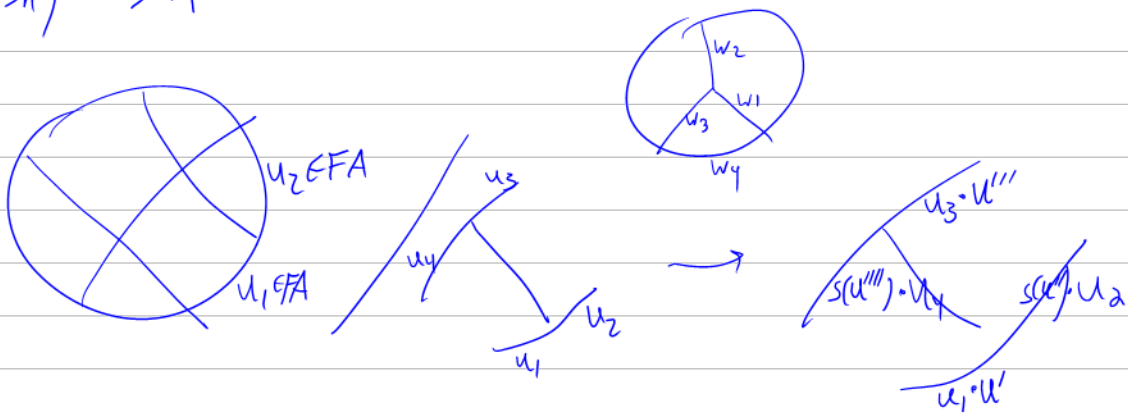
$K^{(n)}$ 



Q) Is \mathbb{A}^1 an expansion



$K(PDS_n) \rightarrow \mathbb{A}^1$



$\pi \left(\text{disk with } d \text{ chords} \right) / \text{transfer} = \pi^{d+1} / \mathbb{A}^1$

\parallel punctured disk n punctures
 $[D : D_n]$
 $D \sim \mathcal{D}_{d+1}$

$\pi(\text{disk with } d \text{ chords}) / \text{transfer} \cong$
 $[\text{disk with } d \text{ punctures} : D_n] = [D : D_n]$

$$\mathcal{D} = \text{span}\{ \text{diagram 1}, \text{diagram 2} \} / \text{rels}$$

$\Psi: FL \rightarrow \mathcal{D}$... symmetrization ...

$\phi: FL(x, y) \rightarrow \mathcal{D}$ Lit morphism $x \mapsto \text{diagram 1}$ $y \mapsto \text{diagram 2}$

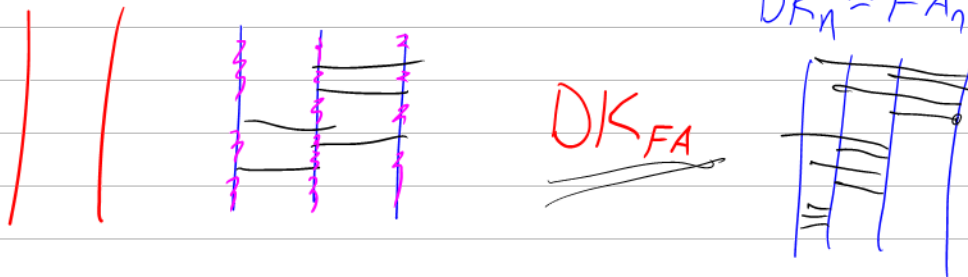
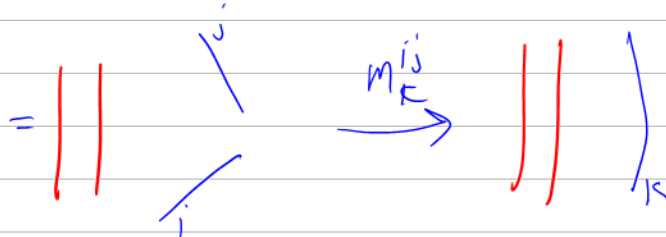
$$\phi(u) = u^{\text{hor}} \quad \Psi(u) = u^{\text{sym}}$$

$$\phi(u) \Psi(u) = \quad u \in FL \quad u = [u_1, u_2]$$

$$\phi(u) = [\phi(u_1), \phi(u_2)]_F$$

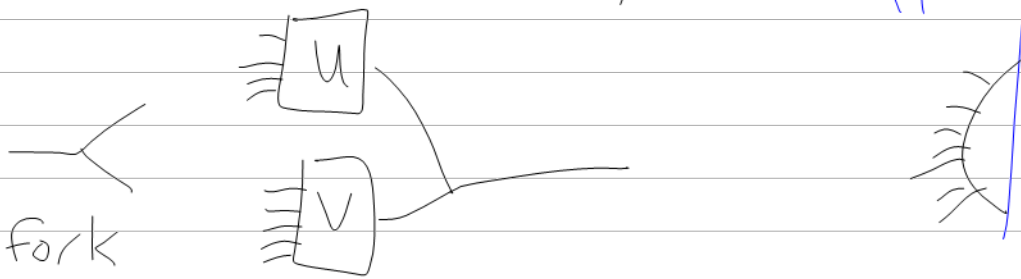
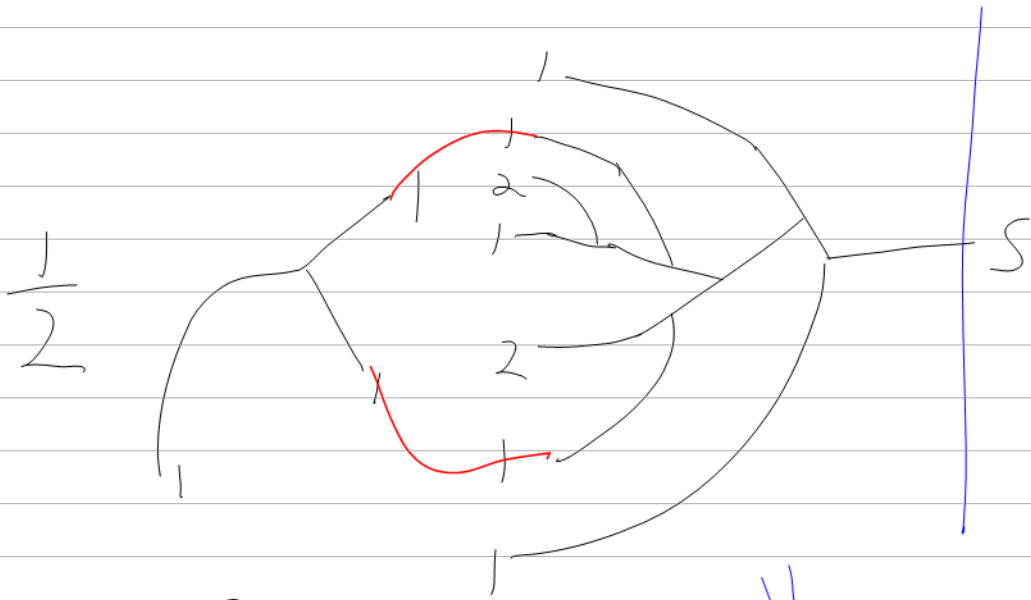
$\Delta_0 \in \mathcal{J}_F$ for diagram 1 $\Delta_0 \in \text{diagram 2}$

All other ops: $\Delta_1, \Delta_2, \dots, \Delta_s, \dots$



$$DK_n = FA_{n-1} \times FA_{n-2} \times \dots \times FA_{n-3}$$

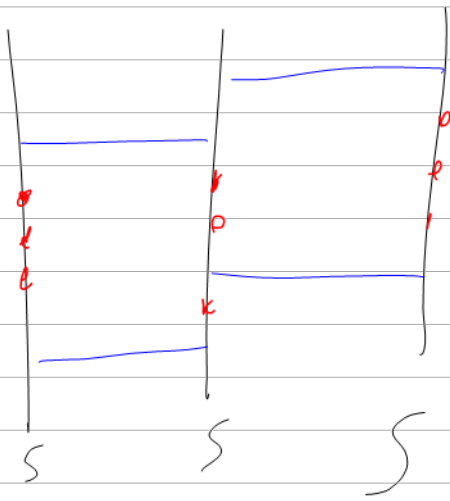
$$R(u) = \frac{1}{2} \sum_i \langle \psi_{x_i}^i, u \rangle$$



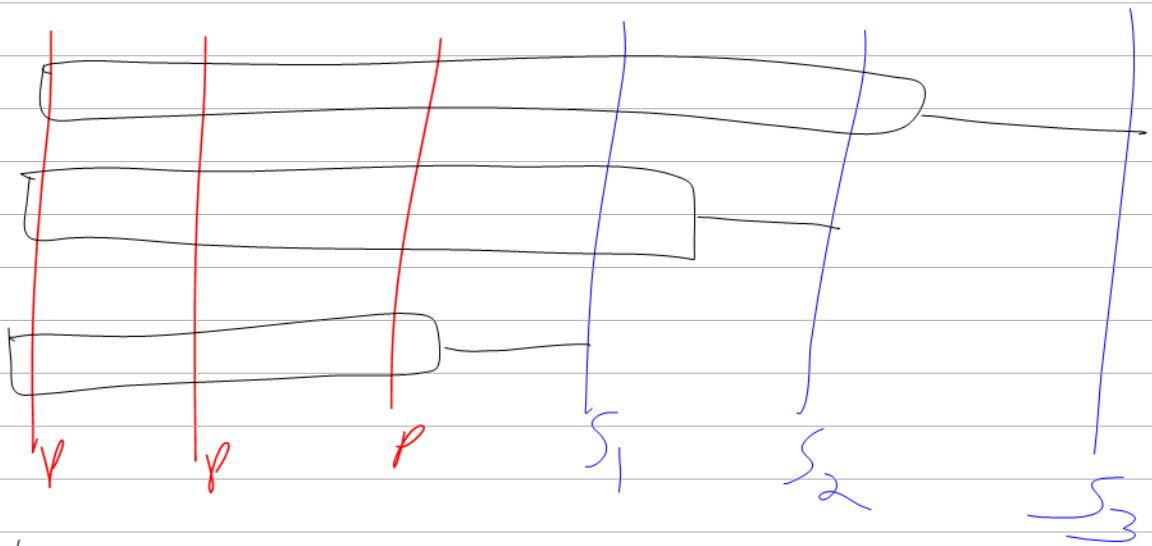
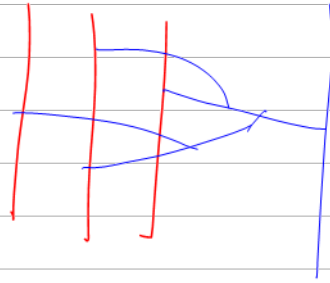
$$\Psi \mapsto \Psi_1 \quad R \mapsto R_1$$

$$L_1 \quad L_2 \quad A_{11} \quad A_{12} \quad A_{22}$$

"Computations for emergent knots in a pole dancing studio".



$FL \rightarrow P(p, p, s)$

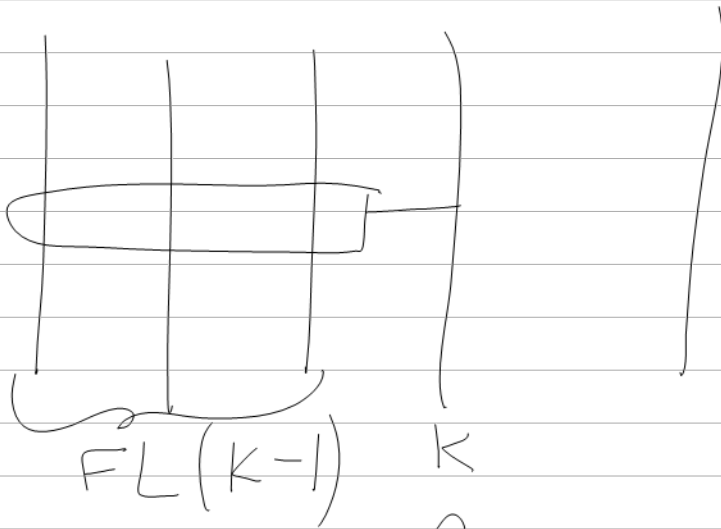
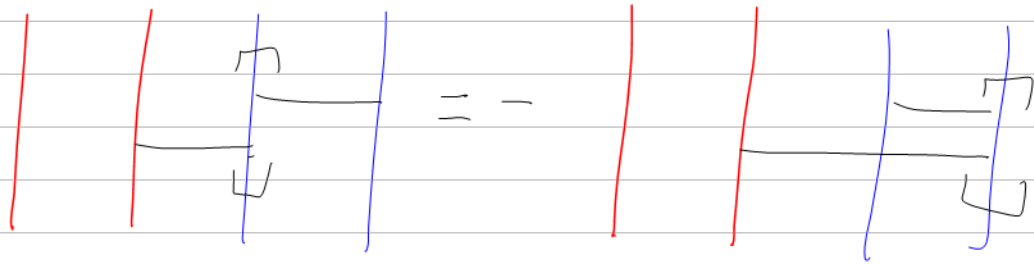
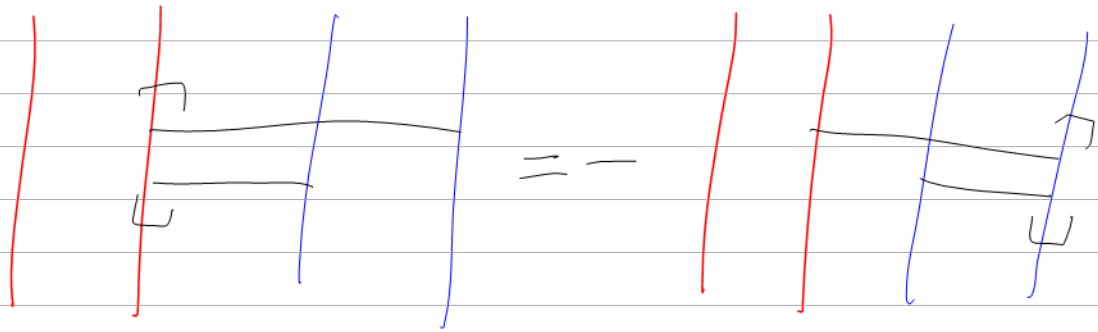


$$FL(p, p, p) \times FL(p, p, p, S_1)_{\deg S_1 \leq 1}$$

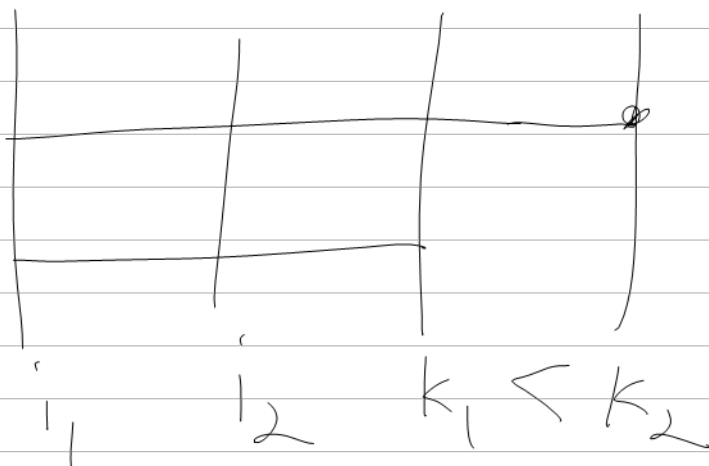
$$\times FL(p, p, p, S_1, S_2)_{\deg S_1 + \deg S_2 \leq 1}$$

$$\left[\begin{matrix} a_{12} \\ FL_2 \end{matrix}, \begin{matrix} a_{13} \\ FL_3 \end{matrix} \right] = - \left[\begin{matrix} a_{23} \\ FL_3 \end{matrix}, \begin{matrix} a_{13} \\ FL_3 \end{matrix} \right] \in FL_3$$

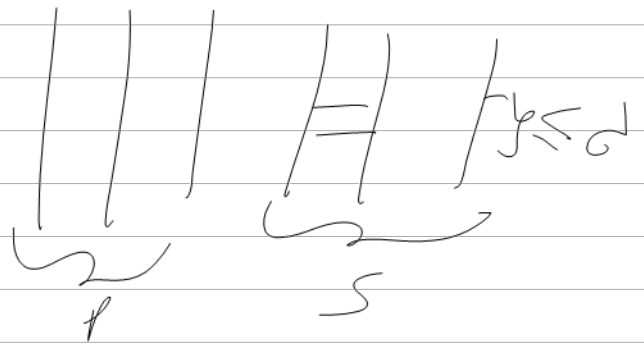
$$\left[\begin{matrix} a_{12} \\ FL_2 \end{matrix}, \begin{matrix} a_{23} \\ FL_3 \end{matrix} \right] = - \left[\begin{matrix} a_{13} \\ FL_3 \end{matrix}, \begin{matrix} a_{23} \\ FL_3 \end{matrix} \right] \in FL_3$$



$$DK_n \stackrel{\text{v.s.}}{=} \bigoplus_{k=2}^n FL_{k-1}$$



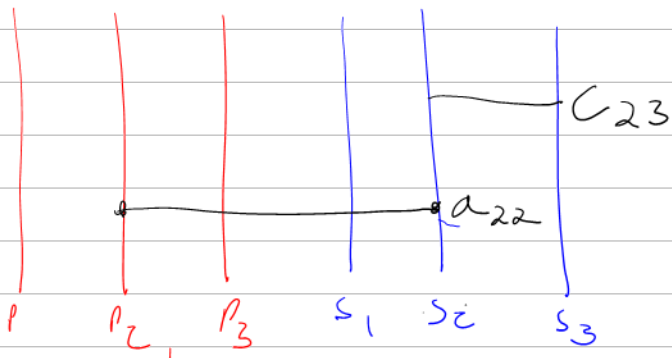
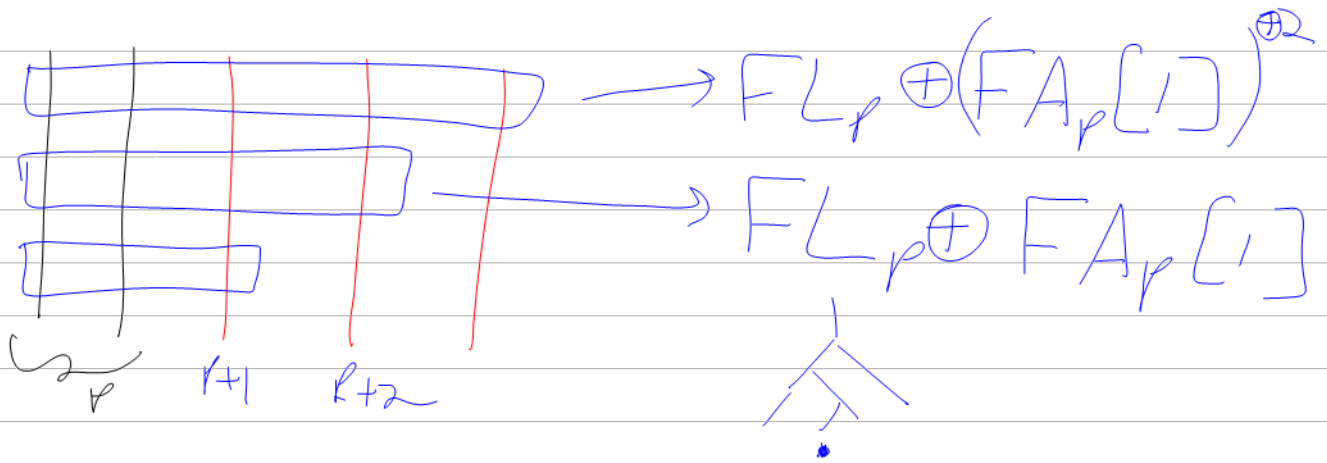
$$DK_{p,s}^{d,t}$$



$$DK^{t,t} \rightarrow \dots \rightarrow DK^{2,*} \rightarrow DK^{1,*} \rightarrow DK^{0,*}$$

$$(FL_p)^S$$

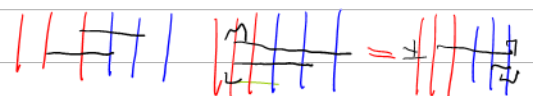
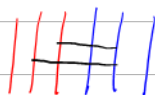
$$DK_{p,s}^{1,*} \stackrel{?}{=} FL_p \times \cancel{FA_p[1]}$$



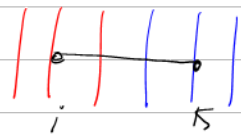
$$DK_{p,s}^{1,*} = FL \left[\begin{array}{ccc} a_{ij} & | & i \leq p, j \leq s \\ a_{ij} & | & i \leq i, j \leq s \end{array} \right]$$

$$/ \text{LT, deg}_C \leq 1$$

$$[a_{i_1 j_1}, a_{i_2 j_2}] =$$



$$L_k: FL(x_1 \dots x_p) \longrightarrow DK_k \quad x_i \longrightarrow a_{ik}$$



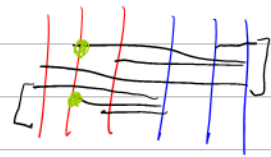
$$A_{jk}: FA(x_1 \dots x_p)[1] \longrightarrow DK \quad W = x_{i_1} x_{i_2} \dots x_{i_\alpha}$$

$$\longrightarrow a_{i_1 k} \dots a_{i_\alpha k} \subset_{jk}$$

claim $DK = \bigoplus_K \text{im } L_k \oplus \bigoplus_{j < k} \bigoplus \text{im } A_{jk} = \bigoplus_K \left[\text{im } L_k \oplus \bigoplus_{j < k} \text{im } A_{jk} \right]$

claim $[A_{j_1 k_1}(w_1), A_{j_2 k_2}(w_2)] = 0$ obvious

$$[L_{k_1}(w_1), A_{j_2 k_2}(w_2)] =$$



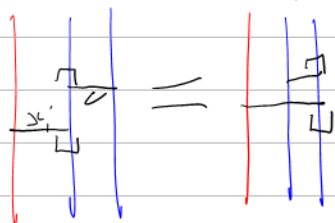
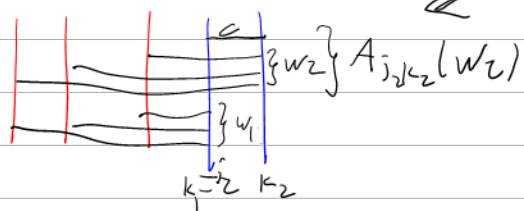
$$\left\{ \begin{array}{l} A_{j_2 k_2}(w_2 w_1) \\ 0 \\ A_{j_2 k_2}(w_1 w_2) \\ 0 \end{array} \right.$$

$$k_1 = j_2 < k_2$$

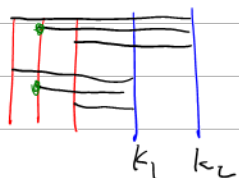
$$k_1 \neq j_2 < k_2$$

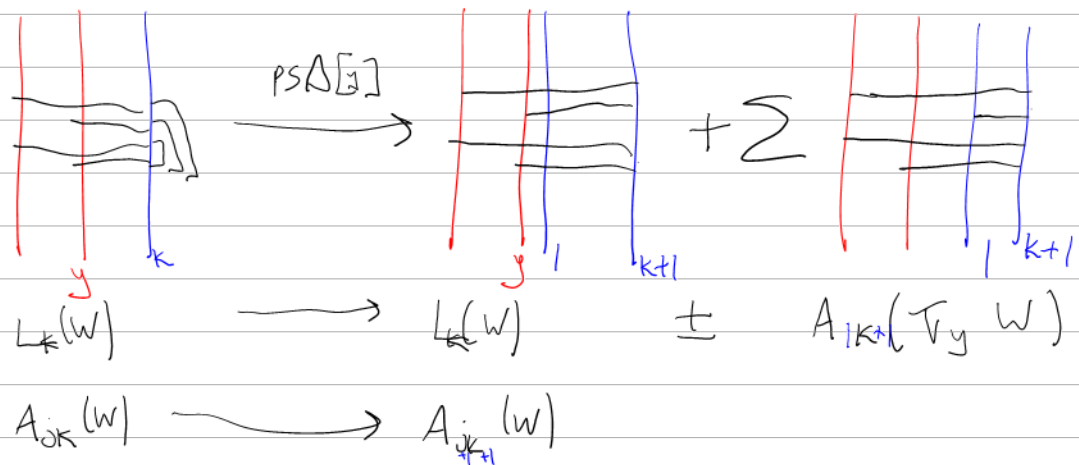
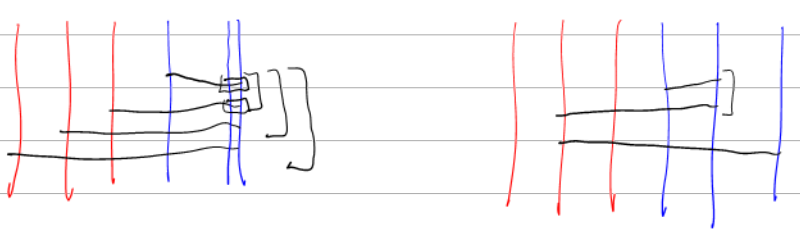
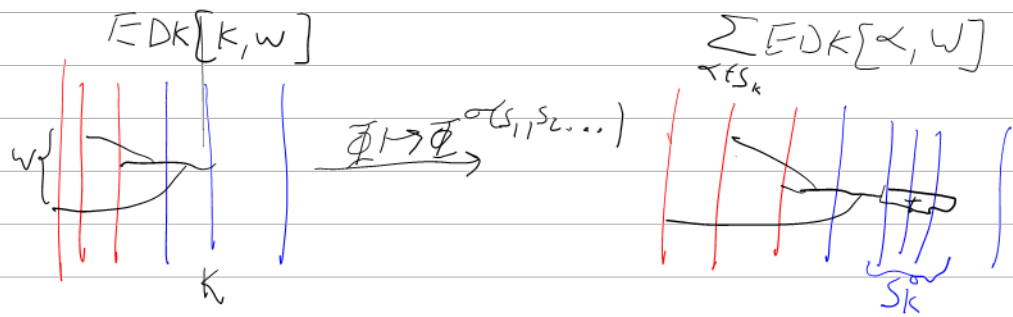
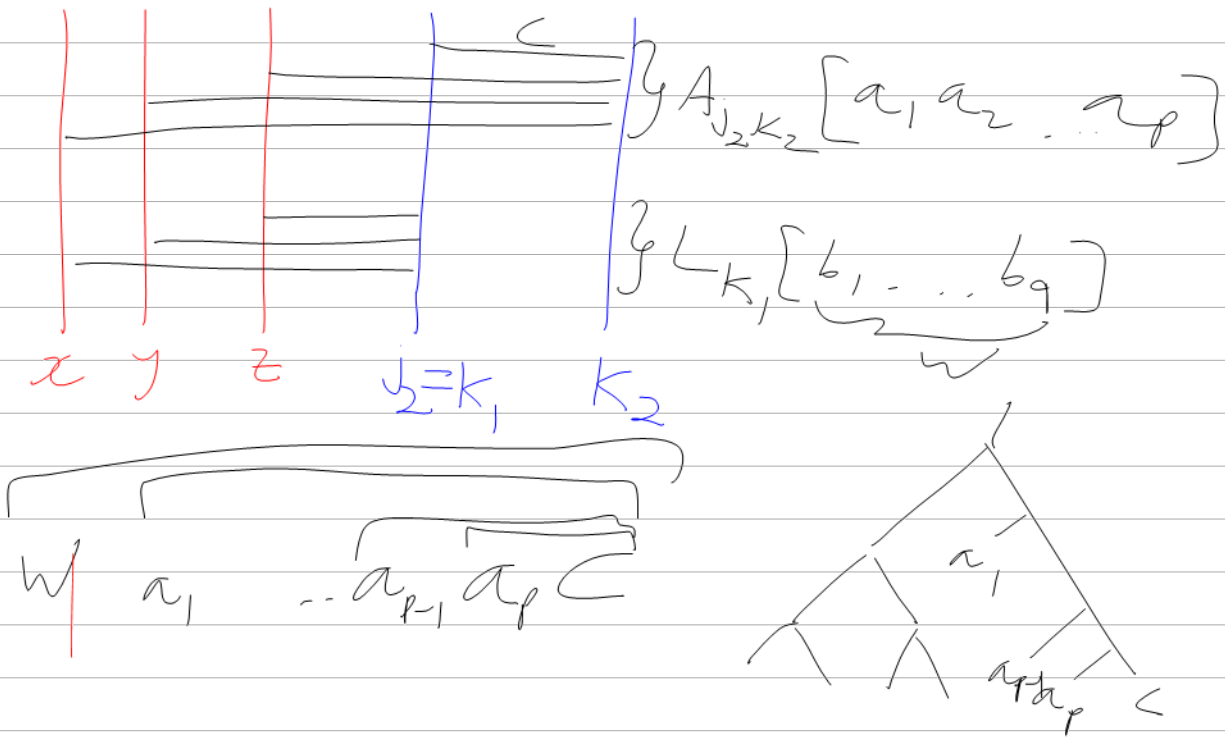
$$k_1 = k_2$$

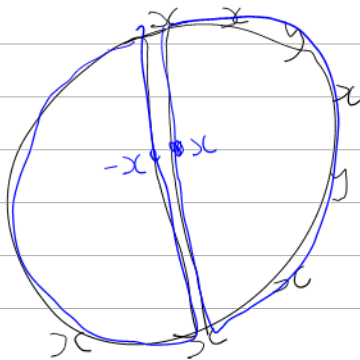
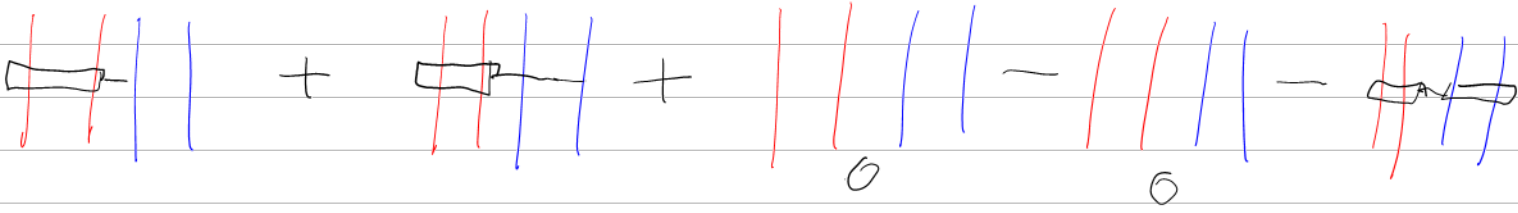
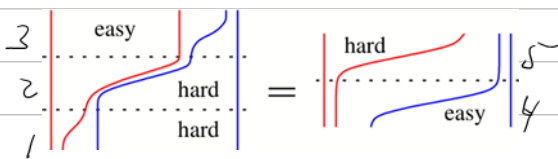
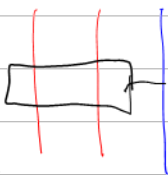
$$k_1 > k_2$$



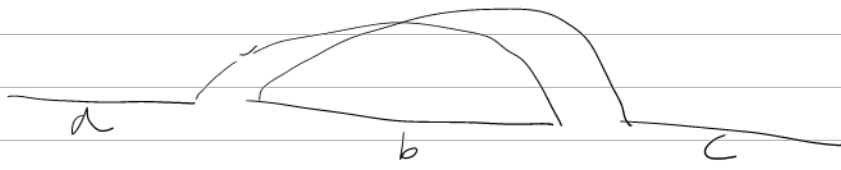
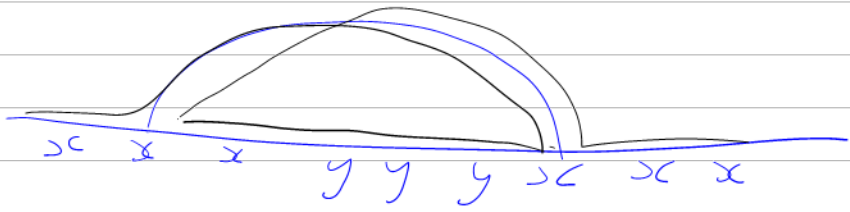
$$k_1 < k_2 \quad [L_{k_1}(w_1), L_{k_2}(w_2)] = \begin{cases} L_{k_1}([w_1, w_2]) & k_1 = k_2 \\ A_{k_1 k_2}(\sum_i (a_i w_1) x_i (a_i w_2)) & k_1 < k_2 \end{cases}$$



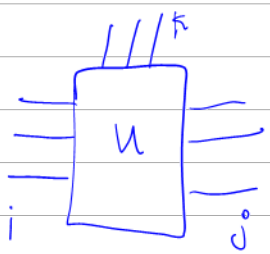




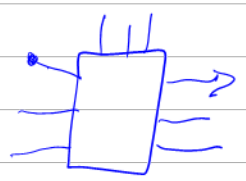
R:



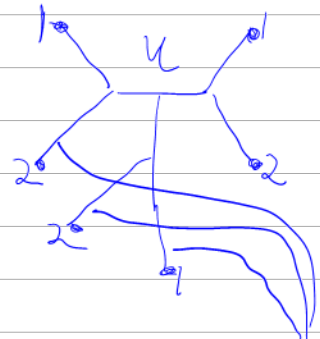
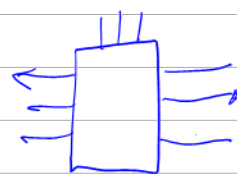
$$R(uv) = R(u) \cdot v + uR(v) + \frac{1}{2}(\partial_x u) \times (\partial_x v)^*$$

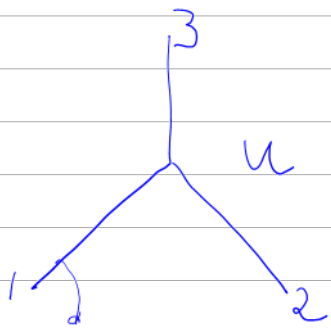


$\partial_i u_j$



$\partial^j u_i$

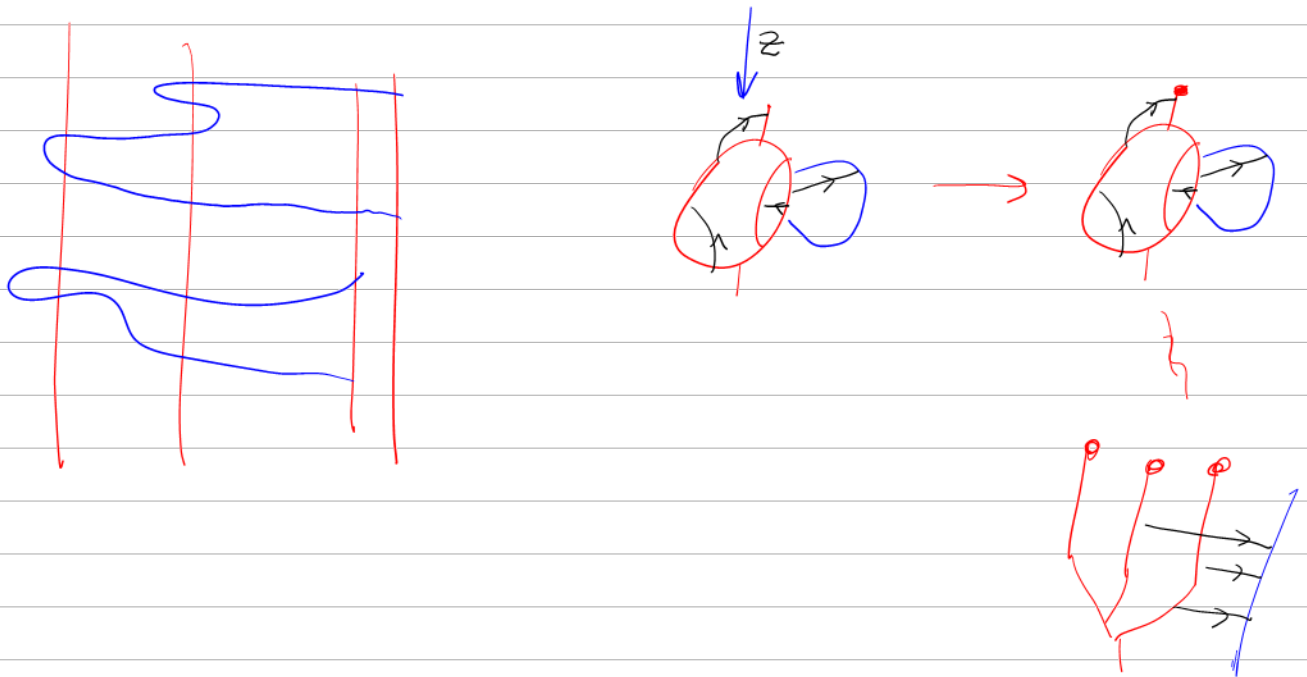
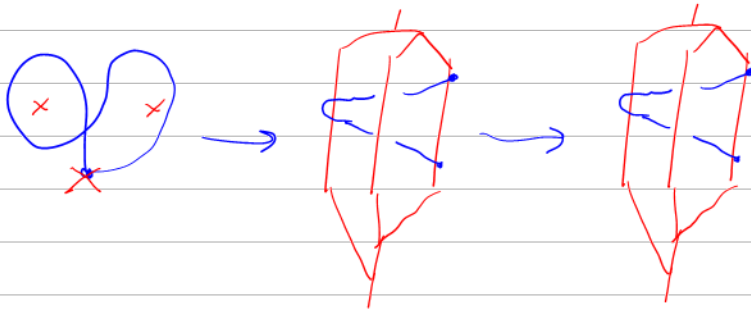
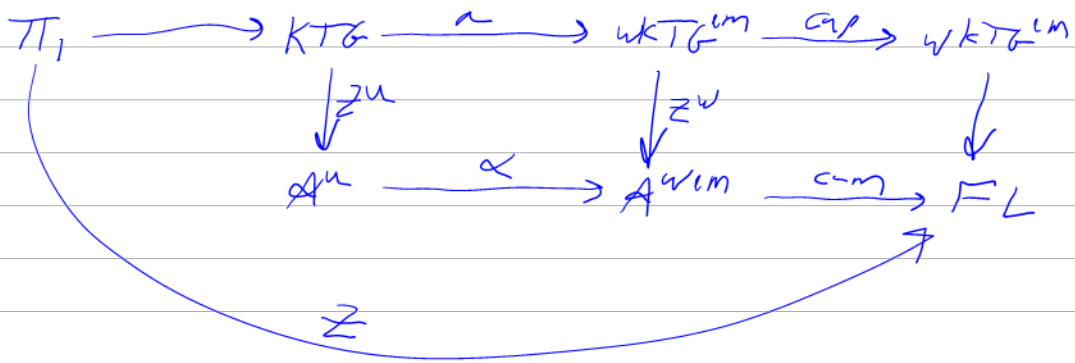




$$u = ([z_3, z_2], [z_1, z_3], [z_2, z_1])$$

$$\partial_1 u_2 = -z_3$$

$$\partial_2 u_1 = z_3$$

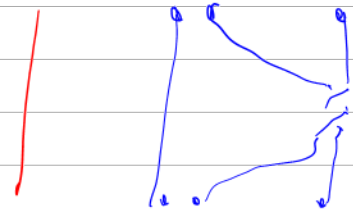


A parametrization is "sorted" if its arcs of the form

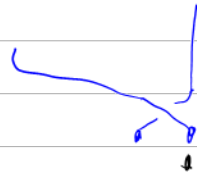
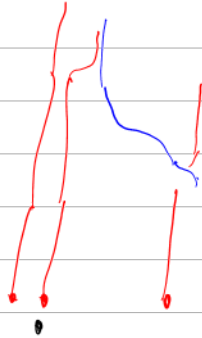
Proof sketch.



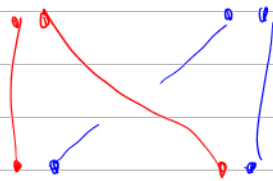
within sorted parametrizations, in $B_{1,3}, T_{2,3}$:



In $B_{3,2}, \alpha_{2,2}$



Finally



using the generators.

Map of the Field:

