In 240510_toward_KV1.pdf:

The emergent linearized $\hexagon$:

1. $\varphi(x + y, 0) - \varphi(y, 0) = 0$.

2. With $R = R(\varphi)$, $R(x + y, 0) - R(y, 0) = 0$.

3. $(\partial_y \varphi)(x, y) + (\partial_y \varphi)(y, 0) - ((\partial_y \varphi)(x + y, 0) = 2R(x, y)$.

$R \colon FL(x, y) \to FA(x, y)$ satisfies $R(x) = R(y) = 0$ and

$$R([u, v]) = [u, R(v)] + [R(u), v]$$
$$+ \frac{1}{2} \left(\begin{array}{c} (\partial_x v) x (\partial_x u)^* - (\partial_x u) x (\partial_x v)^* \\ + (\partial_y v) x (\partial_y u)^* - (\partial_y u) x (\partial_y v)^* \end{array} \right).$$

Lemma 1. $\forall m \geqslant 0$, $\partial_x(\mathrm{ad}_x^m(y)) = -\sum_{i+j=m-1} x^i \mathrm{ad}_x^j(y)$.

Lemma 2. $\partial_x(\mathrm{ad}_x^m(y)) = -\sum_{b+c=m-1} (-1)^c \binom{m}{b} x^b y x^c$.

Lemma 3. $R(\mathrm{ad}_x^m(y)) = \sum_{i+j=m-1} \frac{(-1)^j}{2} \left(\binom{m-1}{j-1} - \binom{m-1}{i-1} \right) x^i y x^j$.

Lemma 4. If $\varphi \in$ SolEmPent is degree m where $m \geqslant 4$ is even then $\varphi_1 = 0$, where φ_j is the y-degree j part of φ.

Lemma 5. If $\varphi \in$ SolEmPent is of degree $\geqslant 3$, then $(\partial_y \varphi)^* = \partial_y \varphi$.

Schneps' Theorem (in *Double Shuffle and Kashiwara-Vergne Lie Algebras*). Let $b \in FL(x, y)$. Then $\exists a \in FL(x, y)$ s.t. $[x, a] + [y, b] = 0$ iff $\partial_y b = \partial^y b \ (= (\partial_y b)^*)$.

Claim 1. If $\varphi \in$ SolEmPent is of degree $\geqslant 3$ and if we learn that $(\partial_x \varphi)^* = (\partial_x \varphi)(y, x)$, then $\nu(\varphi) := (\varphi(y, x), \varphi(x, y)$ satisfies KV1.