Summary:

Let $\varphi \in \mathrm{SolEMPext}$ of degree ≥ 3 & assume that $\overline{\partial_x \varphi} = (\partial_x \varphi)(y,x)$. Then, $\nu(\varphi) := (\varphi(y,x), \varphi(x,y)) \in \mathrm{tder}_2$ satisfies (KV1) & (KV2).

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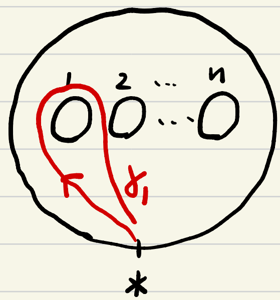
1. the homotopy intersection form & (KV1)

$$\pi = \pi_1(\Sigma_{0,n+1}) \cong \langle \gamma_1, \dots, \gamma_n \rangle$$



$$\eta : \mathbb{Q}\pi \otimes \mathbb{Q}\pi \longrightarrow \mathbb{Q}\pi \quad \text{(Massuyeau–Turaev)}$$

$$\eta(\alpha, \beta) = \sum_{p \in \alpha \cap \beta} \varepsilon_p \, \alpha_{*p} \beta_{p*}$$

η is a Fox pairing : $\eta(a, bc) = \eta(a,b)c + \varepsilon(b)\eta(a,c)$

$\eta(ab, c) = \varepsilon(b)\eta(a,c) + a\,\eta(b,c)$

$\mathrm{gr}\,\mathbb{Q}\pi = \mathrm{Ass}(z_1, \dots, z_n)$ $\quad z_i = [\gamma_i - 1]$ $\quad H := \bigoplus_i \mathbb{Q} z_i \cong H_1(\Sigma_{0,n+1})$

$\zeta : H \times H \longrightarrow H$, $\zeta(z_i, z_j) := \delta_{ij} \cdot z_i$

$$\zeta : H \times H \longrightarrow H, \quad \zeta(z_i, z_j) := \delta_{ij} \cdot z_i$$

$$\eta \rightsquigarrow \eta_{gr} : Ass^{\otimes 2} \longrightarrow Ass$$

$$\eta_{gr}(a_1 \cdots a_m, b_1 \cdots b_n) = a_1 \cdots a_{m-1}\, \zeta(a_m, b_1)\, b_2 \cdots b_n$$

<u>Thm</u> (Massuyeau–Turaev, Naef)

$\varphi \in tAut(Lie(z_1, \dots, z_n))$

$$\varphi \circ \eta_{gr} = \eta_{gr} \circ \varphi \iff \varphi(z_1 + \cdots + z_n) = z_1 + \cdots + z_n$$

Also true: $\varphi \in tAut(Prim(\widehat{\mathbb{Q}\pi}))$

$\varphi \circ \eta = \eta \circ \varphi \iff \varphi($  $) =$ 

logarithm

$$u = (u_1, \dots, u_n) \in tder(Lie(z_1, \dots, z_n))$$

$$u \circ \eta_{gr} = \eta_{gr} \circ u \iff u(z_1 + \cdots + z_n) = 0$$

$$\Updownarrow$$

$$u(\eta_{gr}(z_i, z_j)) = \eta_{gr}(u(z_i), z_j) + \eta_{gr}(z_i, u(z_j))$$

$$u(z_1+\cdots+z_n)=0$$

$$\Updownarrow$$

$$u\big(\eta_{gr}(z_i,z_j)\big)=\eta_{gr}(u(z_i),z_j)+\eta_{gr}(z_i,u(z_j))$$

η

$$\mathrm{LHS}=u(\zeta(z_i,z_j))=\begin{cases}0 & i\neq j\\ u(z_i)=[z_i,u_i] & i=j\end{cases}$$

$$\mathrm{RHS}=\eta(\underbrace{[z_i,u_i]}_{z_iu_i-u_iz_i},z_j)+\eta(z_i,\underbrace{[z_j,u_j]}_{z_ju_j-u_jz_j})$$

$$\left(\eta_{gr}(a_1\cdots a_m,b_1\cdots b_n)=a_1\cdots a_{m-1}\zeta(a_m,b_1)b_2\cdots b_n\right)$$

$$=z_i(\partial_j u_i)z_j-u_i\zeta(z_i,z_j)+\zeta(z_i,z_j)u_j-z_i(\partial^i u_j)z_j$$

$$=\begin{cases}z_i(\partial_j u_i-\partial^i u_j)z_j & i\neq j\\ z_i(\partial_j u_i-\partial^i u_j)z_j+[z_i,u_i] & i=j\end{cases}$$

$$\left[\begin{array}{l}\underline{\text{Lem}}\quad u=(u_1,\cdots,u_n)\in\mathrm{tder}\\ \quad u(z_1+\cdots+z_n)=0\iff \partial_j u_i=\partial^i u_j\quad\forall i,\forall j\end{array}\right]$$

$$\left[\begin{array}{l} \underline{\text{Lem}}\quad u=(u_1,\cdots,u_n)\in \text{tder} \\ u(z_1+\cdots+z_n)=0 \iff \partial_j u_i = \overset{\overline{\partial_i u_j}}{\overset{\parallel}{\partial^i u_j}} \quad \forall i, \forall j \end{array}\right]$$

The case $n=2$ & $u=v(\varphi)=(\overset{u_1}{\varphi(y,x)},\overset{u_2}{\varphi(x,y)})$ $\quad z_1=x,\ z_2=y$

$$u(x+y)=0 \iff \begin{cases} \partial_x(\varphi(y,x)) = \overline{\partial_x\varphi(x,y)} & (1,1) \\ \partial_y(\varphi(x,y)) = \overline{\partial_y(\varphi(x,y))} & \\ \partial_y(\varphi(y,x)) = \overline{\partial_x(\varphi(x,y))} & (2,1) \\ \cdots\cdots & \end{cases}$$

$$\iff \partial_y\varphi = \overline{\partial_y\varphi}\ \ \&\ \ (\partial_x\varphi)(y,x)=\overline{\partial_x\varphi}$$

φ $\quad$ φ ?

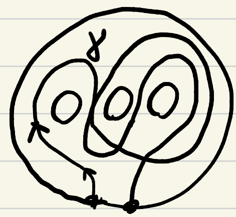
$\varphi\in$ SolEMPent

2. The map R & self-intersection of loop

<u>Recall</u> $R : \mathrm{Lie} \longrightarrow \mathrm{Ass}$, $R(z_i) = 0$ &

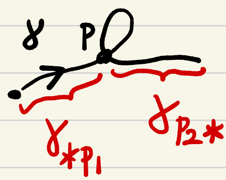
$$R([u,v]) = [u, R(v)] + [R(u), v] + \frac{1}{2}\sum_i \left((\partial_i v) z_i \overline{\partial_i u} - (\partial_i u) z_i \overline{\partial_i v} \right)$$

$\mu_0 : \mathbb{Q}\pi \longrightarrow \mathbb{Q}\pi$



$$\mu_0(\gamma) = \sum_{p \in \mathrm{Self}(\gamma)} \varepsilon_p \, \gamma_{*p_1} \gamma_{p_2 *}$$

rot $= -1/2$ first   second



homotpy int. form

$\downarrow$

<u>product formula</u>: $\mu_0(ab) = \mu_0(a) b + a \mu_0(b) + \eta(a,b)$

ass. gr

$$(\mu_0)_{gr}(a_1 \cdots a_m) = \sum_{j=1}^{m-1} a_1 \cdots a_{j-1} \zeta(a_j, a_{j+1}) a_{j+2} \cdots a_m$$

[**Prop** $R = -\frac{1}{2}(\mu_0)_{gr}$] $\qquad \mu_0 = (\mu_0)_{gr}$

proof $\mu_0(z_i) = 0$

$$\left(\begin{aligned} &\eta_{gr}(a_1 \cdots a_m, b_1 \cdots b_n) = a_1 \cdots a_{m-1}\, \zeta(a_m, b_1)\, b_2 \cdots b_n \\ &\eta_{gr}(a, b) = \sum_i (\partial_i a) z_i (\partial^i b) \underset{\substack{\uparrow \\ a,b \in \mathrm{Lie}}}{=} \sum_i (\partial_i a) z_i \overline{(\partial_i b)} \end{aligned} \right)$$

$a, b \in \mathrm{Lie}$. Then

$$\begin{aligned} \mu_0([a,b]) &= \mu_0(a) b + a \mu_0(b) + \eta(a,b) \\ &\quad - \mu_0(b) a - b \mu_0(a) - \eta(b,a) \\ &= [\mu_0(a), b] + [a, \mu_0(b)] \\ &\quad + \sum_i \left((\partial_i a) z_i \overline{\partial_i b} - (\partial_i b) z_i \overline{\partial_i a} \right) \end{aligned}$$

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μ_0 recovers the Turaev cobracket:

$$a \xmapsto{\Delta} a' \otimes a'' \xmapsto{\mathrm{id}\otimes\mu_0} a' \otimes \mu_0(a'')$$

$$\xmapsto{\mathrm{id}\otimes((\varepsilon\otimes\mathrm{id})\Delta)} a' \otimes \tilde{\Delta}'(\mu_0(a'')) \otimes \tilde{\Delta}''(\mu_0(a''))$$

$$\xmapsto{(|\,|\circ \mathrm{mult})\otimes\mathrm{id}} |a'\tilde{\Delta}'(\mu_0(a''))| \otimes \tilde{\Delta}''(\mu_0(a''))$$

$a \in \mathbb{Q}\pi$

$$= \mu_r(a) \in |\mathbb{Q}\pi| \otimes \mathbb{Q}\pi$$

Then,

$$\delta(|a|) = \mathrm{Alt}\circ(\mathrm{id}\otimes|\,|)(\mu_r(a)) + \underbrace{|a| \wedge \mathbb{1}}$$

δ: (framed) Turaev cobracket

does not contribute to ass. gr.

Thm (AKKN)

$\varphi \in \mathrm{tAut}(\mathrm{Lie}(z_1,\cdots,z_n))$

$$\varphi \in \mathrm{KRV}_n \iff \begin{cases} \varphi\circ\eta_{gr} = \eta_{gr}\circ\varphi & \text{(KV1)} \\ \varphi\circ\delta_{gr} = \delta_{gr}\circ\varphi & \text{(KV2)} \\ & \text{(KV1)} \end{cases}$$

logarithm $\longrightarrow$ $u \in \mathfrak{krv}_n \iff \cdots\cdots$

3. EM 5-gon & KV equations.

$$\left\{ (\partial_y\varphi)(x,y) + (\partial_y\varphi)(y,0) - (\partial_y\varphi)(x+y,0) - 2R(x,y) = 0 \right.$$

$\varphi \in \mathrm{SolEMPent}$ $\quad \nu(\varphi) := (\varphi(y,x), \varphi(x,y))$

$$R(\nu(\varphi)(y)) = R([y, \varphi(x,y)])$$

$$= [y, \underset{R(\varphi)}{R(x,y)}] + \frac{1}{2}\left((\partial_y\varphi)y - y\,\overline{\partial_y\varphi}\right)$$

$\overline{\partial_y\varphi} = \partial_y\varphi$

$$= [y, R(x,y) - \tfrac{1}{2}\partial_y\varphi]$$

$$\underset{\text{EM 5-gon}}{=} \left[y, \tfrac{1}{2}\left((\partial_y\varphi)(y,0) - (\partial_y\varphi)(x+y,0)\right)\right]$$

$$= [y, f(x+y)]$$

the same !

$R(\nu(\varphi)(x))$
$+R(\nu(\varphi)(y))$
$= 0$

$$R(\nu(\varphi)(x)) = R([x, \varphi(y,x)]) = \cdots = [x, f(x+y)]$$

Prop $u \in \mathrm{sder}(\mathrm{Lie}(z_1,\cdots,z_n))$, $\forall a = a_1\cdots a_m \in \mathrm{Ass}$

$$\mu_r(u(a)) - u.(\mu_r(a)) = \sum_{\lambda=1}^{m} (1\otimes a_1\cdots a_{\lambda-1})\mu_r(u(a_\lambda))(1\otimes a_{\lambda+1}\cdots a_m)$$

$\mu_r \rightsquigarrow \mu_0$

$\cap$

To be continued.... $\mathrm{Tr}\otimes\mathrm{Ass}$

$a, b \in \mathrm{Ass}$ $\quad \mu_r(u(ab)) - u.\mu_r(ab) = (1\otimes a)\mu_r(u(b)) + \mu_r(u(a))(1\otimes b)$

$$\left[\begin{array}{l}\underline{\text{Prop 1}}\quad u \in \text{sder}(\text{Lie}(z_1,\cdots,z_n)), \quad \forall a = a_1\cdots a_m \in \text{Ass} \\ \mu_r(u(a)) - u.(\mu_r(a)) = \sum_{\lambda=1}^{m}(1\otimes a_1\cdots a_{\lambda-1})\mu_r(u(a_\lambda))(1\otimes a_{\lambda+1}\cdots a_m)\end{array}\right]$$

ass. gr. version

$$\mu_r(ab) = \mu_r(a)(1\otimes b) + (1\otimes a)\mu_r(b) + (1\!\!1\otimes \mathrm{id})K(a,b)$$

u : special

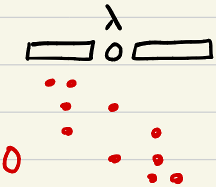
$\Longrightarrow\ u\circ \eta_{gr} = \eta_{gr}\circ u$

$\Longrightarrow\ u\circ K_{gr} = K_{gr}\circ u$

$K : \mathbb{Q}\pi^{\otimes 2} \longrightarrow \mathbb{Q}\pi^{\otimes 2}$ double bracket

$$K(\alpha,\beta) = \sum_{p\in\alpha\cap\beta} \varepsilon_p\, \beta_{*p}\alpha_{p*}\otimes \alpha_{*p}\beta_{p*}$$

$\eta = (\varepsilon\otimes \mathrm{id})K$



$\mu_r(a_j) = 0$

proof $\mu_r(u(a)) = \mu_r\left(\sum_\lambda a_1\cdots a_{\lambda-1}u(a_\lambda)a_{\lambda+1}\cdots a_m\right)$

$$= \sum_\lambda (1\otimes a_1\cdots a_{\lambda-1})\mu_r(u(a_\lambda))(1\otimes a_{\lambda+1}\cdots a_m)$$

$$+(1\!\!1\otimes\mathrm{id})\left(\begin{array}{ll}\sum_{j<k<\lambda}(1\otimes a_1\cdots a_{j-1})K(a_j,a_k)(a_{j+1}\cdots a_{k-1}\otimes a_{k+1}\cdots a_{\lambda-1}u(a_\lambda)a_{\lambda+1}\cdots a_m) & ① \\ +\sum_{j<\lambda}(1\otimes a_1\cdots a_{j-1})K(a_j,u(a_\lambda))(a_{j+1}\cdots a_{\lambda-1}\otimes a_{\lambda+1}\cdots a_m) & ② \\ +\sum_{j<\lambda<k}(1\otimes a_1\cdots a_{j-1})K(a_j,a_k)(a_{j+1}\cdots a_{\lambda-1}u(a_\lambda)a_{\lambda+1}\cdots a_{k-1}\otimes a_{k+1}\cdots a_m) & ③ \\ +\sum_{\lambda<k}(1\otimes a_1\cdots a_{\lambda-1})K(u(a_\lambda),a_k)(a_{\lambda+1}\cdots a_{k-1}\otimes a_{k+1}\cdots a_m) & ④ \\ +\sum_{\lambda<j<k}(1\otimes a_1\cdots a_{\lambda-1}u(a_\lambda)a_{\lambda+1}\cdots a_{j-1})K(a_j,a_k)(a_{j+1}\cdots a_{k-1}\otimes a_{k+1}\cdots a_m) & ⑤\end{array}\right)$$

$u.K(a_j,a_k)$

$$\mu_r(u(a)) = \sum_{\lambda}(1\otimes a_1\cdots a_{\lambda-1})\mu_r(u(a_\lambda))(1\otimes a_{\lambda+1}\cdots a_m)$$

$$+(\mathbb{1}\otimes \mathrm{id})\Big(\sum_{j<k<\lambda}(1\otimes a_1\cdots a_{j-1})K(a_j,a_k)(a_{j+1}\cdots a_{k-1}\otimes a_{k+1}\cdots a_{\lambda-1}u(a_\lambda)a_{\lambda+1}\cdots a_m) \quad \text{①}$$

$$+\sum_{j<\lambda}(1\otimes a_1\cdots a_{j-1})K(a_j,u(a_\lambda))(a_{j+1}\cdots a_{\lambda-1}\otimes a_{\lambda+1}\cdots a_m) \quad \text{②}$$

$$+\sum_{j<\lambda<k}(1\otimes a_1\cdots a_{j-1})K(a_j,a_k)(a_{j+1}\cdots a_{\lambda-1}u(a_\lambda)a_{\lambda+1}\cdots a_{k-1}\otimes a_{k+1}\cdots a_m) \quad \text{③}$$

$$+\sum_{\lambda<k}(1\otimes a_1\cdots a_{\lambda-1})K(u(a_\lambda),a_k)(a_{\lambda+1}\cdots a_{k-1}\otimes a_{k+1}\cdots a_m) \quad \text{④}$$

$$+\sum_{\lambda<j<k}(1\otimes a_1\cdots a_{\lambda-1}u(a_\lambda)a_{\lambda+1}\cdots a_{j-1})K(a_j,a_k)(a_{j+1}\cdots a_{k-1}\otimes a_{k+1}\cdots a_m)\Big) \quad \text{⑤}$$

$a=a_1a_2\cdots a_m$

$$u(\mu_r(a)) = u\Big(\sum_{\lambda}(1\otimes a_1\cdots a_{\lambda-1})\underbrace{\mu_r(a_\lambda)}_{=0}(1\otimes a_{\lambda+1}\cdots a_m)$$

$$+(\mathbb{1}\otimes \mathrm{id})\sum_{j<k}(1\otimes a_1\cdots a_{j-1})K(a_j,a_k)(a_{j+1}\cdots a_{k-1}\otimes a_{k+1}\cdots a_m)\Big)$$

$$=(\mathbb{1}\otimes \mathrm{id})\Big(\sum_{j<k}(1\otimes u(a_1\cdots a_{j-1}))K(a_j,a_k)(a_{j+1}\cdots a_{k-1}\otimes a_{k+1}\cdots a_m) \quad \text{⑥}$$

$$+\sum_{j<k}(1\otimes a_1\cdots a_{j-1})u(K(a_j,a_k))(a_{j+1}\cdots a_{k-1}\otimes a_{k+1}\cdots a_m) \quad \text{⑦}$$

$$+\sum_{j<k}(1\otimes u(a_1\cdots a_{j-1}))K(a_j,a_k)(u(a_{j+1}\cdots a_{k-1})\otimes a_{k+1}\cdots a_m) \quad \text{⑧}$$

$$+\sum_{j<k}(1\otimes u(a_1\cdots a_{j-1}))K(a_j,a_k)(a_{j+1}\cdots a_{k-1}\otimes u(a_{k+1}\cdots a_m))\Big) \quad \text{⑨}$$

① = ⑨, ② + ④ = ⑦, ③ = ⑧, ⑤ = ⑥

u is special

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Prop 2 Let $u \in \mathrm{sder}(\mathrm{Lie}(z_1, \cdots, z_n))$. Assume that

$$\mu_r(u(z_i)) = |b| \otimes [z_i, c] \quad , \quad 1 \leq i \leq n$$

for some $|b| \in \mathrm{tr} = |\mathrm{Ass}|$ & $c \in \mathrm{Ass}$. Then, $\delta \circ u = u \circ \delta$

$\sum_\lambda |b_\lambda| \otimes [z_i, c_\lambda]$

ass. gr. of the framed Turaev cobracket

<u>proof</u> Let $|a| = |a_1 \cdots a_m| \in \mathrm{tr}$. Then,

$$\delta(u(|a|)) - u(\delta(|a|)) = \mathrm{Alt}\,(\mathrm{id} \otimes |\,|)\big(\mu_r(u(a)) - u(\mu_r(a))\big)$$

$$\underset{\text{Prop1}}{=} \mathrm{Alt}\,(\mathrm{id} \otimes |\,|)\Big(\sum_\lambda (1 \otimes a_1 \cdots a_{\lambda-1})\mu_r(u(a_\lambda))(1 \otimes a_{\lambda+1} \cdots a_m)\Big)$$

$$= \mathrm{Alt}\,(\mathrm{id} \otimes |\,|)\Big(\sum_\lambda |b| \otimes a_1 \cdots a_{\lambda-1}[a_\lambda, c]a_{\lambda+1} \cdots a_m\Big)$$

$$= \mathrm{Alt}\,(\mathrm{id} \otimes |\,|)\big(|b| \otimes [a_1 \cdots a_m, c]\big) = 0 \quad //$$

already $(\mathrm{id} \otimes |\,|)(|b| \otimes [a_1 \cdots a_m, c]) = 0$

Going back to the case $n = 2$, let $\varphi \in \mathrm{SolEMPent}$. Recall that $v(\varphi) := (\varphi(y, x), \varphi(x, y)) \in \mathrm{tder}(\mathrm{Lie}(x, y))$ satisfies

$$R(v(\varphi)(x)) = [x, f(x+y)], \quad R(v(\varphi)(y)) = [y, f(x+y)]$$

for some $f(s) \in \mathbb{Q}[[s]]$.

$$\partial_y \varphi = \overline{\partial_y \varphi}$$

$\left[\right.$ <u>Prop 3</u> Assume that $\overline{\partial_x \varphi} = (\partial_x \varphi)(y,x)$. Then $\delta \circ \nu(\varphi) = \nu(\varphi)\circ\delta$.

In particular, $\nu(\varphi) \in \mathrm{krv}_2$. $\left.\right]$

<u>proof</u> From the assumption, $\nu(\varphi)$ is special (see §1).

Set $z = x+y$. Since $R = -\frac{1}{2}\mu_0$, we have

ass. gr. ver.

$$\mu_0(\nu(\varphi)(x)) = [x, f(z)], \qquad \mu_0(\nu(\varphi)(y)) = [y, f(z)]$$

$$\left(\begin{array}{l} \underline{\mu_0\ \&\ \mu_r}: \quad \begin{array}{l} a \xmapsto{\Delta} a' \otimes a'' \xmapsto{\mathrm{id}\otimes\mu_0} a' \otimes \mu_0(a'') \\ \rotatebox{90}{$\in$} \\ \mathbb{Q}\pi \quad \xmapsto{\mathrm{id}\otimes((2\otimes \mathrm{id})\Delta)} a' \otimes \tilde{\Delta}'(\mu_0(a'')) \otimes \tilde{\Delta}''(\mu_0(a'')) \\ \qquad \xmapsto{(|\,|\circ \mathrm{mult})\otimes \mathrm{id}} |a' \tilde{\Delta}'(\mu_0(a''))| \otimes \tilde{\Delta}''(\mu_0(a'')) \\ \qquad\qquad\qquad\qquad \| \\ \qquad\qquad\qquad\qquad \mu_r(a) \in |\mathbb{Q}\pi| \otimes \mathbb{Q}\pi \end{array} \end{array}\right)$$

If $f(z) = z^m$, $\mu_r(\nu(\varphi)(x))$ is given by

$$\nu(\varphi)(x) \xmapsto{\Delta} \nu(\varphi)(x)\otimes 1 + 1 \otimes \nu(\varphi)(x) \xmapsto{\mathrm{id}\otimes\mu_0} 1 \otimes [x, z^m]$$

$$\xrightarrow{(|\,|\otimes \mathrm{mult})\otimes \mathrm{id}\ \circ\ \mathrm{id}\otimes((2\otimes \mathrm{id})\Delta)} \sum_{j=0}^{n} (-1)^{j+1}\binom{m}{j} |z^j| \otimes [x, z^{m-j}],$$

and one can compute $\mu_r(\nu(\varphi)(y))$ similarly. Applying Prop2,

we obtain $\delta\circ\nu(\varphi) = \nu(\varphi)\circ\delta$ //

Comments :

- To prove $\overline{\partial_x \varphi} = (\partial_x \varphi)(y,x)$ might be hard. (As hard as Furusho's thm?)
- Computational result : up to deg 13 ,

$$\begin{cases} \partial_y \varphi = \overline{\partial_y \varphi} \\ \text{2-cycle } \left(\varphi(-x-y,x) = \varphi(-x-y,y) = 0\right) \Rightarrow \overline{\partial_x \varphi} = (\partial_x \varphi)(y,x) \\ \text{3-cycle } \left(\varphi(x,y) + \varphi(y,z) + \varphi(z,x) = 0\right) \\ \qquad z = -x-y \end{cases}$$

- EM 6-gon would imply $U(\varphi)$ is special.
- How about the case for expansions ?
- Formulate the EM par. braids in terms of generators and relations. The associated graded.
- If everything works, we will obtain a factorization of the map of Alekseev-Torossian :

$$\{\text{associators}\} \to \{\text{EM associators}\} \longrightarrow \text{Sol KV}$$

sol. to 5-gon 6-gon

$$\mathfrak{grt}_1 \longrightarrow \mathfrak{grt}_1^{EM} \longrightarrow \mathfrak{krv}_2$$