$$\mathfrak{t}_3 = \mathrm{Lie}(t^{12}, t^{13}, t^{23}) \,/\, \text{relations} \ni c = t^{12} + t^{13} + t^{23} \quad \text{central}$$

$$\begin{array}{ccccc}
\mathrm{Lie}(t^{13}, t^{23}) \subset \mathbb{Q}c \oplus \mathrm{Lie}(t^{13}, t^{23}) & = & \mathfrak{t}^3 & = & \mathbb{Q}c \oplus \mathrm{Lie}(t^{12}, t^{23}) \supset \mathrm{Lie}(t^{12}, t^{23}) \\
\wr\Vert & & & & \wr\Vert \\
\mathrm{Lie}(x,y) & & & & \mathrm{Lie}(x,y) \\
\text{\rotatebox{90}{$\in$}} & & & & \text{\rotatebox{90}{$\in$}} \\
\deg\varphi \geq 2 \quad \varphi(x,y) & \longmapsto & & & \varphi(-x-y, y) \qquad \varphi(x,y) \\
\varphi(-x-y, y) & \longleftarrow & & & \varphi(x,y)
\end{array}$$

EM 5-gon

(P1) $\varphi(y,0) - \varphi(x+y,0) = 0$

(P2) $R(y,0) - R(x+y,0) = 0$

(P3) $\partial_y \varphi(x,y) + \partial_y \varphi(y,0) - \partial_y \varphi(x+y,0)$

$= 2R(x,y)$

$\mathfrak{grt}_1$ rel.

(2-cycle) $\varphi(x,y) + \varphi(y,x) = 0$

(3-cycle) $\varphi(x,y) + \varphi(y,-x-y) + \varphi(-x-y,x) = 0$

(5-gon) $\varphi(t^{1,2}, t^{2,34}) + \varphi(t^{12,3}, t^{3,4})$

$= \varphi(t^{2,3}, t^{3,4}) + \varphi(t^{1,23}, t^{23,4}) + \varphi(t^{1,2}, t^{2,3})$

Question on EM 5-gon and KV equations

AT map $\nu : \mathfrak{grt}_1 \longrightarrow \mathfrak{krv}_2$, $\psi(x,y) \mapsto \left(\psi(-x-y,x), \psi(-x-y,y)\right)$

$\varphi \in \mathrm{SolEMPent}$ $\qquad \nu\left(\varphi(-x-y,y)\right) = \left(\varphi(y,x), \varphi(x,y)\right)$

[Q $\varphi \in \mathrm{SolEMPent} \Longrightarrow \left(\varphi(y,x), \varphi(x,y)\right) \in \mathfrak{krv}_2$?]

$$(\mathrm{KV1}) \iff [x, \varphi(y,x)] + [y, \varphi(x,y)] = 0$$

$$(\mathrm{KV2}) \iff \left| x R(y,x) + y R(x,y) \right| \text{ is DufIo}$$

Extension to $Ass\langle x, y\rangle$ ...

$$R(uv) = uR(v) + R(u)v + \frac{1}{2}(\partial_x v)x(\partial_x u)^* + \frac{1}{2}(\partial_y v)x(\partial_x u)^*$$

or

$$uR(v) + R(u)v - \frac{1}{2}(\partial_x u)x(\partial_x v)^* - \frac{1}{2}(\partial_y u)x(\partial_y v)^*$$

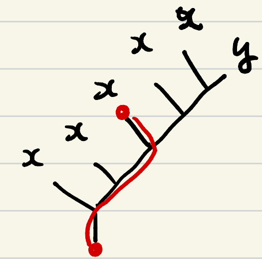
$\cancel{(1-t)\circledast + t\circledast}$?

On EM 5-gon and (KV1)

$R : Lie(x,y) \to \mathbb{Q}\langle\langle x,y\rangle\rangle \qquad R(x) = R(y) = 0$ &

$$R([u,v]) = [u, R(v)] + [R(u), v] + \frac{1}{2}\left(\begin{array}{l} (\partial_x v)x(\partial_x u)^* - (\partial_x u)x(\partial_x v)^* \\ + (\partial_y v)y(\partial_y u)^* - (\partial_y u)y(\partial_y v)^* \end{array} \right)$$

[**Lem1** $\forall m \in \mathbb{Z}_{\geq 0} \quad \partial_x(ad_x^m(y)) = -\sum_{i+j=m-1} x^i ad_x^j(y)$]



proof True for $m = 0$ &

$$\partial_x(ad_x^{m+1}(y)) = \partial_x[x, ad_x^m(y)] = x\,\partial_x(ad_x^m(y)) - ad_x^m(y)\,\partial_x(x)$$

$$\underset{\text{ind.ass.}}{=} -x\left(\sum_{i+j=m-1} x^i ad_x^j(y)\right) - ad_x^m(y) = -\sum_{i+j=m} x^i ad_x^j(y)$$

//

$\left[\underline{\text{Lem2}}\quad \forall m \geq 1 \quad R(\mathrm{ad}_x^m(y)) = -\frac{m-1}{2}x^{m-1}y + \cdots + (-1)^{m-1}\frac{m-1}{2}yx^{m-1}\right]$

<u>proof</u> $m=1$ $\quad R(\mathrm{ad}_x(y)) = R([x,y]) = 0$

$$R(\mathrm{ad}_x^{m+1}(y)) = R([x,\mathrm{ad}_x^m(y)]) = [x, \underbrace{R(\mathrm{ad}_x^m(y))}_{\text{// ind.ass.}}] + \frac{1}{2}\left(\underbrace{(\partial_x \mathrm{ad}_x^m(y))}_{\text{// Lem1}}x - x(\partial_x \mathrm{ad}_x^m(y))^*\right)$$

$$-\frac{m-1}{2}x^{m-1}y + \cdots + (-1)^{m-1}\frac{m-1}{2}yx^{m-1} \qquad -\sum_{i+j=m-1} x^i \mathrm{ad}_x^j(y)$$

☆

The coeff. of $x^m y = -\frac{m-1}{2} - \frac{1}{2} = -\frac{m}{2}$ and the coeff. of yx^m is

$$-(-1)^{m-1}\frac{m-1}{2} - (-1)^{m-1}\frac{1}{2} = (-1)^m\frac{m}{2} \qquad //$$

<u>Rem</u> We can compute the 2nd term of ☆ to get the following recursive formula

$$R(\mathrm{ad}_x^{m+1}(y)) = [x, R(\mathrm{ad}_x^m(y))] + \frac{1}{2}\left((-1)^m yx^m - x^m y\right)$$

$$\left(ad_x^m(y) = \sum_{b+c=m} (-1)^c \binom{m}{b} x^b y x^c \qquad \sum = \sum_{c=0} + \sum_{c\geq 1} \right)$$

(added on 8th May 2024)

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

[Lem 1′ $\partial_x(ad_x^m(y)) = -\sum_{b+c=m-1} (-1)^c \binom{m}{b} x^b y x^c$]

proof

$$\partial_x(ad_x^m(y)) \underset{\text{Lem1}}{=} -\sum_{p+j=m-1} x^p\, ad_x^j(y) = -\sum_{p+q+r=m-1} (-1)^r \binom{q+r}{r} x^{p+q} y x^r$$

$p+q=b$

$r=c$

$$= -\sum_{b+c=m-1} (-1)^c \underbrace{\sum_{p=0}^{b} \binom{b-p+c}{c}}_{=\binom{b+c+1}{c+1}} x^b y x^c = \text{RHS} \quad //$$

$$\binom{b+c+1}{c+1} = \binom{m}{c+1} = \binom{m}{b}$$

Then,

$$\partial_x(ad_x^m(y))x - x\left(\partial_x(ad_x^m(y))\right)^* = \sum_{c=0}^{m-1} (-1)^{c+1} \binom{m}{c+1} x^b y x^{c+1} + \sum_{b+c=m-1} \underbrace{(-1)^c \binom{m}{b} (-1)^m}_{-(-1)^b} x^{c+1} y x^b$$

$$= (-1)^m y x^m - x^m y$$

This proves the recursive formula for $R(ad_x^m(y))$ in the previous page. //

Lem 2' (explicit formula for $R(\mathrm{ad}_x^m(y))$)

$$R(\mathrm{ad}_x^m(y)) = \sum_{i+j=m-1} \frac{(-1)^j}{2} \left(\binom{m-1}{j-1} - \binom{m-1}{j+1} \right) x^i y x^j$$

proof Set $R'_m := \mathrm{RHS}$ and check the recursive formula

$$R'_{m+1} = [x, R'_m] + \frac{1}{2}\left((-1)^m y x^m - x^m y\right)$$

//

$$\binom{m-1}{j+1} = \binom{m-1}{i-1}$$

more symmetric form:

$$R(\mathrm{ad}_x^m(y)) = \sum_{i+j=m-1} \frac{(-1)^j}{2} \left(\binom{m-1}{j-1} - \binom{m-1}{i-1} \right) x^i y x^j$$

Lem 3 $\forall m \geq 1 \quad R(\mathrm{ad}_x^m(y))^{\mathrm{abel}} = \dfrac{(-1)^{m-1} - 1}{2} x^{m-1} y$

proof This follows from the recursive formula in the previous remark, or can be deduced directly from ☆ in the proof of Lem 2. //

Notation Let $\varphi \in \mathrm{SolEMPent}$ be homogeneous of degree m. Write

$$\varphi = \varphi_{m-1,1} + \sum_{2 \leq j \leq m-1} \varphi_{m-j,j} = \varphi_1 + \varphi_{\geq 2} \in W_{m-1,1} \oplus \bigoplus_{2 \leq j \leq m-1} W_{m-j,j},$$

where $W_{a,b}$ is the space of degree $a+b$ and depth b (# of y's $= b$).

Prop 4 Let $\varphi \in \mathrm{SolEMPent}$ be homogeneous of degree m. If m is even and ≥ 3, then $\varphi_1 = 0$.

$\sigma_{2k+1} = \mathrm{ad}_x^{2k}(y) + \cdots\cdots$

<u>proof of Prop 4</u>: We look at the depth 1 part of (P3), which can be computed from φ_1 and $\varphi_{m-2,2}$. One can write $\varphi_1 = c\,\mathrm{ad}_x^{m-1}(y)$ and

$$\varphi_{m-2,2} = \sum_{\substack{p>q \\ p+q=m-2}} c_{p,q}\left[\mathrm{ad}_x^{p}(y),\ \mathrm{ad}_x^{q}(y)\right].$$

The goal is to show that $c = 0$.

$-(m-1)x^{m-2}y$

φ

abelianization

m: even

$$\boxed{(P3)\qquad \partial_y\varphi(x,y) + \partial_y\varphi(y,0) - \partial_y\varphi(x+y,0) - 2R(\varphi) = 0}$$

contribution from $\mathrm{ad}_x^{m-1}(y) = \left(x^{m-1}+y^{m-1}-(x+y)^{m-1}\right)_{m-2,1} - 2R(\mathrm{ad}_x^{m-1}(y))$

to the LHS of $(P3)_{\text{depth}=1}$

∂_y

x^{m-1}

Lem 2

m: even

$$= (-1+m-2)\,x^{m-2}y + \cdots + \left(-1-(m-2)\right)yx^{m-2}$$

$$= (m-3)\,x^{m-2}y + \cdots + (1-m)\,yx^{m-2}$$

$\{x^i y x^j\}_{i+j=m-2}$

$$\boxed{(P3) \qquad \partial_y \varphi(x,y) + \partial_y \varphi(y,0) - \partial_y \varphi(x+y,0) - 2R(\varphi) = 0}$$

contribution from $[ad_x^p(y), ad_x^q(y)]$ is equal to

$$\partial_y [ad_x^p(y), ad_x^q(y)] - 2\Big(R\big([ad_x^p(y), ad_x^q(y)]\big)\Big)_{m-2,1}$$

$$\partial_y [ad_x^p(y), ad_x^q(y)] = ad_x^p(y)\, x^q - ad_x^q(y)\, x^p$$

$$R\big([ad_x^p(y), ad_x^q(y)]\big) = \underbrace{[R(ad_x^p(y)), ad_x^q(y)] + [ad_x^p(y), R(ad_x^q(y))] + \frac{1}{2}\Big(\partial_x (ad_x^q(y))\, x \big(\partial_x(ad_x^p(y))\big)^* - \cdots}_{\text{depth } 2}$$
$$+ \partial_y (ad_x^q(y))\, y \big(\partial_y\, ad_x^p(y)\big)^* - \big(\partial_y\, ad_x^p(y)\big)\, y \big(\partial_y\, ad_x^q(y)\big)^* \Big)$$

$$= ad_x^p(y)\, x^q - ad_x^q(y)\, x^p + (-1)^q x^p y x^q - (-1)^p x^q y x^p$$

Set $v_q := \overbrace{\mathrm{ad}_x^p(y)\,x^q}^{((-1)^p yx^p + \cdots + x^p y)x^q} - \mathrm{ad}_x^q(y)\,x^p + (-1)^q x^p y x^q - (-1)^p x^q y x^p \qquad (p = m-2-q)$

We have

$$v_0 = 2x^{m-2}y + \cdots - yx^{m-2}, \qquad \xrightarrow{\text{abelianization}} \; -x^{m-2}y$$

$$v_q = 0\cdot x^{m-2}y + \cdots + \left((-1)^p - (-1)^q\right) yx^{m-2}$$

$$\varphi_{m-2,2} = \sum_{\substack{p>q \\ p+q=m-2}} c_{p,q}\left[\mathrm{ad}_x^p(y), \mathrm{ad}_x^q(y)\right] \qquad = 0\cdot x^{m-2}y + \cdots + 0\cdot yx^{m-2} \qquad (q>0) \quad \longrightarrow 0$$

Going back to (P3) and looking at the coeff. of $x^{m-2}y$ and yx^{m-2}, we obtain

$$c\,\underbrace{\left((m-3)x^{m-2}y + (1-m)yx^{m-2}\right)}_{\mathrm{ad}_x^{m-1}(y)} + c_{m-2,0}\underbrace{\left(2x^{m-2}y - yx^{m-2}\right)}_{v_0} = 0.$$

Since $\begin{pmatrix} m-3 & 1-m \\ 2 & -1 \end{pmatrix}$ is regular, we conclude that $c = 0$ // Prop 4

$$\left(\begin{array}{l}\text{2-cycle for } \mathfrak{grt}_1 \\ \varphi(x,y)+\varphi(y,x)=0\end{array}\right)$$

<u>Rem</u> Another proof of Prop 4:

$$\begin{cases} \text{the coeff. of } x^{m-2}y \text{ gives } & c(m-3)+2C_{m-2,0}=0 \\ \text{the abelianization gives } & -c(m-1)-C_{m-2,0}=0 \end{cases} \rightsquigarrow c=0$$

[<u>Prop5</u> Let $\varphi \in \mathrm{SolEMPent}$ of degree at least 3. Then, $(\partial_y\varphi)^* = \partial_y \varphi$]

<u>proof</u> (P3) for $\varphi = \varphi_1 + \varphi_{\geq 2} = c\,\mathrm{ad}_x^{m-1}(y) + \varphi_{\geq 2}$ reads

$$(\partial_y\varphi)(x,y) = -c\left(y^{m-1} - (x+y)^{m-1}\right) + 2R(x,y).$$

$$\partial_y\varphi(x,y) = -\partial_y\varphi(y,0) + \partial_y\varphi(x+y,0) + 2R(\varphi)$$

By Prop 4, $c=0$ if m is even. Since $R^*=R$, the RHS is invariant under the antipode $*$, so is the LHS //

<u>Rem</u> $[x,y] \in \mathrm{SolEMPent}$. Thus, the assumption on the degree of φ is necessary.

<u>Rem</u> For $b \in \mathrm{Lie}(x,y)$, write $b = (\partial_x b)x + (\partial_y b)y = x(\partial^x b) + y(\partial^y b)$.

Since $b^* = -b$, one has $\partial^x b = (\partial_x b)^*$ and $\partial^y b = (\partial_y b)^*$. Therefore,

$$(\partial_y \varphi)^* = \partial_y \varphi \iff \partial^y \varphi = \partial_y \varphi.$$

We use the following result:

<u>Thm</u> (Schneps 2012, Double shuffle and Kashiwara-Vergne Lie algebras)

Let $b \in \mathrm{Lie}(x,y)$. Then,

$\exists a \in \mathrm{Lie}(x,y)$ s.t. $[x,a] + [y,b] = 0 \iff \partial_y b = \partial^y b$

Applying this to $b = \varphi(x,y) \in \mathrm{SdEMPent}_{\geq 3}$, we find that $\exists a \in \mathrm{Lie}(x,y)$ s.t.

$$[x, a] + [y, \varphi(x,y)] = 0 \qquad \cdots \star$$

If one can prove that $a = \varphi(y,x)$, we obtain (KV1). Actually,

$$a = \varphi(y,x) \quad \Longleftrightarrow \quad \partial_y a = \partial_y(\varphi(y,x)) \quad \big(= (\partial_x \varphi)(y,x) \big).$$

e.g. Racinet's formula $f = s(\partial_y f)$

Rem ✿ implies

$$x\big((\partial_x a)x + (\partial_y a)y\big) - \big(x(\partial^x a) + y(\partial^y a)\big)x + y\big((\partial_x \varphi)x + (\partial_y \varphi)y\big) - \big(x(\partial^x \varphi) + y(\partial^y \varphi)\big)y = 0,$$

hence $\partial_x a = \partial^x a$, $\partial_y a = \partial^x \varphi = (\partial_x \varphi)^*$ and $\partial^y a = \partial_x \varphi$.

Summary: $\varphi \in \mathrm{SolEMPent}$ (of degree at least 3)

$$\overset{?}{\Longrightarrow} \begin{cases} \text{(i)} \ (\partial_y \varphi(x,y))^* = \partial_y \varphi(x,y) & \text{OK by Prop 5} \\ \text{(ii)} \ (\partial_x \varphi(x,y))^* = (\partial_x \varphi)(y,x) & ? \end{cases}$$

$\Longrightarrow$ (KV1) for $v(\varphi) = \big(\varphi(y,x), \varphi(x,y)\big)$