Pensieve header: Double Integration.

Startup

In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\People\\Frohlich"];
PP_ := Identity;
<< "../../Projects/SL2Portfolio2/Engine-Speedy.m";
<< "../../Projects/SL2Portfolio2/Objects.m";
$k = 1;
HL[ℰ_] := Style[ℰ, Background → Green];
```

In[∘]:= $\overline{R}_{3,4}$

Out[∘]= $\mathbb{E}_{\{\}\to\{3,4\}}\left[-\hbar\, a_4 b_3,\ -\frac{\hbar\, x_4 y_3}{B_3},\ 1+\left(-\frac{\hbar^2 a_4 x_4 y_3}{B_3}-\frac{3\gamma\hbar^3 x_4^2 y_3^2}{4 B_3^2}\right)\epsilon+O[\epsilon]^2\right]$

In[∘]:= $P_{1,2}$

Out[∘]= $\mathbb{E}_{\{1,2\}\to\{\}}\left[\frac{\alpha_2\beta_1}{\hbar},\ \frac{\eta_1\xi_2}{\hbar},\ 1\right]$

In[∘]:= $\overline{R}_{3,4}$ // $P_{4,3}$

Out[∘]= $\mathbb{E}_{\{\}\to\{\}}[0, 0, 1]$

$\{$
int = $\left(\mathbb{E}_{\{i\}\to\{1,2\}}[\beta_i b_1+\alpha_i a_2,\ \eta_i y_1+\xi_i x_2,\ 1] * \overline{R}_{3,4}\right)$ // $bm_{3,1\to1}$ // $am_{4,2\to2}$ // $P_{1,2}$,
$d\Delta_{i\to i,j}$ // int
$\}$ // Column

Out[∘]=
$\mathbb{E}_{\{i\}\to\{\}}\left[\frac{\alpha_i\beta_i}{2\hbar},\ \frac{\sqrt{\mathcal{A}_i}\,\eta_i\xi_i}{\hbar+\hbar\sqrt{\mathcal{A}_i}},\ \frac{\sqrt{\mathcal{A}_i}}{2+2\sqrt{\mathcal{A}_i}}+O[\epsilon]^1\right]$

$\mathbb{E}_{\{i\}\to\{j\}}\left[a_j\alpha_i+b_j\beta_i+\frac{\alpha_i\beta_i}{2\hbar},\ \frac{y_j\eta_i}{\sqrt{\mathcal{A}_i}}+x_j\xi_i+\frac{\sqrt{\mathcal{A}_i}\,\eta_i\xi_i}{\hbar+\hbar\sqrt{\mathcal{A}_i}},\ \frac{\sqrt{\mathcal{A}_i}}{2+2\sqrt{\mathcal{A}_i}}+O[\epsilon]^1\right]$

In[∘]:= $\mathbb{E}_{\{\}\to\{1\}}[\mu\, a_1 b_1,\ \nu\, x_1 y_1,\ 1]$ // $dm_{i,1\to i}$

Out[∘]= $\mathbb{E}_{\{i\}\to\{i\}}\Big[\mu\, a_i b_i+a_i\alpha_i+b_i\beta_i,\ \frac{\nu\, x_i y_i}{\mathcal{A}_i}+y_i\eta_i+\frac{\left(\nu-\nu B_i+\hbar B_i^{\mu/\hbar}\right)x_i\xi_i}{\hbar},$
$1+\Big(-\frac{\nu\, x_i y_i\beta_i}{\mathcal{A}_i}+\gamma\mu B_i^{\mu/\hbar}x_i\xi_i+a_i\left(\nu B_i-\mu B_i^{\mu/\hbar}\right)x_i\xi_i+\frac{1}{2\mathcal{A}_i}\left(\gamma\nu^2-3\gamma\nu^2 B_i+2\gamma\nu\hbar B_i^{\mu/\hbar}\right)x_i^2 y_i\xi_i+$
$\frac{1}{4\hbar}\left(\gamma\nu^2-4\gamma\nu^2 B_i+3\gamma\nu^2 B_i^2-6\gamma\nu\hbar B_i^{1+\frac{\mu}{\hbar}}+2\gamma\nu\hbar B_i^{\mu/\hbar}+4\gamma\mu\hbar B_i^{\frac{2\mu}{\hbar}}\right)x_i^2\xi_i^2\Big)\epsilon+O[\epsilon]^2\Big]$

(171205) (Approx.) On $H^{*cop}\otimes H$ with $R = Id = \rho\otimes r$ (summed), $\int\phi\otimes x := \langle\phi\bar\rho \mid xr\rangle$ is an integral. $\frac{1}{2}$**Pf.** $x_1\int\phi\otimes x_2 = x_1\langle\phi\bar\rho \mid x_2 r\rangle = x_1 r^a r^b\langle\phi\bar\rho\bar\rho^a\rho^b \mid x_2 r\rangle \sim x_1 r_1 r^b\langle\phi\bar\rho\rho^b \mid x_2 r_2\rangle \sim (xr)_1 r^b\langle\phi\bar\rho \mid (xr)_3\rangle\langle\rho^b \mid (xr)_2\rangle \sim (xr)_1(\overline{xr})_2\langle\phi\bar\rho \mid (xr)_3\rangle = \langle\phi\bar\rho \mid xr\rangle = \int\phi\otimes x$. Verify! Attempt in Projects/SL2Portfolio2/DoubleIntegration.nb.

In[∘]:=
```
Block[{$k = 1}, Module[
  {RR = R_{3,4}, AM = am_{2,4→2}, BM = bm_{3,1→1}},
  {
   pint = Simplify /@ (
      E_{}→{1}[μ a_1 b_1, ν x_1 y_1, 1] // dm_{i,1→i} //
        (E_{i}→{1,2}[β_i b_1 + α_i a_2, η_i y_1 + ξ_i x_2, 1] RR) // BM // AM // P_{1,2}
     ),
   pΔint = Simplify /@ (
      E_{}→{1}[μ a_1 b_1, ν x_1 y_1, 1] // dm_{i,1→i} // dΔ_{i→j,i} //
        (E_{i}→{1,2}[β_i b_1 + α_i a_2, η_i y_1 + ξ_i x_2, 1] RR) // BM // AM // P_{1,2}
     ),
   HL[pint E_{}→{j}[0, 0, 1] ≡ pΔint]
  }]] // Column
```

Out[∘]=

$$\mathbb{E}_{\{i\}\to\{\}}\Big[-\frac{\alpha_i\beta_i}{\mu},\ \frac{\mathcal{A}_i\left(-\nu+\nu\mathcal{A}_i^{\hbar/\mu}-\hbar\left(\mathcal{A}_i^{\hbar/\mu}\right)^{\mu/\hbar}\right)\eta_i\xi_i}{\nu\hbar},$$
$$\frac{\hbar^2\mathcal{A}_i^{1-\frac{\hbar}{\mu}}}{\mu\nu}-\frac{1}{4(\mu^2\nu^3)}\Big(\hbar\mathcal{A}_i^{1-\frac{2\hbar}{\mu}}\Big(-4\gamma\mu\nu\hbar^3\mathcal{A}_i+2\nu^2\hbar\mathcal{A}_i^{\hbar/\mu}\left(\gamma\hbar(-3\mu+2\hbar)-2(\mu-\hbar)\beta_i\right)+$$
$$4\gamma\mu\hbar^2\mathcal{A}_i^2\left(\nu+\hbar\left(\mathcal{A}_i^{\hbar/\mu}\right)^{\mu/\hbar}\right)\eta_i\xi_i+4\nu^2\mathcal{A}_i^{1+\frac{2\hbar}{\mu}}\left(\gamma(3\mu-\hbar)\hbar-(\mu+\hbar)\beta_i\right)\eta_i\xi_i-$$
$$4\mu\nu\mathcal{A}_i^{\frac{\mu+\hbar}{\mu}}\left(\gamma\nu\hbar-\left(\nu+2\hbar\left(\mathcal{A}_i^{\hbar/\mu}\right)^{\mu/\hbar}\right)\beta_i\right)\eta_i\xi_i+\gamma\nu^2(3\mu+4\hbar)\mathcal{A}_i^{2+\frac{3\hbar}{\mu}}\eta_i^2\xi_i^2-4\gamma\mu\nu\mathcal{A}_i^{\frac{2(\mu+\hbar)}{\mu}}$$
$$\left(\nu+3\hbar\left(\mathcal{A}_i^{\hbar/\mu}\right)^{\mu/\hbar}\right)\eta_i^2\xi_i^2+\gamma\mu\mathcal{A}_i^{2+\frac{\hbar}{\mu}}\eta_i\xi_i\left(-4\nu\hbar^2+\left(\nu+2\hbar\left(\mathcal{A}_i^{\hbar/\mu}\right)^{\mu/\hbar}\right)^2\eta_i\xi_i\right)\Big)\Big)\epsilon+O[\epsilon]^2\Big]$$

$$\mathbb{E}_{\{i\}\to\{j\}}\Big[-\frac{\alpha_i\beta_i}{\mu},\ \frac{\mathcal{A}_i\left(-\nu+\nu\mathcal{A}_i^{\hbar/\mu}-\hbar\left(\mathcal{A}_i^{\hbar/\mu}\right)^{\mu/\hbar}\right)\eta_i\xi_i}{\nu\hbar},$$
$$\frac{\hbar^2\mathcal{A}_i^{1-\frac{\hbar}{\mu}}}{\mu\nu}-\frac{1}{4(\mu^2\nu^3)}\Big(\hbar\mathcal{A}_i^{1-\frac{2\hbar}{\mu}}\Big(2\nu^2\hbar\mathcal{A}_i^{\hbar/\mu}\left(\gamma\hbar(-3\mu+2\hbar)-2(\mu-\hbar)\beta_i\right)+4\gamma\mu\nu\hbar^3\mathcal{A}_i\left(-1+y_j\eta_i\right)+$$
$$4\gamma\mu\hbar^2\mathcal{A}_i^2\left(\nu+\hbar\left(\mathcal{A}_i^{\hbar/\mu}\right)^{\mu/\hbar}\right)\eta_i\xi_i+4\nu^2\mathcal{A}_i^{1+\frac{2\hbar}{\mu}}\left(\gamma(3\mu-\hbar)\hbar-(\mu+\hbar)\beta_i\right)\eta_i\xi_i-$$
$$4\mu\nu\mathcal{A}_i^{\frac{\mu+\hbar}{\mu}}\left(\gamma\nu\hbar-\left(\nu+2\hbar\left(\mathcal{A}_i^{\hbar/\mu}\right)^{\mu/\hbar}\right)\beta_i\right)\eta_i\xi_i+\gamma\nu^2(3\mu+4\hbar)\mathcal{A}_i^{2+\frac{3\hbar}{\mu}}\eta_i^2\xi_i^2-$$
$$4\gamma\mu\nu\mathcal{A}_i^{\frac{2(\mu+\hbar)}{\mu}}\left(\nu+3\hbar\left(\mathcal{A}_i^{\hbar/\mu}\right)^{\mu/\hbar}\right)\eta_i^2\xi_i^2+4\gamma\mu\nu\hbar^2x_j\left(\nu\hbar y_j-\mathcal{A}_i\left(-\nu+\nu\mathcal{A}_i^{\hbar/\mu}-\hbar\left(\mathcal{A}_i^{\hbar/\mu}\right)^{\mu/\hbar}\right)\xi_i\right)+$$
$$\gamma\mu\mathcal{A}_i^{2+\frac{\hbar}{\mu}}\eta_i\xi_i\left(-4\nu\hbar^2+\left(\nu+2\hbar\left(\mathcal{A}_i^{\hbar/\mu}\right)^{\mu/\hbar}\right)^2\eta_i\xi_i\right)\Big)\Big)\epsilon+O[\epsilon]^2\Big]$$

$$\frac{1}{\mu\nu^2}\mathcal{A}_i^{-\frac{2\hbar}{\mu}}\left(\gamma\epsilon\nu\hbar^4x_jy_j\mathcal{A}_i+\gamma\epsilon\hbar^4y_j\mathcal{A}_i^2\eta_i+\gamma\epsilon\nu\hbar^3x_j\mathcal{A}_i^2\xi_i-\gamma\epsilon\nu\hbar^3x_j\mathcal{A}_i^{2+\frac{\hbar}{\mu}}\xi_i+\gamma\epsilon\hbar^4x_j\mathcal{A}_i^2\left(\mathcal{A}_i^{\hbar/\mu}\right)^{\mu/\hbar}\xi_i\right)==0$$

```
In[∘]:= Block[{$k = 1}, Table[
          pint = Simplify /@ (
              𝔼_{}→{1}[μ a_1 b_1, ν x_1 y_1, 1] // dm_{i,1→i} //
                  (𝔼_{i}→{1,2}[β_i b_1 + α_i a_2, η_i y_1 + ξ_i x_2, 1] RR) // BM // AM // P_{1,2}
            );
          pΔint = Simplify /@ (
              𝔼_{}→{1}[μ a_1 b_1, ν x_1 y_1, 1] // dm_{i,1→i} // dΔ_{i→j,i} //
                  (𝔼_{i}→{1,2}[β_i b_1 + α_i a_2, η_i y_1 + ξ_i x_2, 1] RR) // BM // AM // P_{1,2}
            );
          Echo@HL@TrueQ[pint 𝔼_{}→{j}[0, 0, 1] ≡ pΔint],
          {RR, {(*R_{3,4},*)R_{3,4} // dS_4 // dS_4(*,R_{3,4}//dS_3//dS_3*)}},
          {AM, {am_{2,4→2}(*,am_{4,2→2}*)}}, {BM, {(*bm_{1,3→1},*)bm_{3,1→1}}}
        ]]

True

Out[∘]= {{{True}}}

In[∘]:= Block[{$k = 1}, Table[
          pint = Simplify /@ (
              𝔼_{}→{1}[μ a_1 b_1, ν x_1 y_1, 1] // dm_{i,1→i} //
                  (𝔼_{i}→{1,2}[β_i b_1 + α_i a_2, η_i y_1 + ξ_i x_2, 1] RR) // BM // AM // P_{1,2}
            );
          pΔint = Simplify /@ (
              𝔼_{}→{1}[μ a_1 b_1, ν x_1 y_1, 1] // dm_{i,1→i} // dΔ_{i→j,i} //
                  (𝔼_{i}→{1,2}[β_i b_1 + α_i a_2, η_i y_1 + ξ_i x_2, 1] RR) // BM // AM // P_{1,2}
            );
          Echo@HL@TrueQ[pint 𝔼_{}→{j}[0, 0, 1] ≡ pΔint],
          {RR, {(*R_{3,4},*)R_{3,4} // bS_3 // bS_3}},
          {AM, {am_{2,4→2}, am_{2,4→2}}}, {BM, {(*bm_{1,3→1},*)bm_{3,1→1}}}
        ]]

True

True

Out[∘]= {{{True}, {True}}}

In[∘]:= Dimensions[{{{Style[False, Background → RGBColor[0, 1, 0]],
            Style[True, Background → RGBColor[0, 1, 0]]}},
          {{Style[False, Background → RGBColor[0, 1, 0]],
            Style[False, Background → RGBColor[0, 1, 0]]}}}]

Out[∘]= {2, 1, 2}

In[∘]:= (R̄_{3,4} // bS_3) ≡ (R_{3,4} // bS_3 // bS_3)

Out[∘]= True
```

```
In[•]:= Block[{$k = 0}, Table[
   pint = Simplify /@ (
      𝔼{}→{1}[μ a1 b1, ν x1 y1, 1] // dmi,1→i //
         (𝔼{i}→{1,2}[βi b1 + αi a2, ηi y1 + ξi x2, 1] RR) // BM // AM // P1,2
     );
   pΔint = Simplify /@ (
      𝔼{}→{1}[μ a1 b1, ν x1 y1, 1] // dmi,1→i // dΔi→i,j //
         (𝔼{i}→{1,2}[βi b1 + αi a2, ηi y1 + ξi x2, 1] RR) // BM // AM // P1,2
     );
   Echo@HL@TrueQ[pint 𝔼{}→{j}[0, 0, 1] ≡ pΔint],
   {RR, {(*R3,4,*) R3,4 // bS3 // bS3, R3,4 // b̄S3 // b̄S3, R3,4 // aS4 // aS4,
     R3,4 // āS4 // āS4}}, {AM, {(*am2,4→2,*) am2,4→2}}, {BM, {bm1,3→1, bm3,1→1}}
  ]]

Out[•]= {{{False, False}}, {{False, False}}, {{False, False}}, {{False, False}}}
```

The following verifies $\int\phi\otimes x = x_1\int\phi\otimes x_2$ in this case:

```
n = 0; While[ n ≤ 1,
 Block[{$k = n},
  steps = Simplify /@ {
     𝔼{}→{1,j}[μ a1 b1, ν x1 y1, 1] // dmi,1→i //
        (𝔼{i}→{1,2}[βi b1 + αi a2, ηi y1 + ξi x2, 1] (R3,4 // dS4 // dS4))
       // bm3,1→1 // am2,4→2 // P1,2, (* Integral *)
     𝔼{}→{1,j}[μ a1 b1, ν x1 y1, 1] // dmi,1→i //
           (𝔼{i}→{1,2}[βi b1 + αi a2, ηi y1 + ξi x2, 1] (R3,4 // dS4 // dS4))
          // bm3,1→1 // am2,4→2 // dΔ2→2,3 // dΔ3→3,4 // dS3 // dm2,3→2 // P1,4 // dm2,j→j,
     (* Antipode axiom *)
     𝔼{}→{1,j}[μ a1 b1, ν x1 y1, 1] // dmi,1→i // (𝔼{i}→{1,2}[βi b1 + αi a2,
                 ηi y1 + ξi x2, 1] (R3,4 // dS4 // dS4))
            // bm3,1→1 // am2,4→2 // dΔ2→2,3 // dΔ3→3,4 // Rb1,b2 // dSb2 // dm2,b2→2 // Pb1,3 //
      P1,4 // dm2,j→j, (* R=Id *)
     𝔼{}→{1,j}[μ a1 b1, ν x1 y1, 1] // dmi,1→i // (𝔼{i}→{1,2}[βi b1 + αi a2,
                ηi y1 + ξi x2, 1] (R3,4 // dS4 // dS4))
           // bm3,1→1 // am2,4→2 // dΔ2→2,3 // Rb1,b2 // dSb2 // dmb1,1→1 // dm2,b2→2 // P1,3 //
     dm2,j→j, (* Pairing axiom *)
     𝔼{}→{1,j}[μ a1 b1, ν x1 y1, 1] // dmi,1→i // (𝔼{i}→{1,2}[βi b1 + αi a2,
                 ηi y1 + ξi x2, 1] (R3,4 // dS4 // dS4))
            // bm3,1→1 // dΔ2→21,22 // dΔ4→41,42 // am21,41→2 // dm22,42→3
        // Rb1,b2 // dSb2 // dmb1,1→1 // dm2,b2→2 // P1,3 // dm2,j→j, (* Bialgebra *)
     𝔼{}→{1,j}[μ a1 b1, ν x1 y1, 1] // dmi,1→i // 𝔼{i}→{1,2}[βi b1 + αi a2, ηi y1 + ξi x2, 1] // R3,4
                 // bm3,1→1 // dΔ2→21,22 // dΔ4→41,42 // dS41 //
            dS41 // dS42 // dS42 // am21,41→2 // dm22,42→3
        // Rb1,b2 // dSb2 // dmb1,1→1 // dm2,b2→2 // P1,3 // dm2,j→j,
     (* Antipode anticohomomorphism *)
     𝔼{}→{1,j}[μ a1 b1, ν x1 y1, 1] // dmi,1→i // 𝔼{i}→{1,2}[βi b1 + αi a2,
                   ηi y1 + ξi x2, 1] // R3,4 // Ra1,a2 // dma1,3→3
              // bm3,1→1 // dΔ2→21,22 // dSa2 // dSa2 // dS4 // dS4 // am21,a2→2 // dm22,4→3
        // Rb1,b2 // dSb2 // dmb1,1→1 // dm2,b2→2 // P1,3 // dm2,j→j,
```

```
      (* Quasitriangularity axiom *)
      E_{}→{1,j}[μ a1 b1, ν x1 y1, 1] // dm_{i,1→i} // E_{i}→{1,2}[β_i b1 + α_i a2,
                          η_i y1 + ξ_i x2, 1] // R_{3,4} // R_{a1,a2} // dm_{a1,3→3}
                    // bm_{3,1→1} // dΔ_{2→2_1,2_2} // dS_{a2} // dS4 // dS4 // dm_{2_2,4→3}
            // R_{b1,b2} // dm_{b1,1→1} // am_{b2,a2→2} // dS2 // dm_{2_1,2→2} // P_{1,3} // dm_{2,j→j},
      (* Antipode antihomomorphism *) (* $k=2 passes up to here *)
      E_{}→{1,j}[μ a1 b1, ν x1 y1, 1] // dm_{i,1→i} // E_{i}→{1,2}[β_i b1 + α_i a2, η_i y1 + ξ_i x2, 1]
                    // R_{3,4} // bm_{3,1→1} // dΔ_{2→2_1,2_2} // dS4 // dS4 // dm_{2_2,4→3} E_{}→{2}[0, 0, 1] // dS2 //
         dm_{2_1,2→2} // P_{1,3} // dm_{2,j→j}, (* Reidemeister 2 *)
      E_{}→{1,j}[μ a1 b1, ν x1 y1, 1] // dm_{i,1→i} // E_{i}→{1,2}[β_i b1 + α_i a2, η_i y1 + ξ_i x2, 1]
                 // R_{3,4} // bm_{3,1→1} // dΔ_{2→2_1,2_2} // dS4 // dS4 // dm_{2_2,4→3} // P_{1,3} // dm_{2_1,j→j},
      (* Antipode is unital *)
      (*E_{}→{1}[μ a1b1,ν x1y1,1]//dm_{i,1→i}//
                 dΔ_{i→j,i}
               //E_{j}→{j}[2α_j a_j,ξ_j x_j,1]
              (* Why does this term not affect anything? One can even remove it. *)
              //E_{i}→{1,2}[β_i b1+α_i a2,η_i y1+ξ_i x2,1]//R_{3,4}//dS4//dS4//bm_{3,1→1}//am_{2,4→2}//P_{1,2},
      (* Integral with comultiplicaion *)*)
      E_{}→{1}[μ a1 b1, ν x1 y1, 1] // dm_{i,1→i} // E_{i}→{1,2}[β_i b1 + α_i a2, η_i y1 + ξ_i x2, 1] // dΔ_{2→j,2}
          // R_{3,4} // dS4 // dS4 // bm_{3,1→1} // am_{2,4→2} // P_{1,2}
      (* Integral with comultiplicaion *)
     };
   (*Echo[TrueQ[And@@Table[steps[[$i]]≡steps[[$i+1]],{$i,Length[steps]-1}]]];*)
   If[TrueQ[And @@ Table[steps[[$i]] ≡ steps[[$i + 1]], {$i, Length[steps] - 1}]],
    Echo["$k=" <> ToString[n] <> " passes."],
    Echo["$k=" <> ToString[n] <> " fails."];
    Table[Echo@HL[steps[[$i]]], {$i, Length[steps]}];
    Break[]
   ]; n++]
 ]
```

$k=0 passes.

$k=1 passes.

The following (will) verify $\int \phi \otimes x = \phi_1 \int \phi_2 \otimes x$ in this case :

```
In[∘]:= n = 0; While[ n ≤ 1,
  Block[{$k = n},
   steps = Simplify /@ {
      E_{}→{1}[μ a1 b1, ν x1 y1, 1] // dm_{i,1→i} // E_{i}→{1,2}[β_i b1 + α_i a2, η_i y1 + ξ_i x2, 1] // dΔ_{1→j,1} //
            R_{3,4} // dS̄3 // dS̄3 // bm_{3,1→1} //
        am_{2,4→2} // P_{1,2}, (* Integral with comultiplicaion *)
      E_{}→{1}[μ a1 b1, ν x1 y1, 1] // dm_{i,1→i} // E_{i}→{1,2}[β_i b1 + α_i a2, η_i y1 + ξ_i x2, 1] // dΔ_{1→j,1} //
                  R_{3,4} // dS̄3 // dS̄3 // bm_{3,1→1} // am_{2,4→2} // R_{a1,a2} // dS̄_{a1} // R_{b1,b2} // am_{2,a2→2} //
         am_{2,b2→2} // bm_{a1,b1→ab} // bm_{ab,j→j} // P_{1,2}, (* Reidemeister 2 *)
      E_{}→{1}[μ a1 b1, ν x1 y1, 1] // dm_{i,1→i} // E_{i}→{1,2}[β_i b1 + α_i a2, η_i y1 + ξ_i x2, 1] // dΔ_{1→j,1}
                   // R_{3,4} // dS̄3 // dS̄3 // bm_{3,1→1} // am_{2,4→2} // R_{a1,a2} // dS̄_{a1} // R_{b1,b2} //
            am_{2,a2→2} // am_{2,b2→2} // bm_{a1,b1→ab} // dS̄_{ab} // am_{ab,j→j}
       // P_{1,2}, (* Antipode is unital *)
```

```
    𝔼_{}→{1}[μ a1 b1, ν x1 y1, 1] // dm_{i,1→i} // 𝔼_{i}→{1,2}[βi b1 + αi a2, ηi y1 + ξi x2, 1] // dΔ_{1→j,1}
            // R_{3,4} // dΔ_{3→3_1,3_2} // dS̄_{3_2} // dS̄_{3_2} // bm_{3_2,1→1} // dS̄_{3_1} // am_{2,4→2} // R_{b1,b2} //
        am_{2,b2→2} // bm_{3_1,b1→ab} // dS̄_{ab} // am_{ab,j→j}
      // P_{1,2}, (* R-matrix and strand doubling *)
    (*𝔼_{}→{1}[μ a1b1,ν x1y1,1]//dm_{i,1→i}//𝔼_{i}→{1,2}[βib1+αia2,ηiy1+ξix2,1]//dΔ_{1→j,1}
            //R_{3,4}//dS̄_3//dS̄_3//dΔ_{3→3_1,3_2}//bm_{3_2,1→1}//dS̄_{3_1}//am_{2,4→2}//R_{b1,b2}//am_{2,b2→2}//
        am_{3_1,b1→ab}//dS̄_{ab}//am_{ab,j→j}
      //P_{1,2}, (* THIS LOOKS WRONG YET THIS ROUTINE FINDS NO FAULT *)*)
    𝔼_{}→{1}[μ a1 b1, ν x1 y1, 1] // dm_{i,1→i} // 𝔼_{i}→{1,2}[βi b1 + αi a2, ηi y1 + ξi x2, 1]
              // dΔ_{1→j,1} // R_{3,4} // dΔ_{3→3_1,3_2} // dS̄_{3_2} // dS̄_{3_2} // bm_{3_2,1→1} // dS̄_{3_1} // am_{2,4→2} //
        R_{b1,b2} // am_{2,b2→2} // dS̄_{3_1} // dS̄_{b1} //
      dm_{b1,3_1→ab} // dm_{ab,j→j} // P_{1,2}, (* dS̄ is antimorphism *)
    𝔼_{}→{1}[μ a1 b1, ν x1 y1, 1] // dm_{i,1→i} // 𝔼_{i}→{1,2}[βi b1 + αi a2, ηi y1 + ξi x2, 1]
            // dΔ_{1→j,1} // R_{3,4} // dS̄_3 // dS̄_3 // dΔ_{3→3_1,3_2} // bm_{3_2,1→1} // am_{2,4→2} // R_{b1,b2} //
        am_{2,b2→2} // dS̄_{b1} // dm_{b1,3_1→ab} // dm_{ab,j→j} // P_{1,2}, (* dS̄^2 is coalgebra morphism *)
    𝔼_{}→{1}[μ a1 b1, ν x1 y1, 1] // dm_{i,1→i} // 𝔼_{i}→{1,2}[βi b1 + αi a2,
              ηi y1 + ξi x2, 1] // R_{3,4} // dS̄_3 // dS̄_3
          // dΔ_{1→j,1} // dΔ_{3→3_1,3_2} // bm_{3_2,1→1} // bm_{3_1,j→j}
       // dm_{2,4→2} // R_{b1,b2} // dm_{2,b2→2} // dS̄_{b1} // dm_{b1,j→j} // P_{1,2},
    (* [re-arrangement of commuting terms] *)
    𝔼_{}→{1}[μ a1 b1, ν x1 y1, 1] // dm_{i,1→i} // 𝔼_{i}→{1,2}[βi b1 + αi a2, ηi y1 + ξi x2, 1] //
           R_{3,4} // dS̄_3 // dS̄_3 // dm_{3,1→j} // dΔ_{j→j,1}
       // am_{2,4→2} // R_{b1,b2} // dS̄_{b1} // bm_{b1,j→j} // am_{2,b2→2} // P_{1,2},
    (* Δ is m-morphism *) (* ERROR: One of 2, b2 have B-algebra factors?? *)
    𝔼_{}→{1}[μ a1 b1, ν x1 y1, 1] // dm_{i,1→i} // 𝔼_{i}→{1,2}[βi b1 + αi a2, ηi y1 + ξi x2, 1] //
           R_{3,4} // dS̄_3 // dS̄_3 // dm_{3,1→j} // dΔ_{j→j,1}
       // am_{2,4→2} // R_{b1,b2} // dS̄_{b1} // bm_{b1,j→j} // bΔ_{1→1_1,1_2} // P_{1_2,2} // P_{1_1,b2},
    (* Pairing with (co)multiplication *)
    𝔼_{}→{1,j}[μ a1 b1, ν x1 y1, 1] // dm_{i,1→i} //
       (𝔼_{i}→{1,2}[βi b1 + αi a2, ηi y1 + ξi x2, 1] (R_{3,4} // dS̄_3 // dS̄_3))
      // dm_{3,1→1} // dm_{2,4→2} // P_{1,2} (* Integral *)
  };
 (*Echo[TrueQ[And@@Table[steps[[$i]]≡steps[[$i+1]],{$i,Length[steps]-1}]]];*)
 If[TrueQ[And @@ Table[steps[[$i]] ≡ steps[[$i + 1]], {$i, Length[steps] - 1}]],
  Echo["$k=" <> ToString[n] <> " passes."],
  Echo["$k=" <> ToString[n] <> " fails."];
  Table[Echo@HL[steps[[$i]]], {$i, Length[steps]}];
  Break[]
 ]; n++]
]
```

$k=0 fails.

» $$\mathbb{E}_{\{i\}\to\{j\}}\left[-\frac{\alpha_i\,\beta_i}{\mu},\ \frac{\mathcal{A}_i\left(-\nu+\nu\,\mathcal{A}_i^{\frac{\hbar}{\mu}}-\hbar\left(\mathcal{A}_i^{\frac{\hbar}{\mu}}\right)^{\mu/\hbar}\right)\eta_i\,\xi_i}{\nu\,\hbar},\ \frac{\hbar^2\,\mathcal{A}_i^{1-\frac{\hbar}{\mu}}}{\mu\,\nu}+O[\epsilon]^1\right]$$

» $$\mathbb{E}_{\{i\}\to\{j\}}\left[-\frac{\alpha_i \beta_i}{\mu}, \frac{\mathcal{A}_i \left(-\nu + \nu \mathcal{A}_i^{\frac{\hbar}{\mu}} - \hbar \left(\mathcal{A}_i^{\frac{\hbar}{\mu}}\right)^{\mu/\hbar}\right) \eta_i \xi_i}{\nu \hbar}, \frac{\hbar^2 \mathcal{A}_i^{1-\frac{\hbar}{\mu}}}{\mu \nu} + O[\epsilon]^1\right]$$

» $$\mathbb{E}_{\{i\}\to\{j\}}\left[-\frac{\alpha_i \beta_i}{\mu}, \frac{\mathcal{A}_i \left(-\nu + \nu \mathcal{A}_i^{\frac{\hbar}{\mu}} - \hbar \left(\mathcal{A}_i^{\frac{\hbar}{\mu}}\right)^{\mu/\hbar}\right) \eta_i \xi_i}{\nu \hbar}, \frac{\hbar^2 \mathcal{A}_i^{1-\frac{\hbar}{\mu}}}{\mu \nu} + O[\epsilon]^1\right]$$

» $$\mathbb{E}_{\{i\}\to\{j\}}\left[-\frac{\alpha_i \beta_i}{\mu}, \frac{\mathcal{A}_i \left(-\nu + \nu \mathcal{A}_i^{\frac{\hbar}{\mu}} - \hbar \left(\mathcal{A}_i^{\frac{\hbar}{\mu}}\right)^{\mu/\hbar}\right) \eta_i \xi_i}{\nu \hbar}, \frac{\hbar^2 \mathcal{A}_i^{1-\frac{\hbar}{\mu}}}{\mu \nu} + O[\epsilon]^1\right]$$

» $$\mathbb{E}_{\{i\}\to\{j\}}\left[-\frac{\alpha_i \beta_i}{\mu}, \frac{\mathcal{A}_i \left(-\nu + \nu \mathcal{A}_i^{\frac{\hbar}{\mu}} - \hbar \left(\mathcal{A}_i^{\frac{\hbar}{\mu}}\right)^{\mu/\hbar}\right) \eta_i \xi_i}{\nu \hbar}, \frac{\hbar^2 \mathcal{A}_i^{1-\frac{\hbar}{\mu}}}{\mu \nu} + O[\epsilon]^1\right]$$

» $$\mathbb{E}_{\{i\}\to\{j\}}\left[-\frac{\alpha_i \beta_i}{\mu}, \frac{\mathcal{A}_i \left(-\nu + \nu \mathcal{A}_i^{\frac{\hbar}{\mu}} - \hbar \left(\mathcal{A}_i^{\frac{\hbar}{\mu}}\right)^{\mu/\hbar}\right) \eta_i \xi_i}{\nu \hbar}, \frac{\hbar^2 \mathcal{A}_i^{1-\frac{\hbar}{\mu}}}{\mu \nu} + O[\epsilon]^1\right]$$

» $$\mathbb{E}_{\{i\}\to\{j\}}\left[-\frac{\alpha_i \beta_i}{\mu}, \frac{\mathcal{A}_i \left(-\nu + \nu \mathcal{A}_i^{\frac{\hbar}{\mu}} - \hbar \left(\mathcal{A}_i^{\frac{\hbar}{\mu}}\right)^{\mu/\hbar}\right) \eta_i \xi_i}{\nu \hbar}, \frac{\hbar^2 \mathcal{A}_i^{1-\frac{\hbar}{\mu}}}{\mu \nu} + O[\epsilon]^1\right]$$

» $$\mathbb{E}_{\{i\}\to\{j\}}\left[-\frac{\alpha_i \beta_i}{\mu}, \frac{\mathcal{A}_i \left(-\nu + \nu \mathcal{A}_i^{\frac{\hbar}{\mu}} - \hbar \left(\mathcal{A}_i^{\frac{\hbar}{\mu}}\right)^{\mu/\hbar}\right) \eta_i \xi_i}{\nu \hbar}, \frac{\hbar^2 \mathcal{A}_i^{1-\frac{\hbar}{\mu}}}{\mu \nu} + O[\epsilon]^1\right]$$

» $$\mathbb{E}_{\{i\}\to\{j\}}\Bigg[-\frac{\alpha_i \beta_i}{\mu},$$
$$-\left(\left(\mathcal{A}_i \eta_i \left(\hbar^2 \left(\frac{1}{B_j}\right)^{\frac{\hbar}{\mu}} B_j^{\frac{\hbar}{\mu}} y_j - \hbar^2 y_j \mathcal{A}_i^{\frac{\hbar}{\mu}} - \nu \left(\frac{1}{B_j}\right)^{-\frac{\hbar}{\mu}} B_j^{1-\frac{\hbar}{\mu}} \mathcal{A}_i^{\frac{2\hbar}{\mu}} \xi_i + B_j \mathcal{A}_i^{\frac{\hbar}{\mu}} \left(\nu + \hbar \left(\left(\frac{1}{B_j}\right)^{-\frac{\hbar}{\mu}} B_j^{-\frac{\hbar}{\mu}} \mathcal{A}_i^{\frac{\hbar}{\mu}}\right)^{\mu/\hbar}\right) \xi_i\right)\right)\Big/\right.$$
$$\left.\left(\hbar \left(-\hbar \left(\frac{1}{B_j}\right)^{\frac{\hbar}{\mu}} B_j^{\frac{\hbar}{\mu}} \mathcal{A}_i + \hbar \left(\frac{1}{B_j}\right)^{\frac{\hbar}{\mu}} B_j^{\frac{\mu+\hbar}{\mu}} \mathcal{A}_i + \nu B_j \mathcal{A}_i^{\frac{\hbar}{\mu}}\right)\right)\right),$$
$$\left(\hbar^2 \left(\frac{1}{B_j}\right)^{\frac{\hbar}{\mu}} B_j^{\frac{\hbar}{\mu}} \mathcal{A}_i\right)\Big/\left(-\mu \hbar \left(\frac{1}{B_j}\right)^{\frac{\hbar}{\mu}} B_j^{\frac{\hbar}{\mu}} \mathcal{A}_i + \mu \hbar \left(\frac{1}{B_j}\right)^{\frac{\hbar}{\mu}} B_j^{1+\frac{\hbar}{\mu}} \mathcal{A}_i + \mu \nu B_j \mathcal{A}_i^{\frac{\hbar}{\mu}}\right) + O[\epsilon]^1\Bigg]$$

» $$\mathbb{E}_{\{i\}\to\{j\}}\left[-\frac{\alpha_i \beta_i}{\mu}, \frac{\mathcal{A}_i \left(-\nu + \nu \mathcal{A}_i^{\frac{\hbar}{\mu}} - \hbar \left(\mathcal{A}_i^{\frac{\hbar}{\mu}}\right)^{\mu/\hbar}\right) \eta_i \xi_i}{\nu \hbar}, \frac{\hbar^2 \mathcal{A}_i^{1-\frac{\hbar}{\mu}}}{\mu \nu} + O[\epsilon]^1\right]$$

WARNING : "am" and "bm" are used in the wrong places above. Take care to replace them with “dm”, and do be smart about such transformations.

The following verifies $\int \phi \otimes x =$

$\phi_1 \otimes x_1 \int \phi_2 \otimes x_2$ (this should follow from the above two proofs,

the second of which I cannot quite get) :

```
In[•]:= n = 0; While[n ≤ 1,
  Block[{$k = n},
   steps = Simplify /@ {
      𝔼_{}→{1,j}[μ a1 b1, ν x1 y1, 1] // dm_{i,1→i} //
         (𝔼_{i}→{1,2}[βi b1 + αi a2, ηi y1 + ξi x2, 1] (R3,4 // dS4 // dS4))
        // bm_{3,1→1} // am_{2,4→2} // P1,2, (* Integral *)
      𝔼_{}→{1}[μ a1 b1, ν x1 y1, 1] // dm_{i,1→i} // dΔ_{i→j,i} // 𝔼_{i}→{1,2}[βi b1 + αi a2, ηi y1 + ξi x2, 1]
          // R3,4 // dS̄3 // dS̄3 // bm_{3,1→1} // am_{2,4→2} // P1,2
      (* Integral with comultiplicaion *)
     };
   (*Echo[TrueQ[And@@Table[steps[[$i]]≡steps[[$i+1]],{$i,Length[steps]-1}]]];*)
   If[TrueQ[And @@ Table[steps[[$i]] ≡ steps[[$i + 1]], {$i, Length[steps] - 1}]],
    Echo["$k=" <> ToString[n] <> " passes."],
    Echo["$k=" <> ToString[n] <> " fails."];
    Table[Echo@HL[steps[[$i]]], {$i, Length[steps]}];
    Break[],
    Echo["Other error."]
   ]; n++]
 ]
```

$k=0 passes.

$k=1 passes.

Tests for the R-matrix:

```
In[•]:= (R_{a1,a2} // R_{b1,b2} // dS̄_{a1} // am_{i2,a2→i2} // am_{i2,b2→i2} // bm_{i1,a1→i1} // bm_{i1,b1→i1}) ≡
  (𝔼_{i1,i2}→{i1,i2}[a_{i2} α_{i2} + b_{i1} β_{i1}, y_{i1} η_{i1} + x_{i2} ξ_{i2}, 1])
Out[•]= True
```

```
In[•]:= (R1,4 // R2,3 // dS̄3 // dm_{1,2→1} // dm_{3,4→2}) ≡ (𝔼_{}→{1,2}[0, 0, 1])
Out[•]= True
```

```
In[•]:= (R1,4 // R2,3 // dS̄4 // dm_{1,2→1} // dm_{3,4→2}) ≡ (𝔼_{}→{1,2}[0, 0, 1])
Out[•]= True
```

```
In[•]:= (R1,4 // R2,3 // dS1 // dm_{1,2→1} // dm_{3,4→2}) ≡ (𝔼_{}→{1,2}[0, 0, 1])
Out[•]= True
```

```
In[•]:= (R1,4 // R2,3 // dS2 // dm_{1,2→1} // dm_{3,4→2}) ≡ (𝔼_{}→{1,2}[0, 0, 1])
Out[•]= True
```

In[∘]:= $R_{a1,a2}$ // $R_{b1,b2}$ // dS_{a2} // $dm_{b2,a2\to 2}$ // dS_1 // $dm_{b1,a1\to 1}$

Out[∘]= $$\mathbb{E}_{\{1\}\to\{1,2\}}\Big[-a_1\alpha_1 - b_1\beta_1,\ -\frac{y_1\mathcal{A}_1\eta_1}{B_1} - x_1\mathcal{A}_1\xi_1 + \frac{(\mathcal{A}_1 - B_1\mathcal{A}_1)\eta_1\xi_1}{\hbar B_1},$$
$$1 + \Big(\frac{\gamma\hbar y_1\mathcal{A}_1\eta_1}{B_1} - \frac{y_1\mathcal{A}_1\beta_1\eta_1}{B_1} - \frac{\gamma\hbar y_1^2\mathcal{A}_1^2\eta_1^2}{2B_1^2} - \hbar a_1 x_1\mathcal{A}_1\xi_1 - x_1\mathcal{A}_1\beta_1\xi_1 + \frac{a_1\mathcal{A}_1\eta_1\xi_1}{B_1} -$$
$$\frac{\gamma\hbar x_1 y_1\mathcal{A}_1^2\eta_1\xi_1}{B_1} + \frac{(-\gamma\mathcal{A}_1 + \gamma B_1\mathcal{A}_1)\eta_1\xi_1}{B_1} + \frac{(\mathcal{A}_1 - B_1\mathcal{A}_1)\beta_1\eta_1\xi_1}{\hbar B_1} + \frac{y_1(3\gamma\mathcal{A}_1^2 - \gamma B_1\mathcal{A}_1^2)\eta_1^2\xi_1}{2B_1^2} -$$
$$\frac{1}{2}\gamma\hbar x_1^2\mathcal{A}_1^2\xi_1^2 + \frac{x_1(3\gamma\mathcal{A}_1^2 - \gamma B_1\mathcal{A}_1^2)\eta_1\xi_1^2}{2B_1} + \frac{1}{4\hbar B_1^2}(-3\gamma\mathcal{A}_1^2 + 4\gamma B_1\mathcal{A}_1^2 - \gamma B_1^2\mathcal{A}_1^2)\eta_1^2\xi_1^2\Big)\epsilon + O[\epsilon]^2\Big]$$

In[∘]:= $(R_{a1,a2}$ // $R_{1,2}$ // $dm_{a1,1\to 1}) \equiv (R_{1,2}$ // $d\Delta_{2\to a2,2})$

Out[∘]= True

In[∘]:= $(R_{a1,a2}$ // $R_{1,2}$ // $dm_{a2,2\to 2}) \equiv (R_{1,2}$ // $d\Delta_{1\to 1,a1})$

Out[∘]= True

In[∘]:= $(R_{1,2}$ // $\overline{bS}_1$ // $P_{1,i}) \equiv (\overline{aS}_i$ // $\mathbb{E}_{\{i\}\to\{2\}}[a_2\,\alpha_i,\ x_2\,\xi_i,\ 1])$

Out[∘]= True

In[∘]:= $\{(R_{1,2}$ // $\overline{bS}_1$ // $P_{i,2}) \equiv ((\overline{bS}_i)$ // $\mathbb{E}_{\{i\}\to\{1\}}[b_1\,\beta_i,\ y_1\,\eta_i,\ 1])\}$

Out[∘]= {True}

$\overline{R}_{1,2} \equiv (R_{1,2}$ // $aS_2)$

Out[∘]= $$-\frac{\gamma\in\hbar^2 x_2 y_1}{B_1} == 0$$

In[∘]:= $(\overline{dS}_1$ // $\overline{dS}_2$ // $dm_{1,2\to 1}) \equiv (dm_{2,1\to 1}$ // $\overline{dS}_1)$

Out[∘]= True

In[∘]:= $\{(d\Delta_{1\to 1,2}$ // $\overline{dS}_1$ // $\overline{dS}_2),\ (\overline{dS}_1$ // $d\Delta_{1\to 2,1})\}$

In[∘]:= $d\Delta_{i\to j,k}$ // $\mathbb{E}_{\{j\}\to\{j\}}[(a_j)\,\alpha_j,\ (x_j)\,\xi_j,\ 1]$

Out[∘]= $$\mathbb{E}_{\{i\}\to\{j,k\}}\Big[a_j\alpha_i + a_k\alpha_i + b_k\beta_i,\ y_k\eta_i + x_j\xi_i + x_k\xi_i,\ 1 + \Big(-\hbar a_j x_k\xi_i + \frac{1}{2}\gamma\hbar x_j x_k\xi_i^2\Big)\epsilon + O[\epsilon]^2\Big]$$

In[∘]:= $d\Delta_{i\to j,k}$ // $\overline{dS}_j$ // $dm_{k,j\to i}$

Out[∘]= $\mathbb{E}_{\{i\}\to\{i\}}[0,\ 0,\ 1 + O[\epsilon]^2]$

Finding the right integral (no pun intended)

We begin with the naïve guess that the left integral we have found is also a right integral, and also checking the rest of the options:

In[∘]:=
```
Block[{$k = 0}, Table[
  pint = Simplify /@ (
     𝔼{}→{1}[μ a1 b1, ν x1 y1, 1] // dmi,1→i //
       (𝔼{i}→{1,2}[βi b1 + αi a2, ηi y1 + ξi x2, 1] RR) // BM // AM // P1,2
    );
  pΔint = Simplify /@ (
     𝔼{}→{1}[μ a1 b1, ν x1 y1, 1] // dmi,1→i // dΔi→i,j //
       (𝔼{i}→{1,2}[βi b1 + αi a2, ηi y1 + ξi x2, 1] RR) // BM // AM // P1,2
    );
  Echo@HL@TrueQ[pint 𝔼{}→{j}[0, 0, 1] ≡ pΔint],
  {RR, {(*R3,4,*)R3,4 // dS4 // dS4, R3,4 // dS3 // dS3}},
  {AM, {am2,4→2, am4,2→2}}, {BM, {bm1,3→1, bm3,1→1}}
 ]]
```

Out[∘]= {{{False, False}, {False, False}}, {{False, False}, {False, False}}}

Not surprising. Given that papers cite left integrals and right integrals being related by the antipode, which is a more complicated beast, we can expect the formula for a right integral to be a bit more involved.

In[∘]:=
```
Block[{$k = 0}, Table[
  pint = Simplify /@ (
     𝔼{}→{1}[μ a1 b1, ν x1 y1, 1] // dmi,1→i // SS //
       (𝔼{i}→{1,2}[βi b1 + αi a2, ηi y1 + ξi x2, 1] RR) // BM // AM // P1,2
    );
  pΔint = Simplify /@ (
     𝔼{}→{1}[μ a1 b1, ν x1 y1, 1] // dmi,1→i // dΔi→i,j // SS //
       (𝔼{i}→{1,2}[βi b1 + αi a2, ηi y1 + ξi x2, 1] RR) // BM // AM // P1,2
    );
  Echo@HL@TrueQ[pint 𝔼{}→{j}[0, 0, 1] ≡ pΔint],
  {RR, {R3,4 // dS4 // dS4, R3,4 // dS3 // dS3}}, {AM, {am2,4→2, am4,2→2}},
  {BM, {bm3,1→1, bm1,3→1}}, {SS, {dSi, dS̄i (*aSi,*) (*aS̄i,*) (*bSi,*) (*bS̄i*)}}
 ]]
```

```
False

False

False

False

False

False

False

False

False

False

False

False

False

False

False

False
```

Out[ ]=
```
{{{{False, False}, {False, False}}, {{False, False}, {False, False}}},
 {{{False, False}, {False, False}}, {{False, False}, {False, False}}}}
```

Now it is time to actually review what the literature says on this matter.

The antipode on H^* should be given by $\phi \longmapsto \phi \circ S$, where S denotes the antipode on H, yet the above computation suggests this is still not correct.

Aside: why are these not equal?

In[ ]:= `Simplify@` $\left((x)^{-\hbar/\mu}\right)^{-2\,\mu/\hbar} \equiv x^2$

Out[ ]= $\left(x^{-\frac{\hbar}{\mu}}\right)^{-\frac{2\mu}{\hbar}} \equiv x^2$