DBN: AP: People: Afeke: On the Equivariant Intersection Pairing

https://drorbn.net/AP/People/Afeke/
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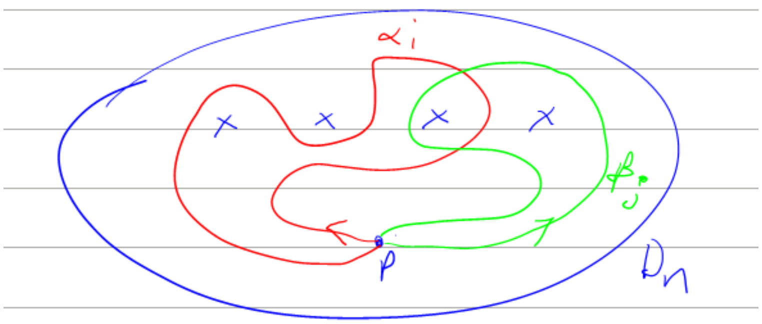

Let D_n be the n-punctured disk and let $\pi\colon \widetilde{D_n} \to D_n$ be its universal Abelian cover. The group $\mathbb{Z}^n = \langle t_1, \dots, t_n\rangle$ acts on $\widetilde{D_n}$ by deck transformations. We take as a given the "standard" (anti-symmetric) intersection pairing $\langle\cdot,\cdot\rangle\colon H_1(\widetilde{D_n}) \otimes H_1(\widetilde{D_n}) \to \mathbb{Z}$ and define the *equivariant* intersection pairing

$$\langle\cdot,\cdot\rangle_e\colon H_1(\widetilde{D_n}) \otimes H_1(\widetilde{D_n}) \to \mathbb{Z}\mathbb{Z}^n = \mathbb{Z}[t_1^{\pm1}, \dots, t_n^{\pm1}] \qquad (1)$$

by

$$\langle a, b\rangle_e = \sum_{k\in\mathbb{Z}^n} \langle t^k a, b\rangle t^k, \qquad (2)$$

where $t^k := t_1^{k_1} t_2^{k_2} \dots t_n^{k_n}$.

Let $p \in D_n$ be a basepoint and let $\tilde{p}$ be a fixed lift of p to $\widetilde{D_n}$ (namely, fix $\tilde{p} \in \widetilde{D_n}$ is such that $\pi(\tilde{p}) = p$). Given a curve α in D_n which begins and ends at p, we denote by $\tilde{\alpha}$ the unique lift of α to $\widetilde{D_n}$ for which $\tilde{\alpha}(0) = \tilde{p}$. We will have then a unique $w(\alpha) \in \mathbb{Z}^n$ for which $\tilde{\alpha}(1) = t^{w(\alpha)}\tilde{p}$.

Every element $a \in H_1(\widetilde{D_n})$ can be written as a finite sum $a = \sum_i f_i\tilde{\alpha}_i$, where $f_i \in \mathbb{Z}\mathbb{Z}^n$ and where the α_i's are curves in D_n which begin and end at p. Similarly write $b = \sum_j g_j\tilde{\beta}_j$. Without loss of generality, all the α_i "lean to the left" near their ends, and all the β_j "lean to the right" near their ends (see the figure). Also without loss of generality the interiors α_i^o and β_j^o of α_i and β_j avoid the basepoint p.

Claim 1. Suppose also that for every i and j the intersection $\alpha_i^o \cap \beta_j^o$ is transverse, and if $q \in \alpha_i^o \cap \beta_j^o$, denote by ϵ_q the sign of that intersection point. Then

$$\langle a, b\rangle_e = \sum_{i,j} \overline{f_i} g_j \sum_{q\in\alpha_i^o\cap\beta_j^o} \epsilon_q t^{-w\left(\widetilde{\alpha_i\#_q\beta_j}\right)}, \qquad (3)$$

where $f \mapsto \bar{f}$ is $\mathbb{Z}$-linear automorphism of $\mathbb{Z}\mathbb{Z}^n$ which extends $t^k \mapsto t^{-k}$ and where $\alpha_i\#_q\beta_j$ is the curve in D_n that follows α_i from its beginning up to the point q, and then follows β_j backwards from q to the beginning of β_j.

Remark 2. If for all i and j the curves $\tilde{\alpha}_i$ and $\tilde{\beta}_j$ are closed, we can write

$$\langle a, b\rangle_e = \sum_{i,j} \overline{f_i} g_j \langle\tilde{\alpha}_i, \tilde{\beta}_j\rangle_e \quad\text{and}\quad \langle\tilde{\alpha}_i, \tilde{\beta}_j\rangle_e = \sum_{q\in\alpha_i^o\cap\beta_j^o} \epsilon_q t^{-w\left(\widetilde{\alpha_i\#_q\beta_j}\right)}, \qquad (4)$$

but in general, $\langle\tilde{\alpha}_i, \tilde{\beta}_j\rangle_e$ may not make sense as a pairing of homology classes because $\tilde{\alpha}_i$ and $\tilde{\beta}_j$ may not be homology classes.

Remark 3. Rather than "α_i leans left and β_j leans right near p" we could say "the intersections of α_i and β_j with every small circle around p are not interlaced".