2023 Plan.

December 16 - 31, 2022: Belmont, Israel.
January to February 19: Groningen (Belmont).

March 3 to April 10: (LA) Sydney (LA).

Mid May: Israel + ?
May 22-25: (Belmont) ICERM (Belmost).

Likely Japan Plan.
June 17 - August 20: Base is Tsuda.
* A week and a half class in the first half of July.
* Visit Kyoto for second half of July.
* Nara conference: A week in first half of August.
August 21 - September 17: Base is Waseda.

Older Japan Plan.
Sunday June 18 – July 22 (5 weeks) – Tsuda with Yuseke.
- First week and a half for class preps and for chatting with Jun and Sakie.
- Then a week and a half of teaching. Possible course (15 hours): Fast computations in Knot Theory.
- Then two weeks for work.

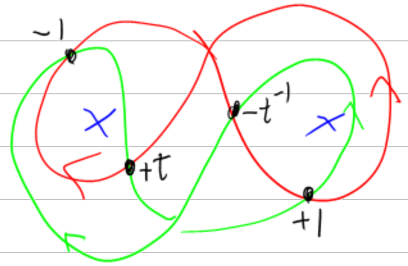
July 23 – August 5 (2 weeks) – Kyoto
August 6 – August 26 (3 weeks) – TiTech with Sakie
August 27 – Saturday September 16 (3 weeks) – Waseda with Jun.

Within October-November: (Belmont) Budapest (Belmont).

Early September 2024: Return to teaching.

$$\begin{matrix} 1 & 0 & -T^s & T^s-1 \\ 0 & 1 & 0 & -1 \end{matrix}$$

$$\begin{pmatrix} t-t^{-1} & a \\ -\bar{a} & t-t^{-1} \end{pmatrix}$$

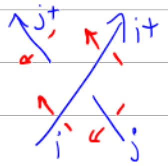


$$(T-T^{-1})(1+1+(T^s-1)(T^{-s}-1))$$
$$+a(-1)+\bar{a}(-1)+(-T^s)(T^{-s}-1)a-(-T^{-s})(T^s-1)\bar{a}$$

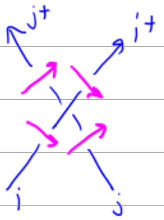
$$(T-T^{-1})(4-T^s-T^{-s})-a(a-\bar{a})+T^s a-\overline{T^s a}$$

$\langle \bullet, \bullet \rangle = t-t^{-1}$

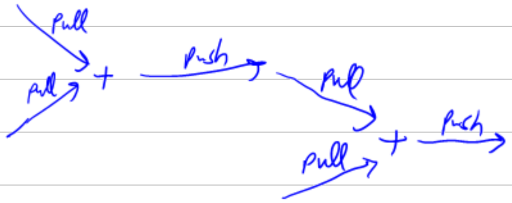
$(t-1)-(t^{-1}-1) = t-t^{-1}$



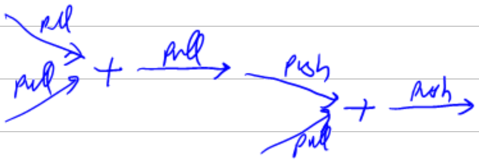
	i	j	i+	j+
$x_i x_{i+}^{-1} = 1$	1	0	-1	0
$x_j x_{i+} x_{j+}^{-1} x_{j+}^{-1} = 1$	0	1	$T-1$	$-T$
$x_{j+}^{-1} x_{i+}^{-1} x_j x_{i+} = 1$	0	T^{-2}	$-T^{-2}+T^{-1}$	$-T^{-1}$
$x_{i+} x_{j+} x_{i+}^{-1} x_j^{-1} = 1$	0	-1	$1-T$	T



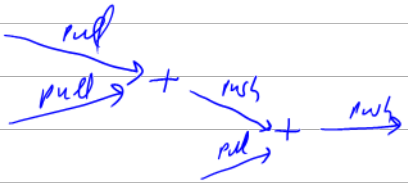
	i	j	i+	j+
$y_i y_{j+} y_{i+}^{-1} y_{j+}^{-1} = 1$	1	0	$-T$	$T-1$
$y_j y_{j+}^{-1} = 1$	0	1	0	-1


pull
pull
push
pull
pull
push

||


pull
pull
pull
push
pull
push

=


pull
pull
push
pull
push

=

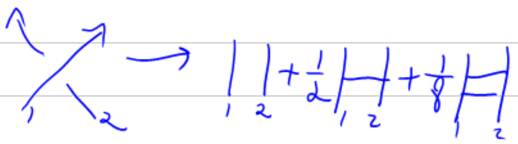
$$\begin{pmatrix} -1 & 0 & c & s \\ 0 & -1 & -s & c \\ 0 & -1 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 0 & c & s \\ 0 & -1 & -s & c \\ 0 & 0 & 1+s & -c \end{pmatrix} \longrightarrow \begin{matrix} -1 & 0 & c & s \\ 0 & -1 & -s & c \\ 0 & 0 & 1 & \frac{-c}{1+s} \end{matrix} \longrightarrow \begin{matrix} -1 & 0 & 0 & s+\frac{c^2}{1+s} \\ 0 & -1 & 0 & c-\frac{cs}{1+s} \\ 0 & 0 & 1 & \frac{-c}{1+s} \end{matrix}$$

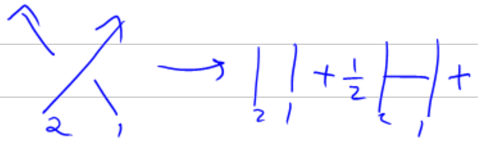
$$= \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & \frac{1}{1+s} \\ 0 & 0 & 1 & \frac{-c}{1+s} \end{pmatrix} \longrightarrow \begin{matrix} -1 & 0 & 1 \\ 0 & -1 & \frac{1}{1+s} \end{matrix}$$

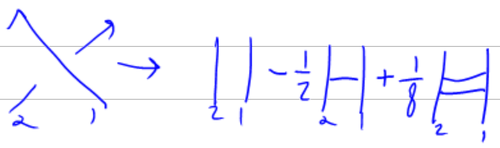
$$s+\frac{c^2}{1+s} = \frac{s(1+s)+c^2}{1+s} = 1$$

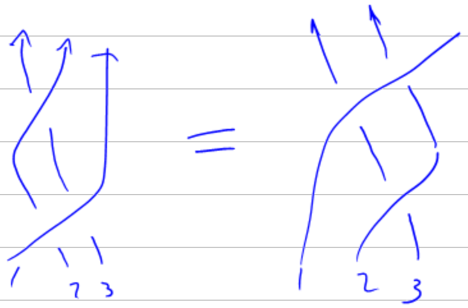
$$c-\frac{cs}{1+s} = \frac{c(1+s)-cs}{1+s} = \frac{1}{1+s}$$

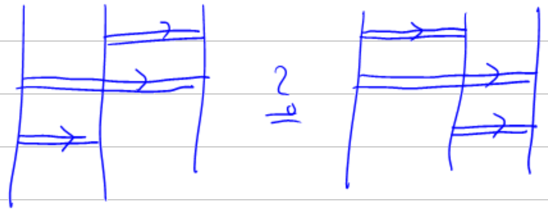
Scheduled Tangles.





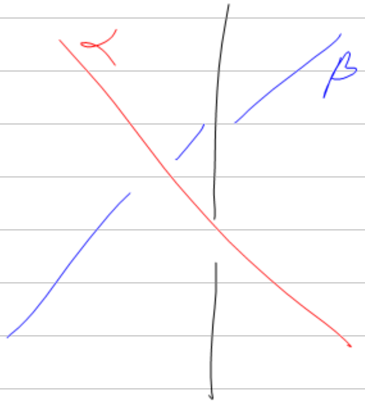






There is no scheduled formula for the Kontsevich integral even to degree 2.

230228b Given a diagram D for a long K, the phase along a curve $\gamma \subset D^c$ multiplies by T^s whenever γ passes over D with sign s. **Conj.** $\mathrm{lk}_K(\alpha,\beta) = \langle$flow: generated by α, measured by $\beta\rangle$ + ~~$\langle$total depth of α over β xings$\rangle$~~. α generates phased flow when it runs over D. β phase~~d~~-measures flow when it runs under D.



$f \sim g$ $\quad \exists \; C_1 \;\; k_1 \;\; N_1$

$\qquad\qquad\;\; C_2 \;\; k_2 \;\; N_2$

$\forall n > N_1 \quad f(n) < C_1 g(n)(\log n)^{k_1}$

$\qquad\qquad\; g(n) < C_2 f(n)(\log n)^{k_2}$

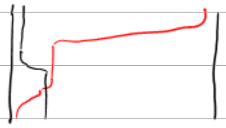
$\Leftrightarrow \exists \; C \;\; k \;\; N$

$$\forall n > N \quad \underbrace{C^{-1} g(n)(\log n)^{-k} < f(n)}_{g \lesssim f} \underbrace{< C g(n)(\log n)^{k}}_{f \lesssim g}$$

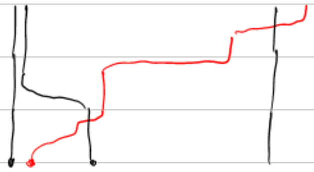
$\cancel{\neg (g \lesssim f) \cong g >> f}$

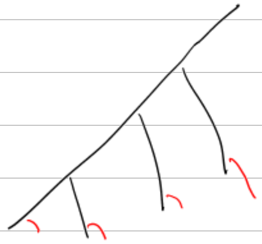
$a \ll b \qquad a < b$

$\exists k$ s.t.



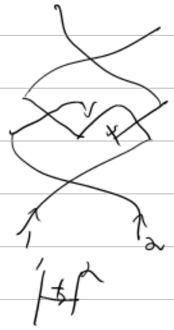
I think we only need the ||| & ||| associators!

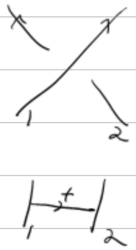


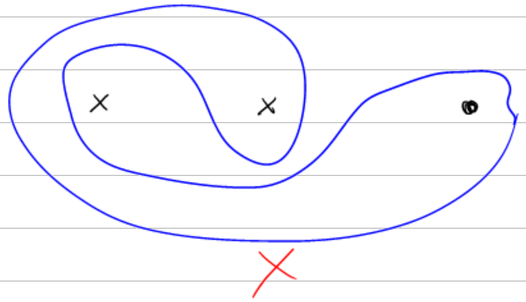


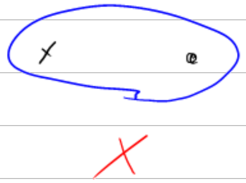
Waseda rooms:

Court Nishiwaseda Twin 1: 52sqm 6330Y has AC, washer-dryer, split bed
Court Nishiwaseda Twin-2: 64sqm 6780Y has AC, washer-dryer, split bed

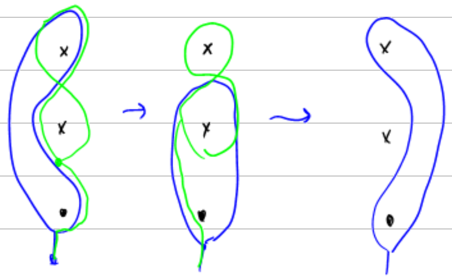


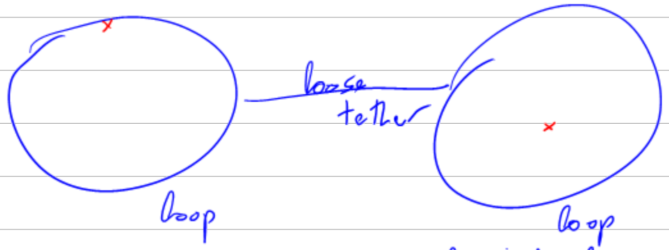




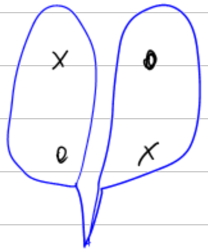


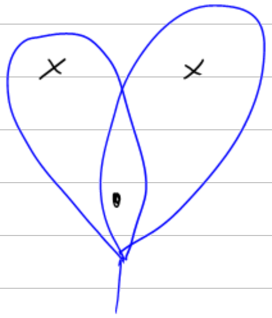
Elements of $H_1(\hat{C})$ must have a leash!

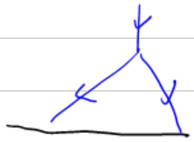
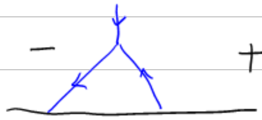
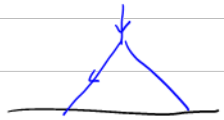
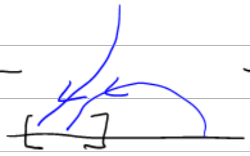
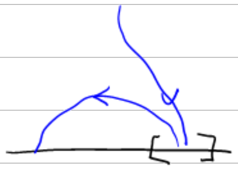


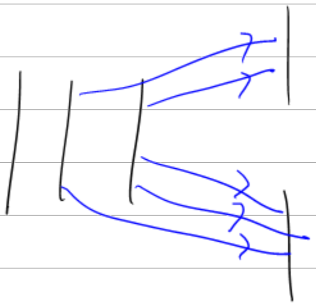


Precisely, linking-0
homological loop.

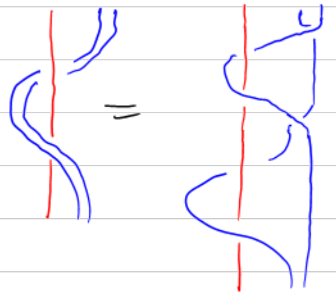


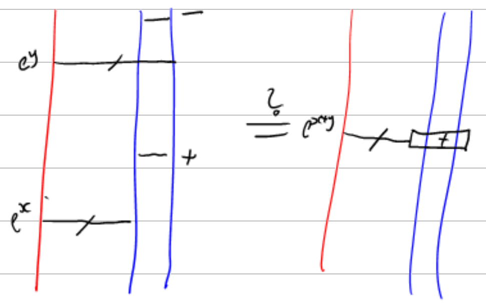


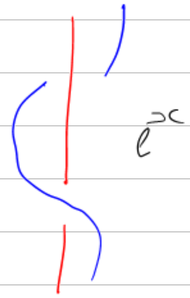
 $=$ $-$  $+$  $=$ $-$  $+$ 

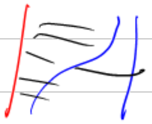


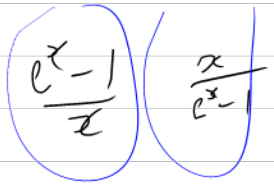
compatibility w/ strand doubling



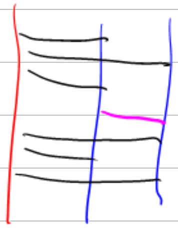


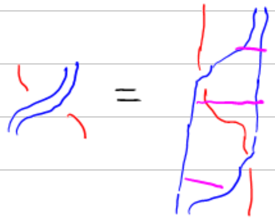


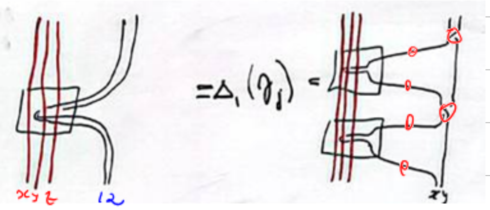




$f(\mathrm{ad}_x)(y)$

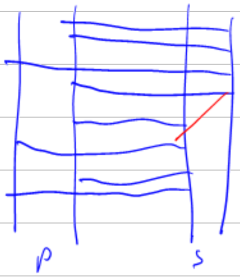




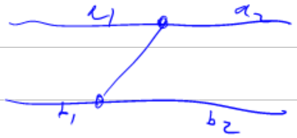


"Expansions for emergent knots"

"Expansions of barely braids".



$$\begin{tikzcd}
A\otimes A \arrow[r, "K_{KKS}"] \arrow[d, equal] & A\otimes A \\
A\otimes A \arrow[r, "\partial^2"] & (\underset{a_1}{A}\otimes\underset{b_1}{A})\otimes_A(\underset{a_2}{A}\otimes\underset{b_2}{A}) \arrow[u, bend right]
\end{tikzcd}$$



$$\Delta_s e^{[a,x]} = e^{[a,x]}\otimes e^{[a,x]}$$
$$\|$$
$$(e^{[a,x]}\otimes 1)(1\otimes e^{[a,x]})$$

$$\partial^2(x_i\otimes x_j)=\delta_{ij}\left(1\otimes x_i\otimes 1\otimes 1 - 1\otimes 1\otimes 1\otimes x_i\right)$$

a second derivative

$$\partial^2(ab\otimes c)=\{\partial^2(a\otimes c)\}(b\otimes 1) + (a\otimes 1)(\partial^2(b\otimes c))$$

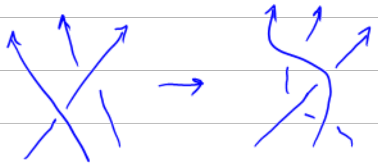
$$\partial^2([a,x]\otimes[a,x])$$

$$\begin{tikzcd}
A \arrow[r, "div"] \arrow[dr, "Div"'] & {|A|} \arrow[d, "D"] \\
 & {|A|\otimes|A|}
\end{tikzcd}$$

Continuing Monoblog/200611a:

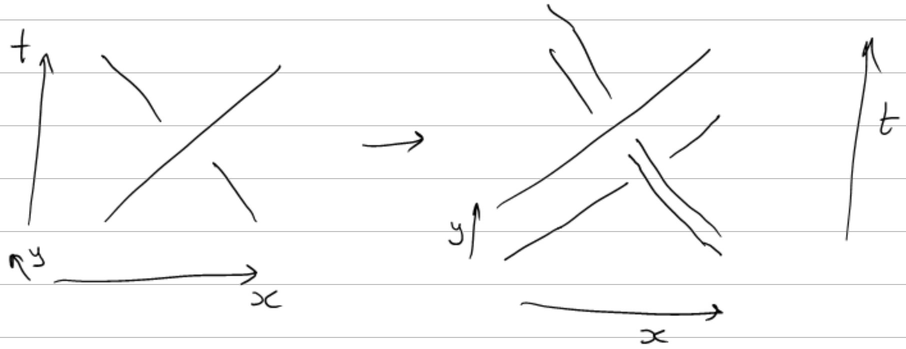
Using OU, given two algebras A & B and $R \in A \otimes B$
I can always make a braid invariant.

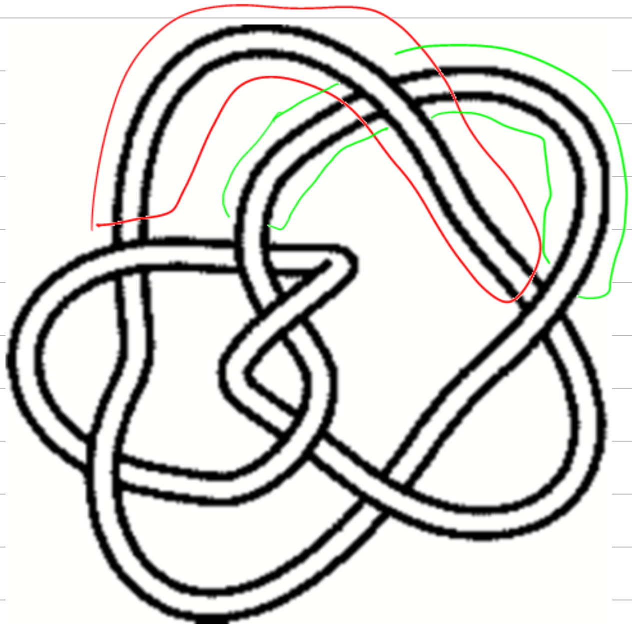
When can I combine the two algebras?

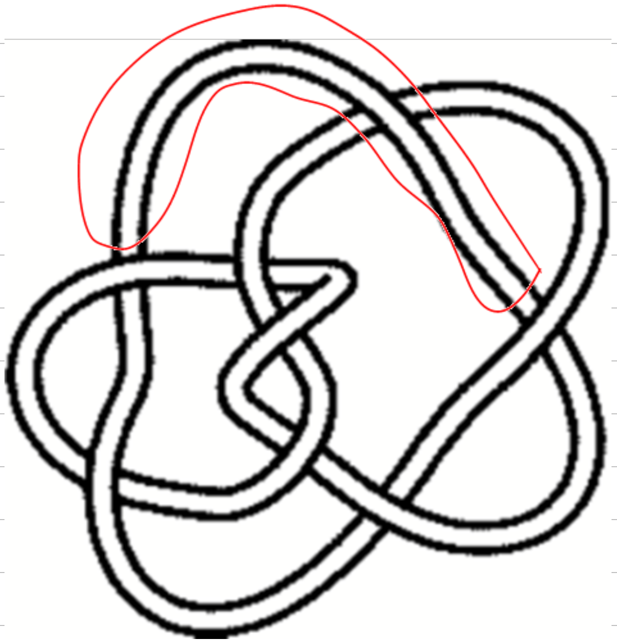


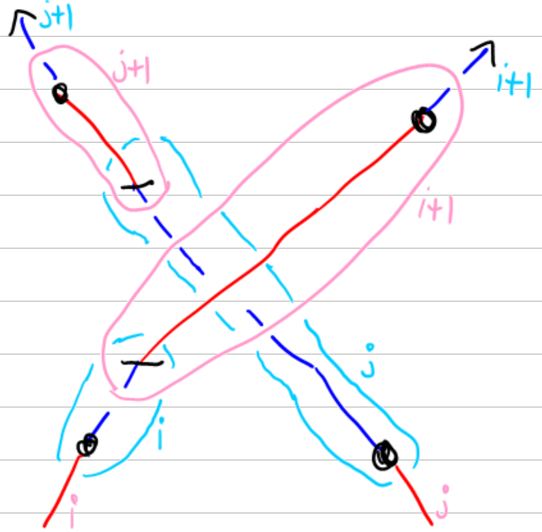
In what way is this "universal"?

why would I want <u>algebras</u> here?


t
y
x
y
x
t





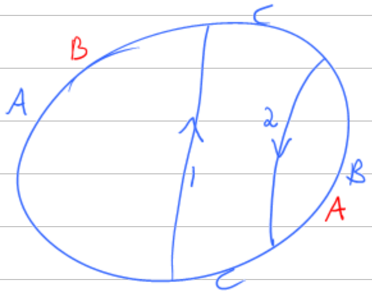


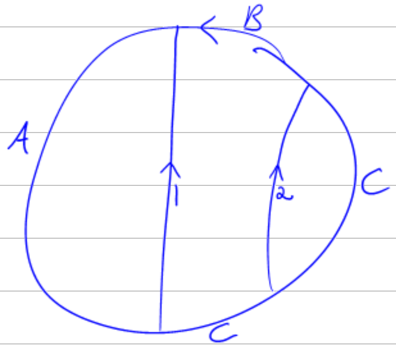
$$\begin{tikzcd}
B_1 \arrow[r, shift left] & A_1 \arrow[l, shift left] \arrow[d, shift left] \arrow[dr] & \\
 & & M \\
B_2 \arrow[r, shift right] & A_2 \arrow[l, shift right] \arrow[u, shift left] \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & R \arrow[d, shift left] \\
R^{g-1} \arrow[r, shift right] & R^g \arrow[l, shift right] \arrow[u, shift left]
\end{tikzcd}$$

$$\det \begin{pmatrix} 0 & z \\ \bar{z} & 0 \end{pmatrix} = -|z|^2$$

$$\begin{pmatrix} 0 & z \\ \bar{z} & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} z+\bar{z} & z \\ \bar{z} & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} z+\bar{z} & 0 \\ 0 & -\frac{\bar{z}z}{\operatorname{Re} z} \end{pmatrix} = \begin{pmatrix} 2\operatorname{Re} z & 0 \\ 0 & -\frac{|z|^2}{2\operatorname{Re} z} \end{pmatrix}$$


B
C
A
2
1
B
A
C


B
A
1
2
C
C

Let $A = U(sl^{\circ}_{2+})$. In $\mathrm{coinv}(A) = A_A$,

$$0 = [x, y^n f(a) x^{n-1}] = n b y^{n-1} f(a) x^{n-1} - y^n \nabla f(a) x^n$$

Aside $[x, f(a)] = -(\nabla f)(a) \cdot x$
as $[x, a] = (-1) \cdot x$

$$\boxed{\phi(z^n f(a)) := \frac{1}{n!} y^n f(a) x^n}$$

So $b\,\phi(z^{n-1} \cdot f(a)) = \phi(z^n \nabla f(a))$

$$0 = [x, y^{n+1} f(a) x^n] = (n+1) b y^n$$

So $b \widehat{z^{-1}} \phi(g(z,a)) = \nabla_a \phi(g(z,a))$

$b^m y^n f(a) x^n$

$tr(e^{\beta b + \alpha a + \xi z}) =$

Let $A = U(sl^{\circ}_{2+})$ and $\phi : \mathbb{Q}[z, a] \to A_A$ by $b^k z^n a^m$
$\to \frac{b^k}{n!} y^n a^m x^n$.

with $f = f(a)$

In A, $[x, f] = -\nabla_a f \cdot x$ so in A_A

$$0 = [x, y^n f x^{n-1}] = n b y^{n-1} f x^{n-1} - y^n \nabla_a f x^n$$

so $\phi(b z^{n-1} f(a)) = \phi(z^n \nabla_a f)$

so $g(tr) = tr(e^{\beta b} e^{\eta y} e^{\alpha a} e^{\xi x}) = \psi(e^{\beta b} e^{\eta \xi z} e^{\alpha a})$

$= \psi(e^{\beta z \nabla_a} e^{\eta \xi z} e^{\alpha a}) = \psi(e^{\alpha a + z(\eta \xi + \beta(1 - e^{-\alpha}))})$

$e^{\xi x} e^{\eta y} = e^{\xi \eta b} e^{\eta y} e^{\xi x}$

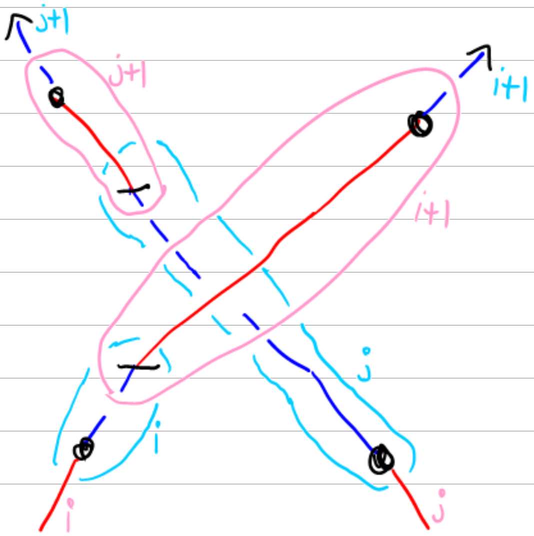
$e^{\xi x} e^{\alpha a} =$

$xa = (a-1)x$

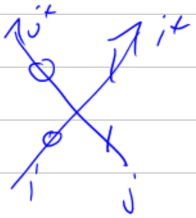
$x e^{\alpha a} = e^{-\alpha} e^{\alpha a} x$

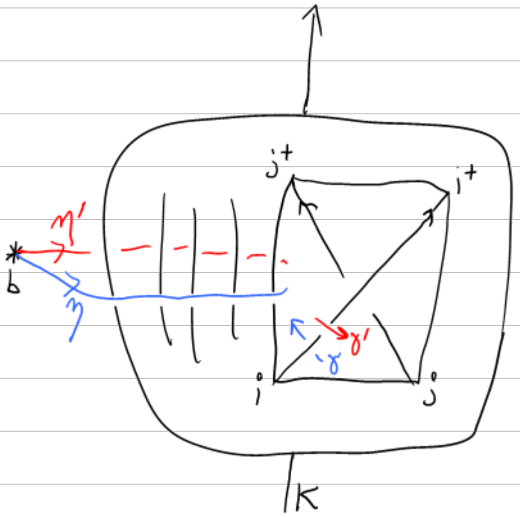
$e^{\xi x} e^{\alpha a} = e^{\alpha a} e^{e^{-\alpha} \xi x}$

I want to axiomatize "knots with Alexander numbering".
I want to axiomatize "ba before yx".
I want these two axiomatizations to be the same.



Alexander Numbering:

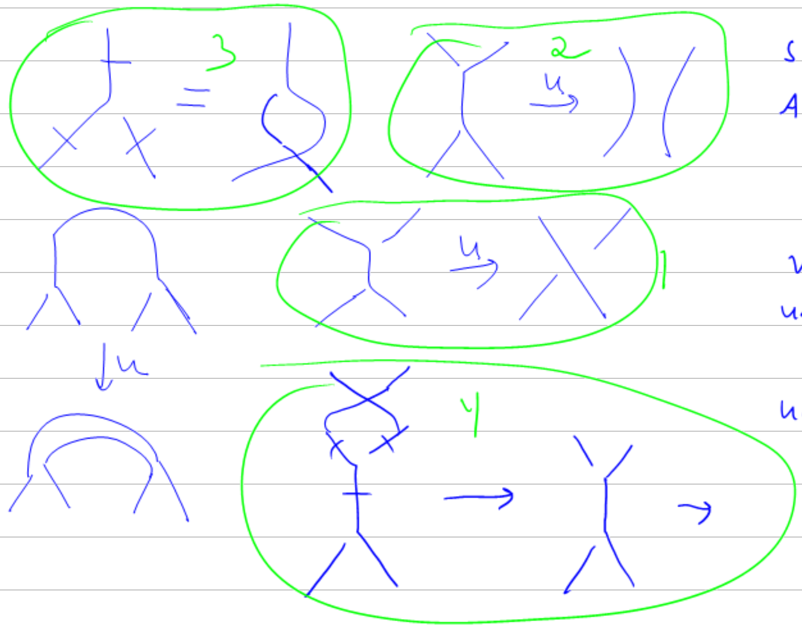




$$\begin{tikzcd}
\langle \rho_i' \rangle & \langle \gamma_i' \rangle & \\
R' \arrow[r, "\delta'"] \arrow[d, "\beta"'] & G' \arrow[d, "\alpha"'] \arrow[r] & A \\
R \arrow[r, "\delta"] & G \arrow[ur] & \\
\langle \rho_i \rangle & \langle \gamma_i \rangle &
\end{tikzcd}$$

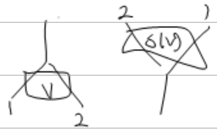
upper wirtinger

lower wirtinger



s: reverse both orientations

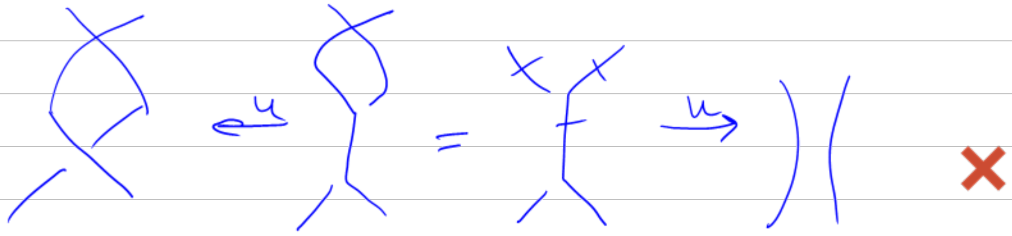
A: reverse only the 1D orientation

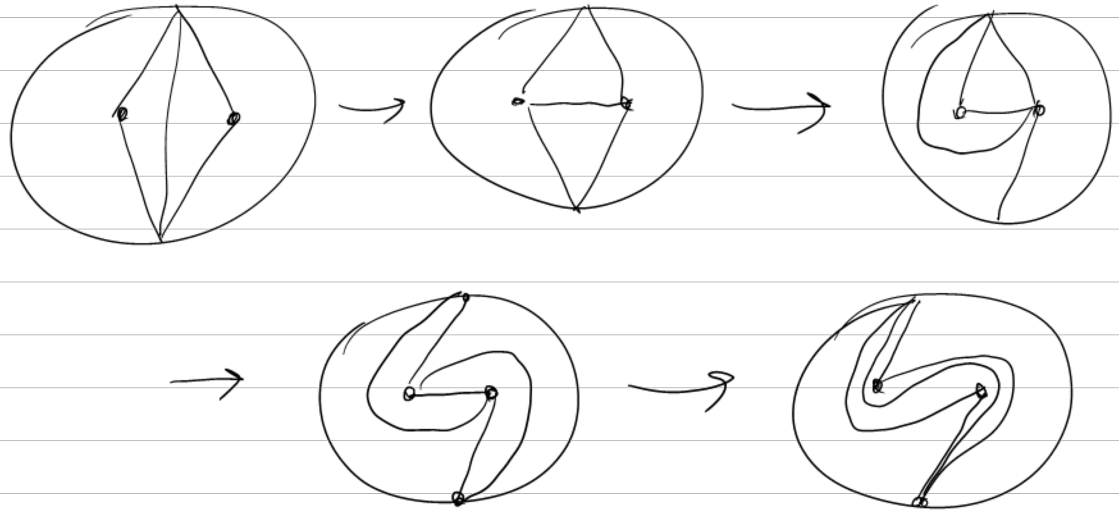


vertices are classical (satisfy both R4s)

unzipping connected stems connects u to u & l to l. with u above l.

unzipping through a wen ~~makes said connection virtual.~~ is defined via conjugation

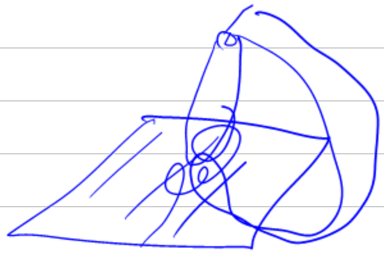


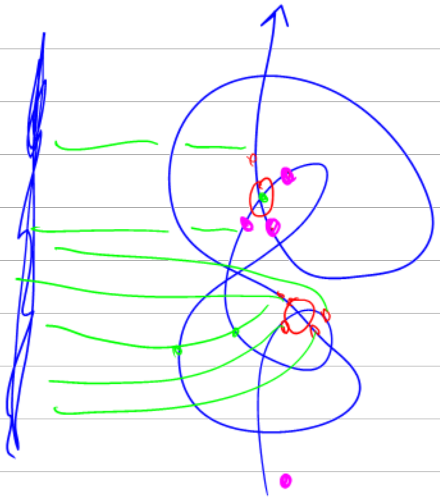


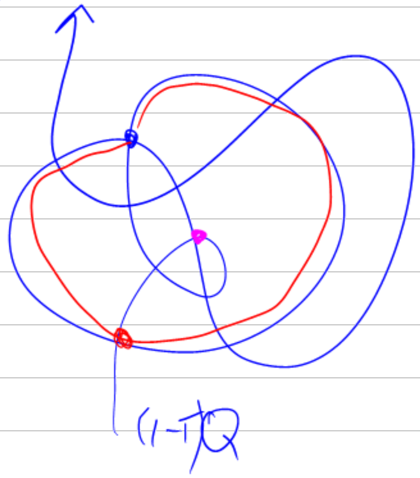
$$\begin{tikzcd} R_1 \arrow[r, "A_1"] & G_1 \arrow[r] \arrow[d, "P"] & M \\ R_2 \arrow[u, "Q"] \arrow[r, "A_2"] & G_2 \arrow[ur] & \end{tikzcd}$$

$PA_1Q = A_2$

~~$P^{-1}A_1P = A_2$~~







Functorial Gaussian Integration and the Alexander Polynomial

Abstract. We develop a fully functorial theory of pushforwards of quadratics ("conditional expectations of Gaussian measures", if you are so inclined) and use it to describe an extension of the Alexander polynomial to tangles (yet another, yet in some ways, better: poly time, happy with closed components, talks to signatures).

Talk by Kasia Rejzner

Budapest colloquium plan

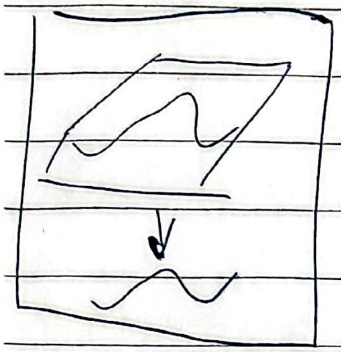
$$\begin{tikzcd} X \arrow[r, "F"] \arrow[dr, dotted, "F^*g"'] & Y \arrow[d, "g"] \\ & \mathbb{R} \end{tikzcd}$$

points push
functions pull

$$\begin{tikzcd} V \arrow[r, "F"] \arrow[dr, dotted, "F^*Q"'] & W \arrow[d, "Q"] \\ & \mathbb{R} \end{tikzcd}$$

quadratics pull

Nobody seems
need to know,
~~Everybody~~
yet everybody
kinda knows,
that quadratics
also push!



If Q is positive definite

$$\int e^{-Q^2/2} = \frac{1}{\sqrt{\det Q}}$$

If Q is anything,

$$\int e^{iQ^2/2} = \frac{e^{\pi/4\,\mathrm{sign}\,Q}}{\sqrt{\det Q}}$$

~~Moral!~~ $= \lim_{\epsilon \to 0}$ regularization

Moral: Pushforwards of quadratics should play well with dets & signatures

$$\boxed{\int e^{ixcy}\,dy \doteq \delta(x)}$$

Moral: Expect to see quadratics on subspaces.

Definition A PQ . . .

Thm $\exists!$ push forward,

properties, relation w/ integration . . .

. . . cont

No.

Date

Page 2: tangles, partial computations, Alexander, signatures, etc.

Page 3: Implementation. "I mean business".

Page 4: results.

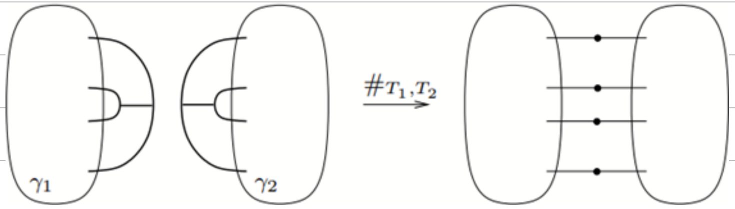
Bail out option: End the talk with a conjecture "everything works" and explain my ~~work~~ ~~com~~ working methodology,

Implementation > Proof.

~~Definition~~ Gauss says: $\Lambda^{top}(V^*) \otimes S^2(V^*)$

$V \to W$

Do dets/signatures say the same?

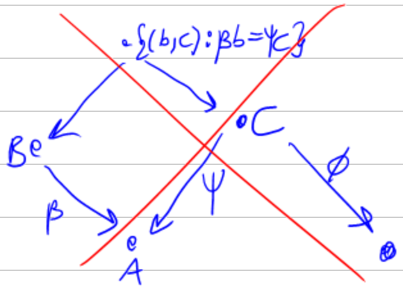

γ_1
γ_2
$\#_{T_1,T_2}$

231013b **Def.** A Measured Partial Quadratic (MPQ) on a v.s. V is a quadratic Q defined on a subspace $D \subset V$ along with a volume form on D.

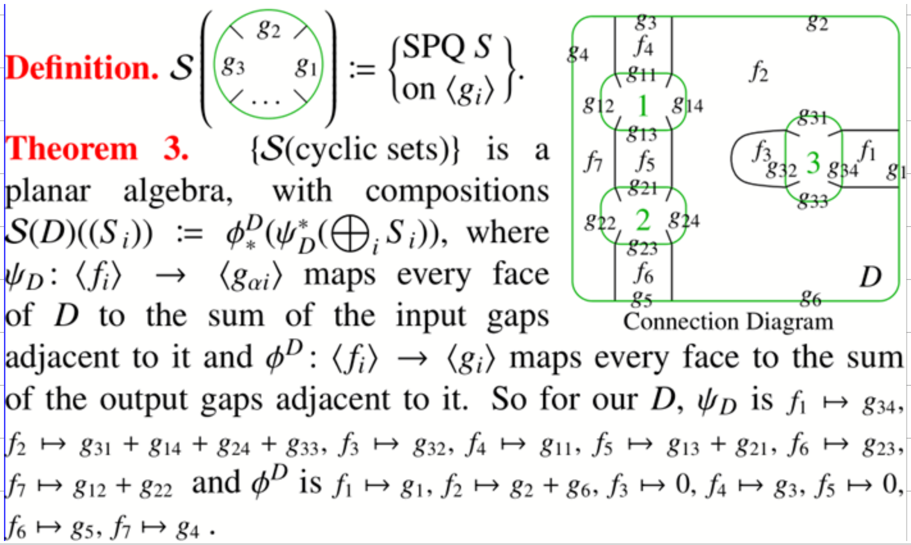
Conj. Given $\phi\colon V \to W$ and an MPQ Q on V there is a unique MPQ ϕ_*Q on W such that for every quadratic U on W, $\det(U + \phi_*Q) = \det(Q + \phi^*U)$ ("~~quadratic reciprocity~~").

231013a Turaev's arXiv:math/0310218 "Virtual Strings" has "based matrices" and sliceness criteria.

Are there subspace-valued planar algebra morphisms to be had from here?



The key seems to be: the kernel of the gaps -> faces maps.



Definition. $\mathcal{S}\left(g_2 \; g_3 \; g_1 \; \ldots \right) := \left\{ \begin{matrix} \text{SPQ } S \\ \text{on } \langle g_i \rangle \end{matrix} \right\}$.

Theorem 3. $\{\mathcal{S}(\text{cyclic sets})\}$ is a planar algebra, with compositions $\mathcal{S}(D)((S_i)) := \phi^D_*(\psi^*_D(\bigoplus_i S_i))$, where $\psi_D\colon \langle f_i \rangle \to \langle g_{\alpha i} \rangle$ maps every face of D to the sum of the input gaps adjacent to it and $\phi^D\colon \langle f_i \rangle \to \langle g_i \rangle$ maps every face to the sum of the output gaps adjacent to it. So for our D, ψ_D is $f_1 \mapsto g_{34}$, $f_2 \mapsto g_{31} + g_{14} + g_{24} + g_{33}$, $f_3 \mapsto g_{32}$, $f_4 \mapsto g_{11}$, $f_5 \mapsto g_{13} + g_{21}$, $f_6 \mapsto g_{23}$, $f_7 \mapsto g_{12} + g_{22}$ and ϕ^D is $f_1 \mapsto g_1$, $f_2 \mapsto g_2 + g_6$, $f_3 \mapsto 0$, $f_4 \mapsto g_3$, $f_5 \mapsto 0$, $f_6 \mapsto g_5$, $f_7 \mapsto g_4$.

Connection Diagram

230811 **Thm** (cf. Lickorish pp. 50). Module presentations $R_i \xrightarrow{\alpha_i} G_i$ $(\to M,\ i = 1,2)$ are equivalent iff $\exists \beta_i, \gamma_i, \eta_i$ as here s.t. $\beta_1\alpha_1 = \alpha_2\gamma_1$, $\beta_2\alpha_2 = \alpha_1\gamma_2$, $\beta_2\beta_1 - I = \alpha_1\eta_1$, and $\beta_1\beta_2 - I = \alpha_2\eta_2$. Then $\begin{pmatrix} I & 0 \\ \beta_1 & \alpha_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 & \beta_2 \\ 0 & I \end{pmatrix} \begin{pmatrix} -\eta_1 & -\gamma_2 \\ \beta_1 & \alpha_2 \end{pmatrix}$ and $\begin{pmatrix} \alpha_1 & \beta_2 \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ \beta_1 & \alpha_2 \end{pmatrix} \begin{pmatrix} \alpha_1 & \beta_2 \\ -\gamma_1 & -\eta_2 \end{pmatrix}$ so elementary ideals make sense. Implement for the Torres-Fox presentations!

$$\begin{tikzcd}
R_1 \arrow[r, shift left, "\alpha_1"] \arrow[d, shift left, "\gamma_1"] & G_1 \arrow[l, shift left, "\eta_1"] \arrow[d, shift left, "\beta_1"] \\
R_2 \arrow[r, shift left, "\alpha_2"] \arrow[u, shift left, "\gamma_2"] & G_2 \arrow[l, shift left, "\eta_2"] \arrow[u, shift left, "\beta_2"]
\end{tikzcd}$$

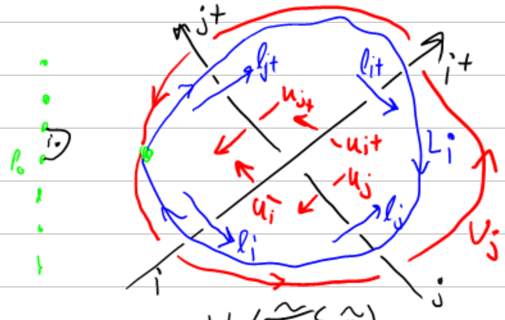
$$\begin{array}{cc|cc}
 & & -\eta_1 & -\gamma_2 \\
 & & \beta_1 & \alpha_2 \\
\hline
\alpha_1 & \beta_2 & -\alpha_1\eta_1+\beta_2\beta_1 & -\alpha_1\gamma_2+\beta_2\alpha_2 \\
-\gamma_1 & -\eta_2 & \gamma_1\eta_1-\eta_2\beta_1 & \gamma_1\gamma_2-\eta_2\alpha_2
\end{array}$$

$$\begin{pmatrix} I & 0 \\ \beta_1 & \alpha_2 \end{pmatrix}^{-1} = \begin{pmatrix} I & 0 \\ -\alpha_2^{-1}\beta_1 & \alpha_2^{-1} \end{pmatrix} \qquad \text{so}$$

$$\begin{pmatrix} \alpha_1^{-1} & \cdot \\ \cdot & \cdot \end{pmatrix} = \begin{pmatrix} \alpha_1 & \beta_2 \\ 0 & I \end{pmatrix}^{-1} = \begin{pmatrix} \alpha_1 & \beta_2 \\ -\gamma_1 & -\eta_2 \end{pmatrix}^{-1} \begin{pmatrix} I & 0 \\ \beta_1 & \alpha_2 \end{pmatrix}^{-1} = \begin{pmatrix} -\eta_1 & -\gamma_2 \\ \beta_1 & \alpha_2 \end{pmatrix} \begin{pmatrix} I & 0 \\ -\alpha_2^{-1}\beta_1 & \alpha_2^{-1} \end{pmatrix}$$

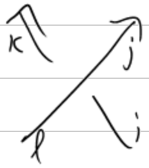
so $\alpha_1^{-1} = -\eta_1 + \gamma_2\alpha_2^{-1}\beta_1$ Indeed, $\alpha_1(-\eta_1 + \gamma_2\alpha_2^{-1}\beta_1) = -\alpha_1\eta_1 + \alpha_1\gamma_2\alpha_2^{-1}\beta_1$
$= I - \beta_2\beta_1 + \beta_2\alpha_2\alpha_2^{-1}\beta_1 = I$ ✓

The case of the positive crossing



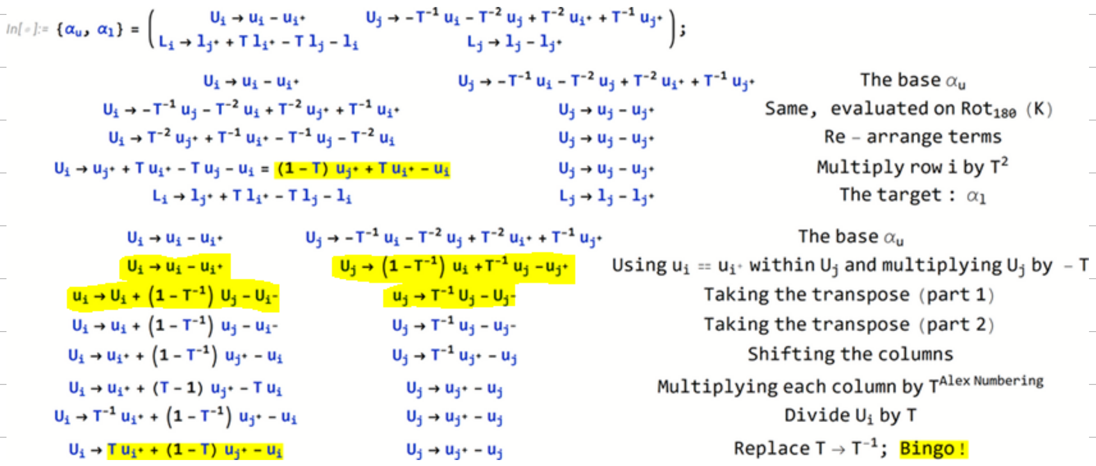
$H_1(\tilde{T}^c, \tilde{p})$

$$\begin{tikzcd}
R_u = \langle U_i, U_j \rangle \arrow[r, shift left, "\alpha_u"] \arrow[d, bend left, "\gamma_u"] & G_u = \langle u_i, u_j, u_{i+}, u_{j+} \rangle \arrow[l, shift left, "\eta_u"] \arrow[d, bend left, "\beta_u"] \\
R_\ell = \langle L_i, L_j \rangle \arrow[r, shift left, "\alpha_\ell"] \arrow[u, bend left, "\gamma_\ell"] & G_\ell = \langle l_i, l_j, l_{i+}, u_{j+} \rangle \arrow[l, shift left, "\eta_\ell"] \arrow[u, bend left, "\beta_\ell"]
\end{tikzcd}$$



$X_{\ell i}$

$\bar{X}_{ji}$



In[◦]:= $\{\alpha_u, \alpha_1\} = \begin{pmatrix} U_i \to u_i - u_{i^+} & U_j \to -T^{-1} u_i - T^{-2} u_j + T^{-2} u_{i^+} + T^{-1} u_{j^+} \\ L_i \to l_{j^+} + T\, l_{i^+} - T\, l_j - l_i & L_j \to l_j - l_{j^+} \end{pmatrix};$

$U_i \to u_i - u_{i^+}$	$U_j \to -T^{-1} u_i - T^{-2} u_j + T^{-2} u_{i^+} + T^{-1} u_{j^+}$	The base α_u
$U_i \to -T^{-1} u_j - T^{-2} u_i + T^{-2} u_{j^+} + T^{-1} u_{i^+}$	$U_j \to u_j - u_{j^+}$	Same, evaluated on Rot_{180} (K)
$U_i \to T^{-2} u_{j^+} + T^{-1} u_{i^+} - T^{-1} u_j - T^{-2} u_i$	$U_j \to u_j - u_{j^+}$	Re – arrange terms
$U_i \to u_{j^+} + T\, u_{i^+} - T\, u_j - u_i = (1-T)\, u_{j^+} + T\, u_{i^+} - u_i$	$U_j \to u_j - u_{j^+}$	Multiply row i by T^2
$L_i \to l_{j^+} + T\, l_{i^+} - T\, l_j - l_i$	$L_j \to l_j - l_{j^+}$	The target : α_1

$U_i \to u_i - u_{i^+}$	$U_j \to -T^{-1} u_i - T^{-2} u_j + T^{-2} u_{i^+} + T^{-1} u_{j^+}$	The base α_u
$U_i \to u_i - u_{i^+}$	$U_j \to \left(1 - T^{-1}\right) u_i + T^{-1} u_j - u_{j^+}$	Using $u_i == u_{i^+}$ within U_j and multiplying U_j by $-T$
$u_i \to U_i + \left(1 - T^{-1}\right) U_j - U_{i^-}$	$u_j \to T^{-1} U_j - U_{j^-}$	Taking the transpose (part 1)
$U_i \to u_i + \left(1 - T^{-1}\right) u_j - u_{i^-}$	$U_j \to T^{-1} u_j - u_{j^-}$	Taking the transpose (part 2)
$U_i \to u_{i^+} + \left(1 - T^{-1}\right) u_{j^+} - u_i$	$U_j \to T^{-1} u_{j^+} - u_j$	Shifting the columns
$U_i \to u_{i^+} + (T - 1)\, u_{j^+} - T\, u_i$	$U_j \to u_{j^+} - u_j$	Multiplying each column by $T^{\text{Alex Numbering}}$
$U_i \to T^{-1} u_{i^+} + \left(1 - T^{-1}\right) u_{j^+} - u_i$	$U_j \to u_{j^+} - u_j$	Divide U_i by T
$U_i \to T\, u_{i^+} + (1 - T)\, u_{j^+} - u_i$	$U_j \to u_{j^+} - u_j$	Replace $T \to T^{-1}$; Bingo !

The stitching of orthogonal is orthogonal: $\begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix}_M \to \begin{pmatrix} \gamma + \frac{\alpha\delta}{1-\beta} & \epsilon + \frac{\delta\theta}{1-\beta} \\ \phi + \frac{\alpha\psi}{1-\beta} & \Xi + \frac{\psi\theta}{1-\beta} \end{pmatrix}$

$$\left(v_1 \middle| \cdots \middle| v_n \right) = \left(\begin{array}{c|c|c|c} v_1' & \cdots & v_{n-1}' & \phi \\ \hline \multicolumn{3}{c|}{\theta} & \gamma \end{array} \right) \to$$

* add $\frac{1}{1-\beta}\begin{pmatrix} \text{col} \\ \text{of} \\ \beta \end{pmatrix}(\text{row of } \beta)$ to M * delete col & row of β

get $\begin{pmatrix} \gamma & \epsilon \\ \phi & \Xi \end{pmatrix} + \frac{1}{1-\beta}\begin{pmatrix} \delta\alpha & \delta\theta \\ \psi\alpha & \psi\theta \end{pmatrix}$

$$\to \left(\underset{\uparrow u_1}{v_1' + \frac{\theta_1\phi}{1-\gamma}} \middle| \underset{\uparrow u_2}{v_2' + \frac{\theta_2\phi}{1-\gamma}} \middle| \cdots \middle| v_{n-1}' + \frac{\theta_{n-1}\phi}{1-\gamma} \right)$$

This is orthogonal! Indeed $\langle u_1, u_2 \rangle = \langle v_1', v_2' \rangle + \frac{\theta_1}{1-\gamma}\langle \phi, v_2' \rangle + \frac{\theta_2}{1-\gamma}\langle v_1', \phi \rangle + \frac{\theta_1\theta_2}{(1-\gamma)^2}\|\phi\|^2$

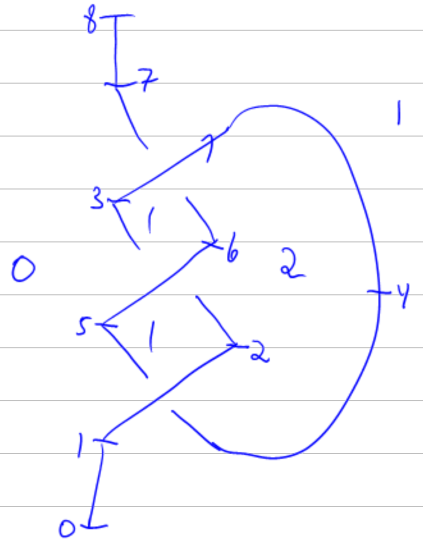
$$= -\theta_1\theta_2 + \frac{\theta_1}{1-\gamma}(-\theta_2\gamma) + \frac{\theta_2}{1-\gamma}(-\theta_1\gamma) + \frac{\theta_1\theta_2}{(1-\gamma)^2}(1-\gamma^2) = \theta_1\theta_2(1-\gamma)(\gamma-1-2\gamma+1+\gamma) = 0$$

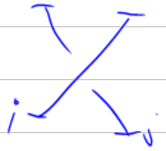
while $\langle u_1, u_1 \rangle = \|v_1'\|^2 + 2\frac{\theta_1}{1-\gamma}\langle v_1', \phi \rangle + \frac{\theta_1^2}{(1-\gamma)^2}\|\phi\|^2 = 1-\theta_1^2 + 2\frac{\theta_1}{1-\gamma}(-\theta_1\gamma) + \frac{\theta_1^2}{(1-\gamma)^2}(1-\gamma^2)$

$$= \frac{1}{1-\gamma}\left(1-\gamma - \theta_1^2\cancel{(1-\gamma)} - 2\cancel{\theta_1^2}\gamma + \theta_1^2\cancel{(1+\gamma)}\right) = \frac{1-\gamma}{1-\gamma} = 1$$

$$0 \to H_1(\tilde{X}) \to H_1(\tilde{X}, \tilde{p}) \xrightarrow{\partial} H_0(\tilde{p}) \to \mathbb{Z} \to 0$$

So $H_1(\tilde{X}) \cong \ker \partial : H_1(\tilde{X}, \tilde{p}) \to H_0(\tilde{p})$





$\longmapsto b_{i,i+1} \quad t_{j,j+1}$

$\to$ bridges: $\underset{}{\overset{\beta_1}{\{0,1,2\}}}, \overset{\beta_2}{\{3,4\}}, \{5,6\}, \{7,8\}$

tunnels: $\underset{\tau_1}{\{2,3\}}, \underset{\tau_2}{\{4,5\}}, \{6,7\}$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7$

Alexander numbering $\nu: \{0 \; 1 \; 0 \; 1 \; 0 \; 1 \; 0\}$

$Q = \sum T^{\nu} \beta \tau \ldots$