Homework Assignment 8

Problem 1 (5 points). For $n > 0$, construct a surjective map $f\colon S^n \to S^n$ whose degree is 0.

Problem 2 (10 points). Let $n > 0$.

1. Use degree theory to show that any non-surjective map $g\colon S^n \to S^n$ must have a fixed point.
2. Use the above to re-prove the Brouwer fixed point theorem: Any map $f\colon D^n \to D^n$ has a fixed point. (Hint: Make a g out of f by mapping both the upper hemisphere of S^n and the lower hemisphere of S^n to the lower hemisphere using f.)

Problem 3 (10 points).

1. Compute the homology over $\mathbb{Z}$ of a the space X obtained from the 2D disk D^2 by identifying each of its boundary points with the point you get from it by applying a 1/3 rotation counterclockwise.
2. Same question, but over $\mathbb{Z}/3$.

Problem 4 (10 points). The statement "all reasonable everyday spaces are at least homotopy equivalent to CW complexes" sounds completely reasonable. At least until you hit the first example where it's hard. Show that the complement X of the trefoil knot  in $\mathbb{R}^3$ is homotopy equivalent to a 3-dimensional CW complex.

Problem 5 (20 points). For the first 3 parts of this problem, sketches are sufficient.

1. Using the same ideas as in the case of $S^2 \to S^2$, define the degree of a map $T^2 \to S^2$, where $T^2 = S^1 \times S^1$ is the two dimensional torus. Similarly define the local degree $\deg_x(f)$ when $f\colon T^2 \to S^2$ when $x \in T^2$ is isolated in $f^{-1}(f(x))$.
2. Show that the degree of maps $T^2 \to S^2$ is invariant under homotopy of such maps.
3. Show that the "local formula" for the degree, $\deg(f) = \sum_{x \in f^{-1}(y)} \deg_x(f)$, if $y \in S^2$ and $F^{-1}(y)$ is finite, holds for $f\colon T^2 \to S^2$.
4. A pair of curves $\gamma_1, \gamma_2\colon S^1 \to \mathbb{R}^3$ whose images are disjoint induces a "direction of sight" map $\lambda\colon T^2 \to S^2$ by $(s_1, s_2) \mapsto \frac{\gamma_1(s_1) - \gamma_2(s_2)}{|\cdot|}$. Show that $\ell(\gamma_1, \gamma_2) \coloneqq \deg(\lambda)$, the so-called "linking number of γ_1 and γ_2", is invariant under homotopies of the pair (γ_1, γ_2) that preserve the disjointness of their images.
5. Show that $\ell(\gamma_1, \gamma_2) = \ell(\gamma_2, \gamma_1)$.
6. Compute ℓ for the following pairs (orient each loop counterclockwise yet pick whatever orientation you want for the ∞-like loop):