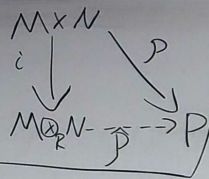


Tensor products

Given M, N modules

- Can Form $M \oplus N$, a new module
- Want multiplication

Def: A tensor product of M, N is a module $M \otimes_R N$ and an R -bilinear map $i: M \times N \rightarrow M \otimes_R N$ s.t. $\forall R$ -module P , and $\hat{p}: M \times N \rightarrow P$ $\exists! \hat{p}: M \otimes_R N \rightarrow P$ s.t. $\hat{p} = \hat{p} \circ i$



Example:

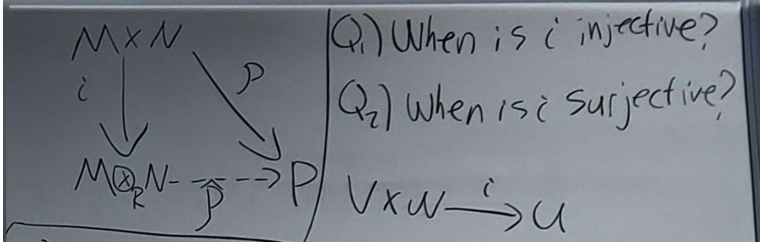
$$M = \mathbb{R}^m, N = \mathbb{R}^n$$

$$i(v, w) = v^t w = (m \times 1)(1 \times n) = m \times n$$

$$M \times N \xrightarrow{i} \text{Mat}(m, n)$$

Thm: Existence and uniqueness
Tensor products exist and are unique up to unique isomorphism,

proof: Existence. Let G be free abelian group generated by $\{m_i n_j\}$. Define $M \otimes N = G/H$. Let $H \subseteq G$ be generated by $\{(m_1 + m_2)n - m_1 n - m_2 n\}$. Let $v \otimes w = [v \cdot w] \in M \otimes N (= G/H)$. Define $i(v, w) = [v \cdot w]$. $r(v \otimes w) = (rv) \otimes w = [rv \cdot w] = [v \cdot rw] = v \otimes (rw)$. $(r+s)(v \otimes w) = (rv + sv) \otimes w = [rv + sv] \cdot w = [rv \cdot w + sv \cdot w] = r(v \otimes w) + s(v \otimes w)$. $1v = v$.



- $i(v, 0) = 0 \Rightarrow$ Never
- Trickier

Lemma: $\text{Span}(\text{Im}(i)) = M \otimes_R N$ $\pi: M \otimes N \rightarrow M \otimes N / \text{Span}(\text{Im}(i))$

proof: $M \times N \xrightarrow{i} M \otimes N \xrightarrow{\pi} M \otimes N / \text{Span}(\text{Im}(i))$. $\pi \circ i(v, w) = 0 = \hat{p}$. By uniqueness, $\pi = \hat{p} \circ i$. \square

$M \otimes N$ is a module.
 $M \otimes N$ is an abelian group \checkmark .
 $(r+s) \cdot (v \otimes w) \stackrel{\text{def}}{=}} (rv + sv) \otimes w = (rv + sv) \cdot w = rv \cdot w + sv \cdot w = r(v \otimes w) + s(v \otimes w)$. \square

$$\begin{aligned} \cdot (r_5) \cdot v \otimes w &= r_5 \cdot v \otimes w \\ &= (r_5 \cdot v) \otimes w = r_5 \cdot (v \otimes w) \\ &= 0 = r_5 \cdot (v \otimes w) \end{aligned}$$

$$\begin{aligned} \cdot r \cdot (v_1 \otimes w_1 + v_2 \otimes w_2) &= r \cdot v_1 \otimes w_1 + r \cdot v_2 \otimes w_2 \\ &= r \cdot (v_1 \otimes w_1) + r \cdot (v_2 \otimes w_2) \end{aligned}$$

• ϕ is bilinear;

$$\iota(V, w_1 + w_2) = V \otimes (w_1 + w_2)$$

$$= V \otimes w_1 + V \otimes w_2 \quad \text{prop 2}$$

$$\rho(rv, w) = rv \otimes w = r \cdot v \otimes w$$

$$\begin{array}{ccc} M \times N & & \\ \downarrow & \searrow P & \\ M \otimes N & \xrightarrow{\beta} & P \end{array}$$

$$\cdot \hat{P}(i(v, w)) = P(v, w)$$

By the lemma, \hat{p} uniquely defined on $M \otimes M$

Uniqueness:

Uniqueness:

$$\begin{array}{ccc}
 & M \times N & \\
 \downarrow i_1 & & \downarrow i_2 \\
 A_1 & \xrightarrow[\hat{i}_1]{\hat{i}_2} & A_2
 \end{array}
 \quad \begin{array}{l}
 \hat{i}_2(i_2(v, w)) = i_2(i_1(v, w)) \\
 i_1(v, w) = \hat{i}_1(i_2(v, w)) \\
 \hat{i}_2(\hat{i}_1(i_2(v, w))) = i_2(i_1(i_2(v, w))) \\
 = (i_2 \circ i_1)(i_2(v, w)) \\
 \hat{i}_2 \circ i_1(x) = x
 \end{array}$$

$$\cdot \mathbb{R}^n \otimes \mathbb{R}^m \cong \mathbb{R}^{nm} (= \text{Mat}_{\mathbb{R}}(n, m))$$

- Ring, $I, J \trianglelefteq R$

$$R/I \times R/J$$

$$R/I \otimes R/J \xrightarrow{\hat{p}} M = R/(I+J)$$

$$\begin{array}{ccc}
 R/I \times R/J & & p([r_1], [r_2]) \\
 \downarrow i & \searrow p & = [r_1, r_2] \text{ (well defined?)} \\
 R/I \otimes R/J & \xrightarrow{\quad p \quad} & R/(I+J) \quad \cdot p \text{ surjective.}
 \end{array}$$

 $\text{Ker } \hat{p}?$

$$\begin{aligned} \hat{p}(C_r \otimes C_1) &= p(C_r, C_1) \\ &\stackrel{\text{suppose}}{=} C_r = [0] \end{aligned}$$

$$(\Leftrightarrow) r \in I + J$$

$$(\Leftarrow) r = x + y, x \in I, y \in J$$

$$[r]_I \otimes [1]_J = [x+y]_I \otimes [1]_J$$

$$= [x]_I \otimes [1]_J + [y]_I \otimes [1]_J$$

$$= [y]_I \otimes [1]_J$$

$$= [y, 1]_I \otimes [1]_J$$

$$= y [1]_I \otimes [1]_J$$

$$= [1]_I \otimes [y]_J = [0]$$

Ker \tilde{p} ? Lemma: Every simple tensor has this form

$$\tilde{p}([r]_I \otimes [1]_J) = p([r], [1])$$

$$= [r] = [0] \quad \text{Suppose}$$

$$\Leftrightarrow r \in I + J$$

$$\Leftrightarrow r = x + y, x \in I, y \in J$$

$$R/I \times R/J \xrightarrow{p} R/(I+J)$$

$$p([r_1], [r_2]) = [r_1, r_2] \text{ (well defined?)}$$

$$p \text{ surjective}$$

$$I = (a), J = (b) \quad R/(a) \otimes R/(b) \simeq R/(\text{ann}(b)) = R/\text{gcd}(a, b)$$

Thm: $(R\text{-mod}, \oplus, \otimes, 0, R)$ "forms a ring"

ve to isomorphism:

- (i) R is an identity for $\otimes \rightarrow R \otimes_R M \cong M \cong M \otimes_R R$
 - (ii) 0 is an identity for $\oplus \rightarrow 0 \oplus M \cong M$
 - (iii) \oplus and \otimes are associative
 - (iv) \otimes distributes over \oplus
 - (v) \oplus is commutative
- I will focus on the \otimes parts

$$(i) R \otimes_R M \cong M$$

we will do this by showing M satisfies the universal property for $R \otimes_R M$, namely,

$$R \times M \rightarrow R \otimes_R M$$

$$\downarrow f \quad \searrow \exists! \tilde{f}$$

$$N$$

If we have, $f: R \times M \rightarrow N$
 f bilinear, note $f(r, m)$
 $= r f(1, m)$
 $= f(1, rm)$

If we consider the map

$$r(m) \mapsto rm$$

$$R \times M \rightarrow M$$

$$\downarrow f \quad \searrow \tilde{f}(m) = f(1, m)$$

$$N \quad f(1, rm) = f(r, m)$$

\tilde{f} is unique, R -linear, and makes the diagram commute, so (M, ϵ) satisfies the universal property. By uniqueness of tensor product, $M \cong R \otimes_R M$

$$A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$$

for each a , we get a bilinear map

$$A \times (B \otimes C) \rightarrow A \otimes (B \otimes C)$$

$$\downarrow \cong (a, b, c) \mapsto (a \otimes b) \otimes c$$

$$(A \otimes B) \times C \rightarrow (A \otimes B) \otimes C$$

$$\downarrow$$

$$(A \otimes B) \otimes C$$

$$\tilde{\epsilon}_{\text{can}}(\sum b_i \otimes c_i) = \sum \tilde{\epsilon}_{\text{can}}(b_i \otimes c_i)$$

$$\sum \tilde{\epsilon}_{\text{can}}(b_i, c_i)$$

so $\tilde{\epsilon}_{\text{can}}$ defines linearity on A

so, we get a bilinear map

$$A \times (B \otimes C) \rightarrow A \otimes (B \otimes C)$$

$$(a, b \otimes c) \mapsto a \otimes (b \otimes c)$$

$$\downarrow \tilde{\epsilon}_{\text{can}}$$

$$(A \otimes B) \otimes C$$

we can also go the other way to get a map

$$(A \otimes B) \otimes C \rightarrow A \otimes (B \otimes C)$$

$$(a \otimes b) \otimes c \mapsto a \otimes (b \otimes c)$$

aside if $f: M_1 \rightarrow M_2$ is a R -module hom.

N is an R -module, there is a map

$$N \otimes_R M_1 \rightarrow N \otimes_R M_2$$

$$(n \otimes m_1) \mapsto n \otimes f(m_1)$$

$$N \otimes M_1 \rightarrow N \otimes M_2$$

$$(n, m) \mapsto (n, f(m))$$

$$N \times M_1 \rightarrow N \times M_2$$

$$\downarrow$$

$$N \otimes M_2$$

$$A \otimes (B \oplus C) \cong (A \otimes B) \oplus (A \otimes C)$$

First, $B \rightarrow B \oplus C$, $C \rightarrow B \oplus C$

we get $A \otimes B \rightarrow A \otimes (B \oplus C)$, $A \otimes C \rightarrow A \otimes (B \oplus C)$

By universal property of (\oplus) , we get

$$\begin{array}{c} A \otimes B \xrightarrow{\quad} A \otimes B \oplus A \otimes C \xleftarrow{\quad} A \otimes C \\ \downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\ A \otimes B \xrightarrow{\quad} A \otimes (B \oplus C) \xleftarrow{\quad} A \otimes C \end{array}$$

$\epsilon_a: B \rightarrow (A \otimes B) \oplus A \otimes C \xrightarrow{\gamma_a} C$

Summarize: use the two universal properties in two different orders.

$\epsilon_a: B \oplus C \rightarrow (A \otimes B) \oplus (A \otimes C)$

