MAT 347 on 260121-23, H43-45: Classes by Matt and Jacob on Tensor Products

Tensor products

Given M, N modules

- Can form $M \oplus N$, a new module
- Want multiplication.

Def: A tensor product of M, N is a module $M \otimes_R N$ and an R-bilinear map $i : M \times N \longrightarrow M \otimes_R N$ s.t. $\forall$ R-module P, and $p : M \times N \to P$ $\exists! \, \hat{p} : M \otimes_R N \longrightarrow P$ s.t. $p = \hat{p} \circ i$

$$\begin{tikzcd} M \times N \arrow[d, "i"'] \arrow[dr, "p"] & \\ M \otimes_R N \arrow[r, dashed, "\hat{p}"'] & P \end{tikzcd}$$

$$\begin{tikzcd} M \times N \arrow[d, "i"'] \arrow[dr, "p"] & \\ M \otimes_R N \arrow[r, dashed, "\hat{p}"'] & P \end{tikzcd}$$

Q_1) When is i injective?

Q_2) When is i surjective?

$V \times W \xrightarrow{i} U$

1) $i(v, 0) = 0 \Rightarrow$ Never
$= i(0, 0)$

2) Trickier

Lemma: $\operatorname{span}(\operatorname{Im}(i)) = M \otimes_R N$

proof:

$$\begin{tikzcd} M \times N \arrow[d, "i"'] \arrow[dr, "{p(v,w) = 0}"] & \\ M \otimes N \arrow[r, dashed, "\hat{p} = 0"] \arrow[r, "\pi"'] & M \otimes N / \operatorname{span}(\operatorname{Im}(i)) \end{tikzcd}$$

$\pi : M \otimes N \longrightarrow M \otimes N / \operatorname{span}(\operatorname{Im}(i))$

$\pi \circ i(v, w) = 0 = p$

By uniqueness, $\pi = \hat{p} = 0$

$\square$

Example:

$M = \mathbb{R}^m$, $N = \mathbb{R}^n$

$i(v, w) = v^t w = (m \times 1)(1 \times n)$
$= m \times n$

v $1 \times m$

w $1 \times n$

$5 \times 4 = 20$

$$\begin{tikzcd} M \times N \arrow[d, "i"'] \\ \operatorname{Mat}_{\mathbb{R}}(m, n) \end{tikzcd}$$

Thm: Existence and uniqueness

Tensor products exist and are unique up to unique isomorphism.

proof: Existence: Let G be free abelian group generated by $\{m \cdot n,\ m \in M, n \in N\}$.

Module axioms:

- $(r+s)v = rv + sv$
- $(rs) \cdot v = r \cdot sv$
- $r(v+w) = r \cdot v + r \cdot w$
- $1v = v$

Let $H < G$ be generated by:

- $(m_1 + m_2) \cdot n - m_1 \cdot n - m_2 n$
- $m \cdot (n_1 + n_2) - m \cdot n_1 - m \cdot n_2$
- $mr \cdot n - m \cdot rn$

Define

- $M \otimes N = G/H$
- $i(v, w) = [v \cdot w]$
- $r \cdot [v \cdot w] := [rv \cdot w] \overset{\text{why?}}{=} [v \cdot rw]$

Let $v \otimes w := [v \cdot w] \in M \otimes N \ (= G/H)$

$M \otimes N$ is a module.

- $M \otimes N$ is an abelian group ✓
- $(r+s) \cdot (v \otimes w) \overset{\text{def}}{=} (r+s)v \otimes w$
$= (rv + sv) \otimes w$
$= rv \otimes w + sv \otimes w$
$= r \cdot (v \otimes w) + s \cdot (v \otimes w)$ $\square$

• $(rs)\cdot v\otimes w = rs.v\otimes w$

$= (r\cdot(s.v))\otimes w$

$= 0 = r\cdot(sv\otimes w)$

□

• $r\cdot(v_1\otimes w_1 + v_2\otimes w_2) = r\cdot v_1\otimes w_1 + r\cdot v_2\otimes w_2$

$= r\cdot(v_1\otimes w_1) + r\cdot(v_2\otimes w_2)$ □

• $1\cdot(v\otimes w) = 1\cdot v\otimes w = v\otimes w$ □

• i is bilinear:

$i(v_1+v_2, w) = (v_1+v_2)\otimes w$

$= v_1\otimes w + v_2\otimes w$ prop 1

$i(v, w_1+w_2) = v\otimes(w_1+w_2)$

$= v\otimes w_1 + v\otimes w_2$ prop 2

$i(rv, w) = rv\otimes w = r\cdot v\otimes w$

$$\begin{tikzcd} M\times N \arrow[d, "i"'] \arrow[dr, "p"] & \\ M\otimes N \arrow[r, dashed, "\hat p"] & P \end{tikzcd}$$

• $\hat p(i(v,w)) = p(v,w)$

• "By the lemma", $\hat p$ uniquely defined on $M\otimes N$

Uniqueness:

$$\begin{tikzcd} & M\times N \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\ A_1 \arrow[rr, dashed, shift left, "\hat i_2"] & & A_2 \arrow[ll, dashed, shift left, "\hat i_1"] \end{tikzcd}$$

• $i_2(v,w) = \hat i_2(\overbrace{i_1(v,w)}^{x})$

• $i_1(v,w) = \hat i_1(i_2(v,w))$

$= \hat i_2(\hat i_1(i_2(v,w)))$

$= (\hat i_2\circ\hat i_1)(\underbrace{i_2(v,w)}_{x})$

$\hat i_2\circ\hat i_1(x) = x$

Examples:

• $\mathbb{R}^n\otimes\mathbb{R}^m \simeq \mathbb{R}^{nm}$ $(= Mat_{\mathbb{R}}(n,m))$

• R a ring, $I, J \trianglelefteq R$

$$\begin{tikzcd} R/I\times R/J \arrow[d] \arrow[dr, "p"] & \\ R/I\otimes R/J \arrow[r, dashed, "\hat p"] & M = R/(I+J) \end{tikzcd}$$

$$\begin{tikzcd} R/I\times R/J \arrow[d, "i"'] \arrow[dr, "p"] & \\ R/I\otimes R/J \arrow[r, dashed, "\hat p"'] & R/(I+J) \end{tikzcd}$$

• $p([r_1],[r_2]) = [r_1 r_2]$ (well defined?)

• $\hat p$ surjective.

Ker $\hat p$?

$\hat p([r]\otimes[1]) = p([r],[1])$

$= [r] \overset{\text{suppose}}{=} [0]$

$\iff r\in I+J$

$\iff r = x+y,\ x\in I,\ y\in J$

$$[r]_I \otimes [1]_J = [x+y]_I \otimes [1]_J$$
$$= \underbrace{[x]_I \otimes [1]_J}_{0} + [y]_I \otimes [1]_J$$
$$= [y \cdot 1]_I \otimes [1]_J$$
$$= y[1]_I \otimes [1]_J$$
$$= [1]_I \otimes [y]_J = 0$$

Ker $\hat{p}$? Lemma: Every simple tensor has this form

$$\hat{p}([r] \otimes [1]) = p([r],[1])$$
$$= [r] \overset{\text{suppose}}{=} [0]$$
$$\Longleftrightarrow r \in I+J$$
$$\Longleftrightarrow r = x+y,\ x \in I,\ y \in J$$

$$\begin{tikzcd} R/I \times R/J \arrow[d, "i"'] \arrow[dr, "p"] & \\ R/I \otimes R/J \arrow[r, dashed, "\hat{p}"'] & R/(I+J) \end{tikzcd}$$

$p([r_1],[r_2])$
$= [r_1 r_2]$ (well defined?)
· $\hat{p}$ surjective.

$I=(a)$, $J=(b)$ $\quad R/(a) \otimes R/b \simeq R/((a)+(b)) = R/\gcd(a,b)$

Thm $(R\text{-mod}, \oplus, \otimes, 0, R)$ "forms a ring"

up to isomorphism:

(i) R is an identity for $\otimes$ $\rightarrow R \otimes_R M \cong M \cong M \otimes_R R$

(ii) 0 is an identity for $\oplus$ $\quad 0 \oplus M \cong M$

(iii) $\oplus$ and $\otimes$ are associative

(iv) $\otimes$ distributes over $\oplus$

(v) $\oplus$ is commutative

I will focus on the $\otimes$ parts

(i) $R \otimes_R M \cong M$

we will do this by showing M satisfies the universal property for $R \otimes_R M$, namely,

$$\begin{tikzcd} R \times M \arrow[r] \arrow[d, "f"'] & R \otimes_R M \arrow[dl, dashed, "\exists! \tilde{f}"] \\ N & \end{tikzcd}$$

If we have, $f: R \times M \to N$ R-bilinear, note $f(r,m) = r f(1,m) = f(1, rm)$

If we consider the map $\quad R \times M \to M$
$\quad (r,m) \mapsto rm$

$$\begin{tikzcd} R \times M \arrow[r, "{(r,m) \mapsto rm}"] \arrow[d, "f"'] & M \arrow[dl, "{\tilde{f}(m) = f(1,m)}"] \\ N & \end{tikzcd}$$

$f(1, rm) = f(r,m)$

$\tilde{f}$ is unique, R-linear, and makes the diagram commute, so (M, ℓ) satisfies the universal property. By uniqueness of tensor product, $M \cong R \otimes_R M$

$A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$

Pf

$$\begin{tikzcd} A \times (B \times C) \arrow[r] \arrow[d, "\cong"] & A \otimes (B \otimes C) \arrow[ddd, bend left] \\ (A \times B) \times C \arrow[d] & \\ (A \otimes B) \times C \arrow[d] & \\ (A \otimes B) \otimes C \arrow[uuur, bend right] & \end{tikzcd}$$

$(a,(b,c)) \mapsto (a \otimes b) \otimes c$

for each a, we get a bilinear map

$$\begin{tikzcd} B \times C \arrow[r] \arrow[d, "\ell_a"'] & B \otimes C \arrow[dl, "\exists! \tilde{\ell}_a"] \\ (A \otimes B) \otimes C & \end{tikzcd}$$

$\tilde{\ell}_{a+a'}(\sum b_i \otimes c_i) = \sum \tilde{\ell}_{a+a'}(b_i \otimes c_i)$
$\sum \ell_{a+a'}(b_i, c_i)$
so $\tilde{\ell}_a$ depends linearly on A

So, we get a bilinear map

$$\begin{tikzcd} A \times (B \otimes C) \arrow[r] \arrow[d, "\tilde{\ell}_a"'] & A \otimes (B \otimes C) \arrow[dl] \\ (A \otimes B) \otimes C & \end{tikzcd}$$

$(a, b \otimes c) \mapsto \tilde{\ell}_a(b \otimes c)$

we can also go the other way to get a map
$(A \otimes B) \otimes C \to A \otimes (B \otimes C)$.
$(a \otimes b) \otimes c \mapsto a \otimes (b \otimes c)$

aside if $f: M_1 \to M_2$ is a R-module hom. N is an R-module, there is a map

$N \otimes_R M_1 \to N \otimes_R M_2$
$n \otimes m_1 \mapsto n \otimes f(m_1)$

(generally, if $N_1 \xrightarrow{g} N_2$ is a map, $N_1 \otimes M_1 \to N_2 \otimes M_2$
$n_1 \otimes m_1 \mapsto g(n_1) \otimes f(m_1)$)

$$\begin{tikzcd} N \times M_1 \arrow[r] \arrow[d, "{(n,m) \mapsto (n, f(m))}"'] & N \otimes M_1 \arrow[d] \\ N \times M_2 \arrow[r] & N \otimes M_2 \end{tikzcd}$$

$A\otimes(B\oplus C)\cong(A\otimes B)\oplus(A\otimes C)$

First, $B\to B\oplus C$, $C\to B\oplus C$

we get $A\otimes B\to A\otimes(B\oplus C)$, $A\otimes C\to A\otimes(B\oplus C)$

By universal property of $\oplus$, we get

$$\begin{tikzcd}
A\times B \arrow[rr] \arrow[dr] & & A\otimes B\oplus A\otimes C \arrow[d] & & A\times C \arrow[ll] \arrow[dl] \\
& A\otimes B \arrow[ur] \arrow[r] & A\otimes(B\oplus C) & A\otimes C \arrow[l] \arrow[ul] &
\end{tikzcd}$$

$$\begin{tikzcd}
A\times(B\oplus C) \arrow[r] \arrow[d] & A\otimes(B\oplus C) \arrow[dl] \\
(A\otimes B)\oplus(A\otimes C) &
\end{tikzcd}$$

$$(a\otimes b,\ a\otimes c)$$
$$\downarrow\ \uparrow$$
$$(a\otimes(b,c))$$

$\ell_a: B\to(A\otimes B)\oplus A\otimes C\xleftarrow{r_a}C$

$\ell_a\oplus r_a: B\oplus C\to(A\otimes B)\oplus(A\otimes C)$

Summarize use the two universal properties in two different orders.

$a\mapsto(a,0)$ $\qquad$ $b\mapsto(0,b)$

$$\begin{tikzcd}
& A\oplus B \arrow[d, "\exists!\ \text{hom.}"] & \\
A \arrow[ur, bend left] \arrow[r, "\text{hom}"'] & P & B \arrow[ul, bend right] \arrow[l, "\text{hom}"]
\end{tikzcd}$$